DTL Online Written Test

Declarations:
1. Please DO NOT disclose this written test to any third party.
2. Please use only pen and paper for working out the questions. You may only use computers to key in the steps/answers for submission.
3. Please write down the detailed steps for getting the final answer. Simply writing down the final answer without steps will get 0 marks.

1. (10p) A mobile phone manufacture got some reports for the phone battery life. Given that
   (1) If the battery works well for the first 3 years, there is 40% chance that it will work well for next 2 years;
   (2) If the battery works well for the first 4 years, there is 15% chance that it will work well for next 2 years;
   (3) If the battery works well for the first 3 years, there is 12% chance that it will work well for next 3 years;

   If the battery works well for the first 4 years, what is the probability that it will work well for next 1 year?

2. (12p) We have the following constraints for 100 real numbers:

$$\begin{aligned} x_1 &= 1 \\ 0 \le x_2 &\le 2x_1 \\ 0 \le x_3 &\le 2x_2 \\ &\ldots\ldots \\ 0 \le x_{99} &\le 2x_{98} \\ 0 \le x_{100} &\le 2x_{99} \end{aligned} \tag{1}$$

   Please find the maximum value of $S = x_1 - x_2 + x_3 - x_4 + ... + x_{99} - x_{100}$.

3. The basic algorithm for computing 3^k(k is a positive integer), is to repeatedly multiply by 3, $k-1$ times. So for example, 3^{10} can be computed in 9 multiplication steps using this method. However, there are more efficient methods for computing it in terms of the number of multiplication steps required. For example, we can use the fact that $3^k * 3^k = 3^{2k}$, to get $3^2, 3^4, 3^8$, in 3 multiplication steps by repeated squaring. In this problem, we can reuse a number that we have already computed as many times as we want in future computations, without adding to our count of multiplication steps. Continuing, $3^{10} = 3^2 * 3^8$ in one more multiplication step, for a total of 4 multiplication steps. Please answer the following:
   (a) (12p) Show that this algorithm has optimal efficiency, in terms of fewest multiplication steps, for computing 3^{10}.
   (b) (12p) Is this algorithm always the most efficient for all positive integer k? If yes, prove it; if no, show an example of k that it is not.

4. Suppose you have a 10×10 checkerboard and a deck of 2×2 cards with squares that match the size of the squares of the checkerboard, so 25 of these cards can be used to completely cover the checkerboard. If we allow the cards to overlap each other, there are many ways to cover the checkerboard. We say that an arrangement of cards is a covering if all of the cards in the arrangement are lined up with the squares on the checkerboard, and they are completely on the checkerboard, possibly overlapping, and every square of the checkerboard has at least one card on top of it. We call a covering of the checkerboard redundant if one of the cards can be removed and the checkerboard is still covered. A covering of the checkerboard is non-redundant if it is no longer a covering if any card is removed. Clearly, the smallest non-redundant covering has 25 cards.
   (a) (12p) Show that there is a non-redundant covering with 35 cards.
   (b) (12p) Show that every covering with 55 cards is redundant.

(To simplify the notation, please use $S(x, y)$ to denote the unit squares on the checkerboard, $S(1, 1)$ is the upper-left corner square and $S(10, 10)$ is the bottom-right corner square; for 2x2 cards, please use $C(x, y)$ to denote the card with the upper-left corner it covered is $S(x, y)$, i.e. it covers the 4 squares $S(x, y), S(x + 1, y), S(x, y + 1), S(x + 1, y + 1)$).

5. A polynomial $f(x)$ (with the coefficients as real numbers) has the factor square property (or FSP) if $f(x)$ is a factor of $f(x^2)$. For example, $g(x) = x - 1$ and $h(x) = x$ have FSP, but $k(x) = x + 2$ does not, because $x - 1$ is a factor of $x^2 - 1$, and x is a factor of x^2 but $x + 2$ is not a factor of $x^2 + 2$. Notice that $Cf(x)$ (C is a real number constant other than 0) has FSP iff $f(x)$ has FSP, so let's only focus on the polynomials that are monic (i.e. have 1 as the highest-degree coefficient).

   (a) (10p) List all the monic FSP polynomials of degree 2. For example, $x^2, x^2 - x, x^2 - 1$ should be on the list.

   (b) (20p) Some monic FSP polynomials are products of lower-degree monic FSP polynomials. For example, x^2 and $x^2 - x$ arise from degree 1 cases, but $x^2 - 1$ does not. List all the monic FSP polynomials of degree 3 that is not a product of lower-degree monic FSP polynomials. For example, x^3 should NOT be on the list.