

## Abstract

Graph pebbling was first introduced as a tool for solving a combinatorial number theory conjecture of Erdős and Lemke. Pebbles are moved throughout a graph by removing two from one vertex to place one on an adjacent vertex. We study a pebbling variant called  $\phi$ -pebbling in which each pebble may move once without another being removed. We compute the  $\phi$ -pebbling number of trees, cycles, and thorn graphs and establish bounds on the  $\phi$ -pebbling number for graphs of diameter two, Cartesian products of graphs, complete  $k$ -partite graphs, paths, hypercubes, grids, and crowns. The results aid our understanding of Graham's Conjecture as  $\phi$ -pebbling represents moving a pebble through both graphs in a Cartesian product simultaneously.

## Graph pebbling

A **graph**  $G = (V, E)$  is a set of vertices  $V$  connected by edges  $E$ . We will study **connected** graphs or those in which there is a sequence of edges between any two vertices. A **path graph**  $P_n$  is one in which two vertices are connected to exactly one other vertex and  $n - 2$  vertices are connected to exactly two other vertices.

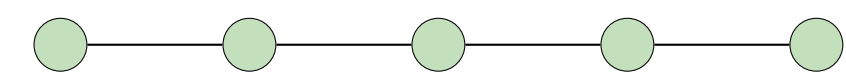
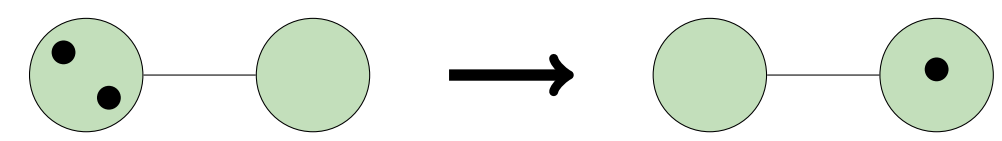


Figure 1. The path graph  $P_3$  arranged in a straight line and connected sequentially.

### What is pebbling?

- Place a non-negative integer number of **pebbles** on a graph.
- Move pebbles according to the following rule: **Take two pebbles away from a vertex, then place one pebble on an adjacent vertex**



If, given a **specific configuration of pebbles**, we can **move  $t$  pebbles to a vertex**, we say that this vertex is  **$t$ -reachable**.

**Definition.** The  **$t$ -pebbling number** of a graph  $G$ , denoted  $\pi_t(G)$ , is the minimum  $k$  such that every vertex is reachable by at least  $t$  pebbles in any configuration of  $k$  pebbles.

## Definition of $\phi$ -pebbling

### What is $\phi$ -pebbling?

- Each pebble is allowed **one free move**.
- Lone pebbles would be unusable in normal pebbling, but in  $\phi$ -pebbling, we can **coalesce lone pebbles onto a mutual neighbor** as free moves, and then proceed normally.

**Definition.** The  **$\phi$ -pebbling number** of a graph  $G$ , denoted  $\phi(G)$ , is the minimum  $k$  such that every vertex is reachable via cut-the-corner moves in any configuration of  $k$  pebbles.

**Proposition.** Asplund et al. [1] prove  $\phi(G) \leq \left\lceil \frac{\pi(G)}{2} \right\rceil$ .

## Motivation for $\phi$ -pebbling

The **Cartesian product** of two graphs  $G$  and  $H$  denoted  $G \square H$  has vertices

$$V(G) \times V(H) = \{(g, h) : g \in V(G), h \in V(H)\}$$

and an edge between  $(g_1, h_1)$  and  $(g_2, h_2)$  if

$$g_1 = g_2 \text{ AND } h_1 \text{ is adjacent to } h_2 \quad \text{OR} \quad g_1 \text{ is adjacent to } g_2 \text{ AND } h_1 = h_2.$$

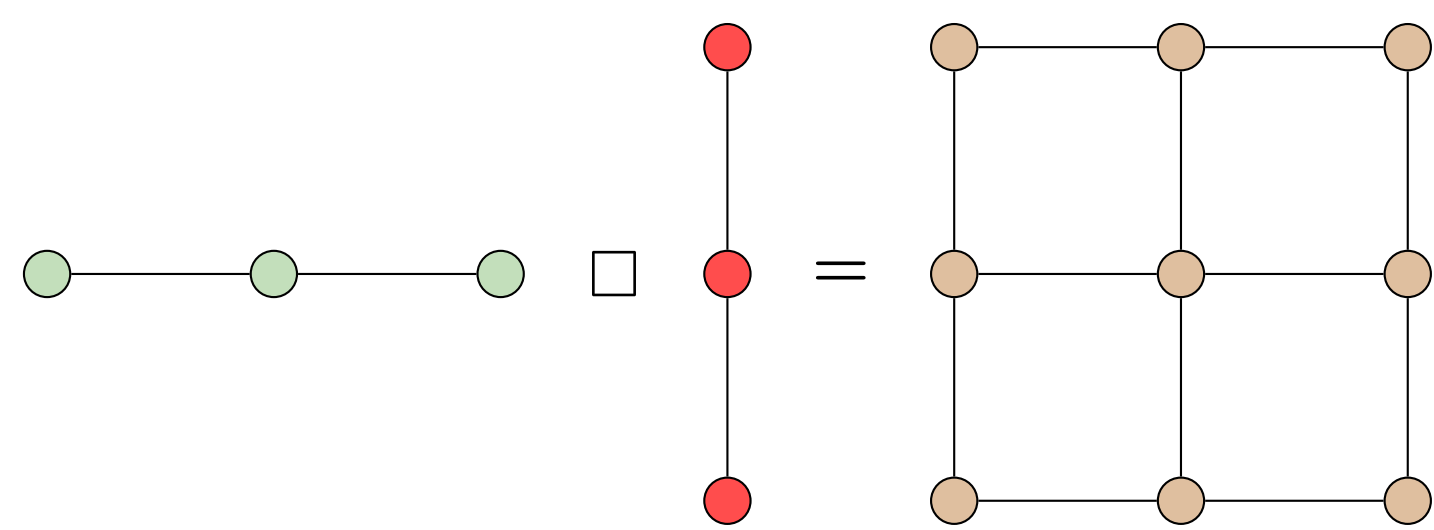


Figure 2. The Cartesian product of  $P_3$  with  $P_3$ , which reveals why this product is also known as the "box product"

**Graham's Conjecture.** Let  $G$  and  $H$  be graphs with Cartesian product  $G \square H$ . Then  $\pi(G \square H) \leq \pi(G)\pi(H)$ .

**Proposition.** Asplund et al. [1] use  $\phi$ -pebbling to prove  $\pi(G \square H) \leq 2\pi(G)\pi(H)$ .

## $\phi$ -pebbling computations

**Definition.** A **tree** is a graph in which there is exactly one sequence of non-repeating edges between any two vertices. A **path partition** of a tree is a set of disjoint paths that lies within the tree. We denote a path partition by a non-increasing list  $\ell_1, \dots, \ell_m$  representing the number of vertices of each path in the partition. Path partition  $\mathcal{L}$  is **larger** than path partition  $\mathcal{L}'$  if it is larger in the first position where they differ [2].

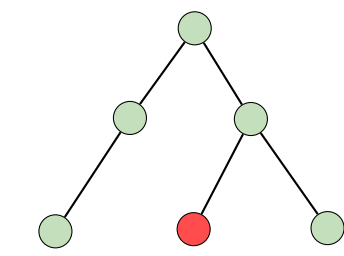


Figure 3. A tree with maximum path partition  $P_3$  and  $P_1$

**Proposition: Trees.** Let  $T$  be a tree with maximum path partition  $\ell_1, \dots, \ell_m$  in non-increasing order. Then

$$\phi(T) = \left( \sum_{i=1}^m 2^{\ell_i-1} \right) - m + 1.$$

### Proof sketch.

- Chung [3] proves the pebbling number of a tree is  $\pi(T) = \left( \sum_{i=1}^m 2^{\ell_i} \right) - m + 1$ .
- By the Bunde et al. No-Cycle Lemma [2], we never need to move a pebble in both directions along an edge.
- We can reduce the  $\phi$ -pebbling problem to a standard pebbling number in which each path partition has one fewer edge.

**Proposition: Cycles.** A **cycle graph** is a connected graph in which each vertex has exactly two connections. Let  $C_n$  be a cycle with  $n$  vertices. Then

$$\phi(C_n) = \begin{cases} \left\lceil \frac{\pi(C_n)}{2} \right\rceil & n \equiv 1 \pmod{4} \\ \frac{\pi(C_n)}{2} & n \text{ is even} \\ \left\lfloor \frac{\pi(C_n)}{2} \right\rfloor & n \equiv 3 \pmod{4}. \end{cases}$$

### Proof sketch.

- In order to pebble any odd cycle graph  $C_{2k+1}$  for some  $k \in \mathbb{N}$ , place  $x$  pebbles on each of two adjacent vertices and attempt to reach the target vertex a distance  $k$  away from both vertices. In the example  $C_5$  below, we place pebbles on the bottom two vertices and attempt to reach the top vertex.

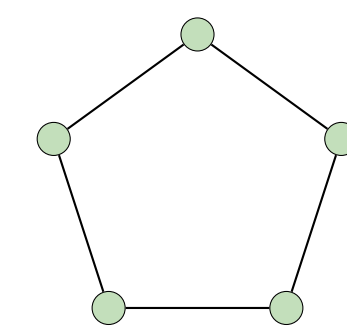


Figure 4. The cycle graph  $C_5$  arranged in a pentagon and connected in a closed loop.

- Then  $x + \left\lfloor \frac{x}{2} \right\rfloor$  is the greatest number of pebbles that can reach a vertex that is distance  $k - 1$  from the target vertex.
- Since  $\pi(P_k) = 2^{k-1}$ , if  $x$  is the largest  $x \in \mathbb{N}$  such that  $2^{k-1} - 1 = x + \left\lfloor \frac{x}{2} \right\rfloor$ , then we cannot reach the root vertex with  $2x$  pebbles.
- Thus,  $\phi(C_{2k+1}) = 2x + 1$ , which is equal to  $\left\lceil \frac{\pi(C_n)}{2} \right\rceil$  if  $k$  is even and  $\left\lfloor \frac{\pi(C_n)}{2} \right\rfloor$  if  $k$  is odd.

**Proposition: Thorns.** For a graph  $G$ , the **thorn graph**  $G^*$  is obtained by attaching vertices (**thorns**) to each  $v \in V(G)$ . A thorn adjacent to  $v$  is denoted  $u$ , and  $v$  is its base. For every connected  $G$  with thorn graph  $G^*$ ,

$$\phi(G^*) = \pi_2(G).$$

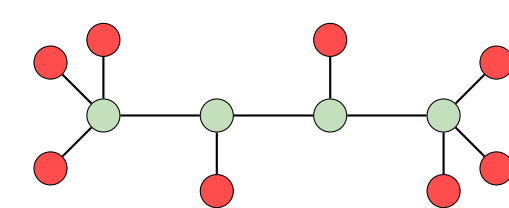


Figure 5. Muthulakshmi et al. [4] shows a thorn graph  $G^*$  constructed from the path  $G = P_4$ . Each green vertex is a base node of  $G$  and the red vertices are attached thorns on  $G^*$ .

### Proof sketch.

- Upper bound:** Place  $\pi_2(G)$  pebbles on  $G^*$ . Move thorn pebbles to their bases, leaving  $\pi_2(G)$  on  $G$ . Since two pebbles can reach any vertex in  $G$ , and thorns are one step away, any thorn can also be reached. Thus  $\phi(G^*) \leq \pi_2(G)$ .
- Lower bound:** First, place  $\pi_2(G) - 1$  pebbles on  $G$  such that two cannot reach a target vertex  $v$ . Second, move each pebble to a thorn. There is a thorn connected to  $v$  that is not reachable via  $\phi$ -pebbling. With the free move, we can place at most  $\pi_2(G) - 1$  on  $G$ , which is insufficient to reach  $v$  with two pebbles or a thorn by one. Thus  $\phi(G^*) \geq \pi_2(G)$ .

## Graphs of diameter two

**Definition.** The **distance** between two vertices  $u$  and  $v$  in a graph  $G$  is the minimum number of edges between them. The **diameter** of a graph  $G$  is the largest distance between any two vertices of  $G$ .

**Theorem.** Let  $G$  be a graph of diameter two with  $n$  vertices. Then

$$\phi(G) \leq \sqrt{4n+5} - 2.$$

**Proof sketch.** Using  $\phi$ -pebbling moves, we can reach a vertex  $v$  in a graph  $G$  with at least one pebble if any of the following holds:

- There is more than one pebble on any single vertex
- A vertex is connected to at least 4 vertices with pebbles
- A vertex with a pebble is connected to at least two other vertices with pebbles
- A vertex connected to  $v$  is connected to at least two vertices with pebbles

Given a number of pebbles  $p$ , we argue that in a graph of diameter two with fewer than  $1 + \frac{3}{2}p + \frac{1}{4}p^2$  vertices each vertex is reachable by at least one pebble. Let  $v$  be a target vertex. We need at least  $p$  vertices connected to  $v$  and  $p$  other vertices each with exactly one pebble. In order for  $G$  to have diameter two, we need an extra  $\frac{1}{4}p^2 - \frac{1}{2}p$  vertices that are not  $v$ , neighbors of  $v$ , or a vertex with a pebble when  $p$  is even. When  $p$  is odd, we need at least  $\frac{1}{4}p^2 - p + \frac{3}{4}$ . By setting  $n = 1 + \frac{3}{2}p + \frac{1}{4}p^2$  and solving for  $p$  as a function of  $n$ , we obtain the desired result.

**Example.** For any integer  $p$ , there is a graph  $G$  with  $\frac{p^2}{2} + \frac{3p}{2} + 1$  vertices such that  $\phi(G) > p$ .

- Create a target vertex  $r$ .
- Denote  $p$  neighbors of  $r$  by  $N_1(r) = \{v_1, \dots, v_p\}$ . Connect any  $v_i$  with any other  $v_j$ .
- Create vertices  $N_2(r) = \{w_1, \dots, w_p\}$ . Connect vertex  $w_i$  with  $v_i$ .
- For each pair of vertices  $\{v_i, v_j\}$ , create a vertex  $x_{i,j}$  that is adjacent to  $v_i, v_j, w_i$ , and  $w_j$ . Connect each  $x_{i,j}$  to each other  $x_{k,\ell}$ .
- If we place one pebble on each vertex of  $N_2(r)$ , we cannot reach  $r$  with a free move and subsequent pebbling moves.

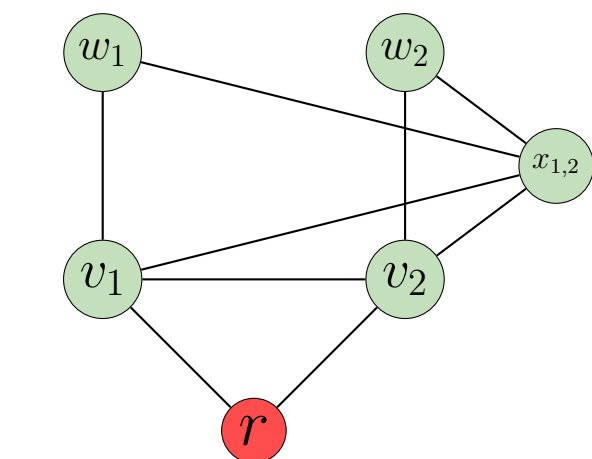


Figure 6. The construction for  $p = 2$

## Other results

**Cartesian Products:**  $\phi(G \square H) \leq \min\{\phi(G)(\pi(H) + |H|), \phi(H)(\pi(G) + |G|)\}$

**Complete  $k$ -Partite Graphs:**  $\phi(K_{a_1, a_2, \dots, a_k}) \leq 2$

**Hypercubes:**  $\phi(Q_n) = \left\lceil \frac{3^n}{2} \right\rceil$

**Grids:**  $\phi(G_{m,n}) = 2^{m+n-3}$

**Crowns:**  $\phi(W_n) \leq 4$

## Future Work

- Determine an upper bound on  $\phi(G)$  that is sharp infinitely often for  $\text{diam}(G) = 2$ .
- Generalize  $\phi$ -pebbling number of a graph to allow for  $k$  free moves per pebble, denoted  $\phi_k(G)$ . Can we prove that  $\phi_{k+1}(G) \leq \left\lfloor \frac{\phi_k(G)+1}{2} \right\rfloor$ ?
- Determine whether computing the  $\phi$ -pebbling number of a graph is NP-complete.

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## References

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