b. If the perimeter of the rectangle is 2 feet more than eight times the width of the rectangle, find the dimensions of the rectangle.

25. On his trip to work each day, Brady pays the same toll, using either all quarters or all dimes. If the number of dimes needed for the toll is 3 more than the number of quarters, what is the toll?

1-4 SOLVING ABSOLUTE VALUE EQUATIONS AND INEQUALITIES

Absolute Value Equations

We know that if a is a positive number, then $|a| = a$ and that $|-a| = a$. For example, if $|x| = 3$, then $x = 3$ or $x = -3$ because $|3| = 3$ and $|-3| = 3$. We can use these facts to solve an **absolute value equation**, that is, an equation containing the absolute value of a variable.

For instance, solve $|2x - 3| = 17$. We know that $|17| = 17$ and $|-17| = 17$. Therefore $2x - 3$ can equal 17 or it can equal -17.

$$\begin{array}{rcl} 2x - 3 = 17 & \quad\text{or}\quad & 2x - 3 = -17 \\ 2x - 3 + 3 = 17 + 3 & & 2x - 3 + 3 = -17 + 3 \\ 2x = 20 & & 2x = -14 \\ x = 10 & & x = -7 \end{array}$$

The solution set of $|2x - 3| = 17$ is $\{-7, 10\}$.

In order to solve an absolute value equation, we must first isolate the absolute value expression. For instance, to solve $|4a + 2| + 7 = 21$, we must first add -7 to each side of the equation to isolate the absolute value expression.

$$\begin{aligned} |4a + 2| + 7 &= 21 \\ |4a + 2| + 7 - 7 &= 21 - 7 \\ |4a + 2| &= 14 \end{aligned}$$

Now we can consider the two possible cases: $4a + 2 = 14$ or $4a + 2 = -14$

$$\begin{array}{rcl} 4a + 2 = 14 & \quad\text{or}\quad & 4a + 2 = -14 \\ 4a + 2 - 2 = 14 - 2 & & 4a + 2 - 2 = -14 - 2 \\ 4a = 12 & & 4a = -16 \\ a = 3 & & a = -4 \end{array}$$

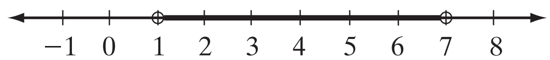
The solution set of $|4a + 2| + 7 = 21$ is $\{-4, 3\}$.

Note that the solution sets of the equations $|x + 3| = -5$ and $|x + 3| + 5 = 2$ are the empty set because absolute value is always positive or zero.

CASE 2 $|x - 4| < 3$

The solution set of this inequality consists of the values of x that are less than 3 units from 4 in either direction, that is, $x - 4$ is less than 3 and greater than -3.

$$x - 4 > -3 \quad \text{and} \quad x - 4 < 3$$
$$x > 1 \quad \text{and} \quad x < 7$$

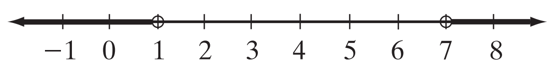


If x is an integer, the solution set is $\{2, 3, 4, 5, 6\}$. Note that these are the integers between the solutions of $|x - 4| = 3$.

CASE 3 $|x - 4| > 3$

The solution set of this inequality consists of the values of x that are more than 3 units from 4 in either direction, that is $x - 4$ is greater than 3 or less than -3.

$$x - 4 < -3 \quad \text{or} \quad x - 4 > 3$$
$$x < 1 \quad \text{or} \quad x > 7$$



If x is an integer, the solution set is $\{\ldots, -3, -2, -1, 0, 8, 9, 10, 11, \ldots\}$. Note that these are the integers that are less than the smaller solution of $|x - 4| = 3$ and greater than the larger solution of $|x - 4| = 3$.

We know that $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$. We can use these relationships to solve inequalities of the form $|x| < k$ and $|x| > k$.

Solve $|x| < k$ for positive k

If $x \geq 0$, $|x| = x$.

Therefore, $x < k$ and $0 \leq x < k$.

If $x < 0$, $|x| = -x$.

Therefore, $-x < k$ or $x > -k$.

This can be written $-k < x < 0$.

The solution set of $|x| < k$ is

$$-k < x < k.$$

Solve $|x| > k$ for positive k

If $x \geq 0$, $|x| = x$.

Therefore, $x > k$.

If $x < 0$, $|x| = -x$.

Therefore, $-x > k$ or $x < -k$.

The solution set of $|x| > k$ is

$$x < -k \text{ or } x > k.$$

- **If $|x| < k$ for any positive number k, then $-k < x < k$.**
- **If $|x| > k$ for any positive number k, then $x > k$ or $x < -k$.**

In general,

$$\begin{aligned}(a + bi)(a - bi) &= a(a - bi) + bi(a - bi)\\ &= a^2 - abi + abi - b^2i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2\end{aligned}$$

or

$$\mathbf{(a + bi)(a - bi) = a^2 + b^2}$$

The product of a complex number $a + bi$ and its conjugate $a - bi$ is a real number, $a^2 + b^2$. Thus, if we the multiply numerator and denominator of $\frac{1}{a + bi}$ by the complex conjugate of the denominator, we will have an equivalent fraction with a denominator that is a real number.

$$\frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

For example, the complex conjugate of $2 - 4i$ is $2 + 4i$ and the multiplicative inverse of $2 - 4i$ is

$$\begin{aligned}\frac{1}{2 - 4i} = \frac{1}{2 - 4i} \cdot \frac{2 + 4i}{2 + 4i} &= \frac{2 + 4i}{2^2 + 4^2} = \frac{2}{2^2 + 4^2} + \frac{4i}{2^2 + 4^2}\\ &= \frac{2}{20} + \frac{4}{20}i\\ &= \frac{1}{10} + \frac{1}{5}i\end{aligned}$$

We can check that $\frac{1}{10} + \frac{1}{5}i$ is the multiplicative inverse of $2 - 4i$ by multiplying the two numbers together. If $\frac{1}{10} + \frac{1}{5}i$ is indeed the multiplicative inverse, then their product will be 1.

$$\begin{aligned}(2 - 4i) \cdot \left(\tfrac{1}{10} + \tfrac{1}{5}i\right) &\stackrel{?}{=} 1\\ 2\left(\tfrac{1}{10} + \tfrac{1}{5}i\right) - 4i\left(\tfrac{1}{10} + \tfrac{1}{5}i\right) &\stackrel{?}{=} 1\\ 2\left(\tfrac{1}{10}\right) + 2\left(\tfrac{1}{5}i\right) - 4i\left(\tfrac{1}{10}\right) - 4i\left(\tfrac{1}{5}i\right) &\stackrel{?}{=} 1\\ \tfrac{2}{10} + \tfrac{2}{5}i - \tfrac{4}{10}i - \tfrac{4}{5}i^2 &\stackrel{?}{=} 1\\ \tfrac{1}{5} + \tfrac{2}{5}i - \tfrac{2}{5}i - \tfrac{4}{5}(-1) &\stackrel{?}{=} 1\\ \tfrac{1}{5} + \tfrac{4}{5} &= 1 \checkmark\end{aligned}$$

► **For any non-zero complex number $a + bi$, its multiplicative inverse is**

$$\frac{1}{a + bi} \quad \text{or} \quad \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

Repeat the process for the following function, using the numbers $-3, -2, -1, 1, 2,$ and 3 for each function. Find the roots of the function.

a. $f(x) = x^3 - 2x^2 - x + 2$

b. $f(x) = x^3 - 3x^2 - 4x + 12$

c. $f(x) = x^3 - 7x - 6 = x^3 + 0x^2 - 7x - 6$ (Include 0 in the list of coefficients.)

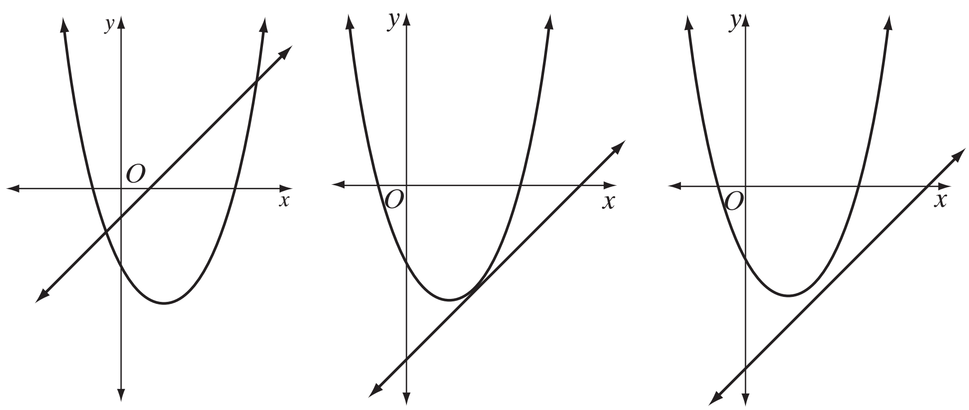
d. $f(x) = x^4 - 5x^2 + 4 = x^4 + 0x^3 - 5x^2 + 0x + 4$

5-9 SOLUTIONS OF SYSTEMS OF EQUATIONS AND INEQUALITIES

A system of equations is a set of two or more equations. The system is **consistent** if there exists at least one common solution in the set of real numbers. The solution set of a consistent system of equations can be found using a graphic method or an algebraic method.

Solving Quadratic-Linear Systems

A system that consists of a quadratic function whose graph is a parabola and a linear function whose graph is a straight line is a **quadratic-linear system** in two variables. In the set of real numbers, a quadratic-linear system may have two solutions, one solution, or no solutions. Each solution can be written as the coordinates of the points of intersection of the graph of the parabola and the graph of the line.



Two Solutions One Solution No Solution

Recall that for the function $y = ax^2 + bx + c$, the x-coordinate of the turning point and the equation of the axis of symmetry is $x = \frac{-b}{2a}$. When a is positive, the parabola opens upward and has a minimum value of y as shown in the figures above. When a is negative, the parabola opens downward and has a maximum value of y as shown in the following example.

Exercises

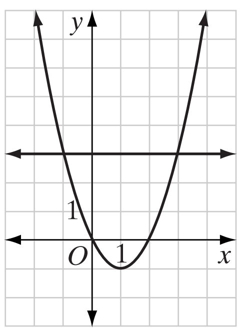
Writing About Mathematics

1. Explain the relationship between the solutions of $y > ax^2 + bx + c$ and the solutions of $0 > ax^2 + bx + c$.

2. Explain why the equations $y = x^2 + 2$ and $y = -2$ have no common solution in the set of real numbers.

Developing Skills

In 3–8, determine each common solution from the graph.

3.


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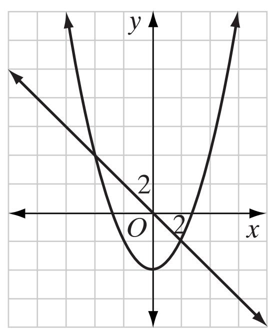

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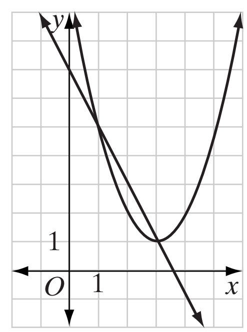

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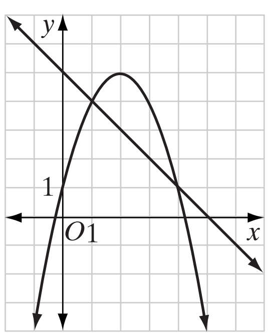

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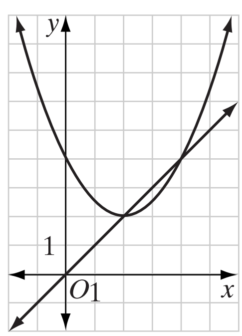

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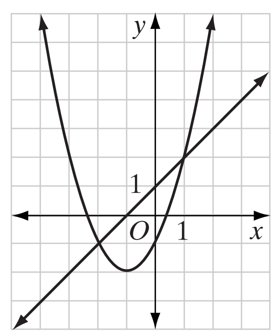


In 9–17, graph each system and determine the common solution from the graph.

9. $y = x^2 - 2x - 1$
$y = x + 3$

10. $y = x^2 + 2x$
$y = 2x + 1$

11. $-x^2 + 4x - y - 2 = 0$
$x + y = 4$

12. $y = -x^2 + 6x - 1$
$y = x + 3$

13. $y = 2x^2 + 2x + 3$
$y - x = 3$

14. $y = 2x^2 - 6x + 5$
$y = x + 2$

15. $x^2 - 4x - y + 4 = 0$
$y = \frac{4x + 7}{4}$

16. $\frac{x - 1}{y} = \frac{6}{x + 12}$
$y = x + 2$

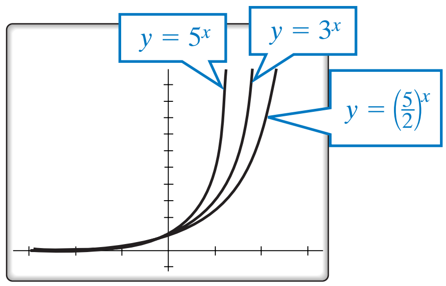
17. $\frac{y}{x} = \frac{x + 7}{5}$
$y = 2x$

ENTER: Y= (5 ÷ 2) ^ X,T,Θ,*n* ENTER

3 ^ X,T,Θ,*n* ENTER

5 ^ X,T,Θ,*n* GRAPH

DISPLAY:



If we press TRACE, the calculator will display the values $x = 0$, $y = 1$ for the first graph. By pressing ▼, the calculator will display this same set of values for the second function and again for the third. The point (0, 1) is a point on every function of the form $y = b^x$. For each function, as x decreases through negative values, the values of y get smaller and smaller but are always positive. We say that as x approaches $-\infty$, y approaches 0. The x-axis or the line $y = 0$ is a **horizontal asymptote**.

EXAMPLE 1

a. Sketch the graph of $y = \left(\frac{3}{2}\right)^x$.

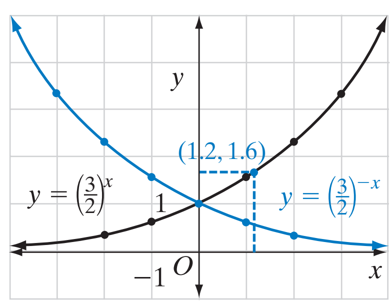
b. From the graph, estimate the value of $\left(\frac{3}{2}\right)^{1.2}$, the value of y when $x = 1.2$.

c. Sketch the graph of the image of the graph of $y = \left(\frac{3}{2}\right)^x$ under a reflection in the y-axis.

d. Write an equation of the graph drawn in part **c**.

Solution **a.**

x	y
-2	$\left(\frac{3}{2}\right)^{-2} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$
-1	$\left(\frac{3}{2}\right)^{-1} = \left(\frac{2}{3}\right)^1 = \frac{2}{3}$
0	$\left(\frac{3}{2}\right)^0 = 1$
1	$\left(\frac{3}{2}\right)^1 = \frac{3}{2}$
2	$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$
3	$\left(\frac{3}{2}\right)^3 = \frac{27}{8}$



8-2 LOGARITHMIC FORM OF AN EXPONENTIAL EQUATION

An exponential equation and a logarithmic equation are two different ways of expressing the same relationship. For example, to change an exponential equation to a logarithmic equation, recall that a logarithm or log is an exponent. In the exponential equation $81 = 3^4$, the exponent or log is 4. The basic statement is:

$$\log = 4$$

Then write the base as a subscript of the word "log" to indicate the log to the base 3:

$$\log_3 = 4$$

Now add the value of the power, that is, log to the base 3 of the power 81:

$$\log_3 81 = 4$$

To change from a logarithmic equation to an exponential statement, we must again look for the exponent. For example, in the logarithmic equation $\log_{10} 0.01 = -2$, the basic statement is that the log or exponent is -2. The base is the number that is written as a subscript of the word "log":

$$10^{-2}$$

The number that follows the word "log" is the power:

$$10^{-2} = 0.01$$

Recall that $10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$.

In general:

► $b^a = c$ **is equivalent to** $\log_b c = a$.

Exponent, Power

$$b^a = c$$

Base

Power, Exponent

$$\log_b c = a$$

Base

EXAMPLE 1

Write $9^3 = 729$ in logarithmic form and express its meaning in words.

Solution In the equation $9^3 = 729$, the logarithm (exponent) is 3:

$$\log = 3$$

Write the base, 9, as a subscript of "log":

$$\log_9 = 3$$

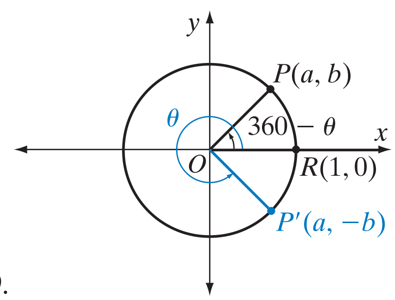
ENTER: 2nd SIN⁻¹ 0.5726) ENTER

DISPLAY: sin⁻¹(0.5726)
34.9317314

To the nearest degree, $(\theta - 180) = 35$. Therefore:

$$\theta = 35 + 180 = 215° \quad \textit{Answer}$$

Fourth-Quadrant Angles



In the diagram, $R(1, 0)$ and $P(a, b)$ are points on the unit circle. Under a reflection in the x-axis, the image of $P(a, b)$ is $P'(a, -b)$ and the image of $R(1, 0)$ is $R(1, 0)$. Since angle measure is preserved under a line reflection, the measure of the acute angle $\angle ROP$ is equal to the measure of the acute angle $\angle ROP'$.

- If $m\angle ROP' = \theta$, then $m\angle ROP = 360 - \theta$.
- If $m\angle ROP' = \theta$, then $\sin\theta = -b$ and $\cos\theta = a$.
- If $m\angle ROP = (360 - \theta)$, then $\sin(360 - \theta) = b$ and $\cos(360 - \theta) = a$.

Therefore:

$$\sin\theta = -\sin(360 - \theta) \qquad \cos\theta = \cos(360 - \theta)$$

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, then:

$$\tan\theta = \frac{-\sin(360 - \theta)}{\cos(360 - \theta)} = -\tan(360 - \theta)$$

Let θ be the measure of a fourth-quadrant angle. Then there exists a first-quadrant angle with measure $360 - \theta$ such that:

$$\boldsymbol{\sin\theta = -\sin(360 - \theta)}$$

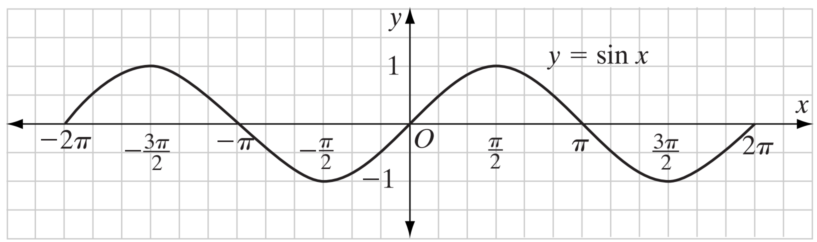
$$\boldsymbol{\cos\theta = \cos(360 - \theta)}$$

$$\boldsymbol{\tan\theta = -\tan(360 - \theta)}$$

The positive acute angle $(360 - \theta)$ is the **reference angle of the fourth-quadrant angle**. When drawn in standard position, the reference angle is in the first quadrant.

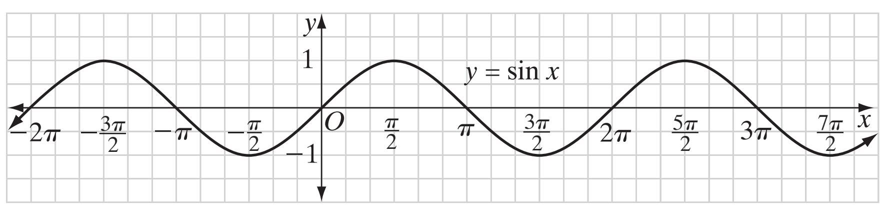
When we plot a larger subset of the domain of the sine function, this pattern is repeated. For example, add to the points given above the point whose x-coordinates are in the interval $-2\pi \leq x \leq 0$.

x	-2π	$-\frac{11\pi}{6}$	$-\frac{5\pi}{3}$	$-\frac{3\pi}{2}$	$-\frac{4\pi}{3}$	$-\frac{7\pi}{6}$	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0
sin x	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0



Each time we increase or decrease the value of the x-coordinates by a multiple of 2π, the basic sine curve is repeated. Each portion of the graph in an interval of 2π is one **cycle** of the sine function.

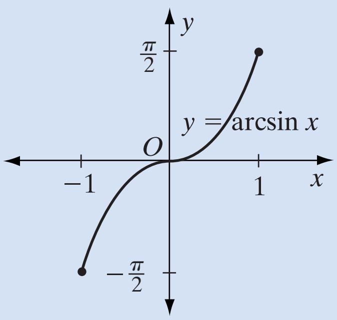
The graph of the function $y = \sin x$ is its own image under the translation $T_{2\pi,0}$. The function $y = \sin x$ is called a **periodic function** with a **period** of 2π because for every x in the domain of the sine function, $\sin x = \sin (x + 2\pi)$.

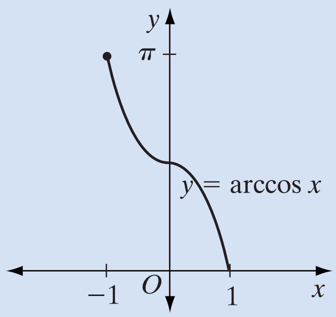


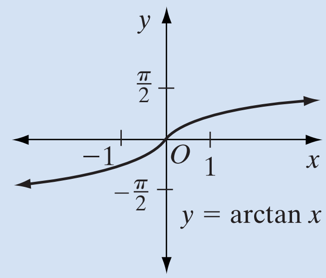
► **The period of the sine function $y = \sin x$ is 2π.**

Each cycle of the sine curve can be separated into four quarters. In the first quarter, the sine curve increases from 0 to the maximum value of the function. In the second quarter, it decreases from the maximum value to 0. In the third quarter, it decreases from 0 to the minimum value, and in the fourth quarter, it increases from the minimum value to 0.

Function	Restricted Domain	Inverse Function
$y = \sin x$	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$y = \arcsin x$ or $y = \sin^{-1} x$
$y = \cos x$	$0 \le x \le \pi$	$y = \arccos x$ or $y = \cos^{-1} x$
$y = \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	$y = \arctan x$ or $y = \tan^{-1} x$







VOCABULARY

11-1 Cycle • Periodic function • Period • Odd function

11-2 Even function

11-3 Amplitude • Phase shift

11-4 Simple harmonic motion • Frequency

REVIEW EXERISES

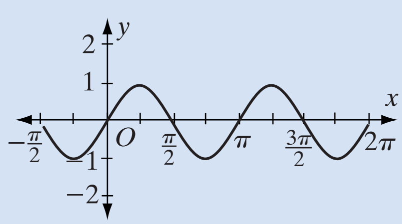
In 1–6, for each function, state: **a.** the amplitude **b.** the period **c.** the frequency **d.** the domain **e.** the range. **f.** Sketch one cycle of the graph.

1. $y = 2 \sin 3x$ **2.** $y = 3 \cos \frac{1}{2}x$ **3.** $y = \tan x$

4. $y = \cos 2\left(x - \frac{\pi}{3}\right)$ **5.** $y = \sin (x + \pi)$ **6.** $y = -2 \cos x$

In 7–10, for each graph, write the equation in the form: **a.** $y = a \sin b(x + c)$ **b.** $y = a \cos b(x + c)$. In each case, choose one cycle with its lower endpoint closest to zero to find the phase shift.

7.



8.