+5 for presentation

Intro to Numerical Computing
Homework 12

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Alex Knigge & Ryan Scherbarth
CS/MATH 375

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1. Accuracy of the trapezoid rule

   (a) Calculate how many points are needed to ensure the composite Trapezoid rule is accurate to 10^6 for

   $$I = \int_0^{\pi} x \sin(x) dx$$

   (b) Write a MATLAB function `integral = comp_trap_int(f,a,b,n)` where f is the function to be integrated, `a` and `b` are the limits of integration, n is the number of intervals and `integral` is the approximation of the integral. This function approximates the definite integral $\int_a^b f(x)dx$ using `n` intervals for the composite trapezoid rule.

   (c) Using your MATLAB function above, approximate the definite integral

   $$\int_0^{\pi} x \sin(x) dx$$

   using $n = 4, 8, 16, 32$ intervals. Tabulate your results by showing three columns: the value of the approximate integral computed by the composite Trapezoid rule, the error, and the observed order of convergence p for each value of n. Recall that the order of convergence p is computed as

   $$p = \frac{\log(err^{(n)}/err^{(n-1)})}{\log(h^{(n)}/h^{(n-1)})}$$

   where $err^{(n)}$ is the error for n intervals and $h(n) = (b-a)/n$ is the sub-interval length.

   Do you see the expected convergence? If not, try to explain your results.

   (d) Approximate (by hand) the above integral using the **composite Simpson** rule with 2 intervals of equal length (take $n = 4$, which means 5 points). How many evaluations of the function $f(x)$ are required? How large is your error and how does this compare to using the composite trapezoid method?

Ans:

(a)

Composite trapezoid rule:

$$T(f;P) = \int_a^b f(x)dx \approx \frac{h}{2}\sum_{i=0}^{n-1}(f(x_i) + f(x_{i+1}))$$

To find how many points should be used to ensure the Composite Trapezoid rule is accurate for 10^{-6} we need

$$\frac{(b-a)h^2 f''(\eta)}{12} \leq 10^{-6}$$

Need to first find the derivatives of $f(x) = x\sin(x)$

$$f(x) = x\sin(x)$$

$$f'(x) = x\cos(x) + \sin(x)$$

$$f''(x) = -x\sin(x) + \cos(x) + \cos(x) = -x\sin(x) + 2\cos(x)$$

How big is $f''(x)$ on this interval? $\sin(x)$ and $\cos(x)$ are periodic, and for $f''(x)$ it will be largest when $x = \pi, 0$ because $\cos(0)$ or $\cos(\pi)$ equals 1 ($\sin(x) = 0$ so disregard that term). Thus,

$$f''(0) = f''(\pi) = 2$$

$$|f''(x)| \leq 2$$

And using the relation $(b-a)/n = h$,

$$(b-a) = \pi$$

$$h = \frac{\pi}{n}$$

Then, the bound becomes

$$\frac{\pi \cdot (\frac{\pi}{n})^2 \cdot 2}{12} \leq 10^{-6}$$

$$\frac{\pi^3}{6n^2} \leq 10^{-6}$$

$$\frac{10^6\pi^3}{6} \leq n^2$$

$$n \geq \sqrt{\frac{10^6\pi^3}{6}} \approx 2273.26$$

$$n > 2274$$

little low -1

(b)
```
function integral = comp_trap_int(f,a,b,n)
    h = (b-a)/n;
    trap_sum = (f(a)+f(b))/2;
    for i = 1 : n - 1
        x_i = a + i * h;
        trap_sum = trap_sum + f(x_i);
    end
    integral = trap_sum * h;
end
```

n	Approximation	Error	p
4	2.978417	1.836824e+00	-
8	3.101116	1.959523e+00	-0.09
16	3.131493	1.989900e+00	-0.02
32	3.139069	1.997476e+00	-0.01

there appears to be a bug in your code... o

(c) We do not see the expected convergence we want. As n increases, the error does not decrease and the p values are negative. Our approximation is getting further away from the exact value of 2. Since our function is a smooth function, the trapezoid rule will struggle with something like $x\sin(x)$ and it will probably be better to use the composite Simpson rule.

(d) Composite Simpson rule:

$$\frac{h}{3}[f(a)+f(b)+4\sum_{i=1}^{n/2} f(a+(2i-1)h+2\sum_{i=1}^{n/2-1} f(a+2ih)$$

Using $n=4$ for $\int_0^\pi x\sin(x)dx$

$$\frac{h}{3}[0+0+4\sum_{i=1}^{n/2} f(0+(2i-1)h)+2\sum_{i=1}^{n/2-1} f(0+2ih)]$$

$$\frac{\pi}{12}[4\sum_{i=1}^{n/2} f((2i-1)\frac{\pi}{4})+2\sum_{i=1}^{n/2-1} f(2i\frac{\pi}{4})]$$

$$\frac{\pi}{12}[4\cdot(f(\frac{\pi}{4})+f(\frac{3\pi}{4}))+2\cdot f(\frac{\pi}{2})]$$

$$\frac{\pi}{12}[4(\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2}+\frac{3\pi}{4}\cdot\frac{\sqrt{2}}{2})+2\cdot\frac{\pi}{2}]$$

$$\frac{\pi}{12}[\frac{\pi\sqrt{2}}{2}+\pi]$$

$$\frac{\pi^2}{12}(\sqrt{2}+1) \approx 1.9856$$

5 total evaluations were required. The error is 0.014389 which is much less than what we got using the composite trapezoid rule where we got the smallest error of 1.99747. Using composite Simpson rule for this function did give us a better approximation.

Error is about 2x too big, and your integral is very off...-1

2. Degree of precision (the other accuracy!). We say that an integration rule has degree of precision p, if the rule can exactly integrate all polynomials of up to degree p exactly.

   (a) Show that Simpson's rule
   $$\int_b^a f(x)dx \approx \frac{b-a}{6}(f(a) + 4f((a+b)/2)_f(b))$$
   is exact for $f(x) = 1, f(x) = x, f(x) = x^2$.
   This can be shown directly, or via the fact that Simpson's rule is constructed using an interpolating polynomial for $f(x)$.

   (b) Show that Simpson's rule is also exact for $f(x) = x^3$. This can be shown directly.

   (c) Show that Simpson's rule is a linear operator. That is, let $S(f(x))$ represent one application of Simpson's rule to the function $f(x)$ in order to approximate the integral $\int_a^b f(x)dx$, i.e.,
   $$\int_a^b f(x)dx \approx S(f(x)) = \frac{b-a}{6}(f(a) + 4f((a+b)/2)_f(b)).$$
   Show that $S(\cdot)$ is a linear operator, i.e., that
   $$S(\alpha f(x) + \beta g(x) = \alpha S(f(x)) = \beta S(g(x)),$$
   for functions $f(x)$ and $g(x)$ and scalars α and β.

   (d) Using the previous three facts, show that Simpson's rule is exact for all polynomials up to degree 3. For your proof, you will likely want to use the linearity of Simpson's rule (proved above), and the well-known linearity of integration, i.e., that
   $$\int_a^b \alpha f(x) = \beta g(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx,$$
   for integrable functions $f(x)$ and $g(x)$ and scalars α and β.

Ans:

(a)

$$\text{When } f(x) = 1$$
$$\frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b)$$
$$\text{Exact answer: } \int_a^b 1dx = b-a$$
$$\frac{b-1}{6}(1 + 4(1) + 1) = \frac{b-a}{6}(6) = b-a$$

When $f(x) = x$

$$\text{Exact answer: } \int_a^b x dx = \frac{b^2}{2} - \frac{a^2}{2}$$

$$\frac{b-a}{6}(a + 2(a+b) + b) = \frac{ab - a^2 + b^2 - ab}{2} = \frac{b^2}{2} - \frac{a^2}{2}$$

When $f(x) = x^2$

$$\text{Exact answer: } \int_a^b x^2 = \frac{b^3}{3} - \frac{a^3}{3}$$

$$\frac{b-a}{6}(a^2 + a^2 + 2ab + b^2 + b^2) = \frac{b-a}{2}(a^2 + ab + b^2) = \frac{a^3}{3} + \frac{b^3}{3}$$

(b)

When $f(x) = x^3$

$$\text{Exact answer: } \int_a^b x^3 = \frac{b^4}{4} - \frac{a^4}{4}$$

$$= \frac{b-a}{6}(a^3 + 4(\frac{a^3 + 3a^2b + 3ab^2 + b^3}{8} + b^3))$$

$$= \frac{b-a}{6}(a^3 + a^2b + ab^2 + b^3) = \frac{b^4}{4} - \frac{a^4}{4}$$

(c)

$$S(f(x)) = \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b))$$

$$\frac{b-a}{6}((\alpha f(a) + \beta g(a))4(\alpha f(m) + \beta g(m)) + (\alpha f(b) + \beta g(b)))$$

$$\text{Solve using } m = \frac{a+b}{2}$$

$$= \frac{b-a}{6}(\alpha(f(a) + 4f(m) + f(b)) + \beta(g(a) + 4g(m) + g(b))$$

$$= \alpha\frac{b-a}{6}(f(a) + 4f(m) + f(b)) + \beta(g(a) + 4g(m) + g(b))$$

$$S(\alpha f(x) + \beta g(x)) = \alpha S(f(x)) + \beta S(g(x))$$

Therefore, we can see both the additivity and homogeneity properties, and conclude that we have a linear operator.

(d)

$$S(c_1) = c_1 \int_a^b a dx = c_1(b - a)$$

$$S(c_2 x) = c_2 \int_a^b a dx = c_2(\frac{b^2}{2} - \frac{a^2}{2})$$

$$\dots$$

$$S(c_i x)^i = c_i \int_a^b x^i dx = \frac{b^i}{i} - \frac{a^i}{i}$$

should be (i+1)?

Using the Integration Linearity property:

$$\int_a^b (c_0 + c_1 x + c_2 x^2 + c_3 x^3) dx$$

Expanding is identical to $S(P(x))$:

$$\int_a^b P(x) dx = c_0 \int_a^b a dx + c_1 \int_a^b x dx + c_2 \int_a^b x^2 dx + c_i \int_a^b x^i dx$$

Simpson's rule will be exact for all $p \leq 3$

you have not proven the case for p=3. It looks as if you hav