

1. (8p) There are  $n$  infinite sequences of positive integers,

$$A_1 = a_{11} \ a_{12} \ a_{13} \ \dots,$$

$$A_2 = a_{21} \ a_{22} \ a_{23} \ \dots,$$

$$A_3 = a_{31} \ a_{32} \ a_{33} \ \dots,$$

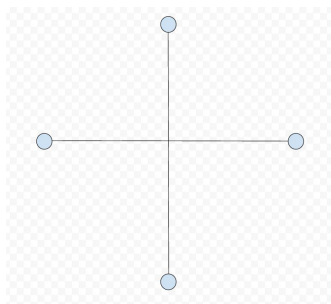
...

$$A_n = a_{n1} \ a_{n2} \ a_{n3} \ \dots,$$

Show that there must exist  $i$  and  $j$  such that  $a_{ki} \leq a_{kj}$  for all  $1 \leq k \leq n$ .

2. (8p) How many different ways are there to paint  $n$  balls with 4 colors (A, B, C and D) such that the number of balls in color A is odd while the number of those in color B is even (including 0)?

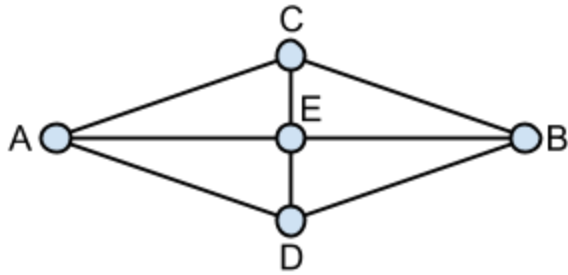
3. (8p) There are 4 lights on a device in the direction of north, south, east and west, as shown in the picture. The device can turn 90 degrees clockwise or counterclockwise each time. Now let's play a game. The player is blindfolded and doesn't know the initial status of each light (on/off). But he/she can ask the host to change the status of one or more lights each time (by indicating the directions). The host will tell the player how many lights are on after he makes the change, and then he rotates the device clockwise or counterclockwise randomly by 90 degrees (without telling the player). If the player manages to turn on all four lights he/she wins the game. If you are to play this game do you have any strategy to win in finite steps? If not, show your strategy that has the highest probability to win.



4. (8p) There is a weird calculator that can only give the result of  $xy + x + y + 1$  for any 2 real numbers  $x$  and  $y$ . Can you use this calculator to calculate  $x^7y + xy + x + 1$  for any given  $x$  and  $y$ ?

5. (10p) Two villages A, B are connected by the roads shown in the graph.

A hurricane breaks some road(s). If the probability of each road broken is  $\frac{1}{2}$  (there are 8 roads in the graph and they are all independent with each other) find the probability that A and B are still connected.



6. (10p) 10 investors bought 3 stocks each and any two of them bought at least one stock in common. For the stock owned by most people, what is the minimum number of people who have bought it?

7. (12p) Let's play a game as follows. Assume that the player starts from  $x=0$  on the number line and can move towards  $+\infty$  each time at a distance of the number rolled out from a six-face even dice with 1,2,3,4,5,6 on the faces. There are two boxes at  $x=10000$  and  $x=10001$  respectively.

7.1) (6p) What is the probability that the player will reach at least one of the 2 boxes?

7.2) (6p) If the two boxes are at  $x=10000$  and  $x=10002$  respectively what is the probability that the player will reach at least one of them?