0.9 Parabola

The graph of

$$y = ax^2 + bx + c$$

where $a \neq 0$, is a *parabola*. The parabola intersects the x-axis at two distinct points if $b^2 - 4ac > 0$. It touches the x-axis (one intersection point only) if $b^2 - 4ac = 0$ and does not intersect the x-axis if $b^2 - 4ac < 0$.

- If $a > 0$, the parabola opens upward and there is a lowest point (called the *vertex* of the parabola).
- If $a < 0$, the parabola opens downward and there is a highest point (*vertex*).

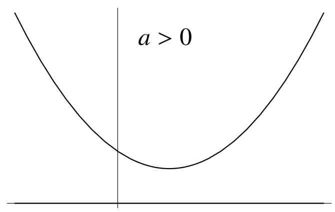


Figure 0.5(a)

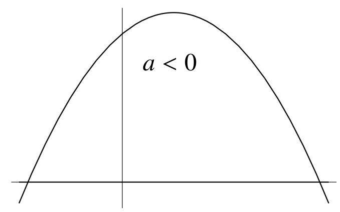


Figure 0.5(b)

The vertical line that passes through the vertex is called the *axis of symmetry* because the parabola is symmetric about this line.

To find the vertex, we can use the completing square method to write the equation in the form

$$y = a(x - h)^2 + k \tag{0.9.1}$$

The vertex is (h, k) because $(x - h)^2$ is always non-negative and so

- if $a > 0$, then $y \geq k$ and thus (h, k) is the lowest point;
- if $a < 0$, then $y \leq k$ and thus (h, k) is the highest point.

Example Consider the parabola given by

$$y = x^2 + 6x + 5.$$

Find its vertex and axis of symmetry.

Solution Using the completing square method, the given equation can be written in the form (0.9.1).

$$\begin{aligned} y &= x^2 + 6x + 5 \\ y &= (x^2 + 6x + 9) - 9 + 5 \\ y &= (x + 3)^2 - 4 \\ y &= (x - (-3))^2 - 4. \end{aligned}$$

The vertex is $(-3, -4)$ and the axis of symmetry is the line given by $x = -3$ (the vertical line that passes through the vertex). □

FAQ In the above example, the coefficient of x^2 is 1, what should we do if it is not 1?

(4) $$\begin{aligned} A \cap (B \cap C) &= \{2,3,5\} \cap \{2\} \\ &= \{2\} \end{aligned}$$ □

Note Given any sets A, B and C, we always have

$$(A \cap B) \cap C = A \cap (B \cap C) \qquad \text{and} \qquad (A \cup B) \cup C = A \cup (B \cup C).$$

Thus we may write $A \cap B \cap C$ and $A \cup B \cup C$ without ambiguity. We say that set intersection and set union are *associative*.

Definition Let A and B be sets. The *relative complement* of B in A, denoted by $A \setminus B$ or $A - B$ (read "*A setminus (or minus) B*"), is the set whose elements are those belonging to A but not belonging to B, that is,

$$A \setminus B = \{x \in A : x \notin B\}.$$

Example Let $A = \{a, b, c\}$ and $B = \{c, d, e\}$. Then we have $A \setminus B = \{a, b\}$.

For each problem, we will consider a set that is "large" enough, containing all objects under consideration. Such a set is called a *universal set* and is usually denoted by U. In this case, all sets under consideration are subsets of U and they can be written in the form $\{x \in U : P(x)\}$.

Example In considering addition and subtraction of whole numbers $(0, 1, 2, 3, 4, \ldots)$, we may use $\mathbb{Z}$ (the set of all integers) as a universal set.

(1) The set of all positive even numbers can be written as $\{x \in \mathbb{Z} : x > 0 \text{ and } x \text{ is divisible by } 2\}$.

(2) The set of all prime numbers can be written as $\{x \in \mathbb{Z} : x > 0 \text{ and } x \text{ has exactly two divisors}\}$.

Definition Let U be a universal set and let B be a subset of U. Then the set $U \setminus B$ is called the *complement* of B (in U) and is denoted by B' (or B^c).

Example Let $U = \mathbb{Z}_+$, the set of all positive integers. Let B be the set of all positive even numbers. Then B' is the set of all positive odd numbers.

Example Let $U = \{1, 2, 3, \ldots, 12\}$ and let

$$\begin{aligned} A &= \{x \in U : x \text{ is a prime number}\} \\ B &= \{x \in U : x \text{ is an even number}\} \\ C &= \{x \in U : x \text{ is divisible by } 3\}. \end{aligned}$$

Find the following sets.

(1) $A \cup B$

(2) $A \cap C$

(3) $B \cap C$

(4) $(A \cup B) \cap C$

(5) $(A \cap C) \cup (B \cap C)$

Solve for x.
$$\begin{aligned} x^2 &= y-2 \\ x &= \pm\sqrt{y-2}. \end{aligned}$$

Note that x can be solved if and only if $y-2 \geq 0$.

The range of f is $\{y \in \mathbb{R} : y-2 \geq 0\} = \{y \in \mathbb{R} : y \geq 2\} = [2, \infty)$.

Alternatively, to see that the range is $[2, \infty)$, we may use the graph of $y = x^2 + 2$ which is a parabola. The lowest point (vertex) is $(0, 2)$. For any $y \geq 2$, we can always find $x \in \mathbb{R}$ such that $f(x) = y$.

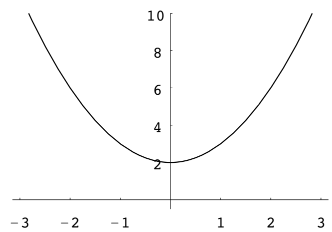


Figure 2.1

(2) Put $y = g(x) = \dfrac{1}{x-2}$.

Solve for x.
$$\begin{aligned} y &= \frac{1}{x-2} \\ x-2 &= \frac{1}{y} \\ x &= \frac{1}{y} + 2. \end{aligned}$$

Note that x can be solved if and only if $y \neq 0$.

The range of g is $\{y \in \mathbb{R} : y \neq 0\} = \mathbb{R} \setminus \{0\}$.

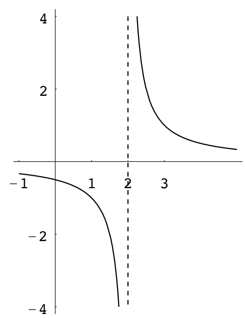


Figure 2.2

(3) Put $y = h(x) = \sqrt{1+5x}$. Note that y cannot be negative.

Solve for x.
$$\begin{aligned} y &= \sqrt{1+5x}, && y \geq 0 \\ y^2 &= 1+5x, && y \geq 0 \\ x &= \frac{y^2-1}{5}, && y \geq 0. \end{aligned}$$

Note that x can always be solved for every $y \geq 0$.

The range of h is $\{y \in \mathbb{R} : y \geq 0\} = [0, \infty)$.

Remark $y = \sqrt{1+5x} \Longrightarrow y^2 = 1+5x$
but the converse is true only if $y \geq 0$.

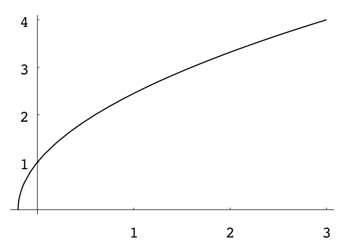


Figure 2.3

□

Example Let $f(x) = \sqrt{x+7} - \sqrt{x^2+2x-15}$. Find the domain of f.

Solution Note that $f(x)$ is defined if and only if $x+7 \geq 0$ and $x^2+2x-15 \geq 0$.
Solve the two inequalities separately:

- $$\begin{aligned} x+7 &\geq 0 \\ x &\geq -7; \end{aligned}$$

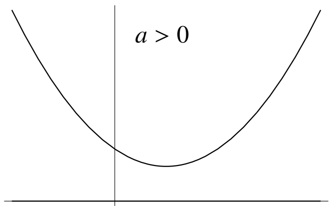


Figure 2.12(a)

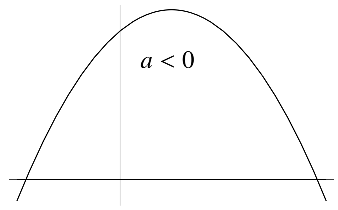


Figure 2.12(b)

Remark Besides using the completing square method to find the vertex, we can also use *differentiation* (see Chapter 5).

(4) **Polynomial Functions** A function f given by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$, is called a *polynomial function* of *degree n*.

If $n = 0$, f is a constant function.
If $n = 1$, f is a linear function.
If $n = 2$, f is a quadratic function.

Example Let $f(x) = x^3 - 3x^2 + x - 1$.

The graph of f is shown in Figure 2.13.

In Chapter 5, we will discuss how to sketch graphs of polynomial functions.

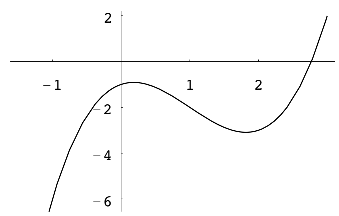


Figure 2.13

The domain of every polynomial function f is $\mathbb{R}$.

There are three possibilities for the range.

(a) If the degree is odd, then $\operatorname{ran}(f) = \mathbb{R}$.

(b) If the degree is even and positive, then

   (i) $\operatorname{ran}(f) = [k, \infty)$ if $a_n > 0$;

   (ii) $\operatorname{ran}(f) = (-\infty, k]$ if $a_n < 0$,

   where k is the y-coordinate of the lowest point for case (i), or the highest point for case (ii), of the graph.

Remark The constant function 0 is also considered to be a polynomial function. However, its degree is assigned to be $-\infty$ (for convenience of a rule for degree of product of polynomials).

(5) **Rational Functions** A *rational function* is a function f in the form

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions.

Can you generalize the results for graphs of polynomial functions of degree $3, 4, \ldots$ *?*

3. Let $f(x) = \frac{2x-1}{x^2+3}$. The graph of f is shown on page 55. Note that there is a highest point and a lowest point. Find the coordinates of these two points. *Hint: consider the range of* f

   The points are called *relative extremum points*. An easy way to find their coordinates is to use *differentiation*, see Chapter 5.

4. An object is thrown upward and its height $h(t)$ in meters after t seconds is given by $h(t) = 1 + 4t - 5t^2$.
   (a) When will the object hit the ground?
   (b) Find the maximum height attained by the object.

5. The manager of an 80-unit apartment complex is trying to decide what rent to charge. Experience has shown that at a rent of \$20000, all the units will be full. On the average, one additional unit will remain vacant for each \$500 increase in rent.
   (a) Let n represent the number of \$500 increases.
   Find an expression for the total revenue R from all the rented apartments.
   What is the domain of R?
   (b) What value of n leads to maximum revenue?
   What is the maximum revenue?

2.5 Compositions of Functions

Consider the function f given by

$$f(x) = \sin^2 x.$$

Recall that $\sin^2 x = (\sin x)^2$. For each input x, to find the output $y = f(x)$,

(1) first calculate $\sin x$, call the resulted value u;

(2) and then calculate u^2.

These two steps correspond to two functions:

(1) $u = \sin x$;

(2) $y = u^2$.

Given two functions, we can "combine" them by letting one function acting on the output of the other.

Definition Let f and g be functions such that the codomain of f is a subset of the domain of g. The *composition* of g with f, denoted by $g \circ f$, is the function given by

$$(g \circ f)(x) = g(f(x)). \tag{2.5.1}$$

The right-side of (2.5.1) is read "*g of f of x*".

Figure 2.32 indicates that f is a function from A to B and g is a function from C to D where $B \subseteq C$.

$(a = 0)$ Note that $f(x) = x^2$ on the left-side and the right-side of 0. Thus we have

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0} x^2 \\ &= 0^2 && \text{Theorem 3.5.1} \\ &= 0 \\ &\neq f(0)\end{aligned}$$

Therefore, f is not continuous at 0.

$(a \neq 0)$ Note that $f(x) = x^2$ on the left-side and the right-side of a. Thus we have

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a} x^2 \\ &= a^2 && \text{Theorem 3.5.1} \\ &= f(a)\end{aligned}$$

Therefore, f is continuous at a.

□

Remark The graph of f is shown in Figure 3.22.

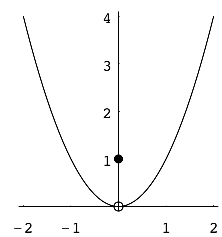


Figure 3.22

In the preceding definition, we consider continuity of a function f at a point a (a real number is considered as a point on the real line). In the next definition, we consider continuity of f on an open interval. Recall that an open interval is a subset of $\mathbb{R}$ that can be written in one of the following forms:

$$\begin{aligned}(\alpha,\beta) &= \{x \in \mathbb{R} : \alpha < x < \beta\} \\ (\alpha,\infty) &= \{x \in \mathbb{R} : \alpha < x\} \\ (-\infty,\beta) &= \{x \in \mathbb{R} : x < \beta\} \\ (-\infty,\infty) &= \mathbb{R}\end{aligned}$$

where α and β are real numbers, and for the first type, we need $\alpha < \beta$.

Definition Let I be an open interval and let f be a function defined on I. If f is continuous at every $a \in I$, then we say that f is *continuous on* I.

Remark

- In the definition, the condition "*f is a function defined on I*" means that f is a function such that $f(x)$ is defined for all $x \in I$, that is, $I \subseteq \text{dom}(f)$.
- Since I is an open interval, we may consider continuity of f at any point a belonging to I.
- If there exists $a \in I$ such that f is not continuous at a, then f is *not continuous on* I.

Proof The proof is similar to that for the product rule. □

Example Let $y = \dfrac{x^2+3x-4}{2x+1}$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution
$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2+3x-4}{2x+1}\right) \\
&= \frac{(2x+1)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x^2+3x-4)-(x^2+3x-4)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(2x+1)}{(2x+1)^2} && \text{Quotient Rule} \\
&= \frac{(2x+1)(2x+3)-(x^2+3x-4)(2)}{(2x+1)^2} && \text{Derivative of Polynomial} \\
&= \frac{(4x^2+8x+3)-(2x^2+6x-8)}{(2x+1)^2} \\
&= \frac{2x^2+2x+11}{(2x+1)^2}
\end{aligned}$$
□

Power Rule for Differentiation (negative integer version) *Let n be a negative integer. Then the power function x^n is differentiable on $\mathbb{R}\setminus\{0\}$ and we have*
$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}.$$

Explanation Since n is a negative integer, it can be written as $-m$ where m is a positive integer. The function $x^n = x^{-m} = \frac{1}{x^m}$ is defined for all $x \neq 0$, that is, the domain of x^n is $\mathbb{R}\setminus\{0\}$.

Proof Let $f : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ be the function given by $f(x) = x^{-m}$, where $m = -n$. By definition, we have
$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x^m} \\
&= \frac{x^m\cdot\frac{\mathrm{d}}{\mathrm{d}x}1 - 1\cdot\frac{\mathrm{d}}{\mathrm{d}x}x^m}{(x^m)^2} && \text{Quotient Rule} \\
&= \frac{x^m\cdot 0 - 1\cdot mx^{m-1}}{x^{2m}} && \text{Derivative of Constant \& Power Rule (positive integer version)} \\
&= -mx^{(m-1)-2m} \\
&= -mx^{-m-1} \\
&= nx^{n-1}
\end{aligned}$$
□

Example Find an equation for the tangent line to the curve $y = \frac{3x^2-1}{x}$ at the point $(1,2)$.

Explanation The curve is given by $y = f(x)$ where $f(x) = \frac{3x^2-1}{x}$. Since $f(1) = 2$, the point $(1,2)$ lies on the curve. To find an equation for the tangent line, we have to find the slope at the point (and then use point-slope form). The slope at the point $(1,2)$ is $f'(1)$. We can use rules for differentiation to find $f'(x)$ and then substitute $x = 1$ to get $f'(1)$.

Solution To find $\frac{\mathrm{d}y}{\mathrm{d}x}$, we can use quotient rule or term by term differentiation.

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

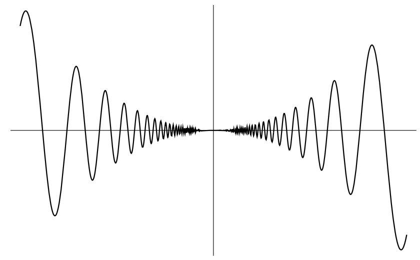


Figure 5.7

Example Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by

$$f(x) = 27x - x^3.$$

Find and determine the nature of the critical number(s) of f.

Explanation The question is to find all the critical numbers of f and for each critical number, determine whether it is a local maximizer, a local minimizer or not a local extremizer.

Solution Differentiating $f(x)$, we get

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(27x - x^3) \\ &= 27 - 3x^2 \\ &= 3(3 + x)(3 - x). \end{aligned}$$

Solving $f'(x) = 0$, we get the critical numbers of f: $x_1 = -3$ and $x_2 = 3$.

- When x is sufficiently close to and less than -3, $f'(x)$ is negative; when x is sufficiently close to and greater than -3, $f'(x)$ is positive. Hence, by the First Derivative Test, $x_1 = -3$ is a local minimizer of f.
- When x is sufficiently close to and less than 3, $f'(x)$ is positive; when x is sufficiently close to and greater than 3, $f'(x)$ is negative. Hence, by the First Derivative Test, $x_2 = 3$ is a local maximizer of f.

□

Remark The function in the above example is considered in the last subsection in which a table is obtained. It is clear from the table that f has a local minimum at -3 and a local maximum at 3. In the next example, we will use the table method to determine nature of critical numbers.

	$(-\infty, -3)$	$(-3, 3)$	$(3, \infty)$
3	+	+	+
$3 + x$	−	+	+
$3 - x$	+	+	−
f'	−	+	−
f	↘	↗	↘

Example Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by

$$f(x) = x^4 - 4x^3 + 5.$$

Find and determine the nature of the critical number(s) of the f.

Explanation The function is considered in an example in the last subsection. Below we just copy the main steps from the solution there.

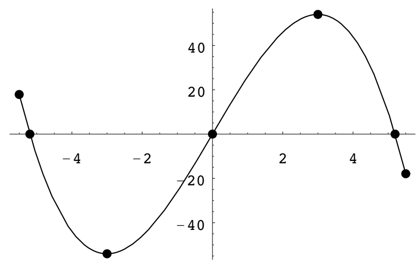
Solution

$$\begin{aligned} f'(x) &= 4x^3 - 12x^2 \\ &= 4x^2(x - 3) \end{aligned}$$

	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
f'	−	−	+
f	↘	↘	↗

- Local maximum point $(3, f(3)) = (3, 54)$
- Intercepts $(0, 0)$, $(3\sqrt{3}, 0)$ and $(-3\sqrt{3}, 0)$
- Endpoints $(-5.5, f(-5.5)) = (-5.5, 17.875)$
  and $(5.5, f(5.5)) = (5.5, -17.875)$

The required graph is shown in the following figure:



□

Remark Since f is an odd function, the graph is symmetric about the origin.

Example Sketch the graph of $y = x^4 - 4x^3 + 5$ for $-1.5 \le x \le 4.2$.

Explanation In two previous examples, we obtain the following:

	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
f'	$-$	$-$	$+$

	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
f''	$+$	$-$	$+$

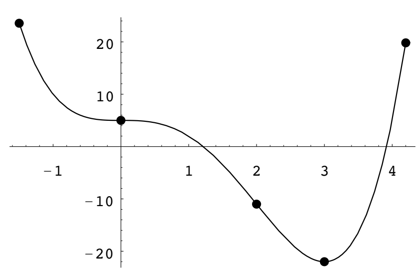
Solution

	$(-\infty, 0)$	$(0, 2)$	$(2, 3)$	$(3, \infty)$
f'	$-$	$-$	$-$	$+$
f''	$+$	$-$	$+$	$+$
f				

On the graph, we have

- Inflection points $(0, f(0)) = (0, 5)$ and $(2, f(2)) = (2, -11)$
- Local minimum point $(3, f(3)) = (3, -22)$
- Endpoints $(-1.5, f(-1.5)) \approx (-1.5, 23.6)$
  and $(4.2, f(4.2)) \approx (4.2, 19.8)$

The required graph is shown in the following figure:



□

Theorem 6.3.1 means that if we can find one antiderivative for a continuous function f on an open interval (a, b), then we can find all. More precisely, if F is an antiderivative for f on (a, b), then all the antiderivatives for f on (a, b) are in the form

$$F(x) + C, \qquad a < x < b \tag{6.3.1}$$

where C is a constant.

Note that (6.3.1) represents a family of functions defined on (a, b)—there are infinitely many of them, with each C corresponds to an antiderivative for f and vice versa. We call the family to be the *indefinite integral* of f (with respect to x) and we denote it by

$$\int f(x)\,\mathrm{d}x.$$

That is,

$$\int f(x)\,\mathrm{d}x = F(x) + C, \qquad a < x < b,$$

where F is a function such that $F'(x) = f(x)$ for all $x \in (a, b)$ and C is an arbitrary constant, called *constant of integration*.

Example Using the two results in the last example, we have the following:

(1) $\displaystyle\int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + C, \quad -\infty < x < \infty,$ where C is an arbitrary constant.

(2) $\displaystyle\int \frac{1}{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{x} + C, \quad x > 0,$ where C is an arbitrary constant.

Remark

- Sometimes, for simplicity, we write $\int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + C$ etc.
  - ◊ The interval $\mathbb{R}$ is omitted because it can be determined easily.
  - ◊ The symbol C is understood to be an arbitrary constant.
- Since we can use any symbol to denote the independent variable, we may also write $\int t^2\,\mathrm{d}t = \frac{1}{3}t^3 + C$ etc.
- Instead of a family of functions, sometimes we write $\int f(x)\,\mathrm{d}x$ to represent a function only. See the discussion in the *Alternative Solution* on page 177.

Terminology

- To *integrate* a function f means to find the indefinite integral of f (that is, to find $\int f(x)\,\mathrm{d}x$ if x is chosen to be the independent variable).
- Same as that for definite integrals, in the notation $\int f(x)\,\mathrm{d}x$, the function f is called the *integrand*.

Integration of Constant (Function) Let k be a constant. Then we have

$$\int k\,\mathrm{d}x = kx + C, \qquad -\infty < x < \infty.$$

Explanation As usual, C is understood to be an arbitrary constant.