Calculus II: Chapter 10 Quiz 2

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1 Find the Taylor Series for $f(x) = \cos(x)$ centered at $a = 0$. For what x-values does the series converge?

The Taylor series for $f(x)$, centered at a can be defined as:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n . \tag{1}$$

Taking $f(x)$ to be defined as $\cos(x)$ and $a = 0$ gives us the first few terms

$$\frac{\cos(0)(x-0)^0}{0!} + \frac{-\sin(0)(x-0)^1}{1!} + \frac{-\cos(0)(x-0)^2}{2!} + \frac{\sin(0)(x-0)^3}{3!} + \cdots , \tag{2}$$

which can be simplified to

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots . \tag{3}$$

This can be generalized in series notation as the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} . \tag{4}$$

To determine the interval of convergence for this series, we must use a convergence test. Since a factorial is involved, we will use the ratio test for convergence.

We define a_n as $\frac{x^{2n}}{(2n)!}$, and a_{n+1} as $\frac{x^{2(n+1)}}{(2(n+1))!}$. Thus, by the definition of the ratio test, we find

$$\begin{aligned}
\mathrm{L} &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^n x^{2n+2}}{(2n+2)!}}{\frac{(-1)^n x^{2n}}{(2n)!}}\right| && (5a)\\
&= \lim_{n\to\infty}\left|\frac{x^{2n+2}\cdot(2n)!}{x^{2n}\cdot(2n+2)!}\right| && (5b)\\
&= \lim_{n\to\infty}\left|\frac{x^2}{(2n+1)(2n+2)}\right| && (5c)\\
&= \lim_{n\to\infty}\left|\frac{x^2}{2(n+1)(2n+1)}\right| && (5d)
\end{aligned}$$

In this limit, the x-values, while variable, do not necessarily increase with n and can be treated as an arbitrary constant of sorts.

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{x^2}{4n^2+6n+2}\right| \to 0 && (6a)\\
&= 0 && (6b)
\end{aligned}$$

Because of this, it follows that a_n converges absolutely, for all values of x. In other words, x converges on the interval $(-\infty, \infty)$.

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2 Give the 4th-order Taylor Polynomial, $p_4(x)$, for $f(x) = \cos(x)$, centered at $a = 0$

The Taylor Polynomial can be derived from the Taylor Series as such:

$$p_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n \,. \tag{7}$$

This allows us to find the fourth Taylor polynomial for the provided function $f(x) = \cos(x)$:

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\begin{aligned}
p_0(x) &= 1 && \text{(8a)}\\
p_1(x) &= 1 - 0x && \text{(8b)}\\
p_2(x) &= 1 - 0x - \frac{1}{2}x^2 && \text{(8c)}\\
p_3(x) &= 1 - 0x - \frac{1}{2}x^2 - 0x^3 && \text{(8d)}\\
p_4(x) &= 1 - 0x - \frac{1}{2}x^2 - 0x^3 + \frac{1}{24}x^4 && \text{(8e)}\\
\rightsquigarrow p_4(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \,. && \text{(8f)}
\end{aligned}
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$\blacksquare$

2.1 Provide the maximum error, $|R_4(x)| = |\cos(x) - p_4(x)|$, for $p_4(x)$ over $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

To find maximum error, we must use Lagrange's formula:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{9}$$

This leads us to the equation

$$R_4(x) = \frac{\overset{\cdot\cdot\cdot\cdot\cdot}{f}(c)}{5!}x^5 = \frac{-\sin c}{120}x^5 \tag{10}$$

over the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Since both $f(x)$ and $p_4(x)$ are symmetric about the y-axis, we can take the interval to be $0 \leq |c| \leq \frac{\pi}{2}$. Looking at the trends of $R_4(x)$ as we vary c from $0 -\frac{\pi}{2}$ to $\frac{\pi}{2}$ allows us to see that the greatest error will occur when $|c|$ is equal to $\frac{\pi}{2}$.

Thus, utilizing this error value for the given equation

$$|R_4(x)| = |\cos(x) - p_4(x)| \tag{11}$$

provides the maximum error over the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ of

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2.2 Compute $p_4(1)$ and $f(1)$ and round to the fourth decimal place. Compare the difference to the upper bound of $|R_4(1)|$.

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