10.9: Convergence of Taylor Series

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26th February 2024

A Taylor Series for $f(x)$ centered at the point a can be defined as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-a)^n$$

1 Example 1

Find the Taylor series for $f(x) = e^x$ centered at $a = 0$.

$$\sum_{n=0} \infty \frac{x^n}{n!}$$

For what x does $\sum \frac{x^n}{n!}$ converge?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \times\right.$$

2 Example 2

Taylor series for $f(x) = \log x$ centered at $a = 1$

$$f(x) = \log x \ f(1) = 0$$
$$f'(x) = \frac{1}{x} \ f'(1) = 1$$
$$f''(x) = -x^-2 \ f''(1) = -1$$
$$f'''(x) = 2x^-3 \ f'''(1) = 2$$
$$f''''(x) = -3 \times 2x^-4 \ f''''(1) = -6$$

$$f^{(n)}(1) = (-1)^{n+1}(n-1)!$$

Thus

$$\sum_{n=1} \infty \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n$$

Does it converge?

Because of the exponentials, it is good to use the ratio test

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{n+1}\times\frac{n}{(x-1)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x-1)n}{n+1}\right| = |x-1| < 1$$

$$-1 < x-1 < 1$$

For $0 < x < 2$, this series converges absolutely.

2.1 Test Endpoints of absolute convergence

Test the endpoints $x = 0$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)^n}{n}$$

$$= -\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n}$$

$$= -\sum_{n=0}^{\infty}\frac{1}{n}$$

Thus diverges at $x = 0$

Test for $x = 2$: Converges by alternating series test.

Taylor Series converges for $0 < x <= 2$

3 Quiz Review

I received a score of $\frac{15}{15}$ on my Chapter 10 Part 1 quiz.

4 Homework Assignment

The homework for today's lesson is 5 problems from chapter 10, section 9: Convergence of Taylor Series, on page 647 in the book.