Weakly proregular sequence and Čech, local cohomology

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2021/10/13

- 可換環論，特に非 Noether 環に注目して研究をしています．可換環論は代数幾何学と歴史的に結びつきの強い分野で（発表者も所属は代数幾何系の研究室です），積極的に研究されているのは Noether 環が中心です．

FAQ

- 非 Noether 環の研究って何をしているの？
- 目的は？
- 応用は？
- ⋯etc.

- Q. 非 Noether 環の研究って何をしているの？
  - A. 大きく分けて，Noether 環の理論を非 Noether に拡張する研究と，非 Noether でしか起こり得ない現象を調べる研究があります．
    $\rightsquigarrow$ Hamilton and Marley (2007), Kim and Walker (2020), Miller (2008) などが CM 環，Gorenstein 環などのホモロジカルな性質を拡張して，非 Noether 環上に一般化する研究を行っています．
    $\rightsquigarrow$ ２次元以上の付値環は決して Noether 環にはなりません．
    また Noether ではない環を含むような環のクラスに Krull 整域（UFD の一般化）があります（Noether 整域が Krull 整域であることと，整閉整域であることは同値です）．
    $\rightsquigarrow$ 少し古いですが，非 Noether 可換環論の話題を集めた本も出ています（Chapman and Glaz (2000)）．

今日は Schenzel (2003) による weakly proregular sequence を紹介して，Noether 環における事実が非 Noether 環に一般化される様子をみていきましょう．そして Schenzel の定理 (Theorem 4.1) の初頭的（？）な証明を発表者の preprint[1] (arXiv:2105.07652) に基づいて紹介します．

[1]Accepted in *Moroccan Journal of Algebra and Geometry with Applications*.

Various cohomologies and homologies are used in (commutative) algebra theory.

Example. A : ring (unitary and commutative), I : ideal of A, and $M, N \in \mathrm{Mod}(A)$.
($\mathrm{Mod}(A)$: the category of A-modules, $\mathrm{mod}(A)$: the category of finitely generated A-modules.)

- $\mathrm{Ext}_A^i(M, -)$ .... Derived functor of $\mathrm{Hom}_A(M, -)$.
- $\mathrm{Tor}_i^A(M, -)$ .... Derived functor of $M \otimes_A -$.
- $H_I^i(-)$ .... Derived functor of $\varinjlim \mathrm{Hom}_A(A/I^n, -)$.
- $H_i(f, -)$ .... Koszul homology defined by the A-linear map $f : N \to A$.
- $\check{H}^i(\underline{a}, -)$ .... Čech cohomology defined by the sequence $\underline{a} = a_1, \ldots, a_r \in A$.
- and more!

Derived functor is obtained from a right (or left) exact functor. For example, let $J^\bullet$ be an injective resolution of N, then $\mathrm{Ext}^i(M, N) := H^i(\mathrm{Hom}(M, J^\bullet))$.

Why are cohomologies used so much?

$\rightsquigarrow$ One of the reasons for this is that ideal theoretic data can be written in a easier form for calculation.

Definition 2.1

A : ring , $M \in \mathrm{Mod}(A)$. $a \in A$ is called M**-regular** if ${}^{\forall}x \neq 0 \in M$, $ax \neq 0$.
A sequence $\underline{a} = a_1, \ldots, a_r \in A$ is called an M**-regular sequence** if;

- $M/(a_1, \ldots, a_r)M \neq 0$,
- $1 \leq {}^{\forall}i \leq r, a_i$ is an $M/(a_1, \ldots, a_{i-1})M$-regular.

Definition 2.2

A : Noetherian ring, $M \in \mathrm{mod}(A)$ and I : ideal with $IM \neq M$.

$$\mathrm{depth}_I(M) := \sup \left\{ r \geq 0 \;\middle|\; {}^{\exists}\underline{a} = a_1, \ldots, a_r \in I, \underline{a} \text{ is an } M\text{-regular sequence.} \right\}$$

is called an **I-depth of M**.

Theorem 2.3 (Rees)

Under the above notation, the length of a maximal regular sequence is constant. Also;

$$\mathrm{depth}_I(M) = \inf \left\{ i \geq 0 \;\middle|\; \mathrm{Ext}^i(A/I, M) \neq 0 \right\}.$$

This theorem shows that the depth of module is calculatable by using a cohomology!

Definition 3.1

A : ring , I : ideal of A.
$H^i_I(-)$: the right derived functor of $\varinjlim \operatorname{Hom}_A(A/I^n, -)$ is called a **local cohomology**.

Note that there are following isomorphisms,

$$H^i_I(M) \cong \varinjlim \operatorname{Ext}^i(A/I^n, M)$$

since taking the inductive limit is an exact functor.

Definition 3.2

A : ring , $\underline{a} = a_1, \ldots, a_r \in A$.
$\{e_i\}$: the standard basis of A^r.
For each $I = \{j_1, \ldots, j_i\}$ $(1 \leq j_1 < \cdots < j_i \leq r)$, let $a_I = a_{j_1} \cdots a_{j_i}$ and $e_I = e_{j_1} \wedge \cdots \wedge e_{j_i}$.
$C^\bullet(\underline{a})$: the complex defined by;

$$C^i(\underline{a}) := \sum_{\#I=i} A_{a_I} e_I,$$

$$d^i : C^i(\underline{a}) \to C^{i+1}(\underline{a}); e_I \mapsto \sum_{j=1}^{r} e_I \wedge e_j.$$

It is called a **Čech complex**.
$\check{H}^i(\underline{a})$: the cohomology of $C^\bullet(\underline{a})$ is called a **Čech cohomology**.

For $M \in \mathrm{Mod}(A)$, we define $C^\bullet(\underline{a}, M) := C^\bullet(\underline{a}) \otimes M, \check{H}^i(\underline{a}, M) := H^i(C^\bullet(\underline{a}, M))$.

Theorem 3.3

A : Noetherian ring, $\underline{a} = a_1, \ldots, a_r \in A$ and $I = (a_1, \ldots, a_r)$. There are isomorphisms;

$$H_I^i(M) \cong \check{H}^i(\underline{a}, M)$$

for any $M \in \mathrm{Mod}(A)$.

What happens if we remove the Noetherian assumption? Can we extend this theorem?
$\rightsquigarrow$ This theorem was extended by Schenzel (2003) by introducing a weakly proregular sequence.

Theorem 4.1 (Schenzel)

A : *ring*, $\underline{a} = a_1, \ldots, a_r \in A$ *and* $I = (a_1, \ldots, a_r)$.

$\underline{a}$ *is a weakly proregular sequence* $\Longleftrightarrow {}^{\forall}i \geq 0, {}^{\forall}M \in \mathrm{Mod}(A), H^i_I(M) \cong \check{H}^i(\underline{a}, M)$.

A weakly proregular sequence is defined using the Koszul complex.

Definition 4.2

A : ring , $\underline{a} = a_1, \ldots, a_r \in A$. $\{e_i\}$: the standard basis of A^r.

$K_\bullet(\underline{a})$ is the complex defined by ;

$$K_i(\underline{a}) = \bigwedge^i A^r$$

$$d_i : K_i(\underline{a}) \to K_{i-1}(\underline{a}); e_I \mapsto \sum_{k=1}^{i} (-1)^{k+1} a_{j_k} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{j_i}.$$

It is called a **Koszul (chain) complex**.

$H_i(\underline{a})$: the homology of $K_\bullet(\underline{a})$ is called a **Koszul homology**.

$\underline{a}^n$: the sequence defined by $a_1^n, \ldots, a_r^n$.
Note that by following morphisms, Koszul complexes constitute an inverse system $\{K_\bullet(\underline{a}^n)\}_{n\geq 0}$;

$$\varphi_{mn} : K_i(\underline{a}^m) \to K_i(\underline{a}^n); e_I \mapsto a_I^{m-n} e_I \ (n \leq m).$$

$\rightsquigarrow$ This induces a morphism between homologies.

Definition 4.3 (Schenzel)

A : ring , $\underline{a} = a_1, \ldots, a_r \in A$.
$\underline{a}$ is called a **weakly proregular sequence** if $1 \leq {}^\forall i \leq r, {}^\forall n \geq 0, {}^\exists m \geq n; \varphi_{mn} : H_i(\underline{a}^m) \to H_i(\underline{a}^n)$ is the zero map.

We will explain that Schenzel's theorem (Theorem 4.1) is an extension of the Noetherian case.

Definition 4.4 (Greenlees, May)

A : ring , $\underline{a} = a_1, \ldots, a_r \in A$.
$\underline{a}$ is called a **proregular sequence** if
$1 \leq {}^{\forall} i \leq r, {}^{\forall} n > 0, {}^{\exists} m \geq n; {}^{\forall} a \in A, aa_i^m \in (a_1^m, \ldots, a_{i-1}^m) \Longrightarrow aa_i^{m-n} \in (a_1^n, \ldots, a_{i-1}^n)$.

The following relations hold;

$$\text{Regular} \Longrightarrow \text{Proregular} \Longrightarrow \text{Weakly proregular.}$$

- The first implication is easy. If $\underline{a}$ is a regular sequence, for each $n > 0$, let $m = n$.
- The second is proved by calculating a Koszul homology.

Proposition 4.5

A: Noetherian ring , $\underline{a} = a_1, \ldots, a_r \in A$. $\underline{a}$ is a proregular sequence.

Proof.

Let $J_m^i = ((a_1^m, \ldots, a_{i-1}^m) : a_i^m A), I_{n,m}^i = ((a_1^n, \ldots, a_{i-1}^n) : a_i^{m-n} A)$.

$$\underline{a} \text{ is a proregular sequence} \iff 1 \leq {}^\forall i \leq r, {}^\forall n > 0, {}^\exists m \geq n; J_m^i \subset I_{n,m}^i.$$

Fix $1 \leq {}^\forall i \leq r$ and omit from the notation.
Fix n, $\{I_{n,m}\}_{m \geq n}$: ascending chain of ideals $\rightsquigarrow {}^\exists m_0 \geq n; {}^\forall m \geq m_0, I_{n,m_0} = I_{n,m}$.
Let $m := m_0 + n$, then ${}^\forall a \in J_{m_0}$, $aa_i^{m-n} = aa_i^{m_0} \in (a_1^{m_0}, \ldots, a_{i-1}^{m_0}) \subset (a_1^n, \ldots, a_{i-1}^n)$.
So $J_{m_0} \subset I_{n,m_0}$. □

Corollary 4.6 (ICYMI : Theorem 3.3)

A : Noetherian ring, $\underline{a} = a_1, \ldots, a_r \in A$ and $I = (a_1, \ldots, a_r)$. There are isomorphisms;

$$H_I^i(M) \cong \check{H}^i(\underline{a}, M)$$

for any $M \in \mathrm{Mod}(A)$.

Another proof of Theorem 3.3.

By above proposition, $\underline{a}$ is proregular. $\rightsquigarrow$ $\underline{a}$ is weakly proregular.
Then according to Schenzel's theorem, $H_I^i(M) \cong \check{H}^i(\underline{a}, M)$. $\square$

Why is a weakly proregular sequence defined by using a Koszul homology?

$\rightsquigarrow$ A Čech cohomology can be written by using a Koszul cohomology!

$K^\bullet(\underline{a}) := \mathrm{Hom}(K_\bullet(\underline{a}), A)$. For $M \in \mathrm{Mod}(A)$, $K^\bullet(\underline{a}, M) := \mathrm{Hom}(K_\bullet(\underline{a}), M) = K^\bullet(\underline{a}) \otimes M$.

$$\begin{tikzcd} K_\bullet(\underline{a}) : \arrow[d, bend right, "\mathrm{Hom}(-,A)"'] & \cdots \arrow[r] & K_1(\underline{a}) \arrow[r] & K_0(\underline{a}) \arrow[r] & 0 \\ K^\bullet(\underline{a}) : & 0 \arrow[r] & K^0(\underline{a}) \arrow[r] & K^1(\underline{a}) \arrow[r] & \cdots \end{tikzcd}$$

The opposition of morphism induces an inductive system $\{K^\bullet(\underline{a}^n)\}_{n\geq 0}$;

$$\varphi^{nm} : K^i(\underline{a}^n) \to K^i(\underline{a}^m); (e_I)^* \mapsto a_I^{m-n}(e_I)^*.$$

Proposition 5.1

A : ring, $\underline{a} = a_1, \ldots, a_r \in A$ and $M \in \mathrm{Mod}(A)$. Then;

$$\check{H}^i(\underline{a}, M) \cong \varinjlim H^i(\underline{a}^n, M).$$

Sketch of the proof.

$\varphi^i : K^i(\underline{a}) \to C^i(\underline{a}); (e_I)^* \mapsto (1/a_I)e_I$ is a morphism of complexes.
So we get $\varphi_n^\bullet : K^\bullet(\underline{a}^n) \to C^\bullet(\underline{a}^n) = C^\bullet(\underline{a})$. It induces $\varphi : \varinjlim K^\bullet(\underline{a}^n) \to C^\bullet(\underline{a})$ and this is an isomorphism.

$$\begin{tikzcd}
\cdots \arrow[r] & K^\bullet(\underline{a}^n) \arrow[r, "\varphi^{nm}"] \arrow[drrr, bend right, "\varphi_n^\bullet"'] & K^\bullet(\underline{a}^m) \arrow[r] \arrow[drr, "\varphi_m^\bullet"] & \cdots \arrow[r] & \varinjlim K^\bullet(\underline{a}^n) \arrow[d, dashed, "\varphi"] \\
 & & & & C^\bullet(\underline{a})
\end{tikzcd}$$

Then ; $\varinjlim H^i(\underline{a}^n, M) = H^i(\varinjlim K^\bullet(\underline{a}^n) \otimes M) \cong H^i(C^\bullet(\underline{a}^n) \otimes M) = \check{H}^i(\underline{a}^n, M)$.

□

Remark 5.2

By Proposition 5.1,

$$\check{H}^i(\underline{a}, M) \cong \varinjlim H^i(\underline{a}^n, M).$$

By the definition,

$$H_I^i(M) \cong \varinjlim \operatorname{Ext}^i(A/I^n, M).$$

$\rightsquigarrow$ Schenzel's theorem holds if $H^i(\underline{a}^n, M) \cong \operatorname{Ext}^i(A/I^n, M)$, which is true when $\underline{a}$ is a regular sequence.

However, if $\underline{a}$ is not regular, it may not work. So we will need to find an another way.

$\rightsquigarrow$ We will show a way to use the δ-functor.

A: ring , I : ideal of A. Let $\Gamma_I(M) := \left\{x \in M \;\middle|\; {}^{\exists}n \geq 0; I^n x = 0\right\}$.

$\rightsquigarrow$ The functor $\Gamma_I(-)$ connects a local cohomology and a Čech cohomology.

Lemma 5.3

A : ring, $\underline{a} = a_1, \ldots, a_r \in A, I = (a_1, \ldots, a_r)$: ideal of A and $M \in \mathrm{Mod}(A)$.

$$H_I^0(M) \cong \Gamma_I(M) \cong \check{H}^0(\underline{a}, M).$$

Proof.

- First isomorphism : $H_I^0(M) = \varinjlim \mathrm{Hom}(A/I^n, M), \mathrm{Hom}(A/I^n, M) \cong \{x \in M \mid I^n x = 0\}$.
- $\check{H}^0(\underline{a}, M)$ is the kernel of $(M \to \bigoplus_{i=1}^r M_{a_i} e_i; x \mapsto (x/1)e_i)$.

  $\rightsquigarrow$ ${}^{\forall}x \in \check{H}^0(\underline{a}, M), 1 \leq {}^{\forall}i \leq r, {}^{\exists}n_i \geq 0; a_i^{n_i} x = 0$. i.e. $x \in \Gamma_I(M)$.

  Similarly $\Gamma_I(M) \subset \check{H}^0(\underline{a}, M)$. $\rightsquigarrow$ $\check{H}^0(\underline{a}, M) = \Gamma_I(M)$ as a submodule of M.

$\square$

Definition 5.4

$\mathscr{A}, \mathscr{B}$: Abelian categories.
$T^\bullet := \{T^i : \mathscr{A} \to \mathscr{B}\}_{i \geq 0}$: family of additive functors.
$T^\bullet$ is called a δ-**functor** if ;

- For each exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathscr{A}$, ${}^\exists \delta^i : T^i(A_3) \to T^{i+1}(A_1)$;

  $$0 \to T^0(A_1) \to T^0(A_2) \to T^0(A_3) \xrightarrow{\delta^0} \cdots \xrightarrow{\delta^{i-1}} T^i(A_1) \to T^i(A_2) \to T^i(A_3) \xrightarrow{\delta^i} \cdots$$

  is exact.
- It transfers a commutative diagram to a commutative diagram.

δ^i is called a **connecting morphism**.

The δ-functor is a generalisation of the derived functor.
It is also useful for proving that the family of functors are form a derived functor!

Definition 5.5

$\mathscr{A}, \mathscr{B}$: Abelian categories, $F : \mathscr{A} \to \mathscr{B}$: additive functor.
F is called **effaceable** if ${}^\forall A \in \mathscr{A}, {}^\exists M \in \mathscr{A}; {}^\exists u : A \to M$: injection; $F(u) = 0$.

Proposition 5.6

$\mathscr{A}, \mathscr{B}$: Abelian categories, $\mathscr{A}$ has enough injectives. $T^\bullet = \{T^i\}_{i \geq 0}$: δ-functor.
${}^\forall i > 0, T^i$ is effaceable. Then;

- *T^0 is left-exact.*
- *${}^\forall i \geq 0, T^i \cong R^i T^0$ (up to unique isomorphism).*

Proposition 5.7

$\check{H}^\bullet(\underline{a}, -)$ is a δ-functor with $\check{H}^0(\underline{a}, -) \cong H^0_I(-)$.

Sketch of the proof.

$0 \to M_1 \to M_2 \to M_3 \to 0$: exact sequence of $\mathrm{Mod}(A)$.
$C^\bullet(\underline{a}, M) = C^\bullet(\underline{a}) \otimes M$ and $C^i(\underline{a})$ is flat. $\rightsquigarrow$ We obtain an exact sequence of complexes by taking tensor products.

$$0 \longrightarrow C^\bullet(\underline{a}, M_1) \longrightarrow C^\bullet(\underline{a}, M_2) \longrightarrow C^\bullet(\underline{a}, M_3) \longrightarrow 0 \ .$$

So there are connecting morphisms. □

Proposition 5.8

A: ring, $\underline{a} = a_1, \ldots, a_r \in A$.

$$\underline{a} \text{ is a weakly proregular seqence} \iff \check{H}^\bullet(\underline{a}, -) \text{ is an effaceable } \delta\text{-functor}.$$

Sketch of the proof.

It is enough to check each injective module J, $\check{H}^i(\underline{a}, J) = 0$ $(^\forall i > 0)$.
Use Proposition 5.1. i.e. $\check{H}^i(\underline{a}, M) \cong \varinjlim H^i(\underline{a}^n, M)$.
$\rightsquigarrow$ Calculate the Koszul (co)homology! (Note that $H^i(\underline{a}^n, J) \cong \mathrm{Hom}(H_i(\underline{a}^n), J)$.) $\square$

Theorem 5.9 (ICYMI : Schenzel's theorem)

A : ring, $\underline{a} = a_1, \ldots, a_r \in A$ and $I = (a_1, \ldots, a_r)$.

$$\underline{a} \text{ is a weakly proregular sequence} \iff {}^{\forall} i \geq 0, {}^{\forall} M \in \mathrm{Mod}(A), H^i_I(M) \cong \check{H}^i(\underline{a}, M).$$

Elementary proof of Schenzel's theorem. A(2021).

It is a combination of what has been said so far.

$$H^0_I(-) \cong \Gamma_I(M) \cong \check{H}^0(\underline{a}, -). \qquad \text{(Lem. 5.3)}$$

$$\underline{a} \text{ is a weakly proregular sequence} \iff \check{H}^{\bullet}(\underline{a}, -) \text{ is an effaceable } \delta\text{-functor.} \qquad \text{(Prop. 5.8)}$$

$\rightsquigarrow$

$$\underline{a} \text{ is a weakly proregular sequence} \iff {}^{\forall} i \geq 0, \check{H}^i(\underline{a}, -) \cong H^i_I(-) = R^i\Gamma_I(-). \qquad \text{(Prop. 5.6)}$$

$\square$

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