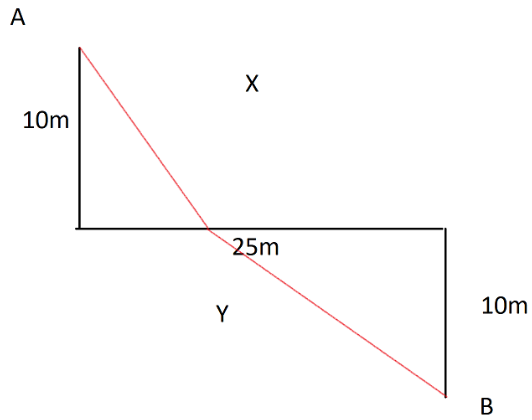
DTL Online Written Test

August 20, 2018

Declaration:
1. Please DO NOT disclose this written test to any third party.
2. Please use only pen and paper for working out the questions. You may only use computers to key in the steps/answers for submission.
3. Please write down the detailed steps for getting the final answer. Simply writing down the final answer without steps will get 0 marks.

1. (6p) Jerry is traveling from point A toward point B. Speed of the Jerry is different for upper region(X Zone) and lower region(Y Zone): 1m/min and 2m/min respectively. What is the distance of shortest-time path?



2. (10p) Prove that any subset of $\{0, 1, \ldots, 2018\}$ with size more than 1009 contains either a power of 2, or two distinct integers whose sum is a power of 2.

3. (10p) If 46 squares are colored red in a 9X9 board, prove that there is a 2X2 block on the board in which at least 3 of the squares are colored red.

4. (10p) There is a group of people. If men will only be honest with men, but lie to women. And women will only be honest with women but lie to men. They are standing around a circle. The total number of people is 13. One person(person1) said to the person(person2) left to him(her):"the number of men is larger than the number of women". Then person2 said to the person(person3) left to him(her):"the number of women is larger than the number of men" and so on. The person13 told the person1:"the number of men is larger than the number of women". What is the possible numbers of men?

5. (12p) Find all pairs of integers (m, n) such that: $m^3 - n^3 = 2mn + 8$.

6. (12p) Now we have an area composed by $21 * 21$ squares. Let's fill it with three colors, red, blue and green (RBG), each color fills 147 squares. Each two neighbouring squares share a common boundary. If the colors on both sides of a boundary are different, this boundary is a 'special boundary'.

   (a) (4p) What is the maximun number of special boundaries? Prove it.

   (b) (8p) What is the minimun number? Prove it.

7. (20p) N coins with face up at first,

   (a) (6p) If put them in a row, first time flip every coins, second time start from the 2nd coin and flip every two coins, 3rd time from the 3rd coin and flip every three coins, and so on. After N-th time, which coins are now face down?

   (b) (7p) If put them in a circle and turn the coins by clockwise, first time flip every coins, second time start from the 2nd coin and flip every two coins, and so on, each time will stop if next coin to be flipped have already flipped during this time. After N-th time, which coins are now face down?

   (c) (7p) If put them in a circle and turn the coins by clockwise, first time flip every coins, second time start from the 2nd coin and flip every two coins, and so on, each time will flip N coins then stop. After N-th time, which coins are now face down?

8. (a) (10p) Show that for any positive integer n, for any integer $m < 2n - 1$ and $m >= n$, we can always construct a set of m integers (integers can repeat) such that the sum of any n integers in the set is not divisible by n.

   (b) (10p) Prove that for any positive integer n, for any set of $m = 2n - 1$ integers (integers can repeat), there must exist n integers in the set whose sum is divisible by n. [hint: you might need to use Fermat's Little Theorem: If p is a prime, and p does not divide a, then $a^{p-1} \equiv 1 \pmod{p}$.]