1.5

Solving Linear Equations - Formulas

Objective: Solve linear formulas for a given variable.

Solving formulas is much like solving general linear equations. The only difference is we will have several varaibles in the problem and we will be attempting to solve for one specific variable. For example, we may have a formula such as $A = \pi r^2 + \pi \mathrm{rs}$ (formula for surface area of a right circular cone) and we may be interested in solving for the varaible s. This means we want to isolate the s so the equation has s on one side, and everything else on the other. So a solution might look like $s = \frac{A - \pi r^2}{\pi s}$. This second equation gives the same information as the first, they are algebraically equivalent, however, one is solved for the area, while the other is solved for s (slant height of the cone). In this section we will discuss how we can move from the first equation to the second.

When solving formulas for a variable we need to focus on the one varaible we are trying to solve for, all the others are treated just like numbers. This is shown in the following example. Two parallel problems are shown, the first is a normal one-step equation, the second is a formula that we are solving for x

Example 74.

$3x = 12$	$wx = z$	In both problems, x is multiplied by something
$\overline{\mathbf{3}} \quad \overline{\mathbf{3}}$	$\overline{\boldsymbol{w}} \quad \overline{\boldsymbol{w}}$	To isolate the x we divide by 3 or w.
$x = 4$	$x = \frac{z}{w}$	Our Solution

We use the same process to solve $3x = 12$ for x as we use to solve $wx = z$ for x. Because we are solving for x we treat all the other variables the same way we would treat numbers. Thus, to get rid of the multiplication we divided by w. This same idea is seen in the following example.

Example 75.

$m + n = p$	for n	Solving for n, treat all other variables like numbers
$\underline{-\boldsymbol{m} \quad -\boldsymbol{m}}$		Subtract m from both sides
$n = p - m$		Our Solution

As p and m are not like terms, they cannot be combined. For this reason we leave the expression as $p - m$. This same one-step process can be used with grouping symbols.

1.7

Solving Linear Equations - Variation

Objective: Solve variation problems by creating variation equations and finding the variation constant.

One application of solving linear equations is variation. Often different events are related by what is called the constant of variation. For example, the time it takes to travel a certain distance is related to how fast you are traveling. The faster you travel, the less time it take to get there. This is one type of variation problem, we will look at three types of variation here. Variation problems have two or three variables and a constant in them. The constant, usually noted with a k, describes the relationship and does not change as the other variables in the problem change. There are two ways to set up a variation problem, the first solves for one of the variables, a second method is to solve for the constant. Here we will use the second method.

The greek letter pi (π) is used to represent the ratio of the circumference of a circle to its diameter.

World View Note: In the 5th centure, Chinese mathematician Zu Chongzhi calculated the value of π to seven decimal places (3.1415926). This was the most accurate value of π for the next 1000 years!

If you take any circle and divide the circumference of the circle by the diameter you will always get the same value, about 3.14159... If you have a bigger circumference you will also have a bigger diameter. This relationship is called **direct variation** or **directly proportional**. If we see this phrase in the problem we know to divide to find the constant of variation.

Example 92.

$$\begin{array}{rl} m \text{ is varies directly as } n & \text{"Directly" tells us to divide} \\ \dfrac{m}{n} = k & \text{Our formula for the relationship} \end{array}$$

In kickboxing, one will find that the longer the board, the easier it is to break. If you multiply the force required to break a board by the length of the board you will also get a constant. Here, we are multiplying the variables, which means as one variable increases, the other variable decreases. This relationship is called **indirect variation** or **inversely proportional**. If we see this phrase in the problem we know to multiply to find the constant of variation.

Example 93.

$$\begin{array}{rl} y \text{ is inversely proportional to } z & \text{"Inversely" tells us to multiply} \\ yz = k & \text{Our formula for the relationship} \end{array}$$

4.6

Systems of Equations - Mixture Problems

Objective: Solve mixture problems by setting up a system of equations.

One application of systems of equations are mixture problems. Mixture problems are ones where two different solutions are mixed together resulting in a new final solution. We will use the following table to help us solve mixture problems:

	Amount	Part	Total
Item 1			
Item 2			
Final			

The first column is for the amount of each item we have. The second column is labeled "part". If we mix percentages we will put the rate (written as a decimal) in this column. If we mix prices we will put prices in this column. Then we can multiply the amount by the part to find the total. Then we can get an equation by adding the amount and/or total columns that will help us solve the problem and answer the questions.

These problems can have either one or two variables. We will start with one variable problems.

Example 191.

A chemist has 70 mL of a 50% methane solution. How much of a 80% solution must she add so the final solution is 60% methane?

	Amount	Part	Total
Start	70	0.5	
Add	x	0.8	
Final			

Set up the mixture table. We start with 70, but don't know how much we add, that is x. The part is the percentages, 0.5 for start, 0.8 for add.

5.1

Polynomials - Exponent Properties

Objective: Simplify expressions using the properties of exponents.

Problems with expoenents can often be simplified using a few basic exponent properties. Exponents represent repeated multiplication. We will use this fact to discover the important properties.

World View Note: The word exponent comes from the Latin "expo" meaning out of and "ponere" meaning place. While there is some debate, it seems that the Babylonians living in Iraq were the first to do work with exponents (dating back to the 23rd century BC or earlier!)

Example 196.

$$\begin{array}{rl} a^3a^2 & \text{Expand exponents to multiplication problem} \\ (aaa)(aa) & \text{Now we have } 5\,a's \text{ being multiplied together} \\ a^5 & \text{Our Solution} \end{array}$$

A quicker method to arrive at our answer would have been to just add the exponents: $a^3a^2 = a^{3+2} = a^5$ This is known as the **product rule of exponents**

$$\textbf{Product Rule of Exponents: } \boldsymbol{a^m a^n = a^{m+n}}$$

The product rule of exponents can be used to simplify many problems. We will add the exponent on like variables. This is shown in the following examples

Example 197.

$$\begin{array}{rl} 3^2\cdot 3^6\cdot 3 & \text{Same base, add the exponents } 2+6+1 \\ 3^9 & \text{Our Solution} \end{array}$$

Example 198.

$$\begin{array}{rl} 2x^3y^5z\cdot 5xy^2z^3 & \text{Multiply } 2\cdot 5\text{, add exponents on } x, y \text{ and } z \\ 10x^4y^7z^4 & \text{Our Solution} \end{array}$$

Rather than multiplying, we will now try to divide with exponents

Example 199.

$$\begin{array}{rl} \dfrac{a^5}{a^2} & \text{Expand exponents} \\ \dfrac{aaaaa}{aa} & \text{Divide out two of the } a's \\ aaa & \text{Convert to exponents} \\ a^3 & \text{Our Solution} \end{array}$$

A quicker method to arrive at the solution would have been to take the exponent of three and put it on each factor in parenthesis, $(ab)^3 = a^3b^3$. This is known as the power of a product rule or exponents.

$$\textbf{Power of } \boldsymbol{a} \textbf{ Product Rule of Exponents: } (\mathbf{ab})^{\boldsymbol{m}} = \boldsymbol{a^m b^m}$$

It is important to be careful to only use the power of a product rule with multiplication inside parenthesis. This property does NOT work if there is addition or subtraction.

Warning 204.

$$(a+b)^m \neq a^m + b^m \qquad \text{These are } \textbf{NOT} \text{ equal, beware of this error!}$$

Another property that is very similar to the power of a product rule is considered next.

Example 205.

$$\left(\frac{a}{b}\right)^3 \qquad \text{This means we have the fraction three timse}$$

$$\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) \qquad \text{Multiply fractions across the top and bottom, using exponents}$$

$$\frac{a^3}{b^3} \qquad \text{Our Solution}$$

A quicker method to arrive at the solution would have been to put the exponent on every factor in both the numerator and denominator, $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$. This is known as the power of a quotient rule of exponents.

$$\textbf{Power of } \boldsymbol{a} \textbf{ Quotient Rule of Exponents: } \left(\frac{\boldsymbol{a}}{\boldsymbol{b}}\right)^{\boldsymbol{m}} = \frac{\boldsymbol{a^m}}{\boldsymbol{b^m}}$$

The power of a power, product and quotient rules are often used together to simplify expressions. This is shown in the following examples.

Example 206.

$$(x^3yz^2)^4 \qquad \text{Put the exponent of 4 on each factor, multiplying powers}$$

$$x^{12}y^4z^8 \qquad \text{Our solution}$$

Example 216.

$\frac{a^3}{a^5}$	Using the quotient rule, subtract exponents
a^{-2}	Our Solution, but we will also solve this problem another way.
$\frac{a^3}{a^5}$	Rewrite exponents as repeated multiplication
$\frac{aaa}{aaaaa}$	Reduce three $a's$ out of top and bottom
$\frac{1}{aa}$	Simplify to exponents
$\frac{1}{a^2}$	Our Solution, putting these solutions together gives:
$a^{-2}=\frac{1}{a^2}$	Our Final Solution

This example illustrates an important property of exponents. Negative exponents yield the reciprocal of the base. Once we take the reciprical the exponent is now positive. Also, it is important to note a negative exponent does not mean the expression is negative, only that we need the reciprocal of the base. Following are the rules of negative exponents

Rules of Negative Exponets:

$$\boldsymbol{a^{-m}=\frac{1}{m}}$$

$$\boldsymbol{\frac{1}{a^{-m}}=a^m}$$

$$\boldsymbol{\left(\frac{a}{b}\right)^{-m}=\frac{b^m}{a^m}}$$

Negative exponents can be combined in several different ways. As a general rule if we think of our expression as a fraction, negative exponents in the numerator must be moved to the denominator, likewise, negative exponents in the denominator need to be moved to the numerator. When the base with exponent moves, the exponent is now positive. This is illustrated in the following example.

Example 217.

$\frac{a^3b^{-2}c}{2d^{-1}e^{-4}f^2}$	Negative exponents on b, d, and e need to flip
$\frac{a^3cde^4}{2b^2f^2}$	Our Solution

7.6

Rational Expressions - Proportions

Objective: Solve proportions using the cross product and use proportions to solve application problems

When two fractions are equal, they are called a proportion. This definition can be generalized to two equal rational expressions. Proportions have an important property called the cross-product.

$$\textbf{Cross Product: If } \frac{\boldsymbol{a}}{\boldsymbol{b}} = \frac{\boldsymbol{c}}{\boldsymbol{d}} \textbf{ then ad} = \textbf{bc}$$

The cross product tells us we can multiply diagonally to get an equation with no fractions that we can solve.

Example 359.

$$\frac{20}{6} = \frac{x}{9} \quad \text{Calculate cross product}$$

9.8 Practice - Teamwork

1) Bills father can paint a room in two hours less than Bill can paint it. Working together they can complete the job in two hours and 24 minutes. How much time would each require working alone?

2) Of two inlet pipes, the smaller pipe takes four hours longer than the larger pipe to fill a pool. When both pipes are open, the pool is filled in three hours and forty-five minutes. If only the larger pipe is open, how many hours are required to fill the pool?

3) Jack can wash and wax the family car in one hour less than Bob can. The two working together can complete the job in $1\,\frac{1}{5}$ hours. How much time would each require if they worked alone?

4) If A can do a piece of work alone in 6 days and B can do it alone in 4 days, how long will it take the two working together to complete the job?

5) Working alone it takes John 8 hours longer than Carlos to do a job. Working together they can do the job in 3 hours. How long will it take each to do the job working alone?

6) A can do a piece of work in 3 days, B in 4 days, and C in 5 days each working alone. How long will it take them to do it working together?

7) A can do a piece of work in 4 days and B can do it in half the time. How long will it take them to do the work together?

8) A cistern can be filled by one pipe in 20 minutes and by another in 30 minutes. How long will it take both pipes together to fill the tank?

9) If A can do a piece of work in 24 days and A and B together can do it in 6 days, how long would it take B to do the work alone?

10) A carpenter and his assistant can do a piece of work in $3\frac{3}{4}$ days. If the carpenter himself could do the work alone in 5 days, how long would the assistant take to do the work alone?

11) If Sam can do a certain job in 3 days, while it takes Fred 6 days to do the same job, how long will it take them, working together, to complete the job?

12) Tim can finish a certain job in 10 hours. It take his wife JoAnn only 8 hours to do the same job. If they work together, how long will it take them to complete the job?

13) Two people working together can complete a job in 6 hours. If one of them works twice as fast as the other, how long would it take the faster person, working alone, to do the job?

14) If two people working together can do a job in 3 hours, how long will it take the slower person to do the same job if one of them is 3 times as fast as the other?

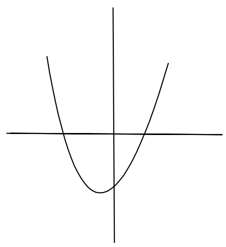
15) A water tank can be filled by an inlet pipe in 8 hours. It takes twice that long for the outlet pipe to empty the tank. How long will it take to fill the tank if both pipes are open?

10.1 Practice - Function Notation

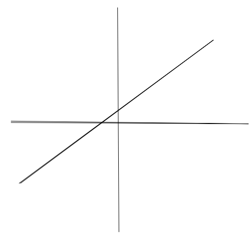
Solve.

1) Which of the following is a function?

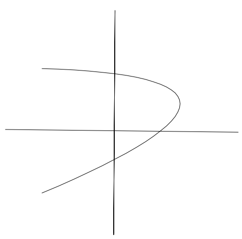
a)



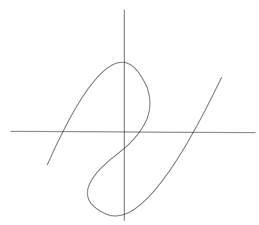
b)



c)



d)



e) $y = 3x - 7$

f) $y^2 - x^2 = 1$

g) $\sqrt{y} + x = 2$

h) $x^2 + y^2 = 1$

Specify the domain of each of the following funcitons.

2) $f(x) = -5x + 1$

3) $f(x) = \sqrt{5 - 4x}$

4) $s(t) = \frac{1}{t^2}$

5) $f(x) = x^2 - 3x - 4$

6) $s(t) = \frac{1}{t^2 + 1}$

7) $f(x) = \sqrt{x - 16}$

8) $f(x) = \frac{-2}{x^2 - 3x - 4}$

9) $h(x) = \frac{\sqrt{3x - 12}}{x^2 - 25}$

10 $y(x) = \frac{x}{x^2 - 25}$

Answers - Point-Slope Form

1) $x = 2$

2) $x = 1$

3) $y - 2 = \frac{1}{2}(x - 2)$

4) $y - 1 = -\frac{1}{2}(x - 2)$

5) $y + 5 = 9(x + 1)$

6) $y + 2 = -2(x - 2)$

7) $y - 1 = \frac{3}{4}(x + 4)$

8) $y + 3 = -2(x - 4)$

9) $y + 2 = -3x$

10) $y - 1 = 4(x + 1)$

11) $y + 5 = -\frac{1}{4}x$

12) $y - 2 = -\frac{5}{4}x$

13) $y + 3 = \frac{1}{5}(x + 5)$

14) $y + 4 = -\frac{2}{3}(x + 1)$

15) $y - 4 = -\frac{5}{4}(x + 1)$

16) $y + 4 = -\frac{3}{2}x(x - 1)$

17) $y = 2x - 3$

18) $y = -2x + 2$

19) $y = -\frac{3}{5}x + 2$

20) $y = -\frac{2}{3}x - \frac{10}{3}$

21) $y = \frac{1}{2}x + 3$

22) $y = -\frac{7}{4}x + 4$

23) $y = -\frac{3}{2}x + 4$

24) $y = -\frac{5}{2}x - 5$

25) $y = -\frac{2}{5}x - 5$

26) $y = \frac{7}{3}x - 4$

27) $y = x - 4$

28) $y = -3$

29) $x = -3$

30) $y = 2x - 1$

31) $y = -\frac{1}{2}x$

32) $y = \frac{6}{5}x - 3$

33) $y - 3 = -2(x + 4)$

34) $y = 3$

35) $y - 1 = \frac{1}{8}(x - 5)$

36) $y - 5 = -\frac{1}{8}(x + 4)$

37) $y + 2 = \frac{3}{2}(x + 4)$

38) $y - 1 = \frac{3}{8}(x + 4)$

39) $y - 5 = \frac{1}{4}(x - 3)$

40) $y + 4 = -(x + 1)$

41) $y + 3 = -\frac{8}{7}(x - 3)$

42) $y + 5 = -\frac{1}{4}(x + 1)$

43) $y = -\frac{3}{4}x - \frac{11}{4}$

44) $y = -\frac{1}{10}x - \frac{3}{2}$

45) $y = -\frac{8}{7}x - \frac{5}{7}$

46) $y = \frac{1}{2}x - \frac{3}{2}$

47) $y = -x + 5$

48) $y = \frac{1}{3}x + 1$

49) $y = -x + 2$

50) $y = x + 2$

51) $y = 4x + 3$

52) $y = \frac{3}{7}x + \frac{6}{7}$

2.5

Answers - Parallel and Perpendicular Lines

1) 2

2) $-\frac{2}{3}$

3) 4

4) $-\frac{10}{3}$

5) 1

6) $\frac{6}{5}$

7) -7

8) $-\frac{3}{4}$

9) 0

10) 2

11) 3

12) $-\frac{5}{4}$

13) -3

14) $-\frac{1}{3}$

15) 2

16) $-\frac{3}{8}$

17) $x = 2$

18) $y - 2 = \frac{7}{5}(x - 5)$

19) $y - 4 = \frac{9}{2}(x - 3)$

20) $y + 1 = -\frac{3}{4}(x - 1)$

21) $y - 3 = \frac{7}{5}(x - 2)$

22) $y - 3 = -3(x + 1)$