Exercises

Writing About Mathematics

1. a. Show that the conditional "If x is divisible by 4, then x is divisible by 2" is true in each of the following cases:

(1) $x = 8$ (2) $x = 6$ (3) $x = 7$

b. Is it possible to find a value of x for which the hypothesis is true and the conclusion is false? Explain your answer.

2. For what truth values of p and q is the truth value of $p \rightarrow q$ the same as the truth value of $q \rightarrow p$?

Developing Skills

In 3–10, for each given sentence: **a.** Identify the hypothesis p. **b.** Identify the conclusion q.

3. If a polygon is a square, then it has four right angles.

4. If it is noon, then it is time for lunch.

5. When you want help, ask a friend.

6. You will finish more quickly if you are not interrupted.

7. The perimeter of a square is $4s$ if the length of one side is s.

8. If many people work at a task, it will be completed quickly.

9. $2x + 7 = 11$ implies that $x = 2$.

10. If you do not get enough sleep, you will not be alert.

In 11–16, write each sentence in symbolic form, using the given symbols.

p: The car has a flat tire.
q: Danny has a spare tire.
r: Danny will change the tire.

11. If the car has a flat tire, then Danny will change the tire.

12. If Danny has a spare tire, then Danny will change the tire.

13. If the car does not have a flat tire, then Danny will not change the tire.

14. Danny will not change the tire if Danny doesn't have a spare tire.

15. The car has a flat tire if Danny has a spare tire.

16. Danny will change the tire if the car has a flat tire.

EXAMPLE 2

The statement "I go to basketball practice on Monday and Thursday" is true. Determine the truth value to be assigned to each statement.

a. If today is Monday, then I go to basketball practice.

b. If I go to basketball practice, then today is Monday.

c. Today is Monday if and only if I go to basketball practice.

Solution Let p: "Today is Monday,"
and q: "I go to basketball practice."

a. We are asked to find the truth value of the following conditional:

$p \to q$: If today is Monday, then I go to basketball practice.

On Monday, p is true and q is true. Therefore, $p \to q$ is true.

On Thursday, p is false and q is true. Therefore, $p \to q$ is true.

On every other day, p is false and q is false. Therefore, $p \to q$ is true.

"If today is Monday, then I go to basketball practice" is always true. *Answer*

b. We are asked to find the truth value of the following conditional:

$q \to p$: If I go to basketball practice then today is Monday.

On Monday, p is true and q is true. Therefore, $q \to p$ is true.

On Thursday, p is false and q is true. Therefore, $q \to p$ is false.

On every other day, p is false and q is false. Therefore, $q \to p$ is true.

"If I go to basketball practice, then Today is Monday" is sometimes true and sometimes false. *Answer*

c. We are asked to find the truth value of the following biconditional:

$p \leftrightarrow q$: Today is Monday if and only if I go to basketball practice.

The conditionals $p \to q$ and $q \to p$ do not always have the same truth value. Therefore, the biconditional "Today is Monday if and only if I go to basketball practice" is not always true. We usually say that a statement that is not always true is false. *Answer*

EXAMPLE 3

Determine the truth value of the biconditional.

$$3y + 1 = 28 \text{ if and only if } y = 9.$$

Exercises

Writing About Mathematics

1. Clovis said that when $p \rightarrow q$ is false and $q \vee r$ is true, r must be true. Do you agree with Clovis? Explain why or why not.

2. Regina said when $p \vee q$ is true and $\sim q$ is true, then $p \wedge \sim q$ must be true. Do you agree with Regina? Explain why or why not.

Developing Skills

In 3–14, assume that the first two sentences in each group are true. Determine whether the third sentence is true, is false, or cannot be found to be true or false. Justify your answer.

3. I save up money or I do not go on the trip.
I go on the trip.
I save up money.

4. If I speed, then I get a ticket.
I speed.
I get a ticket.

5. I like swimming or kayaking.
I like kayaking.
I like swimming.

6. I like swimming or kayaking.
I do not like swimming.
I like kayaking.

7. $x \leq 18$ if $x = 14$.
$x = 14$
$x \leq 18$

8. I live in Pennsylvania if I live in Philadelphia.
I do not live in Philadelphia.
I live in Pennsylvania.

9. If I am late for dinner, then my dinner will be cold.
I am late for dinner.
My dinner is cold.

10. If I am late for dinner, then my dinner will be cold.
I am not late for dinner.
My dinner is not cold.

11. I will go to college if and only if I work this summer.
I do not work this summer.
I will go to college.

12. The average of two numbers is 20 if the numbers are 17 and 23.
The average of two numbers is 20.
The two numbers are 17 and 23.

13. If I am late for dinner, then my dinner will be cold.
My dinner is not cold.
I am not late for dinner.

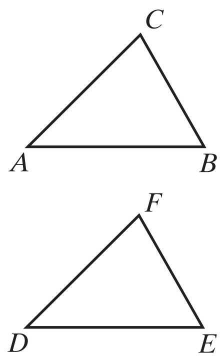
14. If I do not do well in school, then I will not receive a good report card.
I do well in school.
I receive a good report card.

Applying Skills

In 15–27, assume that each given sentence is true. Write a conclusion using both premises, if possible. If no conclusion is possible, write "No conclusion." Justify your answer.

15. If I play the trumpet, I take band.
I play the trumpet.

16. $\sqrt{6}$ is rational or irrational.
$\sqrt{6}$ is not rational.



The correspondence establishes six facts about these triangles: three facts about corresponding sides and three facts about corresponding angles. In the table at the right, these six facts are stated as equalities. Since each congruence statement is equivalent to an equality statement, we will use whichever notation serves our purpose better in a particular situation.

Congruences	Equalities
$\overline{AB} \cong \overline{DE}$	$AB = DE$
$\overline{BC} \cong \overline{EF}$	$BC = EF$
$\overline{AC} \cong \overline{DF}$	$AC = DF$
$\angle A \cong \angle D$	$m\angle A = m\angle D$
$\angle B \cong \angle E$	$m\angle B = m\angle E$
$\angle C \cong \angle F$	$m\angle C = m\angle F$

For example, in one proof, we may prefer to write $\overline{AC} \cong \overline{DF}$ and in another proof to write $AC = DF$. In the same way, we might write $\angle C \cong \angle F$ or we might write $m\angle C = m\angle F$. From the definition, we may now say:

► **Corresponding parts of congruent triangles are equal in measure.**

In two congruent triangles, pairs of corresponding sides are always opposite pairs of corresponding angles. In the preceding figure, $\triangle ABC \cong \triangle DEF$. The order in which we write the names of the vertices of the triangles indicates the one-to-one correspondence.

1. $\angle A$ and $\angle D$ are corresponding congruent angles.
2. $\overline{BC}$ is opposite $\angle A$, and $\overline{EF}$ is opposite $\angle D$.
3. $\overline{BC}$ and $\overline{EF}$ are corresponding congruent sides.

Equivalence Relation of Congruence

In Section 3-5 we saw that the relation "is congruent to" is an equivalence relation for the set of line segments and the set of angles. Therefore, "is congruent to" must be an equivalence relation for the set of triangles or the set of polygons with a given number of sides.

1. Reflexive property: $\triangle ABC \cong \triangle ABC$.
2. Symmetric property: If $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$.
3. Transitive property: If $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle RST$, then $\triangle ABC \cong \triangle RST$.

Therefore, we state these properties of congruence as three postulates:

Postulate 4.11 Any geometric figure is congruent to itself. (Reflexive Property)

Postulate 4.12 A congruence may be expressed in either order. (Symmetric Property)

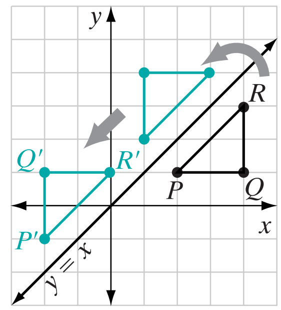
Solution **a.** $r_{y=x}(x, y) = (y, x) \quad T_{-3,-3}(y, x) = (y - 3, x - 3)$

$r_{y=x}(2, 1) = (1, 2) \quad T_{-3,-3}(1, 2) = (-2, -1)$

$r_{y=x}(4, 1) = (1, 4) \quad T_{-3,-3}(1, 4) = (-2, 1)$

$r_{y=x}(4, 3) = (3, 4) \quad T_{-3,-3}(3, 4) = (0, 1)$

The vertices of $\triangle P'Q'R'$ are $P'(-2, -1)$, $Q'(-2, 1)$, and $R'(0, 1)$. *Answer*



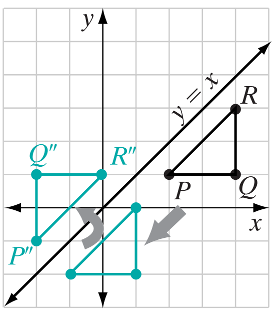
b. $T_{-3,-3}(x, y) = (x - 3, y - 3) \quad r_{y=x}(x - 3, y - 3) = (y - 3, x - 3)$

$T_{-3,-3}(2, 1) = (-1, -2) \quad r_{y=x}(-1, -2) = (-2, -1)$

$T_{-3,-3}(4, 1) = (1, -2) \quad r_{y=x}(1, -2) = (-2, 1)$

$T_{-3,-3}(4, 3) = (1, 0) \quad r_{y=x}(1, 0) = (0, 1)$

The vertices of $\triangle P''Q''R''$ are $P''(-2, -1)$, $Q''(-2, 1)$, and $R''(0, 1)$. *Answer*



c. $\triangle P'Q'R'$ and $\triangle P''Q''R''$ are the same triangle. *Answer*

d. The image of (x, y) under $r_{y=x}$ followed by $T_{-3,-3}$ is

$$(x, y) \rightarrow (y, x) \rightarrow (y - 3, x - 3)$$

The image of (x, y) under $T_{-3,-3}$ followed by $r_{y=x}$ is

$$(x, y) \rightarrow (x - 3, y - 3) \rightarrow (y - 3, x - 3)$$

$r_{y=x}$ followed by $T_{-3,-3}$ and $T_{-3,-3}$ followed by $r_{y=x}$ are the same glide reflection. *Answer*

e. $(x, y) \rightarrow (y - 3, x - 3)$ *Answer*

EXAMPLE 2

Is a reflection in the y-axis followed by the translation $T_{5,5}$ a glide reflection?

Solution The y-axis is a vertical line. The translation $T_{5,5}$ is not a translation in the vertical direction. Therefore, a reflection in the y-axis followed by the translation $T_{5,5}$ is not a glide reflection.

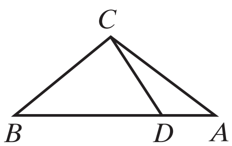
This corresponds to the following **transitive property of inequality**:

Postulate 7.2

If a, b, and c are real numbers such that $a > b$ and $b > c$, then $a > c$.

In arithmetic: If $12 > 7$ and $7 > 3$, then $12 > 3$.

In algebra: If $5x + 1 > 2x$ and $2x > 16$, then $5x + 1 > 16$.



In geometry: If $BA > BD$ and $BD > BC$, then $BA > BC$.
Also, if $m\angle BCA > m\angle BCD$ and $m\angle BCD > m\angle BAC$, then $m\angle BCA > m\angle BAC$.

Substitution Postulate

Consider the substitution postulate as it relates to equality:

► **A quantity may be substituted for its equal in any statement of equality.**

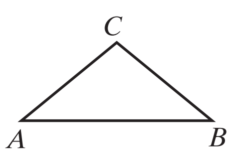
Substitution also holds for inequality, as demonstrated in the following postulate:

Postulate 7.3

A quantity may be substituted for its equal in any statement of inequality.

In arithmetic: If $10 > 2 + 5$ and $2 + 5 = 7$, then $10 > 7$.

In algebra: If $5x + 1 > 2y$ and $y = 4$, then $5x + 1 > 2(4)$.



In geometry: If $AB > BC$ and $BC = AC$, then $AB > AC$.
Also, if $m\angle C > m\angle A$ and $m\angle A = m\angle B$, then $m\angle C > m\angle B$.

The Trichotomy Postulate

We know that if x represents the coordinate of a point on the number line, then x can be a point to the left of 3 when $x < 3$, x can be the point whose coordinate is 3 if $x = 3$, or x can be a point to the right of 3 if $x > 3$. We can state this as a postulate that we call the **trichotomy postulate**, meaning that it is divided into three cases.

Postulate 7.4

Given any two quantities, a and b, one and only one of the following is true:

$$a < b \quad \text{or} \quad a = b \quad \text{or} \quad a > b.$$

Subtraction of inequalities is restricted to a single case:

Postulate 7.7

> If equal quantities are subtracted from unequal quantities, then the differences are unequal in the same order.

However, when unequal quantities are subtracted from unequal quantities, the results may or may not be unequal and the order of the inequality may or may not be the same.

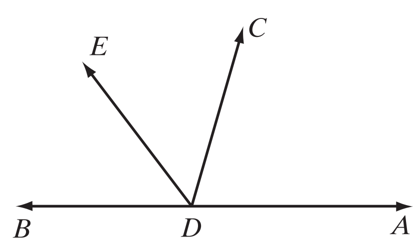
For example:

- $5 > 2$ and $4 > 1$, but it is not true that $5 - 4 > 2 - 1$ since $1 = 1$.
- $12 > 10$ and $7 > 1$, but it is not true that $12 - 7 > 10 - 1$ since $5 < 9$.
- $12 > 10$ and $2 > 1$, and it is true that $12 - 2 > 10 - 1$ since $10 > 9$.

EXAMPLE 1

Given: $m\angle BDE < m\angle CDA$

Prove: $m\angle BDC < m\angle EDA$



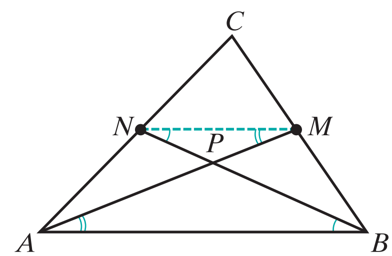
Proof

Statements	Reasons
1. $m\angle BDE < m\angle CDA$	**1.** Given.
2. $m\angle BDE + m\angle EDC < m\angle EDC + m\angle CDA$	**2.** If equal quantities are added to unequal quantities, then the sums are unequal in the same order.
3. $m\angle BDC = m\angle BDE + m\angle EDC$	**3.** The whole is equal to the sum of its parts.
4. $m\angle EDA = m\angle EDC + m\angle CDA$	**4.** The whole is equal to the sum of its parts.
5. $m\angle BDC < m\angle EDA$	**5.** Substitution postulate for inequalities.

Theorem 12.14 Any two medians of a triangle intersect in a point that divides each median in the ratio 2 : 1.

Given $\overline{AM}$ and $\overline{BN}$ are medians of $\triangle ABC$ that intersect at P.

Prove $AP : MP = BP : NP = 2 : 1$



Proof

Statements	Reasons
1. $\overline{AM}$ and $\overline{BN}$ are the medians of $\triangle ABC$.	**1.** Given.
2. M is the midpoint of $\overline{BC}$ and N is the midpoint of $\overline{AC}$.	**2.** The median of a triangle is a line segment from a vertex to the midpoint of the opposite side.
3. Draw $\overline{MN}$.	**3.** Two points determine a line.
4. $\overline{MN} \parallel \overline{AB}$	**4.** The line joining the midpoints of two sides of a triangle is parallel to the third side.
5. $\angle MNB \cong \angle ABN$ and $\angle NMA \cong \angle BAM$	**5.** Alternate interior angles of parallel lines are congruent.
6. $\triangle MNP \sim \triangle ABP$	**6.** AA~.
7. $MN = \frac{1}{2}AB$	**7.** The length of the line joining the midpoints of two sides of a triangle is equal to one-half of the length of the third side.
8. $2MN = AB$	**8.** Multiplication postulate.
9. $AB : MN = 2 : 1$	**9.** If the products of two pairs of factors are equal, the factors of one pair can be the means and the factors of the other the extremes of a proportion.
10. $AP : MP = BP : NP = 2 : 1$	**10.** If two triangles are similar, the ratios of the lengths of the corresponding sides are equal.

Let the intersection of these two lines be Q. Then $\triangle CPQ$ is a right triangle with:

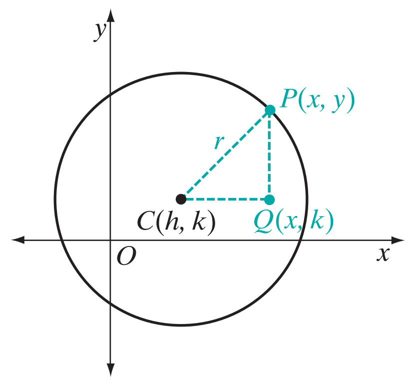
$$CQ = |x - 2| \qquad PQ = |y - 4| \qquad CP = 5$$

We can use the Pythagorean Theorem to write an equation for the circle.

$$CQ^2 \quad + \quad PQ^2 \quad = CP^2$$
$$(x - 2)^2 + (y - 4)^2 = 5^2$$

The points (5, 8), (6, 7), (−1, 8), (−2, 7), (−1, 0), (−2, 1) (5, 0), and (6, 1) appear to be points on the circle and all make $(x - 2)^2 + (y - 4)^2 = 5^2$ true. The points (7, 4), (−3, 4), (2, 9), and (2, −1) also make the equation true, as do points whose coordinates are not integers.

We can write a general equation for a circle with center at $C(h, k)$ and radius r. Let $P(x, y)$ be any point on the circle. From P draw a vertical line and from C draw a horizontal line. Let the intersection of these two lines be Q. Then $\triangle CPQ$ is a right triangle with:



$$CQ = |x - h| \qquad PQ = |y - k| \qquad CP = r$$

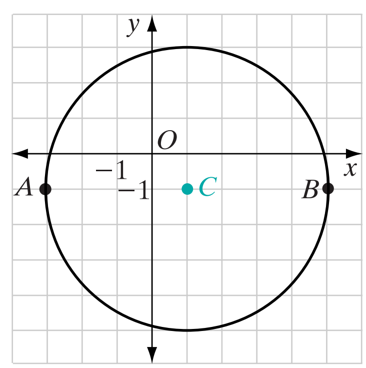
We can use the Pythagorean Theorem to write an equation for the circle.

$$CQ^2 \quad + \quad PQ^2 \quad = CP^2$$
$$(x - h)^2 + (y - k)^2 = r^2$$

In general, the **center-radius equation of a circle** with radius r and center (h, k) is

$$\mathbf{(x - h)^2 + (y - k)^2 = r^2}$$

A circle whose diameter $\overline{AB}$ has endpoints at $A(-3, -1)$ and $B(5, -1)$ is shown at the right. The center of the circle, C, is the midpoint of the diameter. Recall that the coordinates of the midpoint of the segment whose endpoints are (a, b) and (c, d) are $\left(\frac{a + c}{2}, \frac{b + d}{2}\right)$. The coordinates of C are



$$\left(\frac{5 + (-3)}{2}, \frac{-1 + (-1)}{2}\right) = (1, -1).$$

The length of the radius is the distance from C to any point on the circle. The distance between two points on the same vertical line, that is, with the same x-coordinates, is the absolute value of the difference of the y-coordinates. The length of the radius is the distance from $C(1, -1)$ to $A(-3, -1)$. The length of the radius is

$$|1 - (-3)| = 4.$$

EXAMPLE 1

a. Draw the graph of $y = x^2 - 4x - 1$ from $x = -1$ to $x = 5$.

b. Write the coordinates of the turning point.

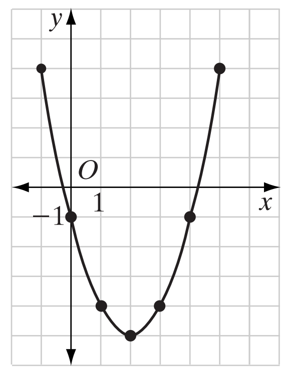
c. Write an equation of the axis of symmetry.

d. What are the coordinates of the fixed point and the fixed line for this parabola?

Solution **a.** (1) Make a table of values using integral values of x from $x = -1$ to $x = 5$.

(2) Plot the points whose coordinates are given in the table and draw a smooth curve through them.

x	$x^2 - 4x - 1$	y
-1	$1 + 4 - 1$	4
0	$0 - 0 - 1$	-1
1	$1 - 4 - 1$	-4
2	$4 - 8 - 1$	-5
3	$9 - 12 - 1$	-4
4	$16 - 16 - 1$	-1
5	$25 - 20 - 1$	4



b. From the graph or from the table, the coordinates of the turning point appear to be $(2, -5)$. We can verify this algebraically:

$$x = \frac{-b}{2a} = \frac{-(-4)}{2(1)} = 2 \qquad y = x^2 - 4x - 1 = (2)^2 - 4(2) - 1 = -5$$

c. The axis of symmetry is the vertical line through the turning point, $x = 2$.

d. Note that the turning point of the parabola is $(2, -5)$. When the turning point of the parabola $y = x^2$ has been moved 2 units to the right and 5 units down, the equation becomes the equation of the graph that we drew:

$$y - (-5) = (x - 2)^2$$
$$y + 5 = x^2 - 4x + 4$$
$$y = x^2 - 4x - 1$$

The turning point of the parabola is the midpoint of the perpendicular segment from the fixed point to the fixed line. Since the coefficient of y in the equation of the parabola is 1, $4d = 1$ or $d = \frac{1}{4}$. The parabola opens