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Asymmetric Statistical Errors

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Abstract

Asymmetric statistical errors arise for experimental results obtained by Maximum Likelihood estimation, in cases where the number of results is finite and the log likelihood function is not a symmetric parabola. This note discusses how separate asymmetric errors on a single result should be combined, and how several results with asymmetric errors should be combined to give an overall measurement. In the process it considers several methods for parametrising curves that are approximately parabolic.

1. Introduction

When an experimental result is presented as $x_{-\sigma^-}^{+\sigma^+}$ this signifies, just as with the usual form $x \pm \sigma$, that x is the value given by a 'best' estimate (i.e. one with good properties of consistency, efficiency, and lack of bias) and that the 68% central confidence region is $[x - \sigma^-, x + \sigma^+]$.

Such asymmetric errors arise through two common causes. The first is when a nuisance parameter a has a conventional symmetric (even Gaussian) probability distribution, but produces a non-linear effect on the desired result x. These errors are generally systematic rather than statistical, and their probability distribution is generally best considered from a Bayesian viewpoint. Their treatment has been considered in a previous note [1].

The second cause of asymmetry is the extraction of a result x through the maximisation of a likelihood function L(x) which is not a symmetric parabola. This occurs because the function is in general only parabolic in the limit when the number of results N, the number of terms contributing to the sum which makes up the log likelihood, is large, and for many results this is not the case. For such a function the errors are conventionally read off the points at which the log likelihood falls by $\frac{1}{2}$ from its peak, though this is not exact [2] and it may be better to obtain the errors from a toy Monte Carlo computation.

Although such asymmetric errors are frequently used in the reporting of particle physics results, constructive analyses of their use are scarce in the literature [3].

2. Two Combination Problems

The two most significant questions on the manipulation of asymmetric errors are the *Combination of Results* and the *Combination of Errors*.

2.1 Combination of Results

The first occurs when one has two results $x_1^{+\sigma_1^+}_{-\sigma_1^-}$ and $x_2^{+\sigma_2^+}_{-\sigma_2^-}$ of the same quantity. This arises when two different experiments measure the same quantity. Assuming that they are compatible (according to some criterion), one wants the propriate value (and errors) that combines the two. This is the equivalent of the well-known expression for symmetric errors

$$\frac{x_1/\sigma_1^2 + x_2/\sigma_2^2}{1/\sigma_1^2 + 1/\sigma_2^2} \pm \sqrt{\frac{1}{1/\sigma_1^2 + 1/\sigma_2^2}} \tag{1}$$

If the log likelihood functions $L_1(x_1)$ and $L_2(x_2)$ are known, then the combined log likelihood is just the sum of the two. The maximum can then be found and the errors read off the $\Delta lnL = -\frac{1}{2}$ points

The question naturally extends to more than two results, and it is clearly a desirably property that the operation be associative: if results are combined pairwise till only one remains, then the pairing strategy should not effect the result. For the addition of likelihoods this obviously holds.

2.2 Combination of Errors

The second question arises when a particular result (taken, without loss of generality, as zero) is subject to several separate (asymmetric) uncertainties, and one needs to quote

the overall uncertainty. An obvious example would be the uncertainty due to background subtraction where the background has several different components, each with asymmetric uncertainties. This is the equivalent of the well-known expression for symmetric errors

If
$$x = x_1 + x_2$$
 then $\sigma^2 = \sigma_1^2 + \sigma_2^2$ (2)

Again, it is desirable that the operation be associative.

If the likelihood functions are known then the joint function $L_1(x_1)L_2(x_2)$ is defined on the (x_1, x_2) plane with its peak at (0,0). The uncertainty on the sum $x_1 + x_2$ is found by the profiling technique: we find $\hat{L}(x_1 + x_2)$, the peak value of the likelihood anywhere on the line $x_1 + x_2 = constant$, and the $\Delta log L = -\frac{1}{2}$ errors can be read off from this [4].

To explain why this works (and when it doesn't), consider first a case where the answer is easily found: suppose x_1 and x_2 are both Gaussian, with the same mean σ . The log likelihood can then be rewritten using $u = x_1 + x_2$ and $v = x_1 - x_2$:

$$-\frac{x_1^2}{2\sigma^2} - \frac{x_2^2}{2\sigma^2} = -\frac{(x_1 + x_2)^2}{4\sigma^2} - \frac{(x_1 - x_2)^2}{4\sigma^2} = -\frac{u^2}{4\sigma^2} - \frac{v^2}{4\sigma^2}$$
(3)

The likelihood is the product of two Gaussians (of width $\sqrt{2}\sigma$), one in the combination of interest u, the other in the ignorable combination v.

Now for some fixed value of v, the likelihood for u is a Gaussian of mean zero, and the 68% central confidence region for u is given by its standard deviation and is of half-width $\sqrt{2}\sigma$. If v is fixed at some other value, the likelihood for u, and the deductions that can be drawn from it, are the same, Thus one can say 'There is a 68% probability that u lies in the region $[-\sqrt{2}\sigma, \sqrt{2}\sigma]$, whatever value of v is chosen', and this can legitimately be shortened by striking out the final condition. And the problem is solved.

To apply this technique in some less transparent case we need to factorise the likelihood into the form $L_1(x_1)L_2(x_2) = L_u(u)L_v(v)$ where we have freedom to choose the functions L_u , L_v , and the form $v(x_1, x_2)$. In some instances this is clearly possible: a double Gaussian with $\sigma_1 \neq \sigma_2$ can be factorised using $v = \sigma_2 x_1 - \sigma_1 x_2$. There are also instances, such as a volcano-crater shaped function, which are manifestly impossible to factorise. These can readily be proposed as counterexamples, but appear somewhat contrived and it is reasonable to hope that they might not occur in practical experience, except for very small N.

On the grounds that if this factorisation is impossible we can get nowhere, let us assume it to be true and see where that leads us. Finding the explicit forms of v and L_v is complicated and one would like to avoid it. This can be done by noting that:

- 1: For fixed v the shape of the total likelihood as a function of u is the same
- 2: For fixed u the shape of the total likelihood as a function of v is the same
- (1) tells us that we can study the properties of $L_u(u)$ by fixing on any value of v. (2) tells us that we can fix the value of v by finding the maximum, the likelihood (as a function of v, with u fixed) will always peak at the same value of v. Thus for a given $u = x_1 + x_2$ one finds the value of $x_1 x_2$ at which L is greatest, as that is always the same value of v.

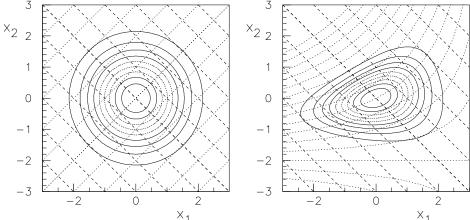


Figure 1: 2-D likelihood functions with lines of constant u and constant v

Figure 1 gives an illustration. The left hand plot shows the standard double Gaussian (shown as a linear function rather than the logarithm, for presentational reasons) as a function of x_1 and x_2 . The lines of constant $u = x_1 + x_2$ run diagonally, from top left to bottom right, and the lines of constant $v = x_1 - x_2$ are orthogonal to them, running from bottom left to top right. For any chosen value of v, the profile of the likelihood as a function of u is the same Gaussian shape, from which 68% limits can be read off, the same in each case. There is a line of constant v = 0 running through the maximum, which follows the maximum for any chosen u.

The right hand plot shows a more interesting function. The lines of constant $u = x_1 + x_2$ are as before. The lines of constant v are such that the likelihood as a function of u along them is the same, up to a constant factor. There is a line of constant v through the maximum which follows the maximum for any chosen u.

This construction shows the limits of the technique. For some given u we plot L as a function of $x_1 - x_2$ and compare it with the same curve for u = 0. Then we map the values of $x_1 - x_2$ onto the corresponding values at u = 0 at which the log likelihood falls off from the peak by the same amount, and these give the lines of constant v. If both curves are single peaks then this is readily done and the mapping is continuous. If there are multiple peaks then this continuous mapping is not possible. Thus for a simple peak the technique will work, but not if there are secondary peaks or valleys.

This generalises readily to the case of several variables. The profile likelihood is a function $\hat{L}(u)$ where $u = \sum x_i$ and \hat{L} is the maximum value of the likelihood in the u = constant hyperplane.

3. Parametrisation of the likelihood function

Thus both questions can be answered if the likelihood functions are known. In general they are not: a quoted result will only give the value and the positive and negative error. We therefore need a way to reconstruct, as best we can, the log likelihood function from them, using a parametrised curve.

This curve must go through the three points, having a maximum at the middle one. This gives four equations, and hence the curve will have four parameters, obtainable from the quoted values of the peak and the positive and negative errors. (The fourth parameter is an additive constant which controls the value of the function at its maximum, which is in fact irrelevant for our purposes.) It must also behave in a 'reasonable' fashion elsewhere.

Various possibilities have been tried, and tested against the log likelihood curves where the true value is known, such as the Poisson and the log of a Gaussian variable. For simplicity in what follows we take the quoted value as zero, and work with just σ_+ and σ_- as input parameters.

3.1 Form 1: a cubic

Adding a cubic term is the obvious step

$$f(x) = -\frac{1}{2}(\alpha x^2 + \beta x^3)$$
 (4)

with the coefficients readily obtained as $\alpha = \frac{\sigma_-^3 + \sigma_+^3}{\sigma_+^2 \sigma_-^2 (\sigma_- + \sigma_+)}$ $\beta = \frac{\sigma_-^2 - \sigma_+^2}{\sigma_+^2 \sigma_-^2 (\sigma_- + \sigma_+)}$. Extension to several values has some consistency, as adding cubics will give another cubic, but associativity is not guaranteed.

This gives curves which will behave sensibly in the $[x-\sigma^-,x+\sigma^+]$ range, but outside that the x^3 term produces an unwanted turning point and the curve does not go to $-\infty$ for large positive and negative x.

3.2 Form 2: A constrained quartic

A quartic curve can be constrained to give only one maximum by making the second derivative a perfect square:

$$f''(x) = -\frac{1}{2}(\alpha + \beta x)^2 \qquad f(x) = -\frac{1}{2}\left(\frac{\alpha^2 x^2}{2} + \frac{\alpha \beta x^3}{3} + \frac{\beta^2 x^4}{12}\right)$$
 (5)

The parameters are given by

$$\beta = \frac{1}{\sigma_{+}\sigma_{-}} \sqrt{\frac{6(\sigma_{-} + \sigma_{+})^{2} \pm 12\sqrt{4\sigma_{+}\sigma_{-}^{3} + 4\sigma_{-}\sigma_{+}^{3} - 2\sigma_{-}^{4} - 2\sigma_{+}^{4}}}{3\sigma_{-}^{2} + 2\sigma_{-}\sigma_{+} + 3\sigma_{+}^{2}}}$$
 (6)

Here the negative sign in the expression for β should be chosen to give a quartic term which is small. In very asymmetric cases (σ_{-} and σ_{+} differing by more than about a factor of 2) the inner square root is negative, indicating that there is no solution of the desired form.

Then one solves for α

$$\alpha = (-)\frac{\beta\sigma}{3} \pm \frac{\sqrt{36 - 2\beta^2\sigma^4}}{6\sigma} \tag{7}$$

for both $\sigma = \sigma_+$ and $\sigma = \sigma_-$, where the (-) minus sign is used for the σ_- case, and selects the solution which is common to both.

Combination again gives closure, in that the sum of two quartics (with second derivative everywhere negative) is a quartic (with second derivative everywhere negative.)

This form gives rather better large x behaviour but is not always satisfactory in the range between σ_{-} and σ_{+} .

3.3 Form 3: Logarithmic

One can also use a logarithmic approximation

$$f(x) = -\frac{1}{2} \left(\frac{\log(1 + \gamma x)}{\log \beta} \right)^2 \tag{8}$$

where

$$\beta = \sigma^{+}/\sigma^{-} \qquad \gamma = \frac{\sigma_{+} - \sigma_{-}}{\sigma_{+}\sigma_{-}} \tag{9}$$

This is easy to write down and work with, and has some motivation, as it describes the expansion/contraction of the abscissa variable at a constant rate. Its unpleasant features are that it is undefined for values of x beyond some point in the direction of the smaller error, as $1 + \gamma x$ goes negative, and that it does not give a parabola in the $\sigma_+ = \sigma_-$ limit.

3.4 Form 4: Generalised Poisson

Starting from the Posson likelihood $L(x) = -x + N \ln x - \ln N!$ one can generalise to

$$f(x) = -\alpha(x+\beta) + \nu \ln \alpha(x+\beta) + const \tag{10}$$

using ν , a continuous variable, to give skew to the function, and then scaling and shifting using α and β . Putting the maximum at the right place requires $\nu = \alpha \beta$ and thus, adjusting the constant for convenience to make the peak value zero:

$$f(x) = -\alpha x + \nu \ln\left(1 + \frac{\alpha x}{\nu}\right) \tag{10a}$$

Writing $\gamma = \alpha/\nu$ the equations at σ_{-} and σ_{+} lead to

$$\frac{1 - \gamma \sigma_{-}}{1 + \gamma \sigma_{+}} = exp^{-\gamma(\sigma_{-} + \sigma_{+})} \tag{11}$$

This has to be solved numerically. It has a solution between $\gamma = 0$ and $\gamma = 1/\sigma_{-}$ which can be found by bifurcation. (Attempts to use more sophisticated algorithms failed.)

Given the value of γ , ν is then found from

$$\nu = \frac{1}{2(\gamma \sigma_+ - \ln(1 + \gamma \sigma_+))} \tag{12}$$

This form did fairly well with many of the tests, but the extraction of the function parameters from σ_{-} and σ_{+} is inelegantly numerical.

3.5 Form 5: Variable Gaussian (1)

Another function is motivated by the Bartlett technique for maximum likelihood errors [2,5]. This assumes (and indeed justifies) that the likelihood function for a result \hat{x} from a true value x is described with good accuracy by a Gaussian whose width depends on the value of x.

$$lnL(\hat{x};x) = -\frac{1}{2} \left(\frac{\hat{x} - x}{\sigma(x)}\right)^2 \tag{13}$$

This does not include the $-\ln \sigma(x)$ term from the denominator of the Gaussian. However it turns out [2] that omitting this term actually improves the accuracy of the $\Delta \ln L = -\frac{1}{2}$ errors, bringing them into line with the Bartlett form.

We make the further assumption that in the neighbourhood of interest this variation in standard devation is linear

$$\sigma(x) = \sigma + \sigma'(x - \hat{x}) \tag{14}$$

$$lnL(\hat{x};x) = -\frac{1}{2} \left(\frac{\hat{x} - x}{\sigma + \sigma'(x - \hat{x})} \right)^2$$
(15)

the requirement that this go through the $-\frac{1}{2}$ points gives

$$\sigma = \frac{2\sigma_{+}\sigma_{-}}{\sigma_{+} + \sigma_{-}} \qquad \sigma' = \frac{\sigma_{+} - \sigma_{-}}{\sigma_{+} + \sigma_{-}} \tag{16}$$

Thus the parameters are easy to find, and when $\sigma_{-} = \sigma_{+}$ the symmetric case is smoothly incorporated.

3.6 Form 6: Variable Gaussian (2)

Still using the Bartlett-inspired form, we could alternatively take the variance as linear

$$V(x) = V + V'(x - \hat{x}) \tag{17}$$

and

$$lnL(\hat{x};x) = -\frac{1}{2} \frac{(\hat{x} - x)^2}{V + V'(x - \hat{x})}$$
(18)

and the parameters are again easy to find, and sensible if $\sigma_{-} = \sigma_{+}$

$$V = \sigma_{-}\sigma_{+} \qquad V' = \sigma_{+} - \sigma_{-} \tag{19}.$$

3.7 Example: Approximating a Poisson likelihood

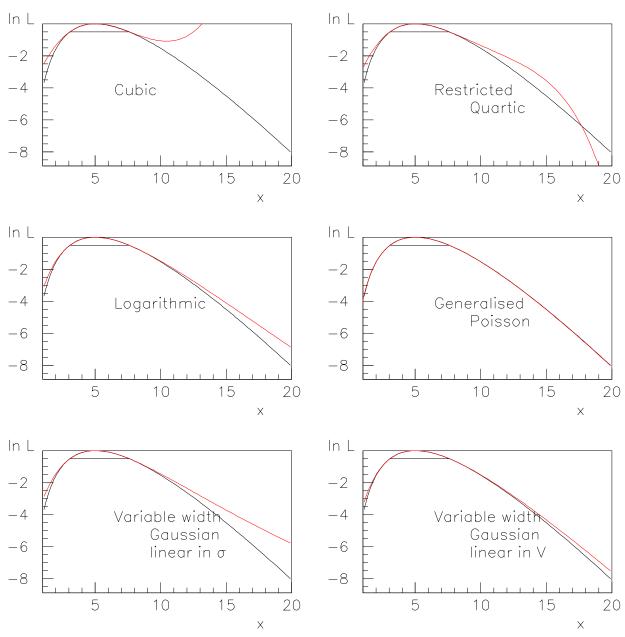


Figure 2: Approximations to a Poisson likelihood

Figure 2 shows in black the likelihood function for Poisson measurement of 5 events. In red are the approximations, constrained to peak at x=5 and to go through the $-\frac{1}{2}$ points, indicated by the horizontal line. They all do well interpolationg in that region, but outside it their behavour is very different. The polynomial forms diverge significantly from the truth. The logarithmic form does fairly well, and the generalised Poisson does perfectly (as it should for a Poisson likelihood). The variable width Gaussian models both do quite well, but the one with linear variance does noticably better than the form linear in the standard deviation

3.8 Example: Approximating a Logarithmic measurement.

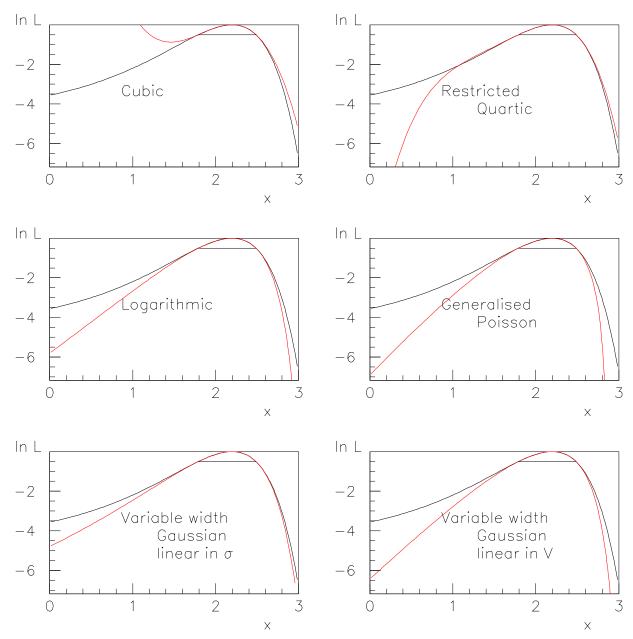


Figure 3: Approximations to the likelihood of the log of a Gaussian measurement

Figure 3 shows the same approximations, fitting a measurement of $x = \ln y$, where y is a Gaussian measurement with the value 8 ± 3 .

Again, all perform well in the central region, and the polynomial forms diverge badly outside that region, though the quartic does adequately on the positive side and down to about $-2\sigma_{-}$ from the peak. The logarithmic curve does fairly well, but the generalised Poisson is not so good. The variable width Gaussians both do well, but in this case the linear σ form does markedly better than the linear variance form.

We can conclude that the variable width Gaussians are the best approximation for our purpose, having good descriptive power together with parameters that are readily obtained from Equations 16 or 19, but that the choice between the linear σ or linear V form is one that the user has to make on a case by case basis. Likelihood functions based on a Poisson measurement will be better represented by the linear V form.

4. Procedure for combination of results

Working with a variable-width Gaussian parametrisation the likelihood function for a set of measurements x_i is

$$lnL = -\frac{1}{2} \sum \left(\frac{\hat{x} - x_i}{\sigma_i(\hat{x})}\right)^2. \tag{20}$$

For the linear σ form, the position of the maximum is given by the equation

$$\hat{x} \sum_{i} w_i = \sum_{i} x_i w_i \quad \text{with} \quad w_i = \frac{\sigma_i}{(\sigma_i + \sigma_i'(\hat{x} - x_i))^3}. \tag{21}$$

For the linear V form the corresponding equation is

$$\hat{x} \sum_{i} w_{i} = \sum_{i} w_{i} (x_{i} - \frac{V_{i}'}{2V_{i}} (\hat{x} - x_{i})^{2}) \quad \text{with} \quad w_{i} = \frac{V_{i}}{(V_{i} + V_{i}' (\hat{x} - x_{i}))^{2}}.$$
 (22)

The algebra is simple, and has been implemented in a Java applet, obtainable under http://www.slac.stanford.edu/~barlow/statistics.html.

Equations 21 and 22 are nonlinear for \hat{x} , and the solution is found by iteration: $\frac{1}{N}\sum_{i}x_{i}$ is taken as a first guess for \hat{x} , and this is used in the right hand side of the equation to give an improved value. The implementation deems it to have converged if the step size is less that 10^{-6} of the total range of interest, defined as from $-3\sigma_{-}$ below the lowest point to $+3\sigma_{+}$ above the highest. In practice such convergence occurs after a few iterations.

The $\Delta log L = -\frac{1}{2}$ points of the function of Equation 20 are also found numerically. The function is reasonably linear over the region where the iteration is performed, and again convergence is rapid: an initial value is taken, inspired by Equation (1), as the inverse root sum of the inverse squares of the positive or negative, as appropriate, errors. A small step is taken, until the $-\frac{1}{2}$ line is crossed, and successive linear interpolation is then done until the value is within 10^{-7} of 0.5. Again, only a few iterations are required for a typical case.

The value of the function at the peak gives the χ^2 for the result, and this can be used to judge the compatibility of the different results. (The number of degrees of freedom is just one less that the number of values being combined.)

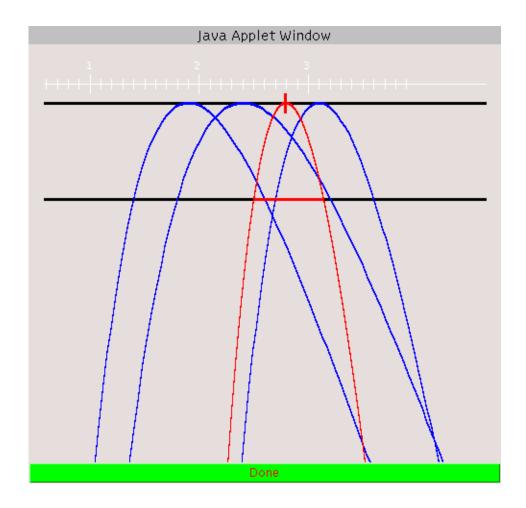


Figure 4: Three parametrised likelihood curves and their sum

Figure 4 shows the graphical result of combining $1.9^{+0.7}_{-0.5}$ with $2.4^{+0.6}_{-0.8}$ and $3.1^{+0.5}_{-0.4}$. The upper black line shows the peak value (which, as mentioned earlier, is not relevant and therefore set to zero). The lower black line shows $\ln L = -\frac{1}{2}$ The 3 blue curves are the three parametrised likelihood curves (using linear σ). It can be seen that they do indeed each go through their 3 known values correctly. Otherwise we have no precise knowledge of what they should look like, but they are apparently well behaved.

The red curve is the sum of the three blue curves (again, adjusted to have a peak value of zero.) The position of the peak, found as described above, is indicated by the short vertical red line, and the horizontal red line indicates the 68% confidence interval, again obtained as described above. One can thus verify by eye that the numerical techniques are giving sensible answers.

Results are also given numerically, as shown in Figure 5. Values and errors are given, and each measurement may be specified as being linear in σ or V using the right hand button. On pressing the bottom left button, the graph above is drawn and the numerical values displayed. There are also facilities to add more values (up to a limit of 10).

Java Applet Window									
Combining Results	value	sigma -		sigma +		linearity			
Experiment 1	1.9į	.5ĭ		.7į		Linear Sigma			
Experiment 2	2.4į	.8 <u>ĭ</u>		.6 <u>ř</u>		Linear Sigma			
Experiment 3	3.1[.4ĭ		.5ĭ		Linear Sigma			
Value	Sigma -		Sigma +		Chi squared				
[2.758	[0.272		(0.293		<u>[</u> 2.42	2			
Go for it! Add valu		lue	Stop		Switch Debug On				

Figure 5: The user interface, showing input values, output values and options

4.1 Example of combination of results

Suppose a counting experiment sees 5 events. The result is quoted (using the $\Delta \ln L = -\frac{1}{2}$ errors, even though this is a case where the full Neyman errors could be given) as $5^{+2.581}_{-1.916}$. Suppose further that it is repeated and the same result is obtained. With the knowledge of the details we can obtain the combined result just by halving the total measurement of $10^{+3.504}_{-2.838}$ to give an exact answer of $5^{+1.752}_{-1.419}$. But in general we would not know this and just be given the measurements, and combine them using the above method. This (using the linear variance model) gives a combined result of $5^{+1.747}_{-1.415}$. So the combined result is exact, with discrepancies only in the fourth decimal place of the errors.

Table 1 shows these, together with the values obtained from other pairs of results with the same sum.

x_1	x_2	Linear σ	Linear V
$5^{+2.581}_{-1.916}$	$5^{+2.581}_{-1.916}$	$5.000^{+1.737}_{-1.408}$	$5.000^{+1.747}_{-1.415}$
$6^{+2.794}_{-2.128}$	$4^{+2.346}_{-1.682}$	$5.000^{+1.778}_{-1.432}$	$5.000^{+1.758}_{-1.425}$
$7^{+2.989}_{-2.323}$	$3^{+2.080}_{-1.416}$	$5.038^{+1.936}_{-1.529}$	$5.009^{+1.793}_{-1.456}$
$8^{+3.171}_{-2.505}$	$2^{+1.765}_{-1.102}$	$5.402^{+2.368}_{-1.826}$	$5.055^{+1.855}_{-1.515}$
$9^{+3.342}_{-2.676}$	$1_{-0.6983}^{+1.358}$	$7.350^{+3.149}_{-2.548}$	$5.203_{-1.605}^{+1.942}$

Table 1: Combining results in a case of two samples from the same Poisson distribution

This shows that the technique, especially with the linear variance model, works very well. There are discrepancies, but these are reasonable given the assumptions that have had to be made. It is worth pointing out that the larger discrepancies of the final two rows are produced by rather unlikely experimental circumstances - the probability of 10 events being split 9:1 or even 8:2 between the two experimental runs is small. (This shows up in their χ^2 values which are large enough to flag a warning.)

5. Procedure for Combination of Errors

To combine errors when the likelihoods are not given in full, and only the errors are available, we again parameterise them by the variable Gaussian model

$$lnL(\vec{x}) = -\frac{1}{2} \sum_{i} \left(\frac{x_i}{\sigma_i + \sigma_i' x_i} \right)^2 \text{ or } \frac{x_i^2}{V_i + V_i' x_i}$$
 (23)

where the x_i represent deviations from the quoted result. Their total is $u = \sum_i x_i$ and to find $\hat{L}(u)$ the sum of Equation 23 is maximised, subject to the constraint $\sum x_i = u$. The method of undetermined multipliers gives the solution as

$$x_i = u \frac{w_i}{\sum_j w_j} \tag{24}$$

where
$$w_i = \frac{(\sigma_i + \sigma_i' x_i)^3}{2\sigma_i}$$
 or $\frac{(V_i + V_i' x_i)^2}{2V_i + V_i' x_i}$ (25)

This is an non-linear set of equations. However a solution can be mapped out, starting at u = 0 for which all the x_i are zero. Increasing u in small amounts, Equation 24 is used to give the small the changes in the x_i , and the weights are then re-evaluated using Equation 25.

This has also been implemented by a Java program obtainable at the web address mentioned above. It has a similar user interface panel, and displays the form of $\hat{L}(u)$ used to read off the total $\Delta \ln L = -\frac{1}{2}$ errors.

5.1 An example of combination of errors

Suppose that N events have been observed in an experiment, and to extract the signal the number of background events must be subtracted. We suppose that there are several such sources, determined by separate experiments, and that, for simplicity, these do not have to be scaled; the backgrounds were determined by running the apparatus, in the absence of signal, for the same period of time as the actual experiment.

Suppose that two backgrounds are measured, one giving 4 events and the other 5. These are reported as $4^{+2.346}_{-1.682}$ and $5^{+2.581}_{-1.916}$. (again using the $\Delta lnL=-\frac{1}{2}$ errors.) This method gives the combined error as $^{+3.333}_{-2.668}$. However in this case where the backgrounds are combined with equal weight, one could just quote the total number of background events as $9^{+3.342}_{-2.676}$. The method's error values are in impressive agreement with this. Further examples are given in table 2

	Linear	σ	Linear	\cdot V
Inputs	σ_{-}	σ_{+}	σ_{-}	σ_{+}
4 + 5	2.653	3.310	2.668	3.333
3 + 6	2.653	3.310	2.668	3.333
2 + 7	2.653	3.310	2.668	3.333
1 + 8	2.654	3.313	2.668	3.333
3 + 3 + 3	2.630	3.278	2.659	3.323
1+1+1+1+1+1+1+1+1+1	2.500	3.098	2.610	3.270

Table 2: Various combinations of Poisson errors which should give $\sigma_{-}=2.676, \sigma_{+}=3.342$

6. Conclusions

If the full likelihood functions are not given, then there is no exact method for combination of errors and results with asymmetric statistical errors. However the procedures decribed here, which work by making an approximation to the likelihood function on the basis of the quoted value and errors, appear to be reasonably accurate and robust. They are also easy to implement and user.

Acknowledgements

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