7. Cubic Splines.

NUMERICAL ANALYSIS. Prof. Y. Nishidate (323-B, nisidate@u-aizu.ac.jp)
http://web-int.u-aizu.ac.jp/∼nisidate/na/

INTERPOLATING CUBIC SPLINES

A polynomial of degree $(n-1)$ can be made to pass through n given data points. In many cases, the resulting curve is not a smooth curve through the points because such a function would not only include the "noise" in the data, but would also very likely oscillate considerably between the data points. A smooth curve can be obtained by connecting data points with a piecewise spline curves which ensures continuity of the first and second derivatives of the function.

Cubic spline interpolant

Suppose that (x_0, y_0), (x_1, y_1), …, (x_N, y_N) are $N+1$ points where $x_0 < x_1 < ... < x_N$. The function $S(x)$ is called a **cubic spline** if there exists N cubic polynomials $S_k(x)$ with the properties:

$S(x) = S_k(x) = s_{k0} + s_{k1}(x-x_k) + s_{k2}(x-x_k)^2 + s_{k3}(x-x_k)^3$ for $k = 0, 1, ..., N-1$

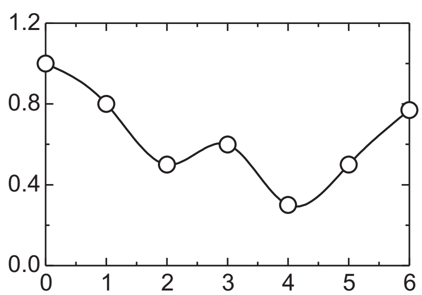
$S(x_k) = y_k$ for $k = 0, 1, ..., N$. The spline goes through each data point.

$S_k(x_{k+1}) = S_{k+1}(x_{k+1})$ for $k = 0, 1, ..., N-2$. The spline forms a continuous function.

$S_k^{'}(x_{k+1}) = S_{k+1}^{'}(x_{k+1})$ for $k = 0, 1, ..., N-2$. The spline forms a smooth function.

$S_k^{''}(x_{k+1}) = S_{k+1}^{''}(x_{k+1})$ for $k = 0, 1, ..., N-2$. The second derivative is continuous.

Example of the spline interpolation is shown below



Cubic spline relations

Since $S(x)$ is piecewise cubic, its second derivative $S^{''}(x)$ is piecewise linear on the interval $[x_0, x_N]$. The linear Lagrange interpolation polynomial gives the following representation for $S^{''}(x) = S_k^{''}(x)$:

$$S_k^{''}(x) = S^{''}(x_k)\frac{x - x_{k+1}}{x_k - x_{k+1}} + S^{''}(x_{k+1})\frac{x - x_k}{x_{k+1} - x_k}$$

After introduction of notation

$$\begin{aligned} m_k &= S^{''}(x_k) \\ m_{k+1} &= S^{''}(x_{k+1}) \\ h_k &= x_{k+1} - x_k \end{aligned}$$

it is possible to rewrite Lagrange interpolation as follows:

$$\begin{aligned} &S_k^{''}(x) = \frac{m_k}{h_k}(x_{k+1} - x) + \frac{m_{k+1}}{h_k}(x - x_k) \\ &k = 0, 1, ..., N-1 \end{aligned}$$

Integrating of this relation two times introduces two constant of integration:

$$\begin{aligned} &S_k(x) = \frac{m_k}{6h_k}(x_{k+1} - x)^3 + \frac{m_{k+1}}{6h_k}(x - x_k)^3 \\ &+p_k(x_{k+1} - x) + q_k(x - x_k) \end{aligned}$$

Substituting x_k and x_{k+1} in the above equations and using the values $y_k = S_k(x_k)$ and $y_{k+1} = S_k(x_{k+1})$ we have the following equations for integration constants p_k and q_k:

$$y_k = \frac{m_k}{6}h_k^2 + p_k h_k \qquad y_{k+1} = \frac{m_{k+1}}{6}h_k^2 + q_k h_k$$

After substitution we arrive at the following equation for the cubic spline function $S_k(x)$:

$$\begin{aligned} &S_k(x) = \frac{m_k}{6h_k}(x_{k+1} - x)^3 + \frac{m_{k+1}}{6h_k}(x - x_k)^3 \\ &+ \left(\frac{y_k}{h_k} - \frac{m_k h_k}{6}\right)(x_{k+1} - x) + \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1}h_k}{6}\right)(x - x_k) \end{aligned}$$

Note that current representation of the spline function involves only the second derivatives m_k.

To find values of the second derivatives we can use the first derivatives of the spline function:

$$\begin{aligned} &S_k^{'}(x) = -\frac{m_k}{2h_k}(x_{k+1} - x)^2 + \frac{m_{k+1}}{2h_k}(x - x_k)^2 \\ &-\left(\frac{y_k}{h_k} - \frac{m_k h_k}{6}\right) + \left(\frac{y_{k+1}}{h_k} - \frac{m_{k+1}h_k}{6}\right) \end{aligned}$$

Evaluating this expression at the point x_k yields

$$\begin{aligned} S_k^{'}(x_k) &= -\frac{m_k}{3}h_k - \frac{m_{k+1}}{6}h_k + d_k \\ d_k &= \frac{y_{k+1} - y_k}{h_k} \end{aligned}$$

To get the derivative for $S_{k-1}^{'}(x_k)$ it is possible to replace k by $k-1$ and to evaluate it at x_k

$$S_{k-1}^{'}(x_k) = \frac{m_k}{3}h_{k-1} + \frac{m_{k-1}}{6}h_{k-1} + d_{k-1}$$

From the above two expression and from the continuity of the first derivative we obtain an important relation connecting m_{k-1}, m_k and m_{k+1}

$$\begin{aligned} &h_{k-1}m_{k-1} + 2(h_{k-1} + h_k)m_k + h_k m_{k+1} = u_k \\ &u_k = 6(d_k - d_{k-1}) \qquad k = 1, 2, ..., N-1 \end{aligned}$$

Construction of cubic splines

The above relations produce $N-1$ equations for the unknown $(N+1)$ second derivatives m_k. Hence two additional equations should be supplied. They are necessary to eliminate m_0 from equation 1 and m_N from equation $(N-1)$. Several possible end-point conditions can be specified. The most widely used are conditions:

$$m_0 = m_N = 0$$

The spline curve with such end-point conditions is called "natural cubic spline".

For the natural cubic spline we can write the following equation system for the unknown second derivative parameters

$$\begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & & ... & & \\ & & a_{N-2} & b_{N-2} & c_{N-2} \\ & & & a_{N-1} & b_{N-1} \end{bmatrix} \begin{Bmatrix} m_1 \\ m_2 \\ ... \\ m_{N-2} \\ m_{N-1} \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ ... \\ u_{N-2} \\ u_{N-1} \end{Bmatrix}$$

$$a_k = h_{k-1} \qquad b_k = 2(h_{k-1} + h_k) \qquad c_k = h_k$$

Finally we have the following expression for the piecewise cubic natural spline:

$$S_k(x) = s_{k0} + s_{k1}(x - x_k) + s_{k2}(x - x_k)^2 + s_{k3}(x - x_k)^3$$
$$s_{k0} = y_k \qquad s_{k1} = d_k - \frac{h_k(2m_k + m_{k+1})}{6}$$
$$s_{k2} = \frac{m_k}{2} \qquad s_{k3} = \frac{m_{k+1} - m_k}{6h_k}$$

Example. Find the natural cubic spline that passes through points (0, 0), (1, 0.5), (2, 2) and (3, 1.5).

Solution. Let us compute the quantities necessary for the equation system:

$$\begin{aligned} h_0 &= h_1 = h_2 = 1 \\ d_0 &= (y_1 - y_0)/h_0 = 0.5 \\ d_1 &= (y_2 - y_1)/h_1 = 1.5 \\ d_2 &= (y_3 - y_2)/h_2 = -0.5 \\ u_1 &= 6(d_1 - d_0) = 6.0 \\ u_2 &= 6(d_2 - d_1) = -12.0 \end{aligned}$$

The equation system for determining the second derivatives of the spline function has the following appearance:

$$\begin{bmatrix} 2(h_0 + h_1) & h_1 \\ h_1 & 2(h_1 + h_2) \end{bmatrix} \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

After substituting of parameters values the system is:

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -12 \end{Bmatrix}$$

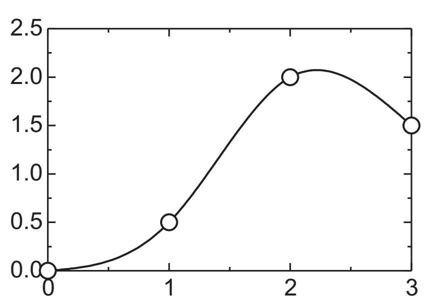
The solution of the equation system is:

$$m_1 = 2.4 \qquad m_2 - 3.6$$

Using this values of the second derivatives we can write down the following expressions for the piecewise spline function

$$\begin{aligned} S_0(x) &= 0.4x^3 + 0.1x, & 0 \le x \le 1 \\ S_1(x) &= -(x-1)^3 + 1.2(x-1)^2 + 1.3(x-1) + 0.5, & 1 \le x \le 2 \\ S_2(x) &= 0.6(x-2)^3 - 1.8(x-2)^2 + 0.7(x-2) + 2, & 2 \le x \le 3 \end{aligned}$$

The natural cubic spline is shown below.



B-SPLINES

Cubic B-splines are similar to the ordinary (interpolating) splines, in that a separate cubic is used for each interval. As previously B-splines provide continuity for the function and its first and second derivatives. However, the B-splines need not pass through any point of the set which is used for its definition.

The B-splines can be presented in terms of parametric equations with a parameter u.

Given the points $p_i = (x_i, y_i),\ i = 0, 1, ..., N$, the cubic B-spline for the interval $(p_i, p_{i+1}),\ i = 1, 2, ..., N-1$ is

$$B_i(u) = \sum_{k=-1}^{2} b_k p_{i+k}$$
$$b_{-1} = \frac{(1-u)^3}{6}$$
$$b_0 = \frac{u^3}{2} - u^2 + \frac{2}{3}$$
$$b_1 = -\frac{u^3}{2} + \frac{u^2}{2} + \frac{u}{2} + \frac{1}{6}$$
$$b_2 = \frac{u^3}{6} \qquad 0 \le u \le 1$$

The coefficients b_k serve as a basis and do not change as we move from one set of four points to the next. They can be considered weighting factors applied to the coordinates of a set of four points.

Explicit equations for x and y are as follows:

$$x(u) = \frac{1}{6}(1-u)^3 x_{i-1} + \frac{1}{6}(3u^3 - 6u^2 + 4)x_i + \frac{1}{6}(-3u^3 + 3u^2 + 3u + 1)x_{i+1} + \frac{1}{6}u^3 x_{i+2}$$

$$y(u) = \frac{1}{6}(1-u)^3 y_{i-1} + \frac{1}{6}(3u^3 - 6u^2 + 4)y_i + \frac{1}{6}(-3u^3 + 3u^2 + 3u + 1)y_{i+1} + \frac{1}{6}u^3 y_{i+2}$$

B-splines provide continuity for the function and its first and second derivatives in the following way:

$$\begin{aligned} B_i(1) &= B_{i+1}(0) = \frac{p_i + 4p_{i+1} + p_{i+2}}{6} \\ B_i'(1) &= B_{i+1}'(0) = \frac{-p_i + p_{i+2}}{2} \\ B_i''(1) &= B_{i+1}''(0) = p_i - 2p_{i+1} + p_{i+2} \end{aligned}$$

One problem with B-splines is that each spline segment requires four points for its definition. With such procedure we can construct B-splines B_1 through B_{N-2}. We need B_0 and B_{N-1}. One simple way to do this is to add fictitious points at each end (minimum is one point at each end; two points at each end make B-spline approximation smoother).

The matrix formulation for cubic B-splines can be convenient. Here it is:

$$B_i(u) = \frac{1}{6}\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{Bmatrix} p_{i-1} \\ p_i \\ p_{i+1} \\ p_{i+2} \end{Bmatrix}$$

Below is the example of B-spline approximation: