input to exactly one output. For example, let's try to find the inverse function for $f(x) = x^2$. Solving the equation $y = x^2$ for x, we arrive at the equation $x = \pm\sqrt{y}$. This equation does not describe x as a function of y because there are two solutions to this equation for every $y > 0$. The problem with trying to find an inverse function for $f(x) = x^2$ is that two inputs are sent to the same output for each output $y > 0$. The function $f(x) = x^3 + 4$ discussed earlier did not have this problem. For that function, each input was sent to a different output. A function that sends each input to a *different* output is called a one-to-one function.

Definition

We say a f is a **one-to-one function** if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

One way to determine whether a function is one-to-one is by looking at its graph. If a function is one-to-one, then no two inputs can be sent to the same output. Therefore, if we draw a horizontal line anywhere in the xy-plane, according to the **horizontal line test**, it cannot intersect the graph more than once. We note that the horizontal line test is different from the vertical line test. The vertical line test determines whether a graph is the graph of a function. The horizontal line test determines whether a function is one-to-one (**Figure 1.38**).

Rule: Horizontal Line Test

A function f is one-to-one if and only if every horizontal line intersects the graph of f no more than once.

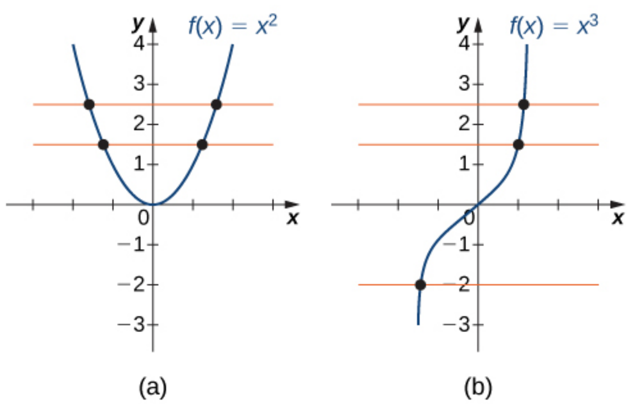


Figure 1.38 (a) The function $f(x) = x^2$ is not one-to-one because it fails the horizontal line test. (b) The function $f(x) = x^3$ is one-to-one because it passes the horizontal line test.

Example 1.28

Determining Whether a Function Is One-to-One

For each of the following functions, use the horizontal line test to determine whether it is one-to-one.

Let $\varepsilon > 0$.

The first part of the definition begins "For every $\varepsilon > 0$." This means we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating "Let $\varepsilon > 0$," we signal our intent to do so.

Choose $\delta = \frac{\varepsilon}{2}$.

The definition continues with "there exists a $\delta > 0$." The phrase "there exists" in a mathematical statement is always a signal for a scavenger hunt. In other words, we must go and find δ. So, where exactly did $\delta = \varepsilon/2$ come from? There are two basic approaches to tracking down δ. One method is purely algebraic and the other is geometric.

We begin by tackling the problem from an algebraic point of view. Since ultimately we want $|(2x+1)-3| < \varepsilon$, we begin by manipulating this expression: $|(2x+1)-3| < \varepsilon$ is equivalent to $|2x-2| < \varepsilon$, which in turn is equivalent to $|2||x-1| < \varepsilon$. Last, this is equivalent to $|x-1| < \varepsilon/2$. Thus, it would seem that $\delta = \varepsilon/2$ is appropriate.

We may also find δ through geometric methods. **Figure 2.40** demonstrates how this is done.

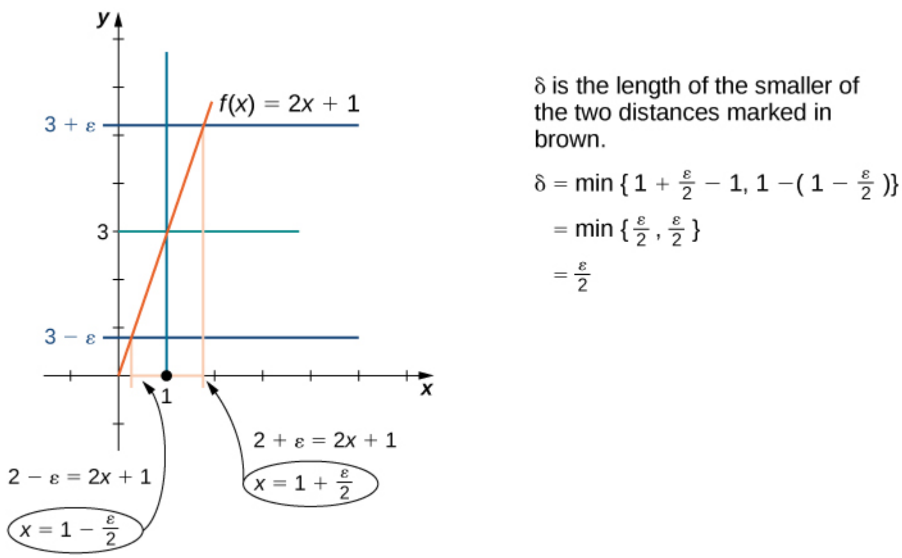


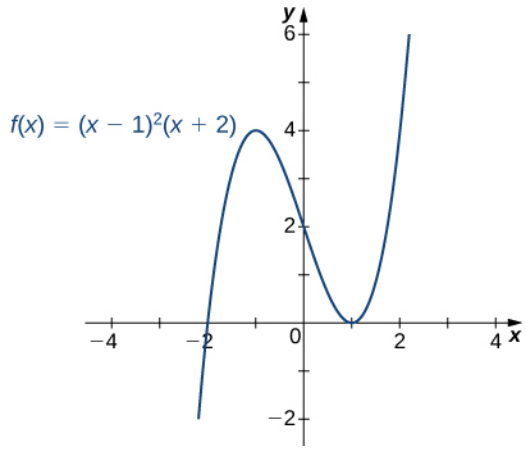
Figure 2.40 This graph shows how we find δ geometrically.

Assume $0 < |x-1| < \delta$. When δ has been chosen, our goal is to show that if $0 < |x-1| < \delta$, then $|(2x+1)-3| < \varepsilon$. To prove any statement of the form "If this, then that," we begin by assuming "this" and trying to get "that."

Thus,

$$\begin{aligned}
|(2x+1)-3| &= |2x-2| && \text{property of absolute value} \\
&= |2(x-1)| && \\
&= |2||x-1| && |2| = 2 \\
&= 2|x-1| && \\
&< 2 \cdot \delta && \text{here's where we use the assumption that } 0 < |x-1| < \delta \\
&= 2 \cdot \frac{\varepsilon}{2} = \varepsilon && \text{here's where we use our choice of } \delta = \varepsilon/2
\end{aligned}$$

Analysis



4.27 Sketch a graph of $f(x) = (x-1)^3(x+2)$.

Example 4.29

Sketching a Rational Function

Sketch the graph of $f(x) = \dfrac{x^2}{\left(1-x^2\right)}$.

Solution

Step 1. The function f is defined as long as the denominator is not zero. Therefore, the domain is the set of all real numbers x except $x = \pm 1$.

Step 2. Find the intercepts. If $x = 0$, then $f(x) = 0$, so 0 is an intercept. If $y = 0$, then $\dfrac{x^2}{\left(1-x^2\right)} = 0$, which implies $x = 0$. Therefore, $(0, 0)$ is the only intercept.

Step 3. Evaluate the limits at infinity. Since f is a rational function, divide the numerator and denominator by the highest power in the denominator: x^2. We obtain

$$\lim_{x \to \pm\infty} \frac{x^2}{1-x^2} = \lim_{x \to \pm\infty} \frac{1}{\frac{1}{x^2} - 1} = -1.$$

Therefore, f has a horizontal asymptote of $y = -1$ as $x \to \infty$ and $x \to -\infty$.

Step 4. To determine whether f has any vertical asymptotes, first check to see whether the denominator has any zeroes. We find the denominator is zero when $x = \pm 1$. To determine whether the lines $x = 1$ or $x = -1$ are vertical asymptotes of f, evaluate $\lim_{x \to 1} f(x)$ and $\lim_{x \to -1} f(x)$. By looking at each one-sided limit as $x \to 1$, we see that

5.2 | The Definite Integral

Learning Objectives

5.2.1 State the definition of the definite integral.
5.2.2 Explain the terms integrand, limits of integration, and variable of integration.
5.2.3 Explain when a function is integrable.
5.2.4 Describe the relationship between the definite integral and net area.
5.2.5 Use geometry and the properties of definite integrals to evaluate them.
5.2.6 Calculate the average value of a function.

In the preceding section we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, real-world problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

Definition

If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral** of *f* from *a* to *b* is given by

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x, \tag{5.8}$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an **integrable function**.

The integral symbol in the previous definition should look familiar. We have seen similar notation in the chapter on **Applications of Derivatives**, where we used the indefinite integral symbol (without the *a* and *b* above and below) to represent an antiderivative. Although the notation for indefinite integrals may look similar to the notation for a definite integral, they are not the same. A definite integral is a number. An indefinite integral is a family of functions. Later in this chapter we examine how these concepts are related. However, close attention should always be paid to notation so we know whether we're working with a definite integral or an indefinite integral.

Integral notation goes back to the late seventeenth century and is one of the contributions of Gottfried Wilhelm Leibniz, who is often considered to be the codiscoverer of calculus, along with Isaac Newton. The integration symbol ∫ is an elongated S, suggesting sigma or summation. On a definite integral, above and below the summation symbol are the boundaries of the interval, $[a, b]$. The numbers *a* and *b* are *x*-values and are called the **limits of integration**; specifically, *a* is the lower limit and *b* is the upper limit. To clarify, we are using the word *limit* in two different ways in the context of the definite integral. First, we talk about the limit of a sum as $n \to \infty$. Second, the boundaries of the region are called the *limits of integration*.

We call the function $f(x)$ the **integrand**, and the *dx* indicates that $f(x)$ is a function with respect to *x*, called the **variable of integration**. Note that, like the index in a sum, the variable of integration is a dummy variable, and has no impact on the computation of the integral. We could use any variable we like as the variable of integration:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$$

$$
\begin{aligned}
\int_{-2}^{2}(t^2-4)dt &= \frac{t^3}{3}-4t\Big|_{-2}^{2} \\
&= \left[\frac{(2)^3}{3}-4(2)\right]-\left[\frac{(-2)^3}{3}-4(-2)\right] \\
&= \left(\frac{8}{3}-8\right)-\left(-\frac{8}{3}+8\right) \\
&= \frac{8}{3}-8+\frac{8}{3}-8 \\
&= \frac{16}{3}-16 \\
&= -\frac{32}{3}.
\end{aligned}
$$

Analysis

Notice that we did not include the "+ C" term when we wrote the antiderivative. The reason is that, according to the Fundamental Theorem of Calculus, Part 2, *any* antiderivative works. So, for convenience, we chose the antiderivative with $C = 0$. If we had chosen another antiderivative, the constant term would have canceled out. This always happens when evaluating a definite integral.

The region of the area we just calculated is depicted in Figure 5.28. Note that the region between the curve and the x-axis is all below the x-axis. Area is always positive, but a definite integral can still produce a negative number (a net signed area). For example, if this were a profit function, a negative number indicates the company is operating at a loss over the given interval.

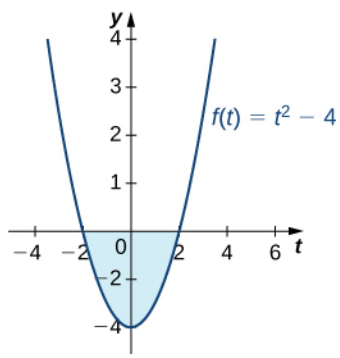


Figure 5.28 The evaluation of a definite integral can produce a negative value, even though area is always positive.

Example 5.21

Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2

Evaluate the following integral using the Fundamental Theorem of Calculus, Part 2:

$$\int_1^9 \frac{x-1}{\sqrt{x}}dx.$$

Solution

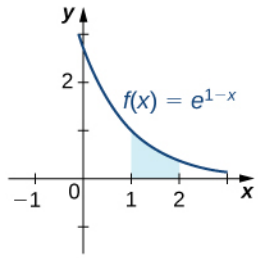


Figure 5.38 The indicated area can be calculated by evaluating a definite integral using substitution.

5.34 Evaluate $\int_0^2 e^{2x}\,dx$.

Example 5.42

Growth of Bacteria in a Culture

Suppose the rate of growth of bacteria in a Petri dish is given by $q(t) = 3^t$, where t is given in hours and $q(t)$ is given in thousands of bacteria per hour. If a culture starts with 10,000 bacteria, find a function $Q(t)$ that gives the number of bacteria in the Petri dish at any time t. How many bacteria are in the dish after 2 hours?

Solution

We have

$$Q(t) = \int 3^t\,dt = \frac{3^t}{\ln 3} + C.$$

Then, at $t = 0$ we have $Q(0) = 10 = \frac{1}{\ln 3} + C$, so $C \approx 9.090$ and we get

$$Q(t) = \frac{3^t}{\ln 3} + 9.090.$$

At time $t = 2$, we have

$$\begin{aligned} Q(2) &= \frac{3^2}{\ln 3} + 9.090 \\ &= 17.282. \end{aligned}$$

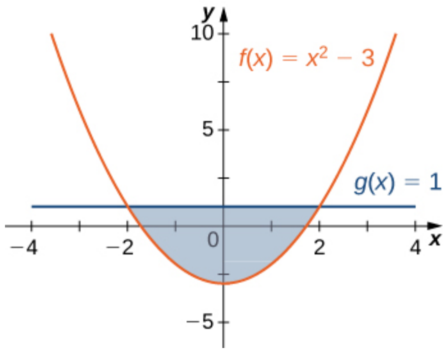
After 2 hours, there are 17,282 bacteria in the dish.

5.35 From **Example 5.42**, suppose the bacteria grow at a rate of $q(t) = 2^t$. Assume the culture still starts with 10,000 bacteria. Find $Q(t)$. How many bacteria are in the dish after 3 hours?

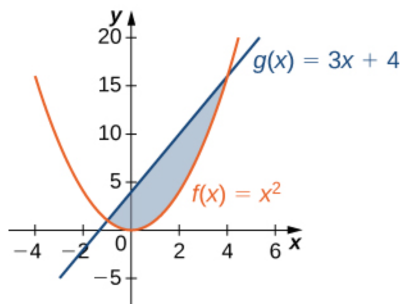
6.1 EXERCISES

For the following exercises, determine the area of the region between the two curves in the given figure by integrating over the x-axis.

1. $y = x^2 - 3$ and $y = 1$

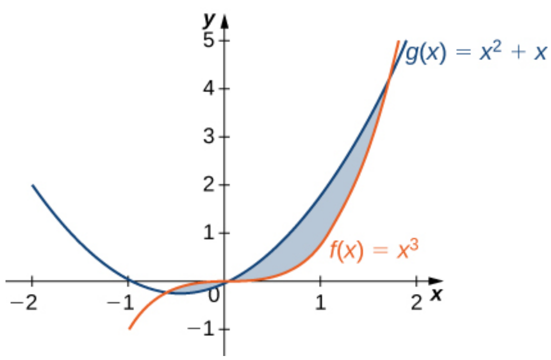


2. $y = x^2$ and $y = 3x + 4$

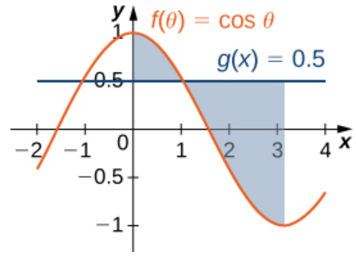


For the following exercises, split the region between the two curves into two smaller regions, then determine the area by integrating over the x-axis. Note that you will have two integrals to solve.

3. $y = x^3$ and $y = x^2 + x$

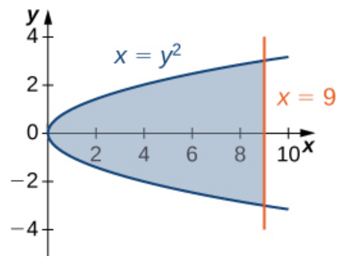


4. $y = \cos\theta$ and $y = 0.5$, for $0 \le \theta \le \pi$

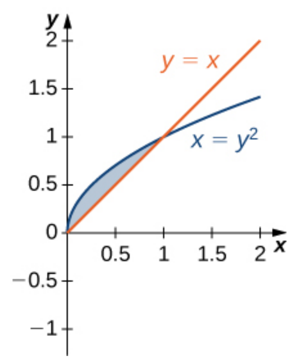


For the following exercises, determine the area of the region between the two curves by integrating over the y-axis.

5. $x = y^2$ and $x = 9$



6. $y = x$ and $x = y^2$



For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis.

7. $y = x^2$ and $y = -x^2 + 18x$

8. $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, and $x = 3$

9. $y = \cos x$ and $y = \cos^2 x$ on $x = [-\pi, \pi]$

10. $y = e^x$, $y = e^{2x-1}$, and $x = 0$

11. $y = e^x$, $y = e^{-x}$, $x = -1$ and $x = 1$

To put it another way, the area under the curve $y = 1/t$ between $t = 1$ and $t = e$ is 1 (**Figure 6.77**). The proof that such a number exists and is unique is left to you. (*Hint*: Use the Intermediate Value Theorem to prove existence and the fact that $\ln x$ is increasing to prove uniqueness.)

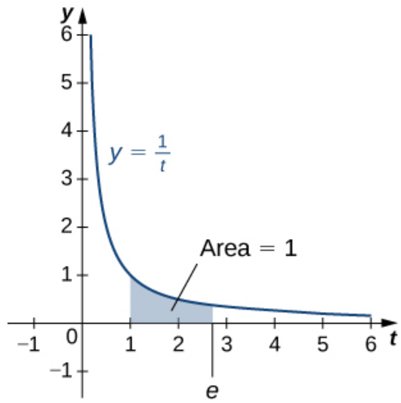


Figure 6.77 The area under the curve from 1 to e is equal to one.

The number e can be shown to be irrational, although we won't do so here (see the Student Project in **Taylor and Maclaurin Series (http://cnx.org/content/m53817/latest/)**). Its approximate value is given by

$$e \approx 2.71828182846.$$

The Exponential Function

We now turn our attention to the function e^x. Note that the natural logarithm is one-to-one and therefore has an inverse function. For now, we denote this inverse function by $\exp x$. Then,

$$\exp(\ln x) = x \text{ for } x > 0 \text{ and } \ln(\exp x) = x \text{ for all } x.$$

The following figure shows the graphs of $\exp x$ and $\ln x$.

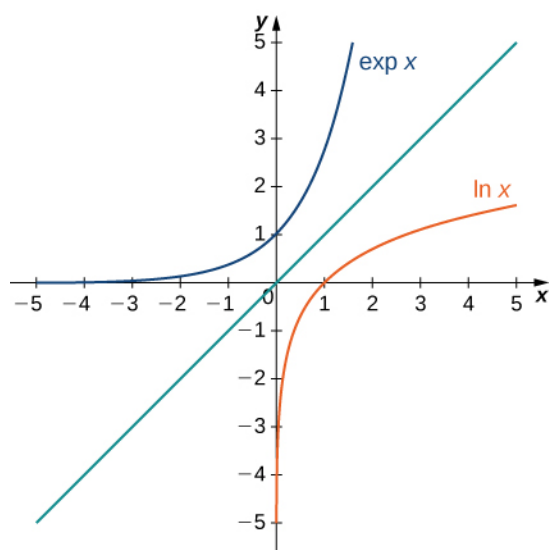
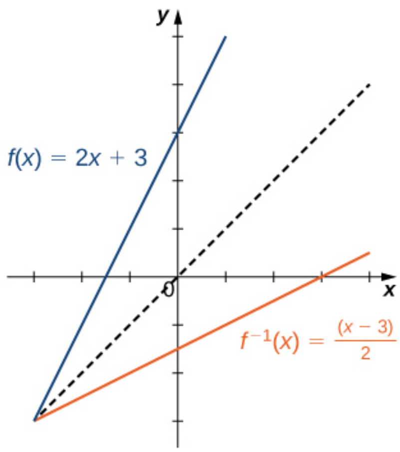


Figure 6.78 The graphs of $\ln x$ and $\exp x$.

We hypothesize that $\exp x = e^x$. For rational values of x, this is easy to show. If x is rational, then we have $\ln(e^x) = x \ln e = x$. Thus, when x is rational, $e^x = \exp x$. For irrational values of x, we simply define e^x as the inverse function of $\ln x$.

1.24. $f^{-1}(x) = \frac{2x}{x-3}$. The domain of f^{-1} is $\{x | x \neq 3\}$. The range of f^{-1} is $\{y | y \neq 2\}$.

1.25.



1.26. The domain of f^{-1} is $(0, \infty)$. The range of f^{-1} is $(-\infty, 0)$. The inverse function is given by the formula $f^{-1}(x) = -1/\sqrt{x}$.

1.27. $f(4) = 900$; $f(10) = 24,\ 300$.

1.28. $x/\left(2y^3\right)$

1.29. $A(t) = 750e^{0.04t}$. After 30 years, there will be approximately $\$2,\ 490.09$.

1.30. $x = \frac{\ln 3}{2}$

1.31. $x = \frac{1}{e}$

1.32. 1.29248

1.33. The magnitude 8.4 earthquake is roughly 10 times as severe as the magnitude 7.4 earthquake.

1.34. $(x^2 + x^{-2})/2$

1.35. $\frac{1}{2}\ln(3) \approx 0.5493$.

Section Exercises

1. a. Domain = $\{-3, -2, -1, 0, 1, 2, 3\}$, range = $\{0, 1, 4, 9\}$ b. Yes, a function

3. a. Domain = $\{0, 1, 2, 3\}$, range = $\{-3, -2, -1, 0, 1, 2, 3\}$ b. No, not a function

5. a. Domain = $\{3, 5, 8, 10, 15, 21, 33\}$, range = $\{0, 1, 2, 3\}$ b. Yes, a function

7. a. -2 b. 3 c. 13 d. $-5x - 2$ e. $5a - 2$ f. $5a + 5h - 2$

9. a. Undefined b. 2 c. $\frac{2}{3}$ d. $-\frac{2}{x}$ e $\frac{2}{a}$ f. $\frac{2}{a+h}$

11. a. $\sqrt{5}$ b. $\sqrt{11}$ c. $\sqrt{23}$ d. $\sqrt{-6x+5}$ e. $\sqrt{6a+5}$ f. $\sqrt{6a+6h+5}$

13. a. 9 b. 9 c. 9 d. 9 e. 9 f. 9

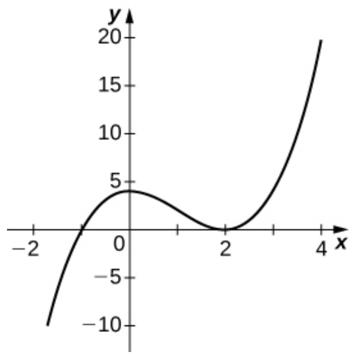
15. $x \geq \frac{1}{8}$; $y \geq 0$; $x = \frac{1}{8}$; no y-intercept

17. $x \geq -2$; $y \geq -1$; $x = -1$; $y = -1 + \sqrt{2}$

19. $x \neq 4$; $y \neq 0$; no x-intercept; $y = -\frac{3}{4}$

21. $x > 5$; $y > 0$; no intercepts

23.



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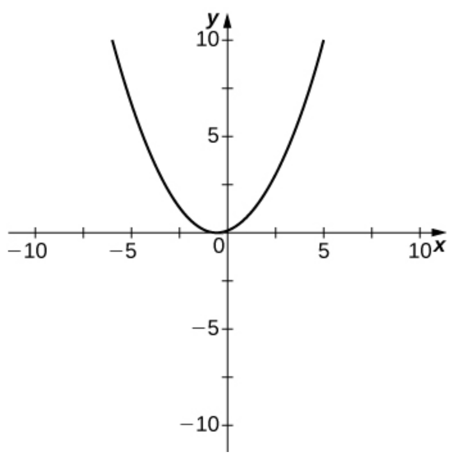

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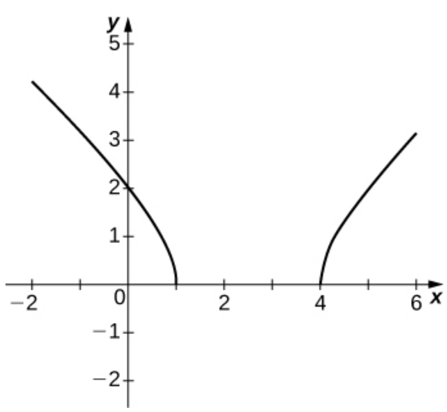

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