Chaotic Behavior Mechanisms in the Bloch-Maxwell Laser Model

Avadis Hacinliyan * **Orhan Ozgur Aybar** **
Ilknur Kusbeyzi *** **E. Eray Akkaya** ****

* *Department of Information Systems and Technologies, Yeditepe University, Kayisdagi, Istanbul, Turkey (e-mail: ahacinliyan@yeditepe.edu.tr).*
** *Department of Information Systems and Technologies, Yeditepe University, Kayisdagi, Istanbul, Turkey (e-mail: oaybar@yeditepe.edu.tr)*
Departments of Mathematics, Gebze Institute of Technology, Turkey
*** *Department of Information Systems and Technologies, Yeditepe University, Kayisdagi, Istanbul, Turkey (e-mail: ikusbeyzi@yeditepe.edu.tr)*
Departments of Mathematics, Gebze Institute of Technology, Turkey
**** *Department of Physics, Yeditepe University, Kayisdagi, Istanbul, Turkey (e-mail: eeakkaya@gmail.com)*

Abstract: Regions of the full parameter space for which chaotic behavior in laser models based on the Maxwell Bloch equation occurs is studied. The range in the parameter space where a maximal positive Lyapunov exponent occurs corresponds to that for Type B lasers. Hopf bifurcation is confirmed for the He-Ne case.

Keywords:
Maxwell-Bloch equations, Lyapunov stability, Hopf Bifurcation, Chaotic behaviour, Optical Nonlinearity, Magnetic Resonance

The pure Bloch model is known to be a linear model, so that chaotic behavior is not ordinarily expected. However, for strong coupling of the AC component, trajectories that are dense in the phase space can result in seemingly chaotic like regions in the trajectory plot, as reported by (6). Chaotic behavior as evidenced by positive Lyapunov exponents requires the addition of nonlinear terms. One way to add this nonlinearity is to couple the Bloch system ro the Maxwell equations in the presence of material media. Such systems serve as laser models including the He-Neon laser. Transition to chaos is based on considering the resonance between the laser cavity frequency and atomic cavity TEM modes. The Maxwell-Bloch equations as given by (1) and (7) involve the coupling of the fundamental cavity mode, E with the collective variables P and Δ, that represent the atomic polarization and the population inversion and are represented by the following equations.

$$\dot{E} = -kE + gP$$
$$\dot{P} = -\gamma_{\perp} P + gE\Delta$$
$$\dot{\Delta} = -\gamma_{\parallel}(\Delta - \Delta_o) - 4gPE$$

For the parameter values $k = \sigma$, $\gamma_{\perp} = g^2/k = 1$, $g^2\Delta_o/k = r$, $\gamma_{\parallel} = b$, the system can be transformed into the Lorenz system about the equilibrium point $\Delta = \Delta_o$ by setting $x = E$, $y = gP/k$, $z = \Delta_o - \Delta$. The meaning of the parameters in the original equations are given by Arrechi, while $\sigma,\ r,\ b$ are the Lorenz parameters.

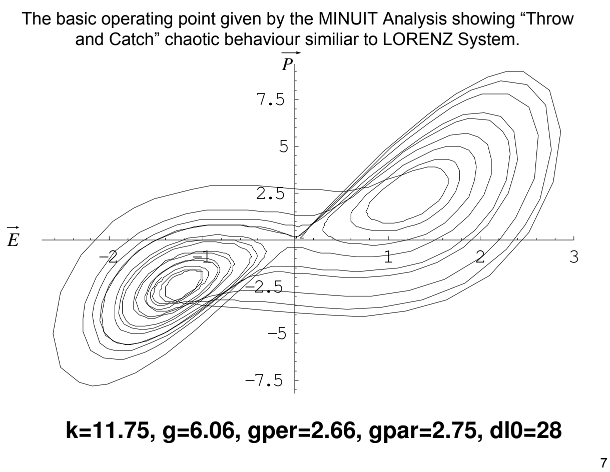

Fig. 1. The Strange Attractor for basic operating point with given parameters

The Maxwell-Bloch equations have more parameters than the Lorenz system; this justifies a more detailed parameter study. Chaotic behavior has been experimentally observed in laser systems (1; 7) and studying the range of parame-

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ters for which chaos can occur is important for controlling chaos in possible applications. A parameter study that would reveal the range of parameters for which chaotic behavior characterized by the well known invariant, a positive maximal Liapunov exponent is of interest and such a study using the Wolf algorithm.(11) will be reported. At the very early stages the system has a spectrum consisting of negative or zero Liapunov Exponents for values of the parameter k less than 6. Whenever k exceeds 6.75, chaotic behavior, as evidenced by the presence of a positive Liapunov exponent occurs for a range of parameter values, between $k = 6.75$ and $k = 11.50$, $\gamma_{\perp} = 2.66$, $\gamma_{\|} = 2.75$, $\Delta_o = 28$, $g = 6.06$. These values are close to the values suggested from a correspondance with the Lorenz model and the closest laser system that they characterize would be a Type B laser.

The prohibitively long computation time for a full parameter space study led us to locate the above mentioned basic operating point by minimizing the negative of the maximal Lyapunov exponent using the CERN MINUIT package. The dominant chaotic behavior exhibited by the system is the so-called Throw and Catch mechanism, in which two linearized equilibrium points with a pair of complex conjugate eigenvalues with slightly positive real parts surround an unstable third equilibrium point with eigenvalues $(+--)$. The chaotic behaviour is predictably similiar to the LORENZ System in Figure 1(8; 13).

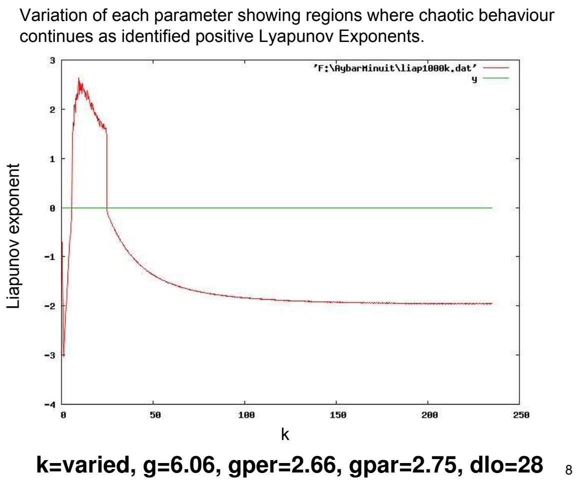

Fig. 2. Variation of parameter k vs Liapunov Exponents Regimes

The variation of the parameter k shows the behaviour of the different regimes of the Liapunov Exponents both positive and negative values in Figure 2. For different values of parameter k, the shape of the attractor are plotted in Figure 3. For values of the parameter k that are greater than 23, the system reverts to a regime with three negative Liaupunov Exponents (3; 9).

The variation of parameter g for the values less than 3, $k = 11.75$, $\gamma_{\perp} = 2.66$, $\gamma_{\|} = 2.75$, $\Delta_o = 28$ does not give rise to positive Liapunov exponents. For the values greater than 3, the strange attractor is plotted for different values of g until the sink reverts to an attractor with unique chaotic properties(8).In Figure 4 and 5 the limit cycle is

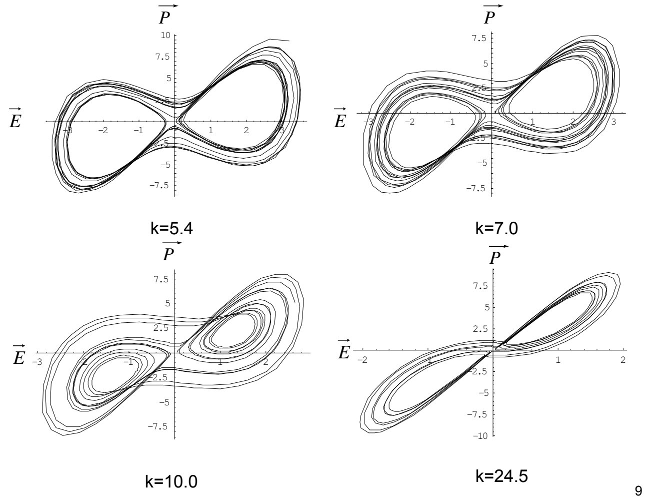

Fig. 3. The Strange Attractor for varied parameter k

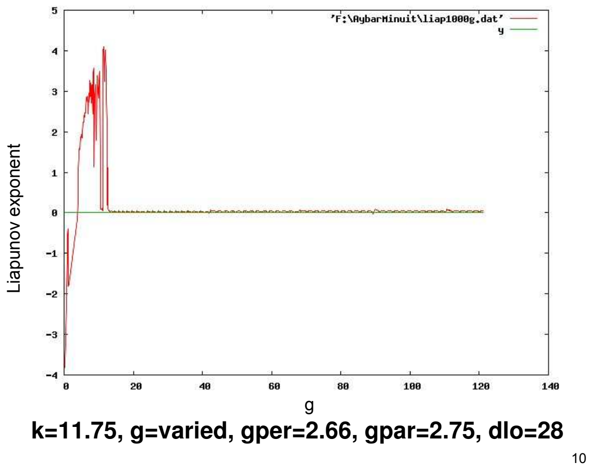

Fig. 4. Variation of parameter g vs Liapunov Exponents Regimes

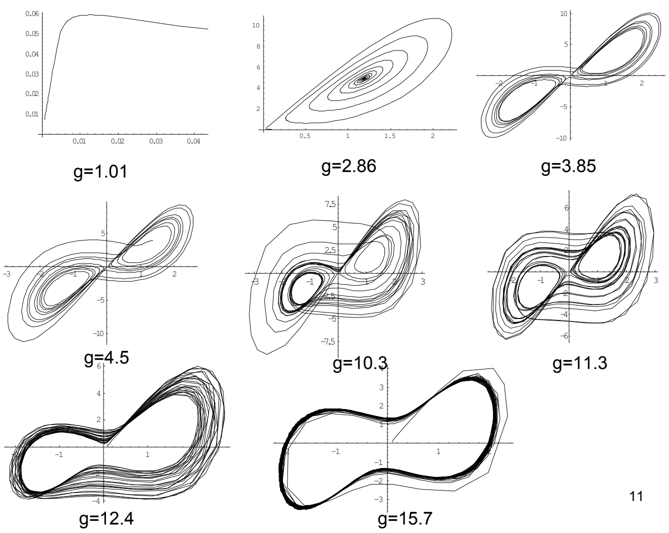

Fig. 5. The Strange Attractor for varied parameter g

observed as orbitally stable with the negative Liapunov coefficient, and therefore the bifurcation is supercritical.

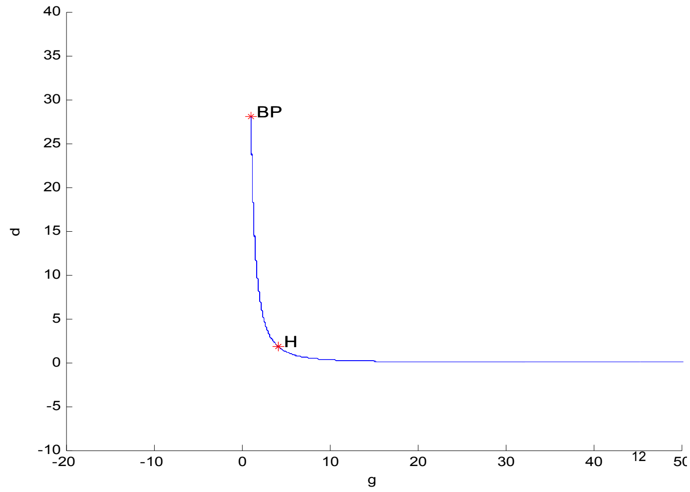

Fig. 6. Bifurcation Plot of variation of parameter g vs d

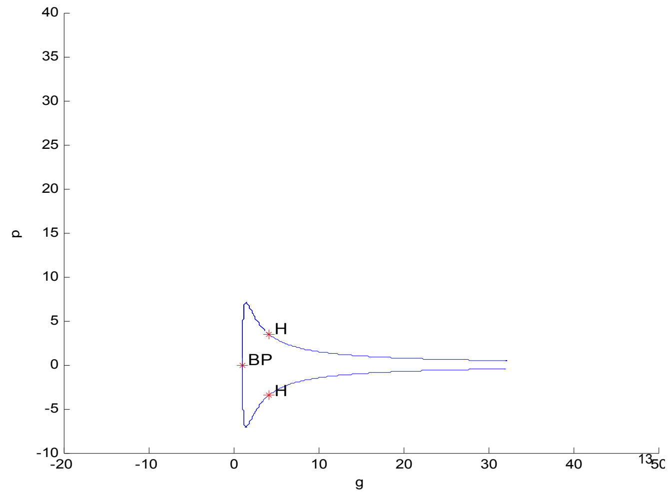

Fig. 7. Bifurcation Plot of variation of parameter g vs p

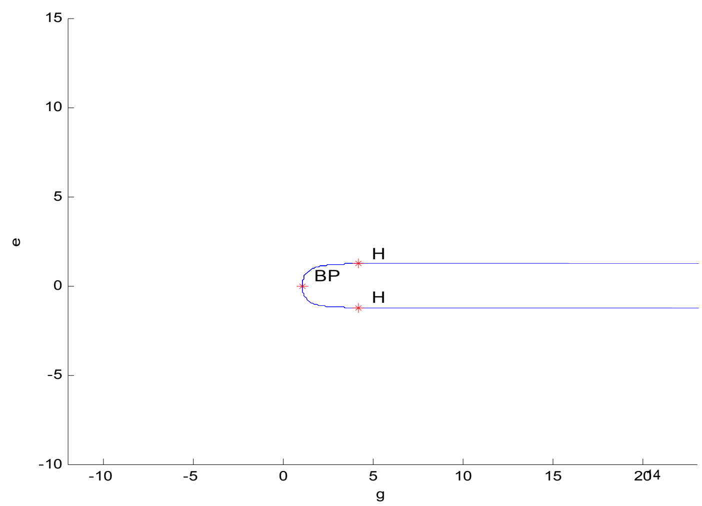

Fig. 8. Bifurcation Plot of variation of parameter g vs e

In Hopf bifurcation, a small amplitude limit cycle bifurcates from the fixed point. The first Liapunov coefficient is important for Hopf bifurcation since it determines whether the bifurcation supercritical or subcritical. The validity of the Hopf bifurcation approximation is investigated numerically by comparing the bifurcation diagrams of the original laser equations and the generated data in Figure 6,7 and 8. Determination of the analytical expressions for all branches of periodic solutions from Hopf bifurcation points are studied for numerically different cases of different regimes. In these figures, The attractor is changing from a limit cycle to a saddle point between these points where Hopf and Branch points are observed for e, d and p values(2; 4; 5).

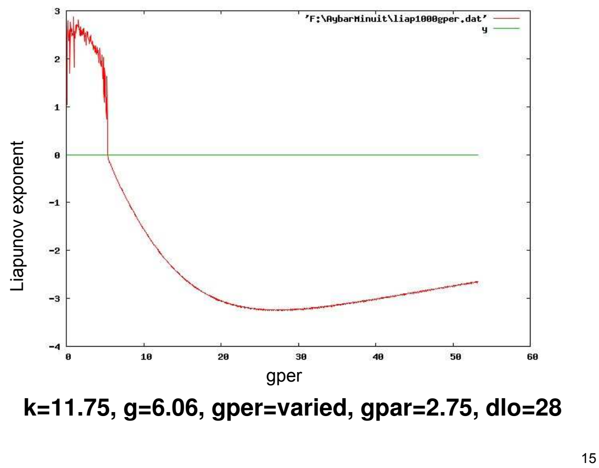

Fig. 9. Variation of parameter gper vs Liapunov Exponents Regimes

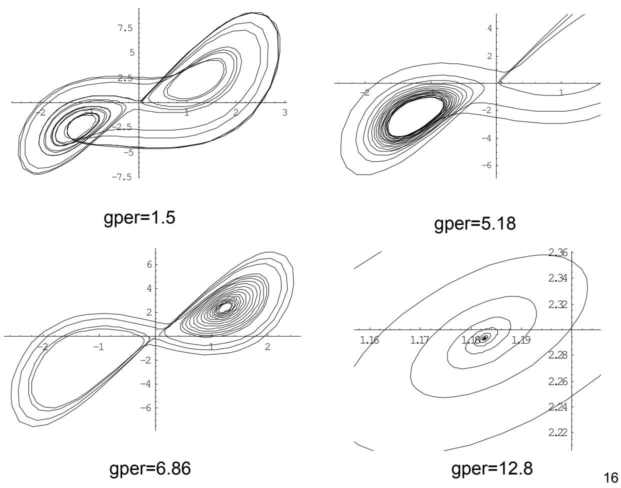

Fig. 10. The Strange Attractor for varied parameter gper

The variation of the parameter gper behaves like the parameter gpar. At the initial stages, both of the paramters have a maximal positive Liapunov Exponents then after a value, that is nearly the same for both parameters, negative Liapunov Exponents are obtained in Figure 9 and 11. The plots of the attractors are consistent with the

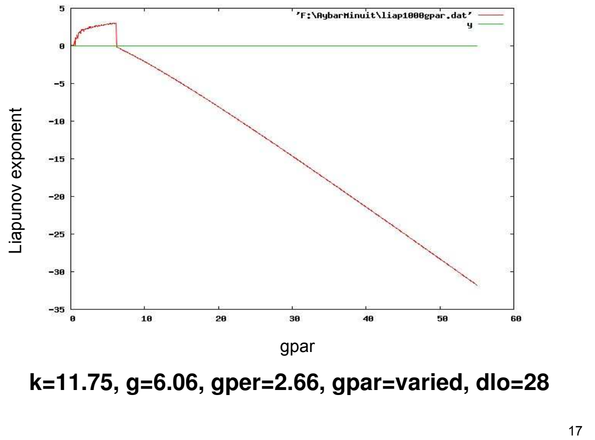

Fig. 11. Variation of parameter gpar vs Liapunov Exponents Regimes

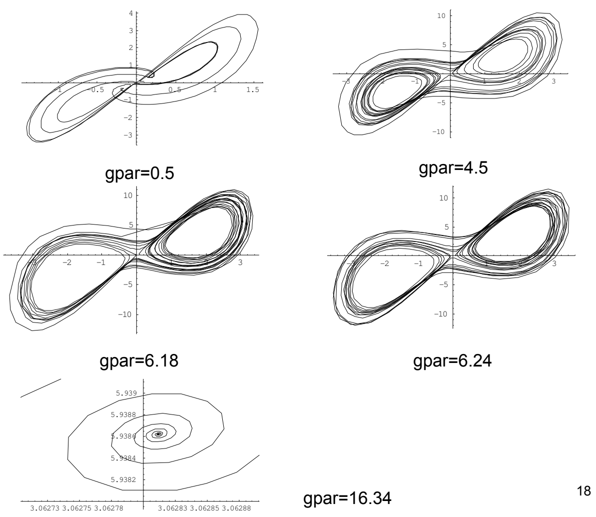

Fig. 12. The Strange Attractor for varied parameter gpar

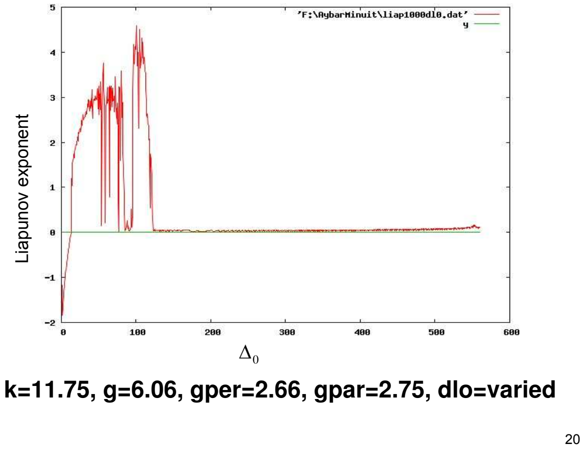

Fig. 13. Variation of parameter dl0 vs Liapunov Exponents Regimes

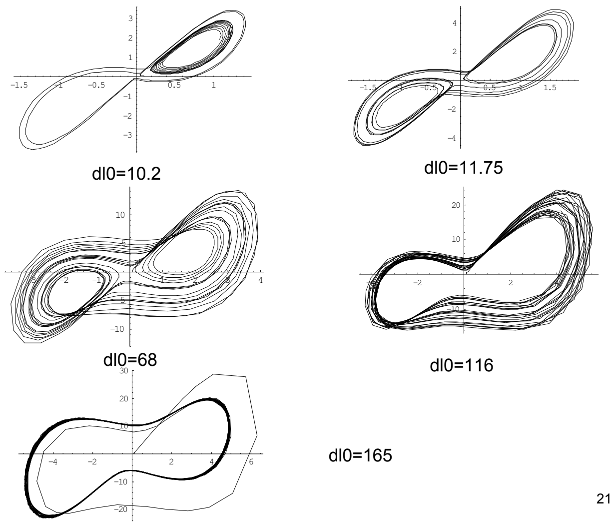

Fig. 14. The Strange Attractor for varied parameter dl0

positive and negative Liapunov Exponents. In Figure 10 and 12, a family of limit cycles are observed in the negative Liapunov Exponent regime where the Hopf point occurs .

The parameter dl0 does have the same scale with respect to the other parameters. The initial movement is starting with positive Liapunov Exponents then it goes into sink part after 120. In Figures 13 and 14, it can be easily observed that behaviour of the strange attractor is changing into the saddle point where the sink appears. In Figures 15 and 16 the branch and Hopf points around $dl0$=15 can be observed.

In conclusion, the Maxwell-Bloch system that is usually used as a model for the Helium-Neon Laser has been studied. This system has a rich structure as a function of its parameters. We have shown that this system exhibits several types of instability. The Liapunov spectrum is compatible with the eigenvalues of the linearized system, which consist of a negative value plus a pair of complex conjugate eigenvalues with slightly negative real part. These parameters correspond to far infrared lasers where Lorenz like chaos

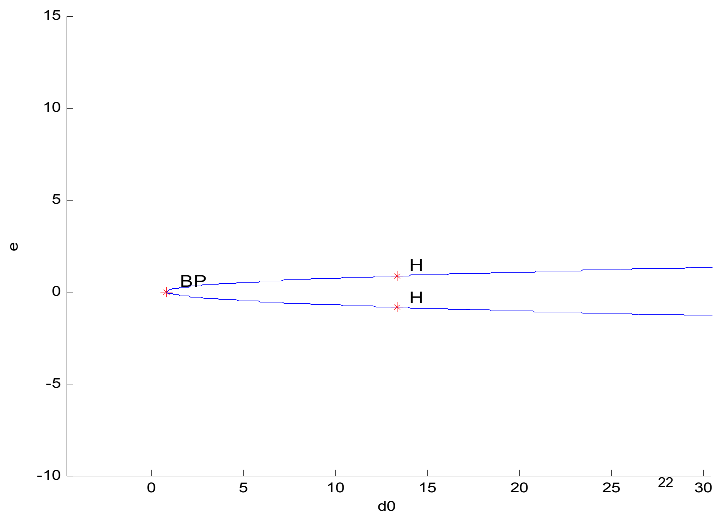

Fig. 15. Bifurcation Plot of variation of parameter dl0 vs e

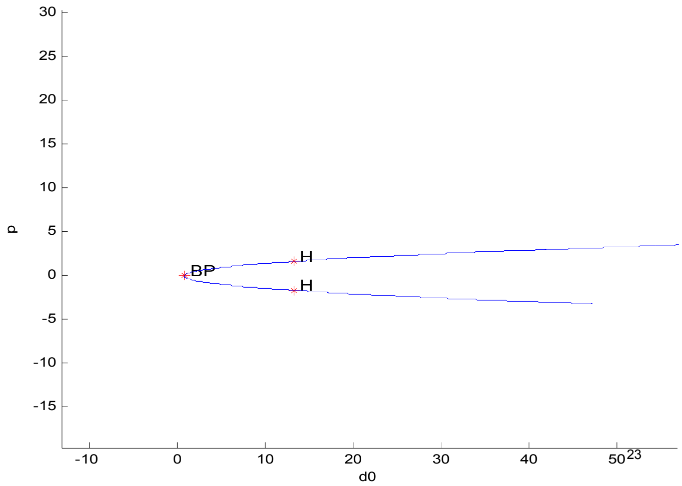

Fig. 16. Bifurcation Plot of variation of parameter dl0 vs p

has been observed(10; 12). The bifurcation mechanism that characterizes the possible transition to chaos from this operating point is studied by the MATCONT(6; 10) package and Hopf bifurcation is identified in several instances. Emphasis has been placed on one operating point where chaotic nature as evidenced by a positive maximal Liapunov exponent is shown while the change of the varying parameters are plotted for each item. However, this behavior is sensitive to variations in the parameters and rapidly changes to steady state. As each parameter is varied, transition to/from chaotic behavior can be seen. We have identified several instances of Hopf bifurcation at this operating point, the operating point is within limits of the parameter range for which chaotic behaviour has been studied in the He-Ne laser(2; 3; 4; 5).

REFERENCES

[1] F. T. Arecchi. Chaos and Generalized Multistability in Quantum Optics. In A.F. Round, editor, *Physica Scripta*, 85-92, 1985.

[2] J. Chrostowski. Noisy Bifurcation in acousto-optic bistability. *Physics Letters A*, 26, 1982.

[3] J. P. Eckmann, S. O. Kamphorst, D. Ruelle. Ergodic Theory of Chaos and Strange Attractors. *Reviews of Modern Physics*, Vol. 57, pages 617–656, 1985.

[4] Thomas Erneux, Braza P. *Singular Hopf Bifurcation to unstable periodic soluttions in NMR Laser. Physics Letters A*, Vol. 40,5, 1989.

[5] Thomas Erneux,Kozyreff G, editor. *Nearly Vertical Hopf Bifurcation for a Passively Q-Switched Microchip Laser*. Vol. 101, 2000.

[6] A. Hacinliyan, N. Z. Perdahçı, G. Şahin, H. A. Yıldırım. *Transforming to Chaos by Normal Forms.* CMDS 10, Shoresh, Israel, 15–20, 2003.

[7] H.Haken. *Light, Volume 2*, 1985.

[8] P. Grassberger, editor. *Non-linear time sequence Analysis.* Int. J. Bifurcation and Chaos, pages 521, 1991.

[9] H. Kantz, editor. *A robust method to estimate the maximal Lyapunov exponent of a time series. Physics Letters A*, Vol. 185, pages 77–87, 1994.

[10] A. Dhooge, W. Govaerts, Yu.A. Kuznetsov, W. Mestrom, A.M. Riet and B. Sautois,. *MATCONT and CL_MATCONT: Continuation toolboxes in matlab. Ghent and Utrecht Universities Preprint*, 2006.

[11] A. Wolf, J. B. Swift, H. L. Swinney and J. A. Vastano. *Routes to chaotic emission in a cw He- Ne Laser. Physics Letters A*, Vol. 28,2, 1983.

[12] A. Wolf, J. B. Swift, H. L. Swinney and J. A. Vastano. *Determining Lyapunov exponents from a time series. Physica D*, pages 285–317, 1985.

[13] A. Umur. *Local Nonperturbative Methods for Algebraic Calculation of Liapunov Exponents. M. S. Thesis, Department of Physics, Boğaziçi University*, 1997.