

Calculus II: Chapter 10 Quiz 2

H. Ryott Glayzer

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1 Find the Taylor Series for $f(x) = \cos(x)$ centered at $a = 0$. For what x -values does the series converge?

The Taylor series for $f(x)$, centered at a can be defined as:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n. \quad (1)$$

Taking $f(x)$ to be defined as $\cos(x)$ and $a = 0$ gives us the first few terms

$$\frac{\cos(0)(x-0)^0}{0!} + \frac{-\sin(0)(x-0)^1}{1!} + \frac{-\cos(0)(x-0)^2}{2!} + \frac{\sin(0)(x-0)^3}{3!} + \cdots, \quad (2)$$

which can be simplified to

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots. \quad (3)$$

This can be generalized in series notation as

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}. \quad (4)$$

To determine the interval of convergence for this series, we must use a convergence test. Since a factorial is involved, we will use the ratio test for convergence.

We define a_n as $\frac{x^{2n}}{(2n)!}$, and a_{n+1} as $\frac{x^{2(n+1)}}{(2(n+1))!}$. Thus, by the definition of the ratio test, we find

$$L = \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}} \right| \quad (5a)$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2} \cdot (2n)!}{x^{2n} \cdot (2n+2)!} \right| \quad (5b)$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+1)(2n+2)} \right| \quad (5c)$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x^2}{2(n+1)(2n+1)} \right| \quad (5d)$$

In this limit, the x -values, while variable, do not necessarily increase with n and can be treated as an arbitrary constant of sorts.

$$L = \lim_{n \rightarrow \infty} \left| \frac{x^2}{4n^2 + 6n + 2} \right| \rightarrow 0 \quad (6a)$$

$$= 0 \quad (6b)$$

Because of this, it follows that a_n converges absolutely, for all values of x . In other words, x converges on the interval $(-\infty, \infty)$. ■

2 Give the 4th-order Taylor Polynomial, $p_4(x)$, for $f(x) = \cos(x)$, centered at $a = 0$

The Taylor Polynomial can be derived from the Taylor Series as such:

$$p_k(x) = \sum_{n=0}^k \frac{f^{(n)}(a)}{n!} (x-a)^n. \quad (7)$$

This allows us to find the fourth Taylor polynomial for the provided function $f(x) = \cos(x)$:

$$p_0(x) = 1 \quad (8a)$$

$$p_1(x) = 1 - 0x \quad (8b)$$

$$p_2(x) = 1 - 0x + \frac{1}{2}x^2 \quad (8c)$$

$$p_3(x) = 1 - 0x + \frac{1}{2}x^2 - 0x^3 \quad (8d)$$

$$p_4(x) = 1 - 0x + \frac{1}{2}x^2 - 0x^3 + \frac{1}{24}x^4 \quad (8e)$$

$$\rightsquigarrow p_4(x) = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4. \quad (8f)$$

■

2.1 Provide the maximum error, $|R_4(x)| = |\cos(x) - p_4(x)|$, for $p_4(x)$ over $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

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2.2 Compute $p_4(1)$ and $f(1)$ and round to the fourth decimal place. Compare the difference to the upper bound of $|R_4(1)|$.

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