Statistical Performance of Convex Tensor Decomposition

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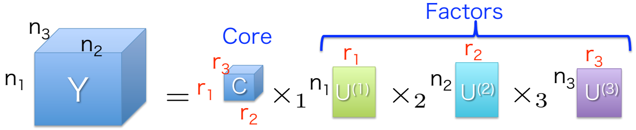
Tucker decomposition [Tucker 66]

- Problem: Given a partially observed approximately low-rank tensor *X*, find



$$\left(Y_{ijk}=\sum_{a=1}^{r_1}\sum_{b=1}^{r_2}\sum_{c=1}^{r_3}C_{abc}U^{(1)}_{ia}U^{(2)}_{jb}U^{(3)}_{kc}\right)$$

- Applications: chemo-/psycho-metrics, signal processing, computer vision, neuroscience
- Estimation: alternate minimization (non-convex)

Convex Tensor Estimation

Matrix

Estimation of *low-rank* matrix (hard) → Convex relaxation → Schatten 1-norm minimization (tractable) [Fazel, Hindi, Boyd 01]

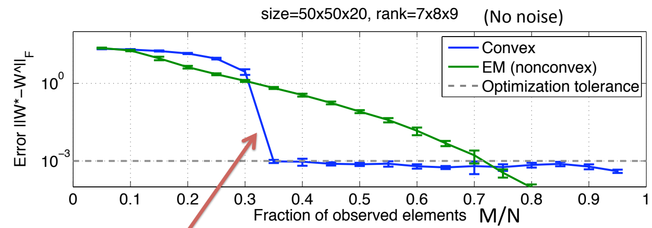
↓ Generalize

Tensor

Estimation of *low-rank tensor* (hard) → Convex relaxation → Overlapped Schatten 1-norm minimization [Liu+09, Signoretto+10, Tomioka+10, Gandy+11]

Motivation: Phase-transition in Convex Tensor Estimation

Tensor completion result [Tomioka et al. 2010]



Goal: Explain this number of samples M from the size of the tensor [n1, n2, n3] and the Tucker rank [r1, r2, r3]

Model: Convex Tensor Estimation

Observation model $\mathcal{W}^*$ true tensor rank-$(r_1,\dots,r_K)$

$$y_i=\langle \mathcal{X}_i,\mathcal{W}^*\rangle+\epsilon_i \quad (i=1,\dots,M)$$

Gaussian noise $N(0,\sigma^2)$

Optimization

$$\hat{\mathcal{W}}=\mathop{\mathrm{argmin}}_{\mathcal{W}\in\mathbb{R}^{n_1\times\cdots\times n_K}}\Big(\underbrace{\frac{1}{2M}\|\boldsymbol{y}-\mathfrak{X}(\mathcal{W})\|_2^2}_{\text{Empirical error}}+\lambda_M\underbrace{\|\mathcal{W}\|_{S_1}}_{\text{Regularization}}\Big)$$

$(N=\prod_{k=1}^{K}n_k)$

Observation model: $\mathfrak{X}:\mathbb{R}^N\rightarrow\mathbb{R}^M$

$\mathfrak{X}(\mathcal{W})=(\langle\mathcal{X}_1,\mathcal{W}\rangle,\dots,\langle\mathcal{X}_M,\mathcal{W}\rangle)^\top$

Reg. Const.

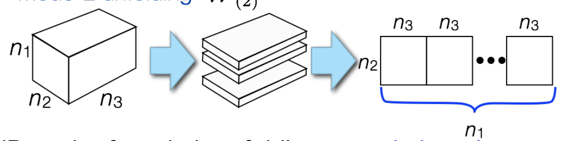
Overlapped Schatten 1-norm for Tensors

$$\|\mathcal{W}\|_{S_1}:=\frac{1}{K}\sum_{k=1}^{K}\|\boldsymbol{W}_{(k)}\|_{S_1}$$

Schatten 1-norm for the mode-k unfolding

Example of unfolding (matricization)

Mode-2 unfolding $\boldsymbol{W}_{(2)}$



NB: rank of mode-k unfolding = mode-k rank r_k

Previous work

Authors	Observation model	Assumption	Target
Recht, Fazel, Parrilo 2007	$y_i=\langle X_i,W\rangle$ $(i=1,\dots,M)$	Restricted Isometry	Matrix
Candès & Recht 2009	$Y_{ij}=W_{ij}$ $((i,j)\in\Omega)$	Incoherence	Matrix
Negahban & Wainwright 2011	$y_i=\langle X_i,W\rangle+\epsilon_i$ $(i=1,\dots,M)$	Restrited Strong Convexity	Matrix
This work	$y_i=\langle X_i,W\rangle+\epsilon_i$ $(i=1,\dots,M)$	Restrited Strong Convexity	Tensor

Restricted strong convexity (RSC)

(cf. Negahban & Wainwright 11)

- Assume that there is a positive constant $\kappa(\mathfrak{X})$ such that for all tensors $\Delta \in C$

$$\frac{1}{M}\|\mathfrak{X}(\Delta)\|_2^2 \geq \kappa(\mathfrak{X})\|\!|\Delta|\!\|_F^2$$

(The set C needs to be defined carefully)

Note:

- If $C=R^N$, $\kappa(X)=\min \mathrm{eig}(X^T X)$ $\quad (X \in R^{M\times N})$
- When M<N, restriction is necessary.
- The smaller C, the weaker the assumption.

Lemma 1: A key inequality

$$\mathcal{W}, \mathcal{X} \in \mathbb{R}^{n_1\times\cdots\times n_K}$$

$$\langle \mathcal{W}, \mathcal{X}\rangle \leq \|\!|\mathcal{W}|\!\|_{S_1}\|\!|\mathcal{X}|\!\|_{\mathrm{mean}}$$

where

$$\|\!|\mathcal{W}|\!\|_{S_1} := \frac{1}{K}\sum_{k=1}^{K}\|\boldsymbol{W}_{(k)}\|_{S_1} \qquad \|\!|\mathcal{X}|\!\|_{\mathrm{mean}} := \frac{1}{K}\sum_{k=1}^{K}\|\boldsymbol{X}_{(k)}\|_{S_\infty}$$

K=2: norm duality (tight)

K>2: not tight

$$\|\boldsymbol{X}\|_{S_1} := \sum_{j=1}^{m}\sigma_j(\boldsymbol{X})$$

$$\|\boldsymbol{X}\|_{S_\infty} := \max_{j\in\{1,\ldots,m\}}\sigma_j(\boldsymbol{X})$$

Theorem 1 (deterministic)

- Solution of the opt. problem $\hat{\mathcal{W}}$
- Reg const λ_M satisfies

$$\lambda_M \geq 2\|\!|\mathfrak{X}^*(\boldsymbol{\epsilon})|\!\|_{\mathrm{mean}}/M$$

where $\mathfrak{X}^*(\epsilon) = \sum_{i=1}^{M}\epsilon_i \mathcal{X}_i$ (noise design correlation)

$$\|\!|\mathcal{X}|\!\|_{\mathrm{mean}} := \frac{1}{K}\sum_{k=1}^{K}\|\boldsymbol{X}_{(k)}\|_{S_\infty}$$

- Under the RSC assumption

$$\|\!|\hat{\mathcal{W}} - \mathcal{W}^*|\!\|_F \leq \frac{32\lambda_M}{\kappa(\mathfrak{X})}\frac{1}{K}\sum_{k=1}^{K}\sqrt{r_k}$$

(cf. Negahban & Wainwright 11)

Two special cases

- Noisy tensor decomposition (M=N)
  - RSC: trivial. $\kappa(\mathfrak{X}) = 1/M$
  - bound on the noise-design correlation term

$$\mathbb{E}\|\!|\mathfrak{X}^*(\boldsymbol{\epsilon})|\!\|_{\mathrm{mean}} \leq \frac{\sigma}{K}\sum_{k=1}^{K}\left(\sqrt{n_k}+\sqrt{N/n_k}\right)$$

- Random Gauss design
  - RSC: more difficult (Lemma 5)
  - bound on the noise-design correlation term

$$\mathbb{E}\|\!|\mathfrak{X}^*(\boldsymbol{\epsilon})|\!\|_{\mathrm{mean}} \leq \frac{\sigma\sqrt{M}}{K}\sum_{k=1}^{K}\left(\sqrt{n_k}+\sqrt{N/n_k}\right)$$

Theorem 2 (noisy tensor decomp.)

When all the elements are observed (M=N) and the regularization const. satisfies

$$\lambda_M \geq \tfrac{2\sigma}{K}\textstyle\sum_{k=1}^{K}\left(\sqrt{n_k}+\sqrt{N/n_k}\right)/N$$

$$\frac{\|\!|\hat{\mathcal{W}} - \mathcal{W}^*|\!\|_F^2}{N} \leq O_p\left(\sigma^2\underbrace{\|\boldsymbol{n}^{-1}\|_{1/2}\|\boldsymbol{r}\|_{1/2}}_{\text{Normalized rank}}\right)$$

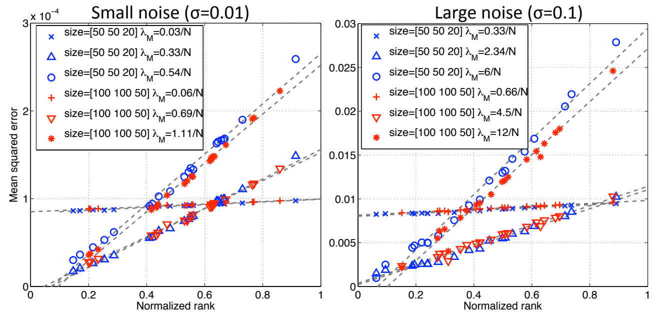
where

$$\|\boldsymbol{n}^{-1}\|_{1/2} := \left(\tfrac{1}{K}\textstyle\sum_{k=1}^{K}\sqrt{1/n_k}\right)^2, \quad \|\boldsymbol{r}\|_{1/2} := \left(\tfrac{1}{K}\textstyle\sum_{k=1}^{K}\sqrt{r_k}\right)^2$$

If n_k=n and r_k=r, normalized rank = r/n

Simulation: Noisy tensor decomposition

Mean squared error $\frac{\|\!|\hat{\mathcal{W}} - \mathcal{W}^*|\!\|_F^2}{N}$



linear relation between MSE and normalized rank!

Theorem 3: random Gauss design

Assume elements of X_i are drown iid from standard normal distribution. Moreover

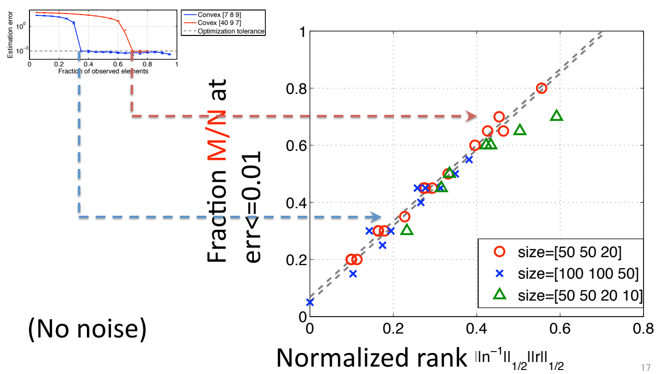
$$\frac{\#\text{samples }(M)}{\#\text{variables }(N)} \geq c_1\underbrace{\|\boldsymbol{n}^{-1}\|_{1/2}\|\boldsymbol{r}\|_{1/2}}_{\text{Normalized rank}}$$

Convergence!

$$\frac{\|\!|\hat{\mathcal{W}} - \mathcal{W}^*|\!\|_F^2}{N} \leq O_p\left(\frac{\sigma^2\|\boldsymbol{n}^{-1}\|_{1/2}\|\boldsymbol{r}\|_{1/2}}{M}\right)$$

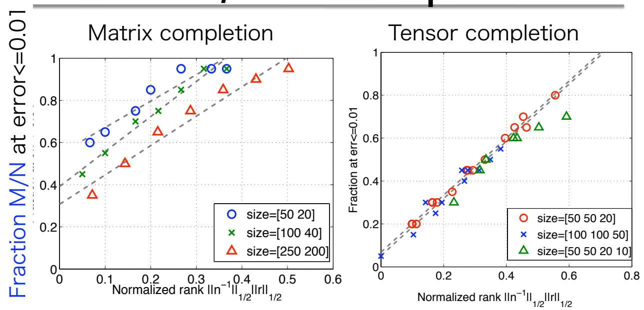
$$\|\boldsymbol{n}^{-1}\|_{1/2} := \left(\tfrac{1}{K}\textstyle\sum_{k=1}^{K}\sqrt{1/n_k}\right)^2, \quad \|\boldsymbol{r}\|_{1/2} := \left(\tfrac{1}{K}\textstyle\sum_{k=1}^{K}\sqrt{r_k}\right)^2$$

Simulation: Tensor Completion



(No noise)

Matrix / tensor completion



Tensor completion *easier* than matrix completion!?

Lemma 5 (RSC for random Gaussian)

Let $\mathfrak{X} : \mathbb{R}^{n_1 \times \cdots \times n_K} \to \mathbb{R}^M$

be a random Gaussisan design. Then

$$\frac{\|\mathfrak{X}(\Delta)\|_2}{\sqrt{M}} \geq \frac{1}{4}|||\Delta|||_F - \frac{1}{K}\sum_{k=1}^{K}\left(\sqrt{\frac{n_k}{M}} + \sqrt{\frac{\bar{n}_{\setminus k}}{M}}\right)|||\Delta|||_{S_1},$$

with probability at least $1 - 2\exp(-N/32)$

Proof: analogous to that of Prop 1 in Negahban & Wainwright 2011 (use Lemma 1)

Conclusion

- Convex tensor decomposition --- now with performance guarantee
- Normalized rank predicts empirical scaling behavior well

Issues

- Why matrix completion more difficult than tensor completion?
- Worst case analysis-> average case analysis
- Analyze tensor completion more carefully
  - Incoherence [Candes & Recht 09]
  - Spikiness [Negahban et al. 10]

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Mode-k unfolding (matricization)