On the Supersingular Locus of the $\mathrm{GU}(2, n-2)$ Shimura Variety

Ryosuke Shimada

Department of Mathematics, the University of Hong Kong

Slides are available at
https://ryosukeshimada.github.io

Motivation

- Simple descriptions of the supersingular (or basic) locus of Shimura varieties have been used towards applications in number theory (the Kudla-Rapoport program, Zhang's Arithmetic Fundamental Lemma, the Tate conjecture for certain Shimura varieties, etc.).
- The study of the perfection of the basic (or supersingular) locus is essentially reduced to the study of the affine Deligne-Lusztig variety (ADLV) via the Rapoport-Zink uniformization.

Table of contents

Definition of ADLVs with Iwahori level

From now on, we will use the following notation.

$$F : \text{a non-archimedean local field with uniformizer } t.$$
$$G/\mathcal{O}_F : \text{an unramified connected reductive group.}$$
$$T \subseteq B : \text{a max. torus, } B : \text{a Borel subgroup.}$$
$$\sigma : \text{the Frobenius automorphism of } L/F.$$
$$L = \widehat{F^{un}}, J_b = \{g \in G(L) \mid g^{-1}b\sigma(g) = b\}.$$
$$I \subseteq G(\mathcal{O}_L) : \text{the standard Iwahori subgroup associated to } T \subset B \subset G.$$
$$W_0 : \text{the Weyl group, } \widetilde{W} : \text{the Iwahori-Weyl group} \cong W_0 \ltimes X_*(T).$$

Definition

For $w \in \widetilde{W}$ and $b \in G(L)$, the *affine Deligne-Lusztig variety* $X_w(b)$ is a scheme locally (perfectly) of finite type$/\overline{\mathbb{F}}_q$ defined as

$$X_w(b) = \{gI \in G(L)/I \mid g^{-1}b\sigma(g) \in IwI\} \subset G(L)/I.$$

Review of Coxeter groups

Let W be a group generated by a subset $S = \{s_1, s_2, \dots, s_r\}$.

Definition

(W, S) is called a *Coxeter system* if there exist $2 \leq m(i,j) \leq \infty$ s.t.

$$W = \langle S \mid s_i^2 = 1, \forall i \text{ and } (s_i s_j)^{m(i,j)} = 1, \forall i \neq j \rangle.$$

A word of min. length among words of $w \in W$ is called *reduced*.
$\ell(w) :=$ the length of any reduced word of w.
$\mathrm{supp}(w) :=$ the subset of S occurring in some reduced word of w.
$v \leq w \Leftrightarrow v$ can be obtained as a subword of a reduced word of w.

(W_0, S) is a Coxeter system with $S =$ the set of simple reflections.

$W_a \subseteq \widetilde{W}$: the affine Weyl group, $\tilde{S} =$ the set of simple affine reflections.

Then $(W_a, \tilde{S})$ is a Coxeter system and $\widetilde{W} \cong W_a \rtimes \Omega$.
Here $\Omega = \{w \in \widetilde{W} \mid \ell(w) = 0\}$.

Definition of closed ADLVs

For $v, w \in W_a$, $\tau, \tau' \in \Omega$, $v\tau \leq w\tau' \overset{\text{def}}{\Longleftrightarrow} v \leq w$ and $\tau = \tau'$.
Let $\mathrm{Adm}(\mu) = \{w \in \widetilde{W} \mid w \leq t^{w_0\mu}$ for some $w_0 \in W_0\}$.

${}^S\widetilde{W}$: the set of minimal length representatives for the cosets in $W_0\backslash\widetilde{W}$.

Set ${}^S\mathrm{Adm}(\mu) = \mathrm{Adm}(\mu) \cap {}^S\widetilde{W}$.

Definition

The *closed affine Deligne-Lusztig variety* in $G(L)/G(\mathcal{O}_L)$ is the closed reduced $\overline{\mathbb{F}}_q$-subscheme defined as

$$X_\mu(b) := \bigsqcup_{w \in {}^S\mathrm{Adm}(\mu)} \pi(X_w(b)),$$

where $\pi\colon G(L)/I \to G(L)/G(\mathcal{O}_L)$ is the projection.

- J_b acts on $X_w(b)$ and $X_\mu(b)$ by left multiplication.
- (Main theorem) In the $\mathrm{GU}(2, n-2)$, p inert case, $X_\mu(b)$ for basic b has a simple description.

Group-theoretic data in the case of GL_n

Example

Let $G = \mathrm{GL}_n$.

T : the torus of diagonal matrices.

B : the subgroup of upper triangular matrices.

$I =$ the inverse image of B^{op} under $G(\mathcal{O}_L) \to G(\overline{\mathbb{F}}_q), \varpi \mapsto 0$

$$W_0 \cong \mathfrak{S}_n, \quad S = \{(1\ 2), \ldots, (n-1\ n)\}.$$

$$X_*(T) \cong \{t^\lambda = \begin{pmatrix} t^{m_1} & & \\ & \ddots & \\ & & t^{m_n} \end{pmatrix} \mid \lambda = (m_1, \ldots, m_n) \in \mathbb{Z}^n\}.$$

Thus $X_*(T) \cong \mathbb{Z}^n$ and $\widetilde{W} \cong \mathfrak{S}_n \ltimes \mathbb{Z}^n$, $\tilde{S} = S \cup \{(1\ n)t^{(-1,0,\ldots,0,1)}\}$.
Set $\tau = t^{(1,0,\ldots,0)} s_1 s_2 \cdots s_{n-1}$. Then $\Omega = \{\tau^m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$.
Set $s_0 = (1\ n)t^{(-1,0,\ldots,0,1)}$ and $s_i = (i\ i+1)$. Then $\tau s_i \tau^{-1} = s_{i+1}$.

Summary of part 1

- $(G, \mu, b) \rightsquigarrow X_\mu(b) := \coprod_{w \in {}^S\mathrm{Adm}(\mu)} \pi(X_w(b))$.
- The study of the perfection of the basic locus is reduced to the study of $X_\mu(b)$ for basic b.

From now on, we pass to the perfection even in the equal characteristic case for simplicity.

Elements with spherical σ-support

Let τ_μ be the image of t^μ under the projection $\widetilde{W} = W_a \rtimes \Omega \to \Omega$. Then τ_μ is basic and $\pi(X_w(\tau_\mu))$ corresponds to the intersection of a global EO stratum with the basic Newton stratum.

Let $w \in W_a\tau, \tau \in \Omega$. Set $\mathrm{supp}_\sigma(w) = \bigcup_{m\in\mathbb{Z}}(\tau\sigma)^m\mathrm{supp}(w\tau^{-1})$.

Theorem (Görtz-He)

If $W_{\mathrm{supp}_\sigma(w)}$ is finite, then $X_w(\tau) = \sqcup_{J_\tau/J_\tau\cap P_{\mathrm{supp}_\sigma(w)}} Y(w)$, where $Y(w)$ is a (classical) Deligne-Lusztig variety in $P_{\mathrm{supp}_\sigma(w)}/I$.

- $Y(w)$ is an irreducible component of $X_w(\tau)$.
- Similar description is true for $\pi(X_w(\tau))$.
- The global EO stratum associated to w is contained in the basic locus if and only if $W_{\mathrm{supp}_\sigma(w)}$ is finite.

Example ($G = \mathrm{GL}_2$)

$X_{s_1}(1) = \bigsqcup_{G(F)/G(\mathcal{O}_F)} \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q), \quad X_1(1) = \bigsqcup_{G(F)/G(F)\cap I}\{\mathrm{pt}\}.$

Fully Hodge-Newton decomposable pairs

Set ${}^S\mathrm{Adm}(\mu)_{\neq\emptyset} = \{w \in {}^S\mathrm{Adm}(\mu) \mid X_w(\tau_\mu) \neq \emptyset\}$.
Then $X_\mu(\tau_\mu) = \bigsqcup_{w \in {}^S\mathrm{Adm}(\mu)_{\neq\emptyset}} \pi(X_w(\tau_\mu))$.
Let $\mathrm{depth}(G, \mu)$ be a certain rational number determined by (G, μ).

Theorem (Görtz-He-Nie)

The pair (G, μ) is fully Hodge-Newton decomposable if and only if the following equivalent conditions are satisfied:

1. *The cocharacter μ is minute $\overset{def}{\iff} \mathrm{depth}(G, \mu) \leq 1$.*
2. *$W_{\mathrm{supp}_\sigma(w)}$ is finite for every $w \in {}^S\mathrm{Adm}(\mu)_{\neq\emptyset}$.*

- The classification of fully HN decomposable cases is known.

Theorem (Görtz-He-Nie)

If (G, μ) is fully Hodge-Newton decomposable, then $X_\mu(\tau_\mu)$ is naturally a disjoint union of Deligne-Lusztig varieties.

Examples of fully Hodge-Newton decomposable cases

Example

The fully Hodge-Newton decomposable cases contain the following cases which have been studied in the context of Shimura varieties:

- The Siegel case of genus 2, which has been studied by Katsura-Oort and Kaiser.
- The $\mathrm{GU}(1, n-1)$, p split case, which has been studied by Harris-Taylor.
- The $\mathrm{GU}(1, n-1)$, p inert case, which has been studied by Vollaard-Wedhorn.
- The $\mathrm{GU}(2, 2)$, p inert case, which has been studied by Howard-Pappas.

Example ($G = \mathrm{GL}_2$)

We have ${}^{S}\mathrm{Adm}((1,-1))_{\neq\emptyset} = \{s_0, 1\}$ and $X_{(1,-1)}(1) = \bigsqcup_{G(F)/G(\mathcal{O}_F)}(\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)) \sqcup \bigsqcup_{G(F)/G(\mathcal{O}_F)} \{\mathrm{pt}\}.$

Beyond fully Hodge-Newton decomposable cases

Example ($G = \mathrm{GL}_2$)

$X_{s_1 t^{(-r,r)}}(1) \cong \coprod_{G(F)/G(\mathcal{O}_F)} (\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)) \times \mathbb{A}^{r-1}$.

Theorem (S.)

Let $G = \mathrm{GL}_n$ and let $\mu \in X_*(T)_+$. If $\mathrm{depth}(G,\mu) < 2$, then $X_\mu(\tau_\mu)$ is naturally a disjoint union of the product of a Deligne-Lusztig variety and a finite-dimensional affine space.

- $\mathrm{depth}(\mathrm{GL}_n, (1^{(2)}, 0^{(n-2)})) = 2 - \frac{4}{n} < 2$.

Theorem (S.-Takamatsu)

Let $G = \mathrm{GSp}_{2n}$ and let $\mu = (1^{(n)}, 0^{(n)})$. If $n = 3, 4$, then $X_\mu(\tau_\mu)$ is naturally a disjoint union of the product of a Deligne-Lusztig variety and a finite-dimensional affine space.

- We have $\mathrm{depth}(\mathrm{GSp}_{2n}, (1^{(n)}, 0^{(n)})) = \frac{n}{2}$.

Summary of part 2

- If $W_{\mathrm{supp}_\sigma(w)}$ is finite, then $X_w(\tau)$ is a union of DLVs.
- (G, μ) is fully HN decomposable $\Leftrightarrow \mathrm{depth}(G, \mu) \leq 1$.
- If this is the case, then $X_\mu(\tau_\mu)$ is a union of DLVs.
- The $\mathrm{depth}(G, \mu) \leq 2$ condition also seems to imply a simple geometric structure on $X_\mu(\tau_\mu)$.

It is easy to check that $\mathrm{depth}(G, \mu) = 2$ for $\mathrm{GU}(2, n-2)$.

Non-emptiness criterion

Let $p := \widetilde{W} \to W_0$ be the projection.
For $w \in \widetilde{W}$, $\mathrm{LP}(w) := \{\textit{length positive elements}\} \subseteq W_0$ (omitted).

Theorem (Görtz-He-Nie, Lim, Schremmer)

Assume that the Dynkin diagram of G is σ-connected. Let $w \in W_a\tau$. Then

$$X_w(\tau) \neq \emptyset \Leftrightarrow \text{(i) } W_{\mathrm{supp}_\sigma(w)} \text{ is finite, or,}$$
$$\text{(ii) } \forall v \in \mathrm{LP}(w), \mathrm{supp}_\sigma(\sigma^{-1}(v^{-1})p(w)v) = S$$

- (i) uses $\widetilde{W} \cong W_a \rtimes \Omega$.
- (ii) uses $\widetilde{W} \cong W_0 \ltimes X_*(T)$.
- This theorem can be seen as the non-emptiness criterion of the intersection of a global EO stratum with the basic Newton stratum in Shimura varieties.

Deligne-Lusztig reduction

Proposition (Deligne-Lusztig, Görtz-He)

Let $w \in \widetilde{W}$ and let $s \in \tilde{S}$ be a simple affine reflection.

1. If $\ell(sw\sigma(s)) = \ell(w)$, then $X_w(b) \cong X_{sw\sigma(s)}(b)$.
2. If $\ell(sw\sigma(s)) = \ell(w) - 2$, then there exists a decomposition $X_w(b) = X_1 \sqcup X_2$ such that
   - X_1 is open and there exists a J_b-equivariant morphism $X_1 \to X_{sw}(b)$, which is a Zariski-locally trivial $\mathbb{G}_m^{\mathrm{pfn}}$-bundle.
   - X_2 is closed and there exists a J_b-equivariant morphism $X_2 \to X_{sw\sigma(s)}(b)$, which is a Zariski-locally trivial $\mathbb{A}^{1,\mathrm{pfn}}$-bundle.

- (General strategy) Reduce the length of w so that $W_{\mathrm{supp}_\sigma(w)}$ is finite.

Main theorem

Let G be the unramified unitary group of degree n. Then $\widetilde{W}_G \cong \widetilde{W}_{\mathrm{GL}_n} \cong \mathfrak{S}_n \ltimes \mathbb{Z}^n$ and $\sigma(s_i) = s_{n-i}$ (by setting $s_n = s_0$).
Set $\mu = (0^{(n-2)}, -1, -1)$, $w_{k,l} := t^\mu s_{n-2}s_{n-3}\cdots s_k s_{n-1}s_{n-2}\cdots s_l$.
Then ${}^S\mathrm{Adm}(\mu) = \{w_{k,l} \mid 1 \leq k < l \leq n\}$ and $\ell(w_{k,l}) = k + l - 3$.
${}^S\mathrm{Adm}(\mu)_{\mathrm{DL}} := \{w \in {}^S\mathrm{Adm}(\mu) \mid \mathrm{supp}_\sigma(w) \neq \tilde{S}\} \subseteq {}^S\mathrm{Adm}(\mu)_{\neq\emptyset}$.
${}^S\mathrm{Adm}(\mu)_{\neq\mathrm{DL}} := {}^S\mathrm{Adm}(\mu)_{\neq\emptyset} \setminus {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}}$.

Theorem (S.)

We have ${}^S\mathrm{Adm}(\mu)_{\mathrm{DL}} = \{w_{k,l} \mid k = 1 \text{ or } l \leq \frac{n+2}{2}\}$. *Moreover,* $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq\mathrm{DL}}$ *if and only if* $3 \leq k < \frac{n+2}{2} < l \leq n-1$ *and one of the following conditions is satisfied:*

1. k *is odd and* $k + l \leq n + 2$.
2. $l \equiv n - 1 \pmod 2$ *and* $k + l \geq n + 3$.

- If $w \in {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}}$, then $\pi(X_w(\tau_\mu))$ is a union of DLVs.
- This was known for $n \leq 5$ (by Howard-Pappas, ABFGGN).

Main theorem

For $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq \mathrm{DL}}$,

$$w'_{k,l} = \begin{cases} w_{k-2,l} & (k+l \leq n+2) \\ w_{k,l-2} & (k+l \geq n+4) \\ w_{k-1,l-1} & (k+l = n+3) \end{cases}$$

Then $w'_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq \emptyset}$.

Theorem (S.)

Let $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq \mathrm{DL}}$. Then $\pi(X_{w_{k,l}}(\tau_\mu))$ is a Zariski-locally trivial $\mathbb{A}^{1,\mathrm{pfn}}$-bundle over $\pi(X_{w'_{k,l}}(\tau_\mu))$. In particular, $X_\mu(\tau_\mu)$ is naturally a disjoint union of iterated fibrations over (irreducible) Deligne-Lusztig varieties, whose fibers are all $\mathbb{A}^{1,\mathrm{pfn}}$.

- The stabilizers of the strata can also be explicitly described.
- The irreducible components of $X_\mu(\tau_\mu)$ were studied by Fox, Howard and Imai in a different way.
- If $n = 4$, then this description coincides with Howard-Pappas.

Example of the main theorem

$$X_\mu(\tau_\mu) = \bigsqcup_{w \in {}^S\mathrm{Adm}(\mu)} \pi(X_w(\tau_\mu)) = \bigsqcup_{w \in {}^S\mathrm{Adm}(\mu)_{\neq\emptyset}} \pi(X_w(\tau_\mu))$$

$$\begin{aligned} {}^S\mathrm{Adm}(\mu) &= \{w_{k,l} \mid 1 \leq k < l \leq n\} \\ {}^S\mathrm{Adm}(\mu)_{\neq\emptyset} &= {}^S\mathrm{Adm}(\mu)_{\neq\mathrm{DL}} \sqcup {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}} \\ {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}} &= \{w_{k,l} \mid k = 1 \text{ or } l \leq \tfrac{n+2}{2}\} \end{aligned}$$

${}^S\mathrm{Adm}(\mu)_{\neq\emptyset}$ for $n = 13$ consists of the following elements:

	$w_{7,12}$		$w_{7,10}$		$w_{7,8}$						
	$w_{6,12}$		$w_{6,10}$			$w_{6,7}$					
	$w_{5,12}$		$w_{5,10}$	$w_{5,9}$	$w_{5,8}$	$w_{5,7}$	$w_{5,6}$				
	$w_{4,12}$					$w_{4,7}$	$w_{4,6}$	$w_{4,5}$			
	$w_{3,12}$	$w_{3,11}$	$w_{3,10}$	$w_{3,9}$	$w_{3,8}$	$w_{3,7}$	$w_{3,6}$	$w_{3,5}$	$w_{3,4}$		
						$w_{2,7}$	$w_{2,6}$	$w_{2,5}$	$w_{2,4}$	$w_{2,3}$	
$w_{1,13}$	$w_{1,12}$	$w_{1,11}$	$w_{1,10}$	$w_{1,9}$	$w_{1,8}$	$w_{1,7}$	$w_{1,6}$	$w_{1,5}$	$w_{1,4}$	$w_{1,3}$	$w_{1,2}$

Example of the main theorem

$$X_\mu(\tau_\mu) = \bigsqcup_{w \in {}^S\mathrm{Adm}(\mu)} \pi(X_w(\tau_\mu)) = \bigsqcup_{w \in {}^S\mathrm{Adm}(\mu)_{\neq\emptyset}} \pi(X_w(\tau_\mu))$$

$$\begin{aligned} {}^S\mathrm{Adm}(\mu) &= \{w_{k,l} \mid 1 \leq k < l \leq n\} \\ {}^S\mathrm{Adm}(\mu)_{\neq\emptyset} &= {}^S\mathrm{Adm}(\mu)_{\neq\mathrm{DL}} \sqcup {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}} \\ {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}} &= \{w_{k,l} \mid k = 1 \text{ or } l \leq \tfrac{n+2}{2}\} \end{aligned}$$

${}^S\mathrm{Adm}(\mu)_{\neq\emptyset}$ for $n = 13$ consists of the following elements:

$$\begin{tikzcd}
 & w_{7,12} \arrow[rr] & & w_{7,10} \arrow[rr] & & w_{7,8} \arrow[dd] & & & & & & \\
 & w_{6,12} \arrow[rr] & & w_{6,10} \arrow[dr] & & & w_{6,7} & & & & & \\
 & w_{5,12} \arrow[rr] & & w_{5,10} \arrow[dd] & w_{5,9} \arrow[dd] & w_{5,8} \arrow[dd] & w_{5,7} & w_{5,6} & & & & \\
 & w_{4,12} \arrow[dr] & & & & & w_{4,7} & w_{4,6} & w_{4,5} & & & \\
 & w_{3,12} \arrow[dd] & w_{3,11} \arrow[dd] & w_{3,10} \arrow[dd] & w_{3,9} \arrow[dd] & w_{3,8} \arrow[dd] & w_{3,7} & w_{3,6} & w_{3,5} & w_{3,4} & & \\
 & & & & & & w_{2,7} & w_{2,6} & w_{2,5} & w_{2,4} & w_{2,3} & \\
w_{1,13} & w_{1,12} & w_{1,11} & w_{1,10} & w_{1,9} & w_{1,8} & w_{1,7} & w_{1,6} & w_{1,5} & w_{1,4} & w_{1,3} & w_{1,2}
\end{tikzcd}$$

Proof: Emptiness

Recall that $w_{k,l} := t^{\mu} s_{n-2}s_{n-3}\cdots s_k s_{n-1}s_{n-2}\cdots s_l$. Then

$$X_w(\tau_\mu) = \emptyset \Leftrightarrow \text{(i) } \mathrm{supp}_\sigma(w) = \tilde{S}, \text{ and,}$$
$$\text{(ii) } \exists v \in \mathrm{LP}(w), \mathrm{supp}_\sigma(\sigma^{-1}(v^{-1})p(w)v) \neq S.$$

Here $p(w_{k,l}) = s_{n-2}s_{n-3}\cdots s_k s_{n-1}s_{n-2}\cdots s_l$, $p(w_{k,l})^{-1} \in \mathrm{LP}(w_{k,l})$.

Lemma

Assume $1 \leq k < l \leq n$ satisfies one of the following conditions:

1. $\frac{n+2}{2} \leq k < l \leq n$.
2. $l \equiv n \pmod 2$, $\frac{n+3}{2} \leq l$ and $n-l+2 \leq k \leq \frac{n+1}{2}$.
3. k is even, $k \leq \frac{n-1}{2}$ and $\frac{n+3}{2} \leq l \leq n-k+2$.

Then $X_{w_{k,l}}(b) = \emptyset$.

Proof. (i) $v = p(w_{k,l})^{-1}$.
(ii)$v =$
$p(w_{k,l})^{-1}(s_{n-l}s_l\cdots s_2 s_{n-2})(s_{n-l-1}s_{l+1}\cdots s_3 s_{n-3})\cdots(s_{\frac{n-l}{2}+1}s_{\frac{n+l}{2}-1})$.

Proof: Key lemma

We write $w \approx_\sigma w'$ if $\exists$ a sequence $w = w_0, w_1, \ldots, w_k = w'$ in $\widetilde{W}$ such that $\forall i$, $\exists t_i \in \tilde{S}$; $w_i = t_i w_{i-1}\sigma(t_i)$ and $\ell(t_i w_{i-1}\sigma(t_i)) = \ell(w_i)$.

Lemma

Assume that $3 \leq k \leq \frac{n+1}{2}$ and $\frac{n+3}{2} \leq l \leq n-1$.

(i) If k is odd and $k + l \leq n+2$, then $\exists s \in \tilde{S}, w' \in \widetilde{W}$ such that $w_{k,l} \approx_\sigma w'$, $sw'\sigma(s) \approx_\sigma w_{k-2,l}$ and $sw' \approx_\sigma w_{k-1,l}$.

(ii) If $l \equiv n-1$ and $k + l \geq n+4$, then $\exists s \in \tilde{S}, w' \in \widetilde{W}$ such that $w_{k,l} \approx_\sigma w'$, $sw'\sigma(s) \approx_\sigma w_{k,l-2}$ and $sw' \approx_\sigma w_{k,l-1}$.

(iii) If $l \equiv n-1$ and $k + l = n+3$, then $\exists s \in \tilde{S}, w' \in \widetilde{W}$ such that $w_{k,l} \approx_\sigma w'$, $sw'\sigma(s) \approx_\sigma w_{k-1,l-1}$ and $sw' \approx_\sigma w_{k,l-1}$.

Proof. Recall that $\sigma(s_i) = s_{n-i}$ and $\tau_\mu s_i \tau_\mu^{-1} = s_{i-2}$ $(\tau_\mu = w_{1,2})$.
For $w_{3,l} = s_0 \cdots s_{l-3} s_{n-1} s_0 \tau_\mu$, $w' := s_{n-2} s_0 w_{3,l} \sigma(s_0 s_{n-2})$, $s := s_{n-1}$.

Summary of part 3

- In the $\mathrm{GU}(2, n-2)$-case, $X_\mu(\tau_\mu)$ is naturally a union of iterated fibrations over Deligne-Lusztig varieties.

Future works are

- Comparison with the Fox-Howard-Imai's description.
- Systematic study of the $\mathrm{depth}(G, \mu) \leq 2$ cases.
- Description before taking perfection.
- Application towards number theory.