On a group-theoretic approach to the supersingular locus of Shimura varieties

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Abstract

In this paper, we give a review of a group-theoretic approach to the supersingular (or basic) locus of Shimura varieties via affine Deligne-Lusztig varieties. Görtz-He-Nie proved that the supersingular locus of fully Hodge-Newton decomposable Shimura varieties admits a simple description. Recently, new simple descriptions have been discovered beyond fully Hodge-Newton decomposable cases. After explaining the work by Görtz-He-Nie, we will introduce these new cases including the Siegel case of genus 3 or 4, which is based on a joint work with Teppei Takamatsu.

1 Introduction

Shimura varieties have been used, with great success, towards applications in number theory. There are many such applications based on the study of integral models and their reductions. It is known that in some cases, the supersingular locus of the reduction of a Shimura variety admits a simple description. For example, Vollaard-Wedhorn [46] described the supersingular locus of the Shimura variety of $\mathrm{GU}(1, n-1)$ at an inert prime as a union of (classical) Deligne-Lusztig varieties. Also in the $\mathrm{GU}(2,2)$-case, Howard-Pappas [25] proved the existence of a similar description. After [46] and [25], Görtz, He and Nie classified the cases where the supersingular locus is naturally a union of Deligne-Lusztig varieties, called the *fully Hodge-Newton decomposable* cases (cf. [13], [15], [16]). The studies by Görtz, He and Nie are based on the fact that the study of the perfection of the supersingular locus can be reduced to a study of an affine Deligne-Lusztig variety via the Rapoport-Zink uniformization.

Recently, new simple descriptions have been discovered in some cases which are not fully Hodge-Newton decomposable. For example, the case of $\mathrm{GU}(2, n-2)$ at an inert prime and the Siegel case of genus 3 or 4 are such cases. In this paper, we summarize the results of these new cases, explaining a group-theoretic approach via affine Deligne-Lusztig varieties. Before this, we also give a review of the works by Görtz, He and Nie.

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2 Preliminaries

From now on, we sometimes drop the adjective "perfect" for notational convenience.

2.1 Notation

Let F be a non-archimedean local field with finite residue field $\mathbb{F}_q$ of prime characteristic p, and let L be the completion of the maximal unramified extension of F. Let σ denote the Frobenius automorphism of L/F. Further, we write $\mathcal{O}$ (resp. $\mathcal{O}_F$) for the valuation ring of L (resp. F). Finally, we denote by ϖ a uniformizer of F and by v_L the valuation of L such that $v_L(\varpi) = 1$.

Let G be an unramified connected reductive group over $\mathcal{O}_F$. Let $B \subset G$ be a Borel subgroup and $T \subset B$ a maximal torus in B, both defined over $\mathcal{O}_F$. For a cocharacter $\mu \in X_*(T)$, let ϖ^μ be the image of $\varpi \in \mathbb{G}_m(F)$ under the homomorphism $\mu\colon \mathbb{G}_m \to T$.

Let $\Phi = \Phi(G,T)$ denote the set of roots of T in G. We denote by Φ_+ (resp. Φ_-) the set of positive (resp. negative) roots distinguished by B. Let Δ be the set of simple roots and $\Delta^\vee$ be the corresponding set of simple coroots. Let $X_*(T)$ be the set of cocharacters, and let $X_*(T)_+$ be the set of dominant cocharacters.

The Iwahori-Weyl group $\tilde{W} = \tilde{W}_G$ is defined as the quotient $N_{G(L)}T(L)/T(\mathcal{O})$. This can be identified with the semi-direct product $W_0 \ltimes X_*(T)$, where W_0 is the finite Weyl group of G. We denote the projection $\tilde{W} \to W_0$ by p. We have a length function $\ell\colon \tilde{W} \to \mathbb{Z}_{\geq 0}$ given as

$$\ell(u\varpi^\lambda) = \sum_{\alpha\in\Phi_+, u\alpha\in\Phi_-} |\langle\alpha,\lambda\rangle + 1| + \sum_{\alpha\in\Phi_+, u\alpha\in\Phi_+} |\langle\alpha,\lambda\rangle|,$$

where $u \in W_0$ and $\lambda \in X_*(T)$.

Let $S \subset W_0$ denote the subset of simple reflections, and let $\tilde{S} \subset \tilde{W}$ denote the subset of simple affine reflections. We often identify Δ and S. The affine Weyl group W_a is the subgroup of $\tilde{W}$ generated by $\tilde{S}$. Then we can write the Iwahori-Weyl group as a semi-direct product $\tilde{W} = W_a \rtimes \Omega$, where $\Omega \subset \tilde{W}$ is the subgroup of length 0 elements. Moreover, $(W_a, \tilde{S})$ is a Coxeter system. We extend the Bruhat order $\leq$ on W_a to $\tilde{W}$ in the usual way: for $w, w' \in W_a$, $\tau, \tau' \in \Omega$, $w\tau \leq w'\tau'$ if and only if $w \leq w'$ and $\tau = \tau'$. For any $J \subseteq \tilde{S}$, let ${}^J\tilde{W}$ (resp. $\tilde{W}^J$) be the set of minimal length elements for the cosets in $W_J\backslash\tilde{W}$ (resp. $\tilde{W}/W_J$), where W_J denotes the subgroup of $\tilde{W}$ generated by J. We write ${}^J\tilde{W}^{J'}$ for ${}^J\tilde{W} \cap \tilde{W}^{J'}$.

For $w \in W_a$, we denote by $\mathrm{supp}(w) \subseteq \tilde{S}$ the set of simple affine reflections occurring in every (equivalently, some) reduced expression of w. Note that $\tau \in \Omega$ acts on $\tilde{S}$ by conjugation. We define the σ-support $\mathrm{supp}_\sigma(w\tau)$ of $w\tau$ as the smallest $\tau\sigma$-stable subset of $\tilde{S}$ which contains $\mathrm{supp}(w)$. We call an element $w\tau \in W_a\tau$ a σ-Coxeter element if exactly one simple reflection from each $\tau\sigma$-orbit on $\mathrm{supp}_\sigma(w\tau)$ occurs in every (equivalently, any) reduced expression of w.

For $\alpha \in \Phi$, let $U_\alpha \subseteq G$ denote the corresponding root subgroup. We set

$$I = T(\mathcal{O}) \prod_{\alpha\in\Phi_+} U_\alpha(\varpi\mathcal{O}) \prod_{\beta\in\Phi_-} U_\beta(\mathcal{O}) \subseteq G(L),$$

which is called the standard Iwahori subgroup associated to the triple $T \subset B \subset G$. For $J \subset \tilde{S}$ with W_J finite, let $P_J \supseteq I$ be the standard parahoric subgroup corresponding to J. We denote by π_J the projection $G(L)/I \to G(L)/P_J$. Set $K = P_S = G(\mathcal{O})$ and $\pi = \pi_S$.

Example 2.1. In the case $G = \mathrm{GL}_n$, we will use the following description. Let T be the torus of diagonal matrices, and we choose the subgroup of upper triangular matrices B as Borel subgroup. Let χ_{ij} be the character $T \to \mathbb{G}_m$ defined by $\mathrm{diag}(t_1, t_2, \ldots, t_n) \mapsto t_i {t_j}^{-1}$. Then we have $\Phi = \{\chi_{ij} \mid i \neq j\}$, $\Phi_+ = \{\chi_{ij} \mid i < j\}$, $\Phi_- = \{\chi_{ij} \mid i > j\}$ and $\Delta = \{\chi_{i,i+1} \mid 1 \leq i < n\}$. Through the isomorphism $X_*(T) \cong \mathbb{Z}^n$, $X_*(T)_+$ can be identified with the set $\{(m_1, \cdots, m_n) \in \mathbb{Z}^n \mid m_1 \geq \cdots \geq m_n\}$. Let us write $s_1 = (1\ 2), s_2 = (2\ 3), \ldots, s_{n-1} = (n-1\ n)$. Set $s_0 = \varpi^{\chi_{1,n}^\vee}(1\ n)$, where $\chi_{1,n}$ is the unique highest root. Then $S = \{s_1, s_2, \ldots, s_{n-1}\}$ and $\tilde{S} = S \cup \{s_0\}$. The Iwahori subgroup $I \subset K$ is the inverse image of B^{op} under the projection $G(\mathcal{O}) \to G(\overline{\mathbb{F}}_q)$ sending ϖ to 0, where B^{op} is the subgroup of lower triangular matrices. Similarly, if $J \subset S$, then P_J is the inverse image of the standard parabolic subgroup (which contains B^{op}) corresponding to J. Finally, $\varpi^{(1,0^{(n-1)})} s_1 s_2 \cdots s_{n-1}$ is a generator of $\Omega \cong \mathbb{Z}$.

Example 2.2. Let us denote by $\mathrm{GSp}_{2n} \subset \mathrm{GL}_{2n}$ the group of symplectic similitudes of dimension $2n$ as in [18, §2.3]. In the case $G = \mathrm{GSp}_{2n}$, we will use the following description. Let T (resp. B) be the intersection of the torus (resp. Borel subgroup) of GL_{2n} as in Example 2.1 with GSp_{2n}. See [19, §8] for the description of the corresponding roots. In particular, $\Delta = \{\frac{1}{2}\chi_{i,i+1} + \frac{1}{2}\chi_{2n-i,2n-i+1} \mid 1 \leq i \leq n-1\} \sqcup \{\chi_{n,n+1}\}$. The cocharacter group $X_*(T)$ can be identified with the set $\{(m_1, \cdots, m_{2n}) \in \mathbb{Z}^{2n} \mid m_1 + m_{2n} = m_2 + m_{2n-1} = \cdots = m_n + m_{n+1}\}$. Set $s_1 = (1\ 2)(2n{-}1\ 2n), s_2 = (2\ 3)(2n{-}2\ 2n{-}1), \ldots, s_{n-1} = (n{-}1\ n)(n{+}1\ n{+}2), s_n = (n\ n+1)$. Then $S = \{s_1, s_2, \ldots, s_n\}$ and the finite Weyl group is the subgroup of the symmetric group of degree $2n$ generated by S. Set $s_0 = \varpi^{\chi_{1,2n}^\vee}(1\ 2n)$. Then $\tilde{S} = S \cup \{s_0\}$. The standard Iwahori subgroup is the intersection of the standard Iwahori subgroup of GL_{2n} as in Example 2.1 with GSp_{2n}. Similarly, if $J \subset S$, then P_J is the inverse image of the standard parabolic subgroup corresponding to J.

2.2 Affine Deligne-Lusztig Varieties

For $w \in \tilde{W}$ and $b \in G(L)$, the affine Deligne-Lusztig variety $X_w(b)$ in the affine flag variety $G(L)/I$ is defined as

$$X_w(b) = \{xI \in G(L)/I \mid x^{-1}b\sigma(x) \in IwI\}.$$

The admissible subset of $\tilde{W}$ associated to μ is defined as

$$\mathrm{Adm}(\mu) = \{w \in \tilde{W} \mid w \leq \varpi^{u\mu} \text{ for some } u \in W_0\}.$$

We fix a rational level structure, i.e., a subset $J \subset \tilde{S}$ such that W_J is finite and $J = \sigma(J)$. The closed affine Deligne-Lusztig variety in $G(L)/P_J$ is the closed reduced $\overline{\mathbb{F}}_q$-subscheme defined as

$$X(\mu, b)_J = \{gP_J \in G(L)/P_J \mid g^{-1}b\sigma(g) \in P_J \,\mathrm{Adm}(\mu) P_J\}.$$

In the equal characteristic case, affine Deligne-Lusztig varieties are schemes, locally of finite type over $\overline{\mathbb{F}}_q$. In the mixed characteristic case, affine Deligne-Lusztig varieties are perfect schemes, locally perfectly of finite type over $\overline{\mathbb{F}}_q$. See [35], [47], [1] and [20, Lemma 1.1]. Left multiplication by $g^{-1} \in G(L)$ induces an isomorphism between affine Deligne-Lusztig varieties corresponding to b and $g^{-1}b\sigma(g)$. Thus the isomorphism class of the affine Deligne-Lusztig variety only depends on the σ-conjugacy class of b. Also, the affine Deligne-Lusztig varieties carry a natural action (by left multiplication) by the σ-centralizer of b

$$\mathbb{J}_b = \{g \in G(L) \mid g^{-1}b\sigma(g) = b\}.$$

Note that $\mathbb{J}_b \cong \mathbb{J}_{g^{-1}b\sigma(g)}$ by sending j to $g^{-1}jg$.

Set ${}^J\mathrm{Adm}(\mu) = \mathrm{Adm}(\mu) \cap {}^J\tilde{W}$. As explained in [17, §2.5], we have

$$X(\mu, b)_J = \bigsqcup_{w \in {}^J\mathrm{Adm}(\mu)} \pi_J(X_w(b)).$$

This is the *EKOR (Ekedahl-Kottwitz-Oort-Rapoport) stratification*, which is the local analogue of the stratification defined in the global context of Shimura varieties in [22]. If (G, μ) arises from a Shimura datum, an EKOR stratum in $X(\mu, b)_J$ actually corresponds to the intersection of a global Ekedahl-Oort stratum with the Newton stratum attached to the σ-conjugacy class $[b]$.

Set $J(w,\sigma) = \max\{J' \subseteq J \mid \mathrm{Ad}(w)\sigma(J') = J'\}$. It follows from [15, Proposition 5.7] that if $W_{\mathrm{supp}_\sigma(w)}$ is finite, then $W_{\mathrm{supp}_\sigma(w)\cup J(w,\sigma)}$ is also finite. Since any two lifts of $\tau \in \Omega$ are $T(\mathcal{O})$-conjugate by [11, Lemma 2.5], we will write $X_w(\tau)$ instead of $X_w(\dot{\tau})$. The following proposition is a combination of [13, Proposition 4.1.1 & Theorem 4.1.2] (see also [16, §2.4]).

Proposition 2.3. Let $\tau \in \Omega$. Let $w \in {}^J\tilde{W} \cap W_a\tau$ such that $W_{\mathrm{supp}_\sigma(w)}$ is finite. Then the projection induces $\pi_J(X_w(\tau)) \cong \pi_{J(w,\sigma)}(X_w(\tau))$ and

$$\pi_{J(w,\sigma)}(X_w(\tau)) = \bigsqcup_{j\in\mathbb{J}_\tau/\mathbb{J}_\tau\cap P_{\mathrm{supp}_\sigma(w)\cup J(w,\sigma)}} j\pi_{J(w,\sigma)}(Y(w)),$$

where

$$\pi_{J(w,\sigma)}(Y(w)) = \{gP_{J(w,\sigma)} \in P_{\mathrm{supp}_\sigma(w)\cup J(w,\sigma)}/P_{J(w,\sigma)} \mid g^{-1}\tau\sigma(g) \in P_{J(w,\sigma)}wP_{\sigma(J(w,\sigma))}\}$$

is (the perfection of) a Deligne-Lusztig variety in the flag variety $P_{\mathrm{supp}_\sigma(w)\cup J(w,\sigma)}/P_{J(w,\sigma)}$. In particular, if $J = \emptyset$, then

$$X_w(\tau) = \bigsqcup_{j\in\mathbb{J}_\tau/\mathbb{J}_\tau\cap P_{\mathrm{supp}_\sigma(w)}} jY(w),$$

where $Y(w) = \{gI \in P_{\mathrm{supp}_\sigma(w)}/I \mid g^{-1}\tau\sigma(g) \in IwI\}$ is a Deligne-Lusztig variety in the flag variety $P_{\mathrm{supp}_\sigma(w)}/I$.

Remark 2.4. Let $w \in {}^J\tilde{W}$. Then $w \in {}^{J(w,\sigma)}\tilde{W}^{\sigma(J(w,\sigma))}$. By [13, Theorem 3.2.1], we have

$$P_{J(w,\sigma)}wP_{\sigma(J(w,\sigma))} = P_{J(w,\sigma)} \cdot_\sigma IwI,$$

where $\cdot_\sigma$ denotes the σ-twisted conjugation action of $G(L)$.

Remark 2.5. Each $Y(w)$ or $\pi_{J(w,\sigma)}(Y(w))$ is irreducible and of dimension $\ell(w)$. See [9] and [3]. Clearly, each $jY(w)$ or $j\pi_{J(w,\sigma)}(Y(w))$ is closed in $X_w(\tau)$ or $\pi_J(X_w(\tau))$, and hence an irreducible component.

2.3 Deligne-Lusztig Reduction Method

The following Deligne-Lusztig reduction method was established in [12, Corollary 2.5.3] ($\mathbb{A}^1$ and $\mathbb{G}_m$ actually mean $\mathbb{A}^{1,\mathrm{pfn}}$ and $\mathbb{G}_m^{\mathrm{pfn}}$ respectively in the mixed characteristic case).

Proposition 2.6. Let $w \in \tilde{W}$ and let $s \in \tilde{S}$ be a simple affine reflection. Then the following two statements hold for any $b \in G(L)$.

(i) If $\ell(sw\sigma(s)) = \ell(w)$, then there exists a $\mathbb{J}_b$-equivariant universal homeomorphism $X_w(b) \to X_{sw\sigma(s)}(b)$.

(ii) If $\ell(sw\sigma(s)) = \ell(w) - 2$, then there exists a decomposition $X_w(b) = X_1 \sqcup X_2$ such that

- X_1 is open and there exists a $\mathbb{J}_b$-equivariant morphism $X_1 \to X_{sw}(b)$, which is the composition of a Zariski-locally trivial $\mathbb{G}_m$-bundle and a universal homeomorphism.
- X_2 is closed and there exists a $\mathbb{J}_b$-equivariant morphism $X_2 \to X_{sw\sigma(s)}(b)$, which is the composition of a Zariski-locally trivial $\mathbb{A}^1$-bundle and a universal homeomorphism.

Let $gI \in X_w(b)$. If $\ell(sw) < \ell(w)$ (we can reduce to this case by exchanging w and $sw\sigma(s)$), then let $g_1 I$ denote the unique element in $G(L)/I$ such that $g^{-1}g_1 \in IsI$ and $g_1^{-1}b\sigma(g) \in IswI$. The set X_1 (resp. X_2) above consists of the elements $gI \in X_w(b)$ satisfying $g_1^{-1}b\sigma(g_1) \in IswI$ (resp. $Isw\sigma(s)I$). All of the maps in the proposition are given as the map sending gI to $g_1 I$.

Remark 2.7. The perfection of a universal homeomorphism is an isomorphism.

2.4 Length Positive Elements

We denote by δ^+ the indicator function of the set of positive roots, i.e.,

$$\delta^+ \colon \Phi \to \{0,1\}, \quad \alpha \mapsto \begin{cases} 1 & (\alpha \in \Phi_+) \\ 0 & (\alpha \in \Phi_-). \end{cases}$$

Note that any element $w \in \tilde{W}$ can be written in a unique way as $w = x\varpi^\mu y$ with μ dominant, $x, y \in W_0$ such that $\varpi^\mu y \in {}^S\tilde{W}$. We have $p(w) = xy$ and $\ell(w) = \ell(x) + \langle \mu, 2\rho \rangle - \ell(y)$. We define the set of *length positive* elements by

$$\mathrm{LP}(w) = \{v \in W_0 \mid \langle v\alpha, y^{-1}\mu \rangle + \delta^+(v\alpha) - \delta^+(xyv\alpha) \geq 0 \text{ for all } \alpha \in \Phi_+\}.$$

Then we always have $y^{-1} \in \mathrm{LP}(w)$. Indeed, y is uniquely determined by the condition that $\langle \alpha, \mu \rangle \geq \delta^+(-y^{-1}\alpha)$ for all $\alpha \in \Phi_+$. Since $\delta^+(\alpha) + \delta^+(-\alpha) = 1$, we have

$$\langle y^{-1}\alpha, y^{-1}\mu \rangle + \delta^+(y^{-1}\alpha) - \delta^+(x\alpha) = \langle \alpha, \mu \rangle - \delta^+(-y^{-1}\alpha) + \delta^+(-x\alpha) \geq 0.$$

Thanks to Kottwitz [30], a σ-conjugacy class $[b]$ of $b \in G(L)$ is uniquely determined by two invariants: the Kottwitz point $\kappa(b) \in \pi_1(G)/((1-\sigma)\pi_1(G))$ and the Newton point $\nu_b \in X_*(T)_{\mathbb{Q},+}$. Clearly, $X_w(b) = \emptyset$ if $\kappa(b) \neq \kappa(\dot{w})$. We say that $b \in G(L)$ is basic if ν_b is central. The following theorem is a refinement of the non-emptiness criterion in [14], which is conjectured by Lim in [31] and proved by Schremmer in [37, Proposition 5].

Theorem 2.8. Assume that the Dynkin diagram of G is σ-connected, i.e., σ acts transitively on the set of irreducible components of Φ. Let $b \in G(L)$ be a basic element with $\kappa(b) = \kappa(\dot{w})$. Then $X_w(b) = \emptyset$ if and only if the following two conditions are satisfied:

(i) $|W_{\mathrm{supp}_\sigma(w)}|$ is infinite.

(ii) There exists $v \in \mathrm{LP}(w)$ such that $\mathrm{supp}_\sigma(\sigma^{-1}(v)^{-1}p(w)v) \subsetneq S$.

2.5 The $\mathbb{J}$-stratification

Let J be the fixed level structure in §2.2. The map $w \mapsto \dot{w}$ induces a bijection

$$W_J\backslash\tilde{W}/W_J \xrightarrow{\sim} P_J\backslash G(L)/P_J.$$

If $J = \emptyset$, then this is the usual Iwahori-Bruhat decomposition. If $J = S$, it is the Cartan decomposition. We denote by inv_J the relative position map

$$\mathrm{inv}_J\colon G(L) \times G(L) \to W_J\backslash\tilde{W}/W_J, \quad (g,h) \mapsto P_J g^{-1} h P_J.$$

We will simply write inv for $\mathrm{inv}_\emptyset$. By definition, two elements $gK, hK \in G(L)/P_J$ lie in the same $\mathbb{J}_b$-stratum if and only if for all $j \in \mathbb{J}_b$, $\mathrm{inv}_J(j,g) = \mathrm{inv}_J(j,h)$. By [11, Theorem 2.10], the $\mathbb{J}_b$-strata are locally closed in $G(L)/P_J$. By intersecting each $\mathbb{J}_b$-stratum with $X(\mu,b)_J$, we obtain the $\mathbb{J}_b$-stratification of it. This stratification is first introduced by Chen-Viehmann [7] in the hyperspecial level. Görtz generalized it to any parahoric level in [11].

As explained in [7, Remark 2.1], the $\mathbb{J}_b$-stratification heavily depends on the choice of b in its σ-conjugacy class. Thus we need to fix a specific representative to compare the $\mathbb{J}_b$-stratification on $X(\mu,b)_J$ to another stratification. As explained in [11, Remark 3.3], a reasonable choice is the unique length 0 element $\tau = \tau_\mu$ such that $X(\mu,\tau)_J \neq \emptyset$. Equivalently, τ is the image of ϖ^μ under the projection $\tilde{W} = W_a \rtimes \Omega \to \Omega$. Note that for any $w \in \tilde{W}$, the $\mathbb{J}_{\dot{w}}$-stratification is independent of the choice of lift $\dot{w}$ in $G(L)$ (cf. [11, Lemma 2.5]). In the sequel, we will mainly focus on the case $b = \tau$ and simply write $\mathbb{J}$ instead of $\mathbb{J}_{\dot{\tau}}$.

3 Fully Hodge-Newton decomposable cases

We now fix $\mu \in X_*(T)_+$. Let τ be as in §2.5.

3.1 Fully Hodge-Newton decomposable pairs

For $\alpha \in \Delta$, we define ω_α to be the rational fundamental weight such that

$$\langle \omega_\alpha, \beta^\vee \rangle = \begin{cases} 1 & (\beta = \alpha) \\ 0 & (\beta \in \Delta \setminus \{\alpha\}). \end{cases}$$

For each σ-orbit $\mathscr{O}$ of Δ, we set

$$\omega_{\mathscr{O}} = \sum_{\alpha \in \mathscr{O}} \omega_\alpha.$$

For a dominant cocharacter $\mu \in X_*(T)$, we define

$$\operatorname{depth}(G, \mu) := \max_{\mathscr{O} \subseteq \Delta} \langle \omega_{\mathscr{O}}, \mu \rangle,$$

where $\mathscr{O}$ runs through all σ-orbits of S.

See [15, Definition 3.1] for the definition of fully Hodge-Newton decomposable pairs (G, μ). Set ${}^J\mathrm{Adm}(\mu)_{\neq \emptyset} = \{w \in {}^J\mathrm{Adm}(\mu) \mid X_w(\tau) \neq \emptyset\}$. We first recall the following characterization proved in [15, Theorem B].

Theorem 3.1. The pair (G, μ) is fully Hodge-Newton decomposable if and only if the following equivalent conditions are satisfied:

(i) The cocharacter μ is minute, which means by definition that $\operatorname{depth}(G, \mu) \leq 1$.

(ii) $W_{\mathrm{supp}_\sigma(w)}$ is finite for every $w \in {}^J\mathrm{Adm}(\mu)_{\neq \emptyset}$.

In particular, the validity of the condition (ii) is independent of the rational level J.

We say that the triple (G, μ, J) is of Coxeter type if (G, μ) is fully Hodge-Newton decomposable and every $w \in {}^J\mathrm{Adm}(\mu)_{\neq \emptyset}$ is a σ-Coxeter element. Unlike the fully Hodge-Newton decomposability, the validity of Coxeter type depends on the parahoric level. See [16, Theorem 1.4] for the classification.

Example 3.2. The fully Hodge-Newton decomposable cases contain the following cases which have been investigated in the context of Shimura varieties (cf. [13, §5.3]):

- The Siegel case of genus 2, which has been studied by Katsura-Oort [28] and Kaiser [27].

- The $\mathrm{GU}(1,n-1)$, p split case, which has been studied by Harris-Taylor [21].
- The $\mathrm{GU}(1,n-1)$, p inert case, which has been studied by Vollaard-Wedhorn [46].
- The $\mathrm{GU}(2,2)$, p inert case, which has been studied by Howard-Pappas [25].

All of these cases concern the hyperspecial level S, and are of Coxeter type.

Example 3.3. The pair $(\mathrm{SL}_n, \chi_{1,n}^\vee)$ is fully Hodge-Newton decomposable, but does not come from a Shimura variety. If $J = S$, then $(\mathrm{SL}_n, \chi_{1,n}^\vee, J)$ is of Coxeter type. On the other hand, it is not of Coxeter type if $J \subsetneq S$.

3.2 The Bruhat-Tits stratification

In the fully Hodge-Newton decomposable cases, we have the following simple description of $X(\mu,\tau)_J$.

Theorem 3.4. If (G,μ) is fully Hodge-Newton decomposable, then $X(\mu,\tau)_J$ is naturally a disjoint union of Deligne-Lusztig varieties.

Proof. This is a combination of Proposition 2.3 and Theorem 3.1. See also [15, §5.11]. □

The disjoint decomposition in Theorem 3.4 is called the *weak Bruhat-Tits stratification*. This is a stratification in the strong sense that the closure of a stratum is a union of strata. Let $j, j' \in \mathbb{J} = \mathbb{J}_{\dot{\tau}}$. The closure of a stratum $j\pi_J(Y(w))$ contains a stratum $j'\pi_J(Y(w'))$ if and only if the following two conditions are both satisfied:

(i) $w \geq_{J,\sigma} w'$, which means by definition that there exists $u \in W_J$ such that $w \geq u^{-1}w'u$.

(ii) $j(\mathbb{J} \cap P_{\mathrm{supp}_\sigma(w) \cup J(w,\sigma)}) \cap j'(\mathbb{J} \cap P_{\mathrm{supp}_\sigma(w') \cup J(w',\sigma)}) \neq \emptyset$.

By [23, §4.7], $\geq_{J,\sigma}$ gives a partial order on ${}^J\tilde{W}$. Let $\mathcal{B}(\mathbb{J}, F)$ denote the rational Bruhat-Tits building of $\mathbb{J}$. Then (2) above is equivalent to requiring that $\kappa(j) = \kappa(j')$ and that the simplices in $\mathcal{B}(\mathbb{J}, F)$ corresponding to $j(\mathbb{J} \cap P_w)j^{-1}$ and $j'(\mathbb{J} \cap P_{w'})j'^{-1}$ are neighbors (i.e., there exists an alcove which contains both of them). See [16, §2.4] for these facts.

If (G,μ,J) is of Coxeter type, then the weak Bruhat-Tits stratification satisfies further nice properties (cf. [16, Proposition 2.8 & Corollary 2.9]). So this stratification is called the *Bruhat-Tits stratification* in this case. In [7, §4], Chen-Viehmann conjectured that the Bruhat-Tits stratification coincides with the $\mathbb{J}$-stratification and verified this conjecture in the Siegel case of genus 2 and the Vollaard-Wedhorn case. In [11], Görtz proved this conjecture in general:

Theorem 3.5. Let (G, μ, J) be of Coxeter type. Then the Bruhat-Tits stratification on $X(\mu, b)_J$ coincides with the $\mathbb{J}$-stratification.

Remark 3.6. The $\mathbb{J}$-stratification is a stratification in the loose sense that in general, the closure of a stratum is not a union of strata. See [7, §2.1].

The following proposition is essential for the proof of Theorem 3.5.

Proposition 3.7. Let $Y(w)$ be as in Proposition 2.3, and let w_0 be the longest element in $W_{\mathrm{supp}_\sigma(w)}$. If w is a σ-Coxeter element, then $Y(w) \subseteq Iw_0I/I$.

Proof. See [32, Corollary 2.5] or [11, Proposition 1.1]. □

The fact that the Bruhat-Tits stratification is finer than the $\mathbb{J}$-stratification easily follows from Proposition 3.7 as follows (cf. [11, §3.3]): Set $\mathbb{J}^0 = \{j \in \mathbb{J} \mid \kappa(j) = 0\}$. It is easy to check that $\mathrm{inv}(j, g) = \mathrm{inv}(j, g')$ for all $j \in \mathbb{J}$ if this is true for $\mathbb{J}^0$. We fix $j \in \mathbb{J}^0$. Let $w \in \tilde{W}$ with $W_{\mathrm{supp}_\sigma(w)}$ finite. Then by [11, Proposition 1.7], there exists $gI \in P_{\mathrm{supp}_\sigma(w)}/I$ with $g \in \mathbb{J} \cap P_{\mathrm{supp}_\sigma(w)}$ such that $\mathrm{inv}(j, y) = \mathrm{inv}(j, g)\mathrm{inv}(g, y)$ for any $y \in P_{\mathrm{supp}_\sigma(w)}/I$. In particular, if $y \in Y(w)$ and w is a σ-Coxeter element, then $\mathrm{inv}(j, y) = \mathrm{inv}(j, g)w_0$ by Proposition 3.7. So $\mathrm{inv}(j, y)$ is independent of $y \in Y(w)$. Thus the value $\mathrm{inv}(j, -)$ (and hence $\mathrm{inv}_J(j, -)$) is constant on each Bruhat-Tits stratum.

The converse is more difficult and relies on combinatorial arguments on the affine root system and the building of G.

Example 3.8. In the case $(\mathrm{SL}_3, \chi_{1,3}^\vee, \emptyset)$ (cf. Example 3.3), the weak Bruhat-Tits stratification does not coincide with the $\mathbb{J}$-straitification. Indeed, we have $\varpi^{\chi_{1,3}^\vee} = s_0s_1s_2s_1$ and hence $s_1s_2, s_2s_1 \in \mathrm{Adm}(\mu)$ (see [2, Theorem 2.2.2] for example). The above argument shows that both $Y(s_1s_2)$ and $Y(s_2s_1)$ are contained in the same $\mathbb{J}$-stratum because $\mathrm{supp}_\sigma(s_1s_2) = \mathrm{supp}_\sigma(s_2s_1) = \{s_1, s_2\}$.

4 A generalization of Coxeter type

In this section, we will treat the cases of positive Coxeter type, which can be considered as a generalization of Coxeter type.

4.1 Elements with positive Coxeter part

We say that $w \in \tilde{W}$ has *positive Coxeter part* if $\sigma^{-1}(v)^{-1}p(w)v$ is a σ-Coxeter element for some $v \in \mathrm{LP}(w)$. Affine Deligne-Lusztig varieties associated to elements of positive Coxeter type were studied in a joint work [38] with Schremmer and Yu. The following theorem is a combination of [38, Theorem 5.7 & Theorem 5.20].

Theorem 4.1. Assume that $w \in \tilde{W}$ has positive Coxeter part and $X_w(b) \neq \emptyset$. Then $X_w(b)$ has only one $\mathbb{J}_b$-orbit of irreducible components, and each irreducible component is an iterated fibration over a Deligne-Lusztig variety of Coxeter type whose iterated fibers are either $\mathbb{A}^1$ or $\mathbb{G}_m$. If b is basic, then all fibers are $\mathbb{A}^1$ and each iterated fibration decomposes into the product of varieties.

An iterated fibration is the composite of some Zariski-locally trivial $\mathbb{A}^1$-bundles (cf. [24, §2G]). If w has positive Coxeter part, we also have an explicit description of the σ-conjugacy classes $[b]$ such that $X_w(b) \neq \emptyset$, the dimension of $X_w(b) (\neq \emptyset)$ and the number of fibers. One of the main ingredients of the proof is the Deligne-Lusztig reduction method (Proposition 2.6), which induces the iterated fibration in Theorem 4.1.

We say that $w \in \tilde{W}$ has *finite Coxeter part* if $\sigma^{-1}(v)^{-1}p(w)v$ is a σ-Coxeter element for $v = y^{-1} \in \mathrm{LP}(w)$ (cf. §2.4). Before [38], He-Nie-Yu [24] studied $w \in \tilde{W}$ with finite Coxeter part and proved the existence of the simple geometric structure as Theorem 4.1. See [40] by the author for a special case of GL_n.

Example 4.2. It follows from [26, Theorem 3.3] that if $G = \mathrm{GL}_2$, then

$$X_{\varpi^{(r,-r)}s_1}(1) \cong (\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)) \times \mathbb{A}^r.$$

More generally, it follows from [4, Theorem 6.17] that if $G = \mathrm{GL}_n$, then

$$X_{\varpi^{((n-1)r,-r,\ldots,-r)}s_1 s_2 \cdots s_{n-1}}(1) \cong \bigsqcup_{G(F)/G(\mathcal{O}_F)} \mathrm{Dr}^{n-1}_{\mathbb{F}_q} \times \mathbb{A}^{\frac{n(n-1)r}{2}-n+1},$$

where $\mathrm{Dr}^{n-1}_{\mathbb{F}_q} := \mathbb{P}^{n-1}_{\mathbb{F}_q} \setminus \bigcup_{H \in \mathcal{H}} H$ and $\mathcal{H}$ = the set of all $\mathbb{F}_q$-rational hyperplanes in $\mathbb{P}^{n-1}_{\mathbb{F}_q}$. This is called the Drinfeld upper half space over $\mathbb{F}_q$ of dimension n, which is isomorphic to the Deligne-Lusztig variety associated to GL_n and $w = s_1 s_2 \cdots s_{n-1}$.

Note that $\varpi^{((n-1)r,-r,\ldots,-r)}s_1 s_2 \cdots s_{n-1}$ has finite Coxeter part. So the above examples are special cases of Theorem 4.1.

4.2 The cases of positive Coxeter type

We say that (G, μ, J) is of *positive Coxeter type* if every $w \in {}^J\mathrm{Adm}(\mu)_{\neq\emptyset}$ satisfies one of the following conditions:

(i) w is a σ-Coxeter element with $W_{\mathrm{supp}_\sigma(w)}$ finite.

(ii) w has positive Coxeter part.

Clearly, this notion is a generalization of Coxeter type. In the case where $G = \mathrm{GL}_n$ and $J = S$, we have the following classification. See [39, Theorem 3.3].

Theorem 4.3. The following assertions on μ are equivalent.

(i) The triple (GL_n, μ, S) is of positive Coxeter type.

(ii) The cocharacter μ is central or one of the following forms modulo $\mathbb{Z}\omega_n^\vee$:

$$
\begin{array}{lr}
\omega_1^\vee, \quad \omega_{n-1}^\vee, & (n \geq 1), \\
\omega_1^\vee + \omega_{n-1}^\vee, \quad \omega_2^\vee, \quad 2\omega_1^\vee, \quad \omega_{n-2}^\vee, \quad 2\omega_{n-1}^\vee, & \\
\omega_2^\vee + \omega_{n-1}^\vee, \quad 2\omega_1^\vee + \omega_{n-1}^\vee, \quad \omega_1^\vee + \omega_{n-2}^\vee, \quad \omega_1^\vee + 2\omega_{n-1}^\vee, & (n \geq 3), \\
\omega_3^\vee, \quad \omega_{n-3}^\vee, & (n = 6, 7, 8), \\
3\omega_1^\vee, \quad 3\omega_{n-1}^\vee, & (n = 3, 4, 5), \\
\omega_1^\vee + \omega_2^\vee, \quad \omega_3^\vee + \omega_4^\vee, & (n = 5), \\
4\omega_1^\vee, \quad \omega_1^\vee + 3\omega_2^\vee, \quad 4\omega_2^\vee, \quad 3\omega_1^\vee + \omega_2^\vee, & (n = 3), \\
m\omega_1^\vee \text{ with } m \in \mathbb{Z}_{>0}, & (n = 2).
\end{array}
$$

Here $\omega_k^\vee$ denotes the cocharacter of the form $(1, \ldots, 1, 0, \ldots, 0)$ in which 1 is repeated k times.

According to [16, Theorem 1.4], (GL_n, μ, S) is of Coxeter type if and only if the cocharacter μ is central or one of the following forms modulo $\mathbb{Z}\omega_n^\vee$:

$$\omega_1^\vee, \quad \omega_{n-1}^\vee \ (n \geq 1), \quad \omega_1^\vee + \omega_{n-1}^\vee \ (n \geq 2), \quad \omega_2^\vee \ (n = 4).$$

The case $\mu = \omega_1^\vee + \omega_{n-1}^\vee$ corresponds to Example 3.3. The case $\mu = \omega_2^\vee$ was studied by Fox [10]. Note that all of these cases are contained in the list of Theorem 4.3.

In the case where $G = \mathrm{GSp}_{2n}$, $\mu = \omega_n^\vee = (1^{(n)}, 0^{(n)})$ and $J = S$, we have the following proposition. See [16, Theorem 1.4] and [43, Proposition 3.1 & §5].

Proposition 4.4. Let $n \geq 2$. The triple $(\mathrm{GSp}_{2n}, \omega_n^\vee, S)$ is of positive Coxeter type if and only if $n = 2, 3$.

According to [16, Theorem 1.4], the triple $(\mathrm{GSp}_{2n}, \omega_n^\vee, S)$ is of Coxeter type if and only if $n = 2$.

By Proposition 2.3 and Theorem 4.1, we can expect a simple geometric structure of $X(\mu, b)_J$ if (G, μ, J) is of positive Coxeter type. This has been verified in the above cases. See [39, Theorem 3.4] and [43, Theorem 3.4].

Theorem 4.5. Assume that (G, μ, S) is one of the triples of positive Coxeter type in Theorem 4.3 or Proposition 4.4. Then the variety $X(\mu, \tau)_S$ is naturally a disjoint union of subvarieties which are universally homeomorphic to the product of a Deligne-Lusztig variety of Coxeter type and a finite-dimensional affine space. Moreover, this stratification coincides with the $\mathbb{J}$-stratification.

The stratification in Theorem 4.5 satisfies the following conditions:

- For $w \in {}^S\mathrm{Adm}(\mu)_{\neq\emptyset}$, $\mathbb{J}$ acts transitively on the set of irreducible components of $X_w(\tau)$.
- For $w \in {}^S\mathrm{Adm}(\mu)_{\neq\emptyset}$, there exists a parahoric subgroup $P_w \subset G(L)$ and an irreducible component $Y(w)$ of $X_w(\tau)$ such that $\pi(X_w(\tau)) = \bigsqcup_{j\in\mathbb{J}/\mathbb{J}\cap P_w} j\pi(Y(w))$.
- $Y(w) \cong \pi(Y(w))$ and each $j\pi(Y(w))$ is a $\mathbb{J}$-stratum of $X(\mu,\tau)_S$.

In this case, we say that the closure relation can be described in terms of the rational Bruhat-Tits building of $\mathbb{J}$ if the $\mathbb{J}$-stratification of $X(\mu,\tau)_S$ is a stratification in the strong sense and $\overline{j\pi(Y(w))} \supseteq j'\pi(Y(w'))$ is equivalent to the following condition:

There exist sequences $w = w_0 \geq_S w_1 \geq_S \cdots \geq_S w_k = w'$ in ${}^S\mathrm{Adm}(\mu)_0$ and $j = j_0, j_1 \ldots, j_k = j'$ in $\mathbb{J}$ such that $j_{i-1}(\mathbb{J}\cap P_{w_{i-1}}) \cap j_i(\mathbb{J}\cap P_{w_i}) \neq \emptyset$ for $1 \leq i \leq k$.

Theorem 4.6. Assume that (G,μ,S) is one of the triples of positive Coxeter type in Theorem 4.3 or Proposition 4.4. Assume moreover that μ is minuscule. Then the closure relation of the stratification of $X(\mu,\tau)_S$ in Theorem 4.5 can be described in terms of the rational Bruhat-Tits building of $\mathbb{J}$.

It seems natural to expect that the closure relation can be described in terms of the rational Bruhat-Tits building of $\mathbb{J}$ in all of the cases in Theorem 4.3 including the non-minuscule cases. If $G = \mathrm{GL}_n$ or GSp_{2n} and μ is minuscule, then $\mathbb{J}$ acts transitively on the set of irreducible components of $X(\mu,\tau)_S$. In general, this is not true for non-minuscule cocharacters, which is a difficulty. See [34, Remark 0.3 & Theorem 0.5] for these facts on irreducible components of $X(\mu,\tau)_S$.

4.3 Comparison of the $\mathbb{J}$-stratification and the Ekedahl-Oort stratification

In view of Theorem 3.5, it would be interesting to study the relationship between the $\mathbb{J}$-stratification and the EKOR stratification. If $J = S$, then the EKOR stratification is called the *EO (Ekedahl-Oort)* stratification.

In general, it is very difficult to study the $\mathbb{J}$-stratification. However, in the case where $G = \mathrm{GL}_n$, $J = S$ and τ is superbasic, i.e., $\kappa(\varpi^\mu) \in \mathbb{Z}$ is coprime to n, the $\mathbb{J}$-stratification coincides with a stratification by semi-modules ([7, Proposition 3.4]). The notion of semi-modules was first considered by de Jong and Oort [8] for minuscule cocharacters. Later Viehmann [44] introduced a notion of extended semi-modules for arbitrary cocharacters, which generalizes the notion of semi-modules. It played a crucial role to prove the dimension formula and the study of irreducible components of $X(\mu, \tau)_S$. This is because for these problems, we can reduce the general case to the case that $G = \mathrm{GL}_n$ and τ is superbasic. In the superbasic case, we have the following characterization of the cases of positive Coxeter type.

Theorem 4.7. Let $G = \mathrm{GL}_n$ and let $\mu \in X_*(T)_+$. Assume that τ is superbasic. Then the following assertions are equivalent.

(i) (GL_n, μ, S) is of positive Coxeter type.

(ii) The $\mathbb{J}$-stratification of $X(\mu, \tau)_S$ gives a refinement of the Ekedahl-Oort stratification.

The main ingredient of the proof is the explicit construction of top-dimensional $\mathbb{J}$-strata (which corresponds to irreducible components) of $X(\mu, \tau)_S$ obtained in [42] by the author. Although the result of [42] only concerns the superbasic case, we can expect a generalization of it.

5 Beyond fully Hodge-Newton decomposable cases

5.1 Weakly fully Hodge-Newton decomposable cases

Recently, Chen-Tong [6] introduced the weak full Hodge-Newton decomposability in the context of p-adic Hodge theory and studied it under the minuscule condition. The weakly admissible locus $\mathcal{F}(G, \mu, \tau)^{wa}$ inside the flag variety $\mathcal{F}(G, \mu)$, attached to G with a minuscule cocharacter μ, is a vast generalization of the Drinfeld upper half plane. The admissible locus $\mathcal{F}(G, \mu, \tau)^a \subseteq \mathcal{F}(G, \mu, \tau)^{wa}$ is a p-adic analogue of the complex analytic period spaces. Chen-Fargues-Shen [5] proved that (G, μ) is fully Hodge-Newton decomposable if and only if $\mathcal{F}(G, \mu, \tau)^a = \mathcal{F}(G, \mu, \tau)^{wa}$. If this is the case, then the Newton stratification of $\mathcal{F}(G, \mu)$ gives a refinement of the Harder-Narashimhan stratification (see [6, §1.4.3] for these stratifications). The main result of [6] states that the weak full Hodge-Newton decomposability, which is a generalization of the full Hodge-Newton decomposability by definition, is equivalent to the condition that the Newton stratification is finer than the Harder-Narashimhan stratification. They also classified the weakly fully Hodge-Newton decomposable cases. Their classification (cf. [6, Remark 2.13]) tells us that for minuscule μ, (GL_n, μ) is weakly fully Hodge-Newton decomposable if and only if τ is superbasic or (GL_n, μ, S) is of positive Coxeter type (cf. Theorem 4.3). In [6, Remark 2.16], they pointed out that it will be an interesting question to investigate the basic affine Deligne-Lusztig varieties associated to a weakly fully Hodge-Newton decomposable pair. For the superbasic case, the geometry of $X_\mu(\tau)$ for GL_n is already studied in [45] to some extent. The answer to Chen-Tong's question for cocharacters of positive Coxeter type is contained in Theorem 4.5 and Theorem 4.6.

5.2 The cases where $\mathrm{depth}(G, \mu) \leq 2$

Recall that (G, μ) is fully Hodge-Newton decomposable if and only if $\mathrm{depth}(G, \mu) \leq 1$. In this point of view, it is natural to think that we can measure the complexity of affine Deligne-Lusztig varieties attached to (G, μ) by $\mathrm{depth}(G, \mu)$. Recently, Schremmer informed the author that there is an upcoming work by He, Schremmer and Viehmann, which classifies the cases where $\mathrm{depth}(G, \mu) < 2$. According to their classification, we have the following proposition in the case of GL_n (which can also be checked by explicit computation as in the examples below).

Proposition 5.1. Let $G = \mathrm{GL}_n$ and let $\mu \in X_*(T)_+$. If $\mathrm{depth}(G, \mu) < 2$, then (G, μ, S) is of positive Coxeter type. If $n \geq 6$, then $\mathrm{depth}(G, \mu) < 2$ if and only if (G, μ, S) is of positive Coxeter type.

Proposition 5.2. Let $n \geq 2$. The triple $(\mathrm{GSp}_{2n}, \omega_n^\vee, S)$ is of positive Coxeter type if and only if $\mathrm{depth}(G, \mu) < 2$.

Example 5.3. Let $G = \mathrm{GL}_n$. Under the notation of Example 2.1, the rational fundamental weights are

$$\omega_{\{\chi_{i,i+1}\}} = (\overbrace{\tfrac{n-i}{n}, \dots, \tfrac{n-i}{n}}^{i}, \overbrace{-\tfrac{i}{n}, \dots, -\tfrac{i}{n}}^{n-i}), \qquad 1 \leq i \leq n-1.$$

Thus $\mathrm{depth}(\mathrm{GL}_n, \omega_1^\vee) = \frac{n-1}{n} < 1$ and $\mathrm{depth}(\mathrm{GL}_n, \omega_2^\vee) = \frac{2(n-2)}{n} < 2$. We also have $\mathrm{depth}(\mathrm{GL}_n, \omega_3^\vee) = \frac{3(n-3)}{n}$. Therefore $\mathrm{depth}(\mathrm{GL}_n, \omega_3^\vee) < 2$ if and only if $n < 9$, which is consistent with Theorem 4.3 and Proposition 5.1.

Let G be the unramified unitary group. Then $\tilde{W} \cong \tilde{W}_{\mathrm{GL}_n}$. Under this identification, we have $\sigma((m_i)_{1 \leq i \leq n}) = (-m_{n+1-i})_{1 \leq i \leq n} \in X_*(T) \cong \mathbb{Z}^n$ and $\sigma(s_i) = s_{n-i} \in \tilde{W}_{\mathrm{GL}_n}$ by setting $s_n = s_0$. Thus the rational fundamental weights are

$$\omega_{\{\chi_{i,i+1}, \chi_{n-i,n-i+1}\}} = (\overbrace{1, \dots, 1}^{i}, 0, \dots, 0, \overbrace{-1, \dots, -1}^{i}), \qquad 1 \leq i \leq \lfloor \tfrac{n}{2} \rfloor.$$

Thus $\mathrm{depth}(G, \omega_1^\vee) = 1$. If $n \geq 4$, then $\mathrm{depth}(G, \omega_2^\vee) = 2$.

Example 5.4. Let $G = \mathrm{GSp}_{2n}$. We follow the notation of Example 2.2. Note that $\Delta = \{\chi_{i,i+1}^\vee + \chi_{2n-i,2n-i+1}^\vee \mid 1 \leq i \leq n-1\} \sqcup \{\chi_{n,n+1}^\vee\}$. So the rational fundamental weights are

$$\omega_{\{\frac{1}{2}\chi_{i,i+1} + \frac{1}{2}\chi_{2n-i,2n-i+1}\}} = (\overbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}^{i}, 0, \dots, 0, \overbrace{-\tfrac{1}{2}, \dots, -\tfrac{1}{2}}^{i}), \qquad 1 \leq i \leq n-1,$$

$$\omega_{\{\chi_{n,n+1}\}} = (\overbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}^{n}, \overbrace{-\tfrac{1}{2}, \dots, -\tfrac{1}{2}}^{n}).$$

Thus $\mathrm{depth}(\mathrm{GSp}_{2n}, \omega_n^\vee) = \frac{n}{2}$.

By Example 5.3 and Example 5.4, we have $\operatorname{depth}(G, \mu) = 2$ in the following cases:

- $G = \mathrm{GSp}_8$ and $\mu = \omega_4^\vee$ (the Siegel case of genus 4).
- $G =$ the unramified unitary group and $\mu = \omega_2^\vee$.

Interestingly, we also found a simple geometric structure in the these cases.

Theorem 5.5. Let (G, μ) be one of the above cases. Then the variety $X(\mu, \tau)_S$ is naturally a disjoint union of subvarieties which are universally homeomorphic to iterated fibrations over Deligne-Lusztig varieties. The index set of this stratification can be described in terms of the rational Bruhat-Tits building of $\mathbb{J}$.

Proof. This is [43, Theorem 4.3] and [41, Theorem C]. $\square$

In these cases, we cannot expect a nice closure relation such as the cases of positive Coxeter type. Indeed, in the $\mathrm{GU}(2,3)$-case, there exists a stratum whose closure is not a union of strata. See [41, Example 4.11].

6 Relationship to Shimura varieties

Assume that $F = \mathbb{Q}_p$. In this mixed characteristic case, affne Deligne-Lusztig varieties are related to the reduction of certain Shimura varieties, or more directly to moduli spaces of p-divisible groups. These moduli spaces are often called Rapoport-Zink spaces. The relation relies on the Dieudonné theory, which classifies p-divisible groups. A Dieudonné module a free module of finite rank over $\mathcal{O}(= W(\overline{\mathbb{F}}_q))$ together with a σ-linear operator $\mathbf{F}$ (Frobenius) and a σ^{-1}-linear operator $\mathbf{V}$ (Verschiebung) such that $\mathbf{FV} = \mathbf{VF} = p$.

Fix a p-divisible group $\mathbb{X}$ over $\overline{\mathbb{F}}_q$. Let M be its Dieudonné module, and let $N = M \otimes_{\mathcal{O}} L$ be its rational Dieudonné module. We fix a basis of M over $\mathcal{O}$ and write $\mathbf{F}$ as $b\sigma, b \in \mathrm{GL}_n(L)$, where $n = \mathrm{rk}_{\mathcal{O}} M$. Lattices inside N which are stable under $\mathbf{F}$ and $\mathbf{V}$ correspond to quasi-isogenies $\mathbb{X} \to X$ of p-divisible groups over $\overline{\mathbb{F}}_q$. A lattice $gM, g \in \mathrm{GL}_n(L)$ is stable under $\mathbf{F}$ and $\mathbf{V}$ if and only if $p(gM) \subseteq \mathbf{F}(gM) \subseteq gM$. This is also equivalent to saying $g^{-1} b\sigma(g) \in K\varpi^{\omega_i^\vee} K$, where $i = \kappa_{\mathrm{GL}_n}(b) = v_L(\det(b))$. Therefore we can identify the set of $\overline{\mathbb{F}}_q$-valued points of the moduli space of quasi-isogenies attached to $\mathbb{X}$ with the set of closed points of the affine Deligne-Lusztig variety attached to GL_n, $\omega_i^\vee$ and b.

More generally, if (G, μ, b) arises from a Rapoport-Zink datum of Hodge type, then $\mathcal{M}(G, \mu, b)_{\overline{\mathbb{F}}_q}^{\mathrm{pfn}} \cong X(\mu, b)_J$, where $\mathcal{M}(G, \mu, b)_{\overline{\mathbb{F}}_q}$ denotes the special fiber of the corresponding Rapoport-Zink space. This is proved in [47, Proposition 0.4]. Although in [47] it was assumed that $J = S$, the same arguments work in any parahoric level. See [15, §7.2] and [16, §5.3].

Similarly, one obtains a relationship to Shimura varieties, or more precisely to the Newton strata in the special fiber of the corresponding moduli space of abelian varieties. Indeed, by the uniformization theorem by Rapoport-Zink [36] (see also [29] for the case of Shimura varieties of Hodge type), basic Rapoport-Zink spaces are related in an explicit way to the basic locus of Shimura varieties. Although the connection is more complicated for non-basic $[b]$ (cf. [33]), it is true in general that the global EKOR stratum associated to w and the Newton stratum associated to $[b]$ have non-empty intersection if and only if $X_w(b) \neq \emptyset$. See [15, Lemma 7.6].

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