MAT3220 Optimization II Tutorial 1

Unconstrained Optimization

Runze You

SDS, CUHK-SZ

September 12, 2023

Outline

Global/Local Minimum and Maximum

- **Definition**: L2, slides 4 and 8.
- We need optimality conditions to help us identify global/local minimum/maxmimum points, if any exists. Without optimality conditions, we can sometimes utilize the special structure in an optimization problem and use some graphical methods to find the global/local min/max. (cf. Example 1 on slide 6 of L2).
- (**Practice**) Find the global minimum and maxmimum point(s) of the function $f(x, y) = x^2 + y^2 + 2x - 3y$ over the unit ball $S = B[0, 1] = \{(x, y) : x^2 + y^2 \leq 1\}$. (This is the first question in last year's midterm exam.)

Definite Matrices - Recap

- **Definition**: Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be **symmetric**.
  - $\mathbf{A} \succeq 0$ if $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0, \forall \mathbf{x} \in \mathbb{R}^n$.
  - $\mathbf{A} \succ 0$ if $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0, \forall \mathbf{x} \in \mathbb{R}^n \backslash \{0\}$.
  - $\mathbf{A} \preceq 0$ if $-\mathbf{A} \succeq 0$.
  - $\mathbf{A} \prec 0$ if $-\mathbf{A} \succ 0$.
  - $\mathbf{A}$ is indefinite if there exists $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ s.t. $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ and $\mathbf{y}^T \mathbf{A} \mathbf{y} < 0$.
- **Necessary condition**: Let $\mathbf{A}$ be PD (PSD). Then the diagonal elements of $\mathbf{A}$ are positive (nonnegative).

Definite Matrices - Recap (Cont'd)

- Three methods of showing that a symmetric matrix $\mathbf{A}$ is PD (PSD). (3 equivalent definitions of PD/PSD matrices.)
  - Use the definitions.
  - All eigenvalues of $\mathbf{A}$ are positive (nonnegative).
  - For PD, all leading principal minors of $\mathbf{A}$ are positive. (For PSD, we require all principal minors of $\mathbf{A}$ are nonnegative.)
- Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. f is called **coercive** if

$$\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = \infty.$$

- **Coerciveness of quadratic functions**: Let $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$, where $\mathbf{A}$ is symmetric. Then f is coercive iff $\mathbf{A} \succ 0$.

Definite Matrices - Practice Problem

For each of the following functions, determine whether it is coercive or not and explain:

- $f(x_1, x_2) = 2x_1^2 - 8x_1x_2 + x_2^2$;
- $f(x_1, x_2) = 4x_1^2 + 2x_1x_2 + 2x_2^2$;
- $f(\mathbf{x}) = \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\|\mathbf{x}\|+1}$, where $\mathbf{A} \succ 0$.

(This is also a problem in last year's midterm exam.)

Optimality Conditions - Recap

Let $f : U \to \mathbb{R}, U \subseteq \mathbb{R}^n$.

- **FONC**: Suppose all partial derivatives of f exist at $\mathbf{x}^* \in \text{int}(U)$. Then if $\mathbf{x}^*$ is a local optimum point, $\nabla f(\mathbf{x}^*) = 0$.
- $\mathbf{x}^* \in \text{int}(U)$ is called a **stationary point** of f if $\nabla f(x^*) = 0$.
    - Suppose $f : U \to \mathbb{R}$ is differentiable on $\text{int}(U)$. Then the stationary points and the boundary points are the possible candidates for global minimum/maximum points, if any exists.
    - A stationary point can be a local minimum point, a local maxmimum point, or a saddle point. So we need some tools to classify the stationary points.

Optimality Conditions - Recap (Cont'd)

Let $f : U \to \mathbb{R}$. U is an open subset of $\mathbb{R}^n$. Suppose $f \in C^2(U)$ and $\mathbf{x}^*$ is a stationary point.

- **SONC**:
  - if $\mathbf{x}^*$ is a local minimum point, then $\nabla^2 f(\mathbf{x}^*) \succeq 0$.
  - if $\mathbf{x}^*$ is a local maximum point, then $\nabla^2 f(\mathbf{x}^*) \preceq 0$.
- **SOSC**:
  - if $\nabla^2 f(\mathbf{x}^*) \succ 0$, then $\mathbf{x}^*$ is a strict local minimum point of f over U.
  - if $\nabla^2 f(\mathbf{x}^*) \prec 0$, then $\mathbf{x}^*$ is a strict local maximum point of f over U.
  - if $\nabla^2 f(\mathbf{x}^*)$ is indefinite, then $\mathbf{x}^*$ is a saddle point of f over U.

Global optimality condition: Let $f \in C^2(\mathbb{R}^n)$. Suppose $\nabla^2 f(\mathbf{x}) \succeq 0, \forall \mathbf{x} \in \mathbb{R}^n$. If $\mathbf{x}^* \in \mathbb{R}^n$ is a stationary point of f, then $\mathbf{x}^*$ is a global minimum point of f.

Optimality Conditions - Practice Problems

- For each of the following functions, find all the stationary points and classify them according to whether they are saddle points, strict/nonstict local/global minimum/maxmimum points:
  1. $f(x_1, x_2) = (4x_1^2 - x_2^2)^2$;
  2. $f(x_1, x_2) = x_1^4 + 2x_1^2 x_2 + x_2^2 - 4x_1^2 - 8x_1 - 8x_2$.

  (The second question is also from last year's midterm.)

Attainment of Minimal Maximal Points

- **Weierstrass Thm**: Let f be a **continuous** function over a **nonempty compact** set $C \subseteq \mathbb{R}^n$. Then there exists a global minimum point of f over C and a global maximum point of f over C.
- **Global min for coercive functions**: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a **continuous** and **coercive** function and let $S \subseteq \mathbb{R}^n$ be a **nonempty closed** set. Then f attains a global minimum point on S.