CS/MATH 375, Fall 2024 — HOMEWORK # 4
Due : Sept. 20th at 10:00pm on Canvas

Instructions

- **Report:** In general, your report needs to read coherently. That is, start off by answering question 1. Fully answer the question, and provide all the information needed to understand your answer. If Matlab code or output is part of the question, include that code or output (e.g., screenshot) alongside your narrative answer. If discussion is required for a question, include that. Overall, your report is your narrative explanation of what was done, your answers to the specific questions, and how you arrived at your answers. *Your report should include your Matlab scripts, code output, and any figures.*

- **What to hand in:** Submission must be one **single PDF** document, containing your entire report, submitted on Canvas.

- **Partners:** You are allowed to (even encouraged) to **work in pairs**. If you work with a partner, only one member of the group should need to submit a report. On Canvas, both partners should join a group (numbered 1 through 15). Then either member can upload the report for the entire group. Groups of more than 2 students are not allowed.

- **Typesetting:** If you write your answers by hand, then make sure that your handwriting is readable. Otherwise, I cannot grade it.

- **Plots:** All plots/figures in the report must be generated in Matlab or Python and not hand drawn (unless otherwise specified in the homework question).

  In general, make sure to (1) title figures, (2) label both axes, (3) make the curves nice and thick to be easily readable, and (4) include a legend for the plotted data sets. The font-size of all text in your figures must be large and easily readable.

Reading: This week was mostly review, so no required reading. If you want to review a bit, read Appendix A. If you want to get ahead, reach chapter 2.1.

Problems:

1. *FLOPS*

   On the course Canvas page (attached to this homework assignment), you will find a script `test_flops.m` that generates a plot for the run-time of a mat-vec operation over different problem sizes.

(a) Modify this file in order to plot a similar timing graph for matrix-matrix multiplication. Make sure to use appropriate axis scaling and readable labels.

Use the Matlab command $A * B$ to perform the matrix-matrix multiply where A, B are $n \times n$ matrices. Find the exhibited big-Oh notation for the runtime, e.g., $O(n)$, $O(n^2)$, $O(n^3)$,... Justify your answer from your plot.

(b) Repeat the previous problem, but now instead of a single matrix operation $(A * B)$, consider multiple Mat-Mat operations. Time the cost of the calculation $A * B + C * D$ for $n \times n$ matrices A, B, C, and D and generate the relevant plot. Does the operation still have the same big-Oh runtime as in part (a)? Explain.

2. *Numerical Invertibility*

(a) Show that the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

is singular. Find a vector in the null-space of A (i.e. a nonzero vector such that $Ay = 0$).

Does the equation $Ax = b$ have no solutions, one solution, or infinitely many solutions if

$$b = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} ?$$

(b) Solve the linear system $Ax = b$ in MATLAB with backslash. Does MATLAB think the matrix is singular? If not, describe why.

(c) Now define a new problem $\tilde{A}x = \tilde{b}$ where

$$\tilde{A} = \frac{1}{10}A, \quad \tilde{b} = \frac{1}{10}b.$$

This problem should have the same solution x. If you were to try to solve this problem by hand, you would encounter the same issue as part (b). Try to solve this problem in MATLAB using the backslash command. Does MATLAB think the matrix is singular? If not, describe why.

3. *Linear Algebra Exercises*

(a) Prove that if L is lower-triangular and invertible, then the matrix $A = L^{-1}$ is also lower triangular.

(b) A square matrix is symmetric if transposing it (or reflecting the entries across the diagonal) doesn't change the matrix. Put another way, A is symmetric if the entries obey $a_{ij} = a_{ji}$. Prove the following statement, or find a counter-example: if A and B are both symmetric, then $C = AB$ is also symmetric.

4. *Matrix Inversion*
   Consider the system of equations

$$x + 2y - z = 10,$$

$$2x - y + z = 3,$$

$$3x - 2y + 2z = 3.$$

(a) Solve this system by hand using techniques from high-school algebra and keep track of the "flops" required (anytime you multiply, add, subtract, or divide two numbers).

(b) Now reform the problem as a matrix equation $A\vec{x} = \vec{b}$.

(c) Calculate the inverse matrix A^{-1} and perform the multiplication $A^{-1}\vec{b}$. Count the number of flops required and compare with part a).