3.2 Domain and Range

Learning Objectives

In this section, you will:

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

Horror and thriller movies are both popular and, very often, extremely profitable. When big-budget actors, shooting locations, and special effects are included, however, studios count on even more viewership to be successful. Consider five major thriller/horror entries from the early 2000s—*I am Legend*, *Hannibal*, *The Ring*, *The Grudge*, and *The Conjuring*. Figure 1 shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

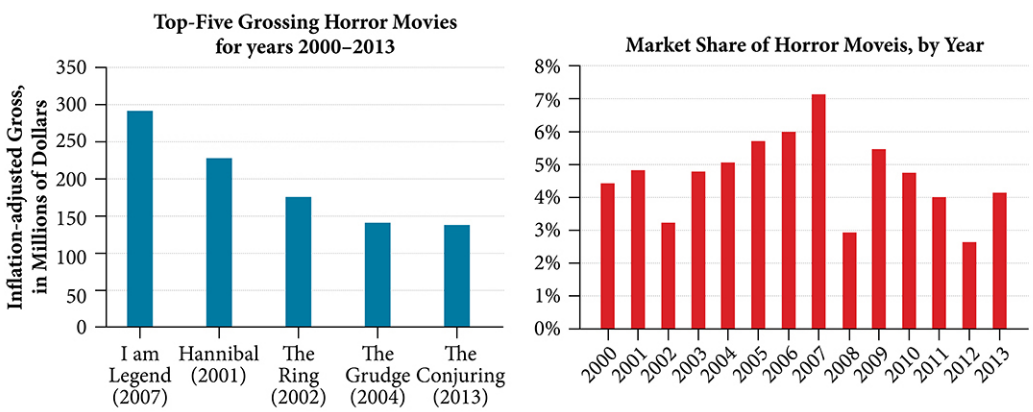

Figure 1 Based on data compiled by www.the-numbers.com.[3]

Finding the Domain of a Function Defined by an Equation

In Functions and Function Notation, we were introduced to the concepts of domain and range. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a “holding area” that contains “raw materials” for a “function machine” and the range as another “holding area” for the machine’s products. See Figure 2.

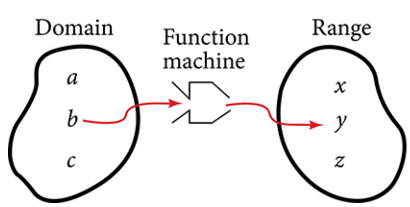

Figure 2

We can write the domain and range in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, they would need to express the interval that is more than 0 and less than or equal to 100 and write $(0,\ 100]$. We will discuss interval notation in greater detail later.

Let’s turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain

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3 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014

Exercises

Review Exercises

Functions and Function Notation

For the following exercises, determine whether the relation is a function.

1. $\{(a, b), (c, d), (e, d)\}$

2. $\{(5, 2), (6, 1), (6, 2), (4, 8)\}$

3. $y^2 + 4 = x$, for x the independent variable and y the dependent variable

4. Is the graph in Figure 1 a function?

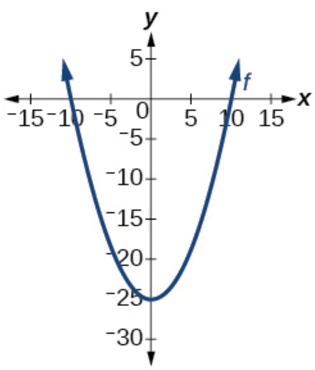


Figure 1

For the following exercises, evaluate $f(-3)$; $f(2)$; $f(-a)$; $-f(a)$; $f(a+h)$.

5. $f(x) = -2x^2 + 3x$

6. $f(x) = 2\,|3x - 1|$

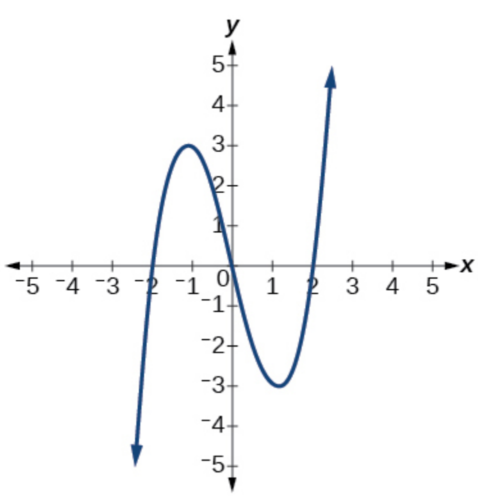
For the following exercises, determine whether the functions are one-to-one.

7. $f(x) = -3x + 5$

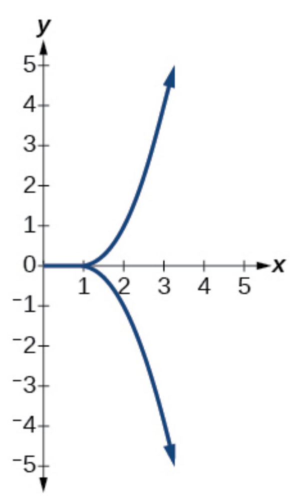
8. $f(x) = |x - 3|$

For the following exercises, use the vertical line test to determine if the relation whose graph is provided is a function.

9.



10.



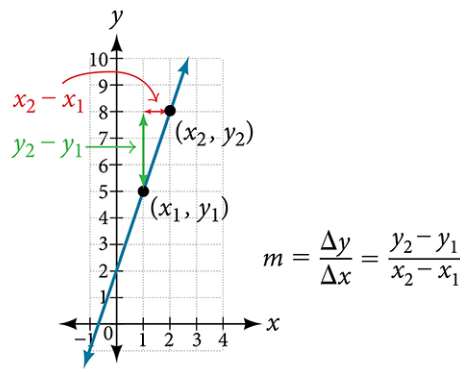


Figure 6 The slope of a function is calculated by the change in y divided by the change in x. It does not matter which coordinate is used as the (x_2, y_2) and which is the (x_1, y_1), as long as each calculation is started with the elements from the same coordinate pair.

Q&A **Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?**

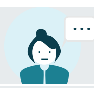
Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

Calculate Slope

The slope, or rate of change, of a function m can be calculated according to the following:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where x_1 and x_2 are input values, y_1 and y_2 are output values.



HOW TO

Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

EXAMPLE 3

Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3, -2)$ and $(8, 1)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution

The coordinate pairs are $(3, -2)$ and $(8, 1)$. To find the rate of change, we divide the change in output by the change in input.

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1 - (-2)}{8 - 3} = \frac{3}{5}$$

We could also write the slope as $m = 0.6$. The function is increasing because $m > 0$.

Marcus will have 380 songs in 12 months.

Analysis

Notice that *N* is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

EXAMPLE 9

Using a Linear Function to Calculate Salary Based on Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income I, depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

Solution

The given information gives us two input-output pairs: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change.

$$\begin{aligned} m &= \frac{920-760}{5-3} \\ &= \frac{\$160}{2 \text{ policies}} \\ &= \$80 \text{ per policy} \end{aligned}$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy. Therefore, Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value.

$$\begin{aligned} I(n) &= 80n + b \\ 760 &= 80(3) + b \qquad \text{when } n = 3, I(3) = 760 \\ 760 &- 80(3) = b \\ 520 &= b \end{aligned}$$

The value of b is the starting value for the function and represents Ilya's income when $n = 0$, or when no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

$$I(n) = 80n + 520$$

Our final interpretation is that Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold.

EXAMPLE 10

Using Tabular Form to Write an Equation for a Linear Function

Table 1 relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

number of weeks, *w*	0	2	4	6
number of rats, *P(w)*	1000	1080	1160	1240

Table 1

Solution

We can see from the table that the initial value for the number of rats is 1000, so $b = 1000$.

EXAMPLE 11

Using Local Extrema to Solve Applications

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic and then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

Solution

We will start this problem by drawing a picture like that in Figure 22, labeling the width of the cut-out squares with a variable, w.

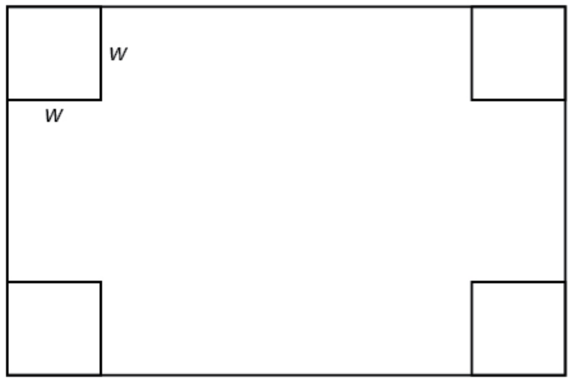


Figure 22

Notice that after a square is cut out from each end, it leaves a $(14 - 2w)$ cm by $(20 - 2w)$ cm rectangle for the base of the box, and the box will be w cm tall. This gives the volume

$$\begin{aligned} V(w) &= (20 - 2w)(14 - 2w)w \\ &= 280w - 68w^2 + 4w^3 \end{aligned}$$

Notice, since the factors are w, $20–2w$ and $14–2w$, the three zeros are 10, 7, and 0, respectively. Because a height of 0 cm is not reasonable, we consider the only the zeros 10 and 7. The shortest side is 14 and we are cutting off two squares, so values w may take on are greater than zero or less than 7. This means we will restrict the domain of this function to $0 < w < 7$. Using technology to sketch the graph of $V(w)$ on this reasonable domain, we get a graph like that in Figure 23. We can use this graph to estimate the maximum value for the volume, restricted to values for w that are reasonable for this problem—values from 0 to 7.

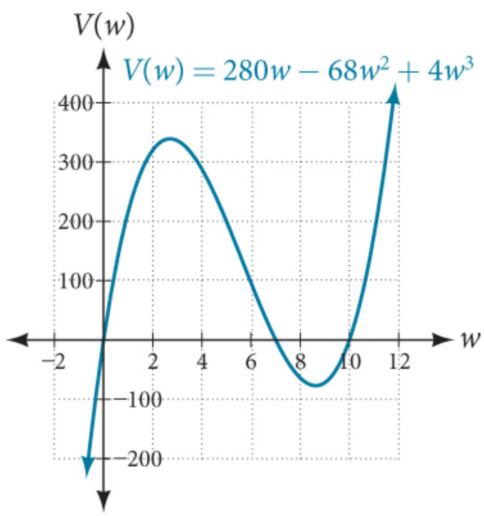


Figure 23

From this graph, we turn our focus to only the portion on the reasonable domain, $[0, \ 7]$. We can estimate the maximum value to be around 340 cubic cm, which occurs when the squares are about 2.75 cm on each side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce Figure 24.

$$\begin{array}{rcll} y & = & \sqrt{x-4} & \text{Replace } f(x) \text{ with } y. \\ x & = & \sqrt{y-4} & \text{Interchange } x \text{ and } y. \\ x & = & \sqrt{y-4} & \text{Square each side.} \\ x^2 & = & y-4 & \text{Add 4.} \\ x^2+4 & = & y & \text{Rename the function } f^{-1}(x). \\ f^{-1}(x) & = & x^2+4 & \end{array}$$

Recall that the domain of this function must be limited to the range of the original function.

$$f^{-1}(x) = x^2 + 4, x \geq 0$$

Analysis

Notice in Figure 8 that the inverse is a reflection of the original function over the line $y = x$. Because the original function has only positive outputs, the inverse function has only positive inputs.

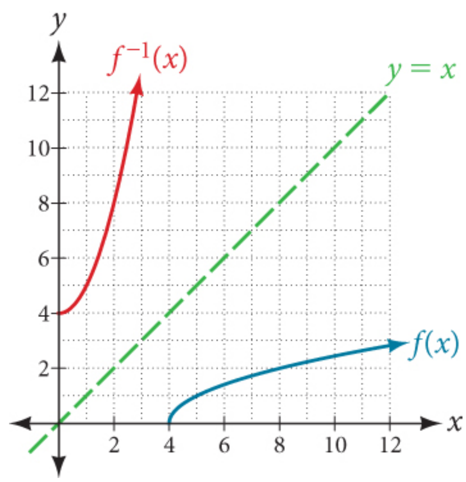


Figure 8

TRY IT #4 Restrict the domain and then find the inverse of the function $f(x) = \sqrt{2x+3}$.

Solving Applications of Radical Functions

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

EXAMPLE 6

Solving an Application with a Cubic Function

Park rangers construct a mound of gravel in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

$$V = \frac{2}{3}\pi r^3$$

Find the inverse of the function $V = \frac{2}{3}\pi r^3$ that determines the volume V of a cone and is a function of the radius r. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi = 3.14$.

Solution

Start with the given function for V. Notice that the meaningful domain for the function is $r > 0$ since negative radii would not make sense in this context nor would a radius of 0. Also note the range of the function (hence, the domain of the inverse function) is $V > 0$. Solve for r in terms of V, using the method outlined previously. Note that in real-world applications, we do not swap the variables when finding inverses. Instead, we change which variable is considered to be the independent variable.

Figure 3 represents the graph of the equation. On the graph, the *x*-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3 \approx 20$. A calculator gives a better approximation: $e^3 \approx 20.0855$.

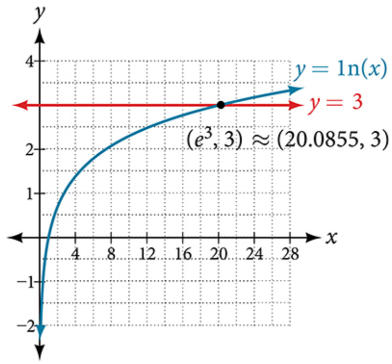


Figure 3 The graphs of $y = \ln x$ and $y = 3$ cross at the point $(e^3, 3)$, which is approximately (20.0855, 3).

TRY IT #11 Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to 2 decimal places.

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for x, and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2 (9-1) = \log_2 (8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x :

$$\begin{array}{ll}
\log(3x-2) - \log(2) = \log(x+4) & \\
\log\left(\frac{3x-2}{2}\right) = \log(x+4) & \text{Apply the quotient rule of logarithms.} \\
\frac{3x-2}{2} = x + 4 & \text{Apply the one to one property of a logarithm.} \\
3x - 2 = 2x + 8 & \text{Multiply both sides of the equation by 2.} \\
x = 10 & \text{Subtract } 2x \text{ and add 2.}
\end{array}$$

To check the result, substitute $x = 10$ into $\log(3x-2) - \log(2) = \log(x+4)$.

$$\begin{array}{ll}
\log(3(10)-2) - \log(2) = \log((10)+4) & \\
\log(28) - \log(2) = \log(14) & \\
\log\left(\frac{28}{2}\right) = \log(14) & \text{The solution checks.}
\end{array}$$

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

For any algebraic expressions S and T and any positive real number b, where $b \neq 1$,

$$
\begin{array}{ll}
-5x + 15y - 5z = -20 & (1) \text{ multiplied by } -5 \\
5x - 13y + 13z = 8 & (3) \\
\hline
2y + 8z = -12 & (5)
\end{array}
$$

Then, we multiply equation (4) by 2 and add it to equation (5).

$$
\begin{array}{l}
-2y - 8z = 14 (4) \text{ multiplied by } 2 \\
2y + 8z = -12 \ (5) \\
\hline
0 = 2
\end{array}
$$

The final equation $0 = 2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

TRY IT #2 Solve the system of three equations in three variables.

$$
\begin{aligned}
x + y + z &= 2 \\
y - 3z &= 1 \\
2x + y + 5z &= 0
\end{aligned}
$$

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

EXAMPLE 5

Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$$
\begin{aligned}
2x + y - 3z &= 0 \quad (1) \\
4x + 2y - 6z &= 0 \quad (2) \\
x - y + z &= 0 \quad (3)
\end{aligned}
$$

Solution

First, we can multiply equation (1) by -2 and add it to equation (2).

$$
\begin{array}{l}
-4x - 2y + 6z = 0 \ \text{ equation}(1) \text{ multiplied by } -2 \\
4x + 2y - 6z = 0 \quad (2) \\
\hline
0 = 0
\end{array}
$$

We do not need to proceed any further. The result we get is an identity, $0 = 0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2, and adding it to equation (1). We then perform the same steps as above and find the same result, $0 = 0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have

$$
\begin{array}{c}
2x + y - 3z = 0 \\
x - y + z = 0 \\
\hline
3x - 2z = 0
\end{array}
$$

For the following exercises, use the given information to find the equation for the ellipse.

9. Center at $(0, 0)$, focus at $(3, 0)$, vertex at $(-5, 0)$

10. Center at $(2, -2)$, vertex at $(7, -2)$, focus at $(4, -2)$

11. A whispering gallery is to be constructed such that the foci are located 35 feet from the center. If the length of the gallery is to be 100 feet, what should the height of the ceiling be?

The Hyperbola

For the following exercises, write the equation of the hyperbola in standard form. Then give the center, vertices, and foci.

12. $\frac{x^2}{81} - \frac{y^2}{9} = 1$

13. $\frac{(y+1)^2}{16} - \frac{(x-4)^2}{36} = 1$

14. $9y^2 - 4x^2 + 54y - 16x + 29 = 0$

15. $3x^2 - y^2 - 12x - 6y - 9 = 0$

For the following exercises, graph the hyperbola, labeling vertices and foci.

16. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

17. $\frac{(y-1)^2}{49} - \frac{(x+1)^2}{4} = 1$

18. $x^2 - 4y^2 + 6x + 32y - 91 = 0$

19. $2y^2 - x^2 - 12y - 6 = 0$

For the following exercises, find the equation of the hyperbola.

20. Center at $(0, 0)$, vertex at $(0, 4)$, focus at $(0, -6)$

21. Foci at $(3, 7)$ and $(7, 7)$, vertex at $(6, 7)$

The Parabola

For the following exercises, write the equation of the parabola in standard form. Then give the vertex, focus, and directrix.

22. $y^2 = 12x$

23. $(x + 2)^2 = \frac{1}{2}(y - 1)$

24. $y^2 - 6y - 6x - 3 = 0$

25. $x^2 + 10x - y + 23 = 0$

For the following exercises, graph the parabola, labeling vertex, focus, and directrix.

26. $x^2 + 4y = 0$

27. $(y - 1)^2 = \frac{1}{2}(x + 3)$

28. $x^2 - 8x - 10y + 46 = 0$

29. $2y^2 + 12y + 6x + 15 = 0$

Residents of the Southeastern United States are all too familiar with charts, known as spaghetti models, such as the one in Figure 1. They combine a collection of weather data to predict the most likely path of a hurricane. Each colored line represents one possible path. The group of squiggly lines can begin to resemble strands of spaghetti, hence the name. In this section, we will investigate methods for making these types of predictions.

Constructing Probability Models

Suppose we roll a six-sided number cube. Rolling a number cube is an example of an **experiment**, or an activity with an observable result. The numbers on the cube are possible results, or **outcomes**, of this experiment. The set of all possible outcomes of an experiment is called the **sample space** of the experiment. The sample space for this experiment is $\{1, 2, 3, 4, 5, 6\}$. An **event** is any subset of a sample space.

The likelihood of an event is known as **probability**. The probability of an event p is a number that always satisfies $0 \leq p \leq 1$, where 0 indicates an impossible event and 1 indicates a certain event. A **probability model** is a mathematical description of an experiment listing all possible outcomes and their associated probabilities. For instance, if there is a 1% chance of winning a raffle and a 99% chance of losing the raffle, a probability model would look much like Table 1.

Outcome	Probability
Winning the raffle	1%
Losing the raffle	99%

Table 1

The sum of the probabilities listed in a probability model must equal 1, or 100%.

HOW TO

Given a probability event where each event is equally likely, construct a probability model.

1. Identify every outcome.
2. Determine the total number of possible outcomes.
3. Compare each outcome to the total number of possible outcomes.

EXAMPLE 1

Constructing a Probability Model
Construct a probability model for rolling a single, fair die, with the event being the number shown on the die.

Solution
Begin by making a list of all possible outcomes for the experiment. The possible outcomes are the numbers that can be rolled: 1, 2, 3, 4, 5, and 6. There are six possible outcomes that make up the sample space.

Assign probabilities to each outcome in the sample space by determining a ratio of the outcome to the number of possible outcomes. There is one of each of the six numbers on the cube, and there is no reason to think that any particular face is more likely to show up than any other one, so the probability of rolling any number is $\frac{1}{6}$.

Outcome	Roll of 1	Roll of 2	Roll of 3	Roll of 4	Roll of 5	Roll of 6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Table 2

1 The figure is for illustrative purposes only and does not model any particular storm.