DTL Online Written Test

October 19, 2017

Declarations:
1. Please DO NOT disclose this written test to any third party.
2. Please use only pen and paper for working out the questions. You may only use computers to key in the steps/answers for submission.
3. Please write down the detailed steps for getting the final answer. Simply writing down the final answer without steps will get 0 marks.

1. (10p) There are two fleas on an infinitely large chessboard where each square is either red or black (there are at least one red square and one black square). One flea (called R) is on a red square and another (called B) is on a black square. R can only jump to any red squares which are in the same row or in the same column as its current square. Similarly, B can only jump to any black squares which are in the same row or in the same column as its current square. What is the minimum value of n such that under any circumstances R and B will meet (when two fleas are in two neighboring squares which are in the same column or in the same row) within n jumps (n = no. of times R jumps + no. of times B jumps)? Please prove your conclusion.

2. (10p) Given $0 \leq x, y, z \leq 1$, find the maximum value of $\sqrt[n]{|x-y|} + \sqrt[n]{|y-z|} + \sqrt[n]{|z-x|}$ (n is a positive integer).

3. (12p) Is there a ten-digit positive integer such that its digits (from left to right) respectively represent the number of $0, 1, 2, ..., 9$ that appear in the integer? If yes, please list all the integers satisfying the condition; if not, please prove it.

4. (12p) Consider the following simplified election:
A country has 3 regions and each region has 3 candidates. These 3 candidates will vote to choose a president candidate from themselves to represent the region. Then the 3 president candidates will vote again to choose a president from themselves. In each election, the person getting 2 votes out of 3 will win. Suppose that out of the 9 candidates, 4 of them belong to the minority group, and the other 5 belong to the majority group. They always vote for candidates in their own group. Given that candidates are randomly and uniformly distributed, what is the probability of one member of the minority group being elected president?
(Tips: There are $\binom{n}{a}\binom{n-a}{b}/K$ ways to divide n persons into 3 groups with a, b, c persons in each group. When a, b, c are all different with each other, $K = 1$; when $a = b = c$, $K = 3! = 6$; for other cases, $K = 2! = 2$. $\binom{n}{m}$ here means the number of combinations when choosing m persons from n.)

5. (12p) A, B and C signed up for a game. Each one of them has a label attached to their back with the number of either 0 or 1. They can see the other two players' numbers but not their own. Now each one of them has two choices: to guess their own number or to give up. If at least one person chooses to guess his/her number and gets it right, all three of them will be rewarded with 10,000 dollars; otherwise they get nothing. The numbers are generated randomly. Once the game starts, they are not allowed to communicate in any means, and they cannot see other players' reaction. However, they are allowed to discuss their strategy before the game. Please device a strategy for them to maximize their chance of winning and calculate the expectation of their reward.

6. (14p) There is a set of weights, each of a different mass. Their markings were somehow mixed up (e.g. 1g weight might have been marked as 2g) but the set of markings is the same as the set of actual masses (e.g. 1/2/3g weights may be marked as 2/1/3g, but not 2/1/4g). Provided that you are not to sense the mass of the weights by holding them or using any tools other than a balance, which can only tell you which side of the balance is heavier (or both sides have equal masses). Solve the following problems:

   (a) (7p) If the set of masses is {1g, 2g, 3g, 4g, 5g}, how to determine whether all weights are correctly marked by using the balance only twice?

(b) (7p) If the set of masses is {1g, 2g, 3g, 4g, 5g, 6g}, how to determine whether all weights are correctly marked by using the balance only twice?

7. (15p) Fill the numbers $1, 2, 3, ..., n^2$ randomly into a $n \times n$ grid, each cell with one number and no duplicates. A is the minimum value among the maximum values of each row; B is the maximum value among the minimum values of each column. Please calculate the probabilty that $A > B$.

8. (15p) We have several ropes with some knots in each of them (the number of knots could be different for each rope). Two players A and B play a game as following: They pick some continous knots from a chosen rope in turn, for example, a rope with 4 knots denoted by $a-b-c-d$, the player can take a or b or $b-c$ or $b-c-d$ or $a-b-c-d$ away, but cannot take away knots a and c within one turn; and if knot b is taken, the rope is cut into two ropes a and $c-d$ which can be chosen in the remaining turns. For each turn, the player can only pick one rope and take at least one knot. The player who picks the last knot will lose the game. The game starts from player A. If A and B are both very smart, in what condition that A will win for sure; and in what condition that B will win for sure?