Light Amplification by Stimulated Emission of Radiation - Exercise set

Project for the Course on Dynamical Systems

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1 Purpose and problem formulation

As described in the project description, the dynamical model of the laser is given by the Maxwell-Bloch equations:

$$\begin{aligned} \dot{E} &= -\gamma_1 E + c_1 P \\ \dot{P} &= -\gamma_2 P + c_2 E D \\ \dot{D} &= -\gamma_3 (D - \lambda) - c_3 E P \end{aligned} \tag{1}$$

where $E(t)$ is the energy, $P(t)$ the medium polarization, $D(t)$ the population inversion at time $t \geq 0$. The constants c_1, c_2 and c_3 are all positive; γ_1, γ_2 and γ_3 are (positive) decay rates of energy, medium polarization and population inversion, respectively. Furthermore, λ is the energy pumping parameter that may take either positive or negative values.

The aim of the project is to investigate lasers of different classes. A *class A* laser is a laser in which the coefficients $\gamma_2, \gamma_3 \gg \gamma_1$. A laser of *class B* is defined by the property that $\gamma_2 \gg \gamma_1, \gamma_3$. Class B lasers are the most common ones and occur for example as semiconducting lasers. For *class C* lasers we have $\gamma_1 > \gamma_2 + \gamma_3$. Class C lasers are less often used for reasons that will become clear in this project. From each laser class, we consider a laser with the physical parameters as specified in Table 1. For each of these lasers, λ is viewed as a pumping parameter that controls the actual behavior of the system.

Laser class	γ_1	γ_2	γ_3	c_1	c_2	c_3
A	0.1	2	3	0.25	0.2	1
B	0.1	10	0.25	1	0.5	1
C	1	0.1	0.25	1	0.1	1

Table 1: Physical laser parameters

2 Questions

Stability and symmetry

For the common class B lasers the parameter γ_2 is much larger than γ_1 and γ_3. This means that the atomic polarization P decays much more rapidly than the energy E and the population inversion D. In this case, it is common to consider the quasi-static approximation $\dot{P} = 0$ in the Maxwell-Bloch equations (1).

1. Given this approximation, express P as function of D and E and derive a second order dynamic model in the variables D and E. (This substitution procedure is called an *adiabatic elimination*).
2. Determine the fixed point(s) of this second order approximate model and classify their stability properties.
3. Draw the bifurcation diagram of the approximate model in view of variations in the pumping parameter λ when ranging over $-\infty < \lambda < +\infty$. Classify the type of bifurcation(s) that occur in this system.

We next continue with the full order model (1).

4. Prove that the system admits symmetry in the sense that if (E, P, D) is a solution to (1), then so is $(-E, -P, D)$.
5. Determine, for arbitrary values of the pumping parameter λ, the fixed points of the class B laser specified in Table 1. Find for this laser the Jacobian $J(E, P, D)$ of the full order model (1) and linearize the system at its fixed point(s). Use the linearized system to determine all possible phase portraits near the fixed point(s) of the nonlinear system. In particular, determine the stability properties of the fixed point(s) and try to assess the stable and/or unstable manifolds of the fixed points.
6. Show that the non-lasing state $E^* = 0$ loses its stability if λ is larger than some critical threshold value λ_{crit} of the pumping parameter. Determine this value and classify the type of bifurcation at this laser threshold.
7. Prove that for all type B lasers the bifurcation value λ_{crit} of item 6 exactly coincides with the bifurcation value of the approximate model that is discussed in item 3.

Positive invariance

Suppose that λ is fixed and consider for an arbitrary point (E, P, D) in the state space of the model (1) the quadratic function

$$H(E, P, D) = \frac{a}{2}E^2 + \frac{b}{2}P^2 + \frac{c}{2}(D - \lambda)^2$$

where a, b and c are positive constants. Any such function defines an ellipsoidal level set

$$\mathcal{H}_\rho := \{(E, P, D) \in \mathbb{R}^3 \mid H(E, P, D) \leq \rho\}$$

centered at the point $(0, 0, \lambda)$. Here, $\rho \geq 0$ is a parameter. For small values of ρ, the *volume* V of a small volumetric element $\mathcal{H}_\rho$ in the flow evolves with time according to the differential equation $\dot{V} = \int_V \text{div}(f)\text{d}V$ where div is the *divergence operator* and f is the vector field defined by the Maxwell-Bloch equations (1).

8. Prove that for suitable positive constants a, b and c, the ellipsoidal level set $\mathcal{H}_\rho$ is positive invariant for the system (1) for some $\rho \geq 0$.
9. Show that small volumetric elements $V = \mathcal{H}_\rho$ in the phase space of (1) contract exponentially fast when time evolves. Discuss what this implies about the solutions of (1).

Simulation

Next, we simulate solutions of the equations (1) to get more insight in the phase portrait and the physical behavior of the system. Consider the physical parameters specified in Table 1. The time unit is in micro-seconds μsec.

10. Implement the system in Matlab and compute numerical solutions $(E(t), P(t), D(t))$ on a sufficiently large time interval and for arbitrary initial conditions for each of the laser classes given in Table 1. Make a 2D-plot of the energy and medium polarization $(E(t), P(t))$ in the laser as functions of time t. For the class B laser, compare the result with a simulation of a solution $(E(t), D(t))$ of the first order approximate model of item 1. For the class C laser, discuss what you observe if λ ranges in the interval $20 \leq \lambda \leq 26$.

Chaos and Lyapunov exponents

To understand the behavior of the class C laser, let us focus on the parameter values in Table 1 with $20 \leq \lambda \leq 26$. We will verify whether solutions show *sensitive dependence on initial conditions* in this range of pumping parameters. Let $w(t) = (E(t), P(t), D(t))$ denote an arbitrary trajectory of the class C laser that is given here.

11. Choose a small $\delta_0 > 0$ and simulate trajectories w' and w'' (just primes, no differentiation is meant here!) of the class C laser with initial conditions w'_0 and w''_0 that are at most δ_0 apart. Verify the hypothesis that the distance $\delta(t) = w'(t) - w''(t)$ grows exponentially in some (finite) time window according to

    $$\|\delta(t)\| \approx |\delta_0|e^{Lt}$$

    for some Lyapunov exponent $L > 0$. Do this by either plotting $\log(\|\delta(t)\|/|\delta_0|)$ against t or by applying an algorithm for the estimation of L.

12. Put your notebook at work and try to plot the Lyapunov exponent L as function of the pumping parameter λ in the interval $(20, 26)$ for a very fine grid of parameter values. Can you decide when chaos occurs on the basis of this plot?

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