On a group-theoretic approach to the supersingular locus of Shimura varieties

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Motivation

- Simple descriptions of the supersingular (or basic) locus of Shimura varieties have been used towards applications in number theory (the Kudla-Rapoport program, Zhang's Arithmetic Fundamental Lemma, the Tate conjecture for certain Shimura varieties, etc.).
- The study of the perfection of the basic locus is essentially reduced to the study of the affine Deligne-Lusztig variety (ADLV) via the Rapoport-Zink uniformization.

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Definition of DLVs

$$\begin{aligned} G/\mathbb{F}_q &: \text{a connected reductive group.} \\ B &: \text{a Borel subgroup.} \\ \sigma &: \text{the Frobenius automorphism of } \overline{\mathbb{F}}_q/\mathbb{F}_q. \\ W_0 &: \text{the finite Weyl group} \end{aligned}$$

$$G(\mathbb{F}_q) = \{g \in G(\overline{\mathbb{F}}_q) \mid g^{-1}\sigma(g) = 1\}.$$

Definition

For $w \in W_0$, the *(classical) Deligne-Lusztig variety* X_w is a scheme of finite type$/\overline{\mathbb{F}}_q$ defined as

$$X_w = \{gK \in G/B \mid g^{-1}\sigma(g) \in BwB\} \subset G/B.$$

- $G(\mathbb{F}_q)$ acts on X_w by left multiplication.
- DLVs play a crucial role in the DL theory for finite groups of Lie type.

Review of Coxeter groups and examples of DLVs

Let W be a group generated by a subset $S = \{s_1, s_2, \ldots, s_r\}$.

Definition

(W, S) is called a *Coxeter system* if there exist $2 \le m(i,j) \le \infty$ s.t.

$$W = \langle S \mid s_i^2 = 1, \forall i \text{ and } (s_i s_j)^{m(i,j)} = 1, \forall i \ne j \rangle.$$

A word of min. length among words of $w \in W$ is called *reduced*.
$\ell(w) :=$ the length of any reduced word of w.
$\mathrm{supp}(w) =$ the subset of S occurring in some reduced word of w.

(W_0, S) is a Coxeter system with $S =$ the set of simple reflections.

Example

$G = \mathrm{GL}_n$. Then $W_0 \cong \mathfrak{S}_n$ and $S = \{(1\ 2), \ldots, (n-1\ n)\}$.

$$X_{(1\ 2\ \cdots\ n)} \cong \mathrm{Dr}^{n-1}_{\mathbb{F}_q} := \mathbb{P}^{n-1}_{\mathbb{F}_q} \setminus \bigcup_{\substack{H: \\ \text{rational hyperplane}}} H.$$

Definition of ADLVs with Iwahori level

From now on, we will use the following notation.

F : a non-archimedean local field with uniformizer t.

$G/\mathcal{O}_F$: an unramified connected reductive group.

$T \subseteq B$: a max. torus, B : a Borel subgroup.

σ : the Frobenius automorphism of L/F.

$L = \widehat{F^{un}}, K = G(\mathcal{O}_L), J_b = \{g \in G(L) \mid g^{-1}b\sigma(g) = b\}$.

$I \subseteq K$: the standard Iwahori subgroup associated to $T \subset B \subset G$.

$\widetilde{W}$: the Iwahori-Weyl group $\cong W_0 \ltimes X_*(T)$.

Definition

For $w \in \widetilde{W}$ and $b \in G(L)$, the *affine Deligne-Lusztig variety* $X_w(b)$ is a scheme locally (perfectly) of finite type$/\overline{\mathbb{F}}_q$ defined as

$$X_w(b) = \{gI \in G(L)/I \mid g^{-1}b\sigma(g) \in IwI\} \subset G(L)/I.$$

Definition of closed ADLVs with arbitrary parahoric level

Set $\Omega = \{w \in \widetilde{W} \mid \ell(w) = 0\}$.

$W_a \subseteq \widetilde{W}$: the affine Weyl group, $\tilde{S} =$ the set of simple affine reflections.

Then $(W_a, \tilde{S})$ is a Coxeter system and $\widetilde{W} \cong W_a \rtimes \Omega$.

For $v, w \in W_a$, $\tau, \tau' \in \Omega$, we define $v\tau \leq w\tau' \overset{\text{def}}{\Longleftrightarrow} v \leq w$ and $\tau = \tau'$.

Let $\mathrm{Adm}(\mu) = \{w \in \widetilde{W} \mid w \leq t^{w_0\mu} \text{ for some } w_0 \in W_0\}$.

Definition

Let $J \subset \tilde{S}$ with $J = \sigma(J)$. The *closed affine Deligne-Lusztig variety* in $G(L)/P_J$ is the closed reduced $\overline{\mathbb{F}}_q$-subscheme defined as

$$X(\mu, b)_J = \{gP_J \in G(L)/P_J \mid g^{-1}b\sigma(g) \in P_J \mathrm{Adm}(\mu) P_J\},$$

where $P_J \supseteq I$ is the standard parahoric subgroup associated to J.

- J_b acts on $X_w(b)$ and $X(\mu, b)_J$ by left multiplication.
- $P_\emptyset = I$ and $P_S = K$.

Group-theoretic data in the case of GL_n

Example

Let $G = \mathrm{GL}_n$.

T : the torus of diagonal matrices.

B : the subgroup of upper triangular matrices.

$\tilde{S} = \{(1\ 2), \dots, (n-1\ n), (1\ n)t^{(-1,0,\dots,0,1)}\}$.

$$X_*(T) \cong \{t^\lambda = \begin{pmatrix} t^{m_1} & & \\ & \ddots & \\ & & t^{m_n} \end{pmatrix} \mid \lambda = (m_1, \dots, m_n) \in \mathbb{Z}^n\}.$$

Thus $X_*(T) \cong \mathbb{Z}^n$ and hence $\widetilde{W} \cong \mathfrak{S}_n \ltimes \mathbb{Z}^n$.
Set $\tau = t^{(1,0,\dots,0)} s_1 s_2 \cdots s_{n-1}$. Then $\Omega = \{\tau^m \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$.
Set $s_0 = (1\ n)t^{(-1,0,\dots,0,1)}$ and $s_i = (i\ i+1)$. Then $\tau s_i \tau^{-1} = s_{i+1}$.

Relationship to Shimura varieties

Assume that $F = \mathbb{Q}_p$. A *Dieudonné module* a free module of finite rank over $\mathcal{O}(= W(\overline{\mathbb{F}}_q))$ together with a σ-linear operator $\mathbf{F}$ and a σ^{-1}-linear operator $\mathbf{V}$ such that $\mathbf{FV} = \mathbf{VF} = p$.

$\mathbb{X}$: a fixed p-divisible group over $\overline{\mathbb{F}}_q$.

M : the Dieudonné module attached to $\mathbb{X}$, $N := M \otimes_{\mathcal{O}} L$.

Fix a basis of M over $\mathcal{O}$ and write $\mathbf{F} = b\sigma, b \in \mathrm{GL}_n(L)$, $n = \mathrm{rk}_{\mathcal{O}} M$. Lattices inside N which are stable under $\mathbf{F}$ and $\mathbf{V}$ correspond to quasi-isogenies $\mathbb{X} \to X$ of p-divisible groups over $\overline{\mathbb{F}}_q$.

$gM, g \in \mathrm{GL}_n(L)$ is stable under $\mathbf{F}$ and $\mathbf{V}$ $\Leftrightarrow$ $p(gM) \subseteq \mathbf{F}(gM) \subseteq gM$,

i.e., $g^{-1} b\sigma(g) \in K p^{\mu} K = K \mathrm{Adm}(\mu) K$ for some μ of the form $(1, \ldots, 1, 0, \ldots, 0)$ (minuscule cocharacters).
More generally, if (G, μ, b) arises from a Rapoport-Zink datum of Hodge type, then $\mathcal{M}(G, \mu, b)^{\mathrm{pfn}}_{\overline{\mathbb{F}}_q} \cong X(\mu, b)_J$, where $\mathcal{M}(G, \mu, b)_{\overline{\mathbb{F}}_q}$ denotes the special fiber of the corresponding Rapoport-Zink space.

Summary of part 1

- $(G, \mu, J, b) \rightsquigarrow X(\mu, b)_J$.
- Affine Deligne-Lusztig varieties are p-adic analogue of classical Deligne-Lusztig varieties.
- The perfection of a Rapoport-Zink space is an affine Deligne-Lusztig variety.

From now on, we pass to the perfection even in the equal characteristic case for simplicity.

The EKOR stratification

Let $J \subseteq \tilde{S}$ and let W_J be the subgroup of $\widetilde{W}$ generated by J.

${}^J\widetilde{W}$: the set of minimal length representatives for the cosets in $W_J \backslash \widetilde{W}$.

Set ${}^J\mathrm{Adm}(\mu) = \mathrm{Adm}(\mu) \cap {}^J\widetilde{W}$.

Proposition

We have the EKOR stratification

$$X(\mu, b)_J = \bigsqcup_{w \in {}^J\mathrm{Adm}(\mu)} \pi_J(X_w(b)),$$

where $\pi_J \colon G(L)/I \to G(L)/P_J$ is the projection.

- If $J = \emptyset$, we speak of the KR stratification instead.
- If $J = S$, we speak of the EO stratification instead.
- This is the local analogue of the stratification defined by He-Rapoport in the global context of Shimura varieties.

Elements with spherical σ-support

We say that $b \in G(L)$ is basic if the Newton vector of b is central. Let τ_μ be the image of t^μ under the projection $\widetilde{W} = W_a \rtimes \Omega \to \Omega$. Then τ_μ is basic and $\pi_J(X_w(\tau_\mu))$ corresponds to the intersection of a global EKOR stratum with the basic Newton stratum.

Theorem (Görtz-He)

Let $\tau \in \Omega$. Let $w \in W_a\tau$ such that $W_{\mathrm{supp}_\sigma(w)}$ is finite. Then $X_w(\tau)$ is a disjoint union of (classical) Deligne-Lusztig varieties.

- The same is true for $\pi_J(X_w(\tau))$.
- The global EKOR stratum associated to w is contained in the basic locus if and only if $W_{\mathrm{supp}_\sigma(w)}$ is finite.

Example

Let $G = \mathrm{GL}_2$. Then

$$X_{s_0}(1) = \bigsqcup_{G(F)/G(\mathcal{O}_F)} \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q).$$

Let Δ denote the set of simple roots. For $\alpha \in \Delta$, we define ω_α to be the rational fundamental weight such that

$$\langle \omega_\alpha, \beta^\vee \rangle = \begin{cases} 1 & (\beta = \alpha) \\ 0 & (\beta \in \Delta \setminus \{\alpha\}). \end{cases}$$

For each σ-orbit $\mathscr{O}$ of Δ, we set

$$\omega_{\mathscr{O}} = \sum_{\alpha \in \mathscr{O}} \omega_\alpha.$$

For a dominant cocharacter $\mu \in X_*(T)$, we define

$$\mathrm{depth}(G,\mu) := \max_{\mathscr{O} \subseteq \Delta} \langle \omega_{\mathscr{O}}, \mu \rangle,$$

where $\mathscr{O}$ runs through all σ-orbits of S.

Fully Hodge-Newton decomposable pairs

Set ${}^J Adm(\mu)_{\neq\emptyset} = \{w \in {}^J Adm(\mu) \mid X_w(\tau_\mu) \neq \emptyset\}$.

Theorem (Görtz-He-Nie)

The pair (G, μ) is fully Hodge-Newton decomposable if and only if the following equivalent conditions are satisfied:

1. *The cocharacter μ is minute $\overset{def}{\iff} \mathrm{depth}(G, \mu) \leq 1$.*
2. *$W_{\mathrm{supp}_\sigma(w)}$ is finite for every $w \in {}^J Adm(\mu)_{\neq\emptyset}$.*

In particular, the validity of the condition (ii) is independent of the rational level J.

Theorem (Görtz-He-Nie)

If (G, μ) is fully Hodge-Newton decomposable, then $X(\mu, \tau_\mu)_J$ is naturally a disjoint union of Deligne-Lusztig varieties.

- This is called the *weak Bruhat-Tits stratification*.
- The closure relation can be described in terms of the Bruhat-Tits building of J_b.

The cases of Coxeter type

We call an element $w \in \widetilde{W}$ a σ-*Coxeter element* if w can be written as the product of elements of $\tilde{S}$ which lie in different σ-orbits.

Definition

We say that (G, μ, J) is of Coxeter type if every $w \in {}^J\mathrm{Adm}(\mu)_{\neq\emptyset}$ is a σ-Coxeter element with $W_{\mathrm{supp}_\sigma(w)}$ finite.

Theorem (Görtz-He, Görtz-He-Nie)

If (G, μ, J) is of Coxeter type, then $X(\mu, \tau_\mu)_J$ is naturally a disjoint union of Deligne-Lusztig varieties of Coxeter type.

- This is called the *Bruhat-Tits stratification* because it satisfies some nicer properties.
- Deligne-Lusztig varieties of Coxeter type are especially important in the (classical) Deligne-Lusztig theory.
- The classification of fully Hodge-Newton decomposable cases and the cases of Coxeter type are known.

Examples of Coxeter type

Example

The fully Hodge-Newton decomposable cases contain the following cases which have been studied in the context of Shimura varieties:

- The Siegel case of genus 2, which has been studied by Katsura-Oort and Kaiser.
- The $\mathrm{GU}(1, n-1)$, p split case, which has been studied by Harris-Taylor.
- The $\mathrm{GU}(1, n-1)$, p inert case, which has been studied by Vollaard-Wedhorn.
- The $\mathrm{GU}(2, 2)$, p inert case, which has been studied by Howard-Pappas.

All of these cases concern $J = S$, and are of Coxeter type.

Example ($G = \mathrm{GL}_2$)

$$X((1,-1),1)_S = \bigsqcup_{G(F)/G(\mathcal{O}_F)} \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q) \sqcup \bigsqcup_{G(F)/G(\mathcal{O}_F)} \{\mathrm{pt}\}.$$

Summary of part 2

- $\mathrm{depth}(G, \mu) \in \mathbb{Q}$.
- (G, μ) is fully HN decomposable $\Leftrightarrow \mathrm{depth}(G, \mu) \leq 1$.
- If this is the case, then $X(\mu, \tau_\mu)_J$ is a union of DLVs.
- The cases of Coxeter type are special cases of fully HN decomposable cases.

From now on, we will mainly focus on the case $J = S$.

Examples beyond the cases of Coxeter type

Example (Ivanov)

Let $G = \mathrm{GL}_2$. Then

$$X_{s_1 t^{(-r,r)}}(1) \cong \coprod_{G(F)/G(\mathcal{O}_F)} (\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)) \times \mathbb{A}^{r-1}.$$

Theorem (Chan-Ivanov)

Let $G = \mathrm{GL}_n$. *Set* $c_r = (1\ 2\ \cdots\ n)t^{(-r,\ldots,-r,(n-1)r)}$. *Then*

$$X_{c_r}(1) \cong \coprod_{G(F)/G(\mathcal{O}_F)} \mathrm{Dr}_{\mathbb{F}_q}^{n-1} \times \mathbb{A}^{\frac{n(n-1)r}{2}-n+1}.$$

- Using this description, Chan-Ivanov gave a geometric realization of the local Langlands correspondence in many interesting cases.

Length positive elements and non-emptiness criterion

Let $p := \widetilde{W} \to W_0$ be the projection. For $w = p(w)t^{\mu} \in \widetilde{W}$, the set of *length positive elements* is

$$\mathrm{LP}(w) := \{v \in W_0 \mid \langle v\alpha, \mu\rangle + \delta^{+}(v\alpha) - \delta^{+}(p(w)v\alpha) \geq 0 \text{ for all } \alpha \in \Phi_{+}\}.$$

Here $\delta^{+} \colon \Phi \to \{0, 1\}$ denotes the charcteristic function of Φ_{+}. It is easy to check that $\mathrm{LP}(w) \neq \emptyset$.

Theorem (Görtz-He-Nie, Lim, Schremmer)

Assume that the Dynkin diagram of G is σ-connected. Let $w \in W_a\tau$. Then

$$X_w(\tau) \neq \emptyset \Leftrightarrow \text{(i) } W_{\mathrm{supp}_{\sigma}(w)} \text{ is finite, or,}$$
$$\text{(ii) } \forall v \in \mathrm{LP}(w), \mathrm{supp}_{\sigma}(\sigma^{-1}(v^{-1})p(w)v) = S$$

- "σ-Coxeter" is a Coxeter condition in (i) focusing on $W_a \rtimes \Omega$.
- What happens if we consider a Coxeter condition in (ii) focusing on $W_0 \ltimes X_*(T)$?

Elements with positive Coxeter part

Theorem (S., He-Nie-Yu, Schremmer-S.-Yu)

If $w \in \widetilde{W}$ has positive Coxeter part, i.e., $\exists v \in \mathrm{LP}(w)$ such that $\sigma^{-1}(v^{-1})p(w)v$ is a σ-Coxeter element, then $X_w(b)$ is a disjoint union of iterated fibrations over Deligne-Lusztig varieties of Coxeter type whose iterated fibers are $\mathbb{A}^1$ or $\mathbb{G}_m$. If b is basic, then all fibers are $\mathbb{A}^1$ and each iterated fibration is the product of varieties.

Proposition (Deligne-Lusztig, Görtz-He)

Let $w \in \widetilde{W}$ and let $s \in \tilde{S}$ be a simple affine reflection.

1. If $\ell(sw\sigma(s)) = \ell(w)$, then $X_w(b) \cong X_{sw\sigma(s)}(b)$.
2. If $\ell(sw\sigma(s)) = \ell(w) - 2$, then there exists a decomposition $X_w(b) = X_1 \sqcup X_2$ such that
   - X_1 is open and there exists a J_b-equivariant morphism $X_1 \to X_{sw}(b)$, which is a Zariski-locally trivial $\mathbb{G}_m^{\mathrm{pfn}}$-bundle.
   - X_2 is closed and there exists a J_b-equivariant morphism $X_2 \to X_{sw\sigma(s)}(b)$, which is a Zariski-locally trivial $\mathbb{A}^{1,\mathrm{pfn}}$-bundle.

The definition of positive Coxeter type

Definition

We say that (G, μ, J) is of *positive Coxeter type* if every $w \in {}^J Adm(\mu)_{\neq \emptyset}$ satisfies one of the following conditions:

1. w is a σ-Coxeter element with $W_{\mathrm{supp}_\sigma(w)}$ finite.
2. w has positive Coxeter part.

- Clearly, this notion is a generalization of Coxeter type.

Theorem (S.)

If (GL_n, μ, S) is of positive Coxeter type, then $X(\mu, \tau_\mu)_S$ is naturally a disjoint union of the product of a Deligne-Lusztig variety of Coxeter type and a finite-dimensional affine space.

- The index set of this stratification and can be described in terms of the Bruhat-Tits building of J_b. If μ is minuscule, the same is true for the closure relations between strata. So this is a natural generalization of the Bruhat-Tits stratification.

Classification of Coxeter type

Let $\omega_k^\vee$ denote the cocharacter of the form $(1, \ldots, 1, 0, \ldots, 0)$ in which 1 is repeated k times.

Theorem

The following assertions on μ are equivalent.

1. (i) *The pair (GL_n, μ, S) is of Coxeter type.*
2. (ii) *The cocharacter μ is central or one of the following forms modulo $\mathbb{Z}\omega_n^\vee$:*

$$\omega_1^\vee, \quad \omega_{n-1}^\vee \ (n \geq 1), \quad \omega_1^\vee + \omega_{n-1}^\vee \ (n \geq 2), \quad \omega_2^\vee \ (n = 4).$$

- The $(\mathrm{GL}_4, \omega_2^\vee, S)$-case was studied by Fox.

Classification of positive Coxeter type

Theorem (S.)

The following assertions on μ are equivalent.

1. *The pair (GL_n, μ, S) is of positive Coxeter type.*
2. *The cocharacter μ is central or one of the following forms modulo $\mathbb{Z}\omega_n^\vee$:*

$$
\begin{aligned}
&\omega_1^\vee, \quad \omega_{n-1}^\vee, \ (n \geq 1),\\
&\omega_1^\vee + \omega_{n-1}^\vee, \quad \omega_2^\vee, \quad 2\omega_1^\vee, \quad \omega_{n-2}^\vee, \quad 2\omega_{n-1}^\vee,\\
&\omega_2^\vee + \omega_{n-1}^\vee, \quad 2\omega_1^\vee + \omega_{n-1}^\vee, \quad \omega_1^\vee + \omega_{n-2}^\vee, \quad \omega_1^\vee + 2\omega_{n-1}^\vee, \ (n \geq 3),\\
&\omega_3^\vee, \quad \omega_{n-3}^\vee, \ (n = 6, 7, 8),\\
&3\omega_1^\vee, \quad 3\omega_{n-1}^\vee, \ (n = 3, 4, 5),\\
&\omega_1^\vee + \omega_2^\vee, \quad \omega_3^\vee + \omega_4^\vee, \ (n = 5),\\
&4\omega_1^\vee, \quad \omega_1^\vee + 3\omega_2^\vee, \quad 4\omega_2^\vee, \quad 3\omega_1^\vee + \omega_2^\vee, (n = 3),\\
&m\omega_1^\vee \text{ with } m \in \mathbb{Z}_{>0}, (n = 2).
\end{aligned}
$$

The Siegel case

Consider $(\mathrm{GSp}_{2n}, \omega_n^\vee, S)$. If $n = 2$, then this is of Coxeter type.

Theorem (S.-Takamatsu)

If $n = 3$, then it is of positive Coxeter type. The corresponding $X(\mu, \tau_\mu)_S$ is naturally a disjoint union of the product of a Deligne-Lusztig variety of Coxeter type and a finite-dimensional affine space. The index set and the closure relation can be described in terms of the Bruhat-Tits building of J_b.

Although it is not of positive Coxeter type if $n \geq 4$, we still have a simple geometric structure in the case $n = 4$.

Theorem (S.-Takamatsu)

If $n = 4$, then the corresponding $X(\mu, \tau_\mu)_S$ is naturally a disjoint union of the product of a Deligne-Lusztig variety and a finite-dimensional affine space. The index set can be described in terms of the Bruhat-Tits building of J_b.

$\mathrm{depth}(G, \mu) \leq 2$

It is easy to check the following proposition.

Proposition

Let $G = \mathrm{GL}_n$ and let $\mu \in X_*(T)_+$. If $\mathrm{depth}(G, \mu) < 2$, then (G, μ, S) is of positive Coxeter type. If $n \geq 6$, then $\mathrm{depth}(G, \mu) < 2$ if and only if (G, μ, S) is of positive Coxeter type.

- We have $\mathrm{depth}(\mathrm{GSp}_{2n}, \omega_n^\vee) = \frac{n}{2}$.

Conjecture

If $\mathrm{depth}(G, \mu) < 2$, then (G, μ, S) is of positive Coxeter type and hence $X(\mu, \tau_\mu)_S$ has a simple geometric structure.

Conjecture

If $\mathrm{depth}(G, \mu) = 2$, then $X(\mu, \tau_\mu)_S$ has a simple geom. structure.

I recently proved the existence of a simple geometric structure in the $\mathrm{GU}(2, n-2)$, p inert case. This case also has $\mathrm{depth}(G, \mu) = 2$.

Summary of part 3

- The case of positive Coxeter type is a generalization of Coxeter type.
- The $\mathrm{depth}(G, \mu) \leq 2$-cases seem to imply a simple geometric structure on $X(\mu, \tau)_S$.