Ordinary Differential Equations

Snapshot

- **Major Concept:** Ordinary differential equations are the foundation of modern science. Equations of this type describe unknown functions in terms of their instantaneous rates of change.
- **Before You Begin:** Familiarize yourself with the basic vocabulary and examples from Section 7.2 and Sections 9.1–9.3 of the textbook.
- **Standards for Practice and Evaluation:** Each of the exercises on this worksheet is designed to be representative of a corresponding question type that might appear on the third midterm or on the final exam. In such contexts, your work will be judged not only for accuracy, but for clarity as well, meaning that you should demonstrate your knowledge of process by describing or annotating each step of the process you are carrying out.

Worksheet Objective

In this worksheet, you will practice graphical, numerical, and algebraic/analytical techniques for solving first-order ordinary differential equations (ODEs). As you are working, be sure to reflect as a group on the pros and cons of each of these different approaches.

Definition

An **ordinary differential equation (ODE)** is an equation which relates an unknown function (often called y) and its various derivatives. These equations may or may not also involve the independent variable of the unknown function. The **order** of an ODE is equal to the order of the highest derivative (of y) that appears in the equation. Examples:

- $u'' + 3u' + 2u = x$ is a second-order ODE for the unknown function $u(x)$
- $\int_0^x y(s)ds = y + 2$ is not an ODE.
- $(y')^2 + y^2 = 2 - t$ is a first-order ODE for the unknown function $y(t)$.

Generally speaking, ODEs are difficult to solve. However, there are certain kinds of ODEs which may be solved more easily than others, and there are general methods for finding solutions graphically or approximately which work well for even broader classes of ODEs.

Remember **Understand** Apply Analyze Evaluate Create

Review the material in Section 7.2 and Sections 9.1–9.3 to give definitions of the following terms in your own words:

- Separable first-order ODE

- Linear first-order ODE

- General Solution of an ODE

- Initial Value Problem

Slope Fields

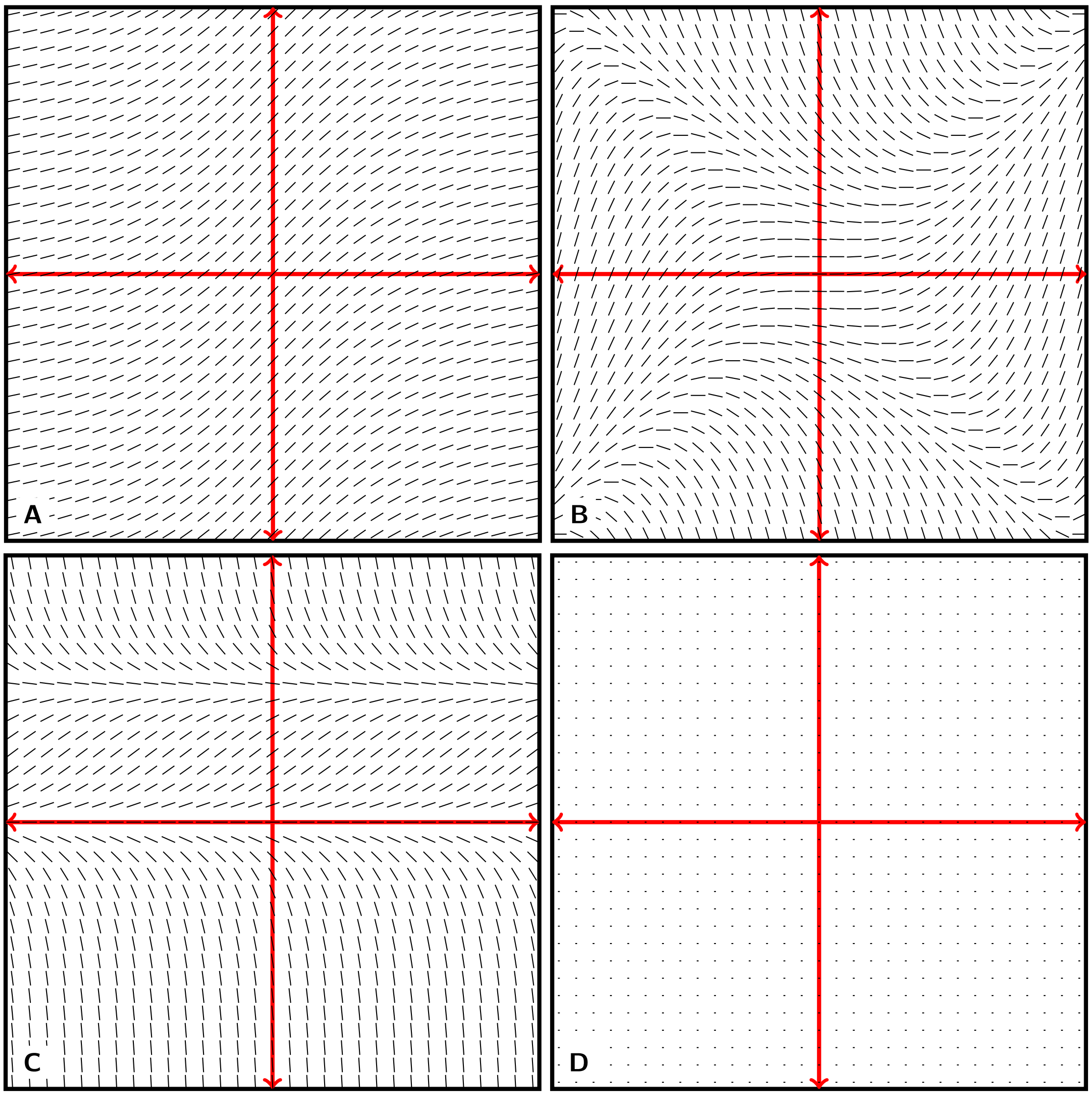
Remember **Understand** Apply Analyze Evaluate Create

What features should you look for in a slope field to help identify its associated differential equation?

Remember Understand **Apply** Analyze Evaluate Create

Match three of the ODEs I-IV to the correct slope fields A–C. For the unmatched ODE, sketch its slope field in box D.

I: $y' = xy$ II: $y' = x^2 - y^2$ III: $y' = \dfrac{1}{1+x^2}$ IV: $y' = 3y(1-y)$



Briefly summarize your reasoning.

For each of the slope fields A–D, sketch three different solution curves.

Solving Separable ODEs

Remember Understand **Apply** Analyze Evaluate Create

Determine which two of the following three ODEs are separable. For those that are separable, find the exact formula for general solutions and then determine the formula for the solution to the initial value problem with $y(0) = 1$.

$$\text{I:} \quad y' = xy \qquad \text{II:} \quad y' = x^2 - y^2 \qquad \text{III:} \quad y' = \frac{1}{1+x^2}$$

Remember Understand **Apply** Analyze Evaluate Create

Use Euler's method with step size $dx = 0.5$ to approximate the solution of the initial value problem

$$y' = xy, \qquad y(0) = 1$$

between $x = 0$ and $x = 2$ (inclusive).

Discuss what you expect to happen to the approximate solution as dx is either increased or decreased. Specifically address the issues of accuracy and roundoff error.

First-Order Linear ODEs

First-Order Linear ODEs MUST be in Standard Form

When using techniques associated with first-order linear ODEs, for those techniques to work properly it is mandatory that the ODE first be written in standard form $y' + P(x)y = Q(x)$.

Remember | **Understand** | Apply | Analyze | Evaluate | Create

Determine which two of the following four ODEs are linear. For those that are linear, put them in standard form and identify the corresponding functions P and Q. For those that are not, briefly explain why linearity fails.

I: $y' = xy + x$ II: $y' = x^2 - y^2$ III: $xy' = -2y + \dfrac{1}{1+x^2}$ IV: $y' = 3y(1-y)$

Definition

An **integrating factor** $I(x)$ is a function which is carefully chosen for a given $P(x)$ so that the expression

$$I(x)y'(x) + I(x)P(x)y(x) \tag{†}$$

equals the derivative $(I(x)y(x))'$. To solve for $I(x)$, identify the two terms of (†) with the terms of the derivative $(I(x)y(x))'$ that are given by the product formula. For example, e^x is an integrating factor for $y' + y$ since $e^x y' + e^x y = (e^x y)'$.

Remember | Understand | **Apply** | Analyze | Evaluate | Create

For the first-order linear ODEs above, find the general solution using the method of integrating factors.

Review and Summary

Focus future review on refining and practicing the following skills:

- Interpreting and generating slope fields and using them to graphically solve first-order ODEs
- Identifying (without prompting) and solving separable ODEs
- Using Euler's method to approximate solutions to ODEs; Ensuring desired levels of accuracy
- Identifying first-order linear ODEs and using integrating factors to solve them