CS/MATH 375, Fall 2024 — HOMEWORK # 9
Due : Nov. 1st at 10:00pm on Canvas

Instructions

- **Report:** In general, your report needs to read coherently. That is, start off by answering question 1. Fully answer the question, and provide all the information needed to understand your answer. If Matlab code or output is part of the question, include that code or output (e.g., screenshot) alongside your narrative answer. If discussion is required for a question, include that. Overall, your report is your narrative explanation of what was done, your answers to the specific questions, and how you arrived at your answers. *Your report should include your Matlab scripts, code output, and any figures.*

- **What to hand in:** Submission must be one **single PDF** document, containing your entire report, submitted on Canvas.

- **Partners:** You are allowed to (even encouraged) to **work in pairs**. If you work with a partner, only one member of the group should need to submit a report. On Canvas, both partners should join a group (numbered 1 through 15). Then either member can upload the report for the entire group. Groups of more than 2 students are not allowed.

- **Typesetting:** If you write your answers by hand, then make sure that your handwriting is readable. Otherwise, I cannot grade it.

- **Plots:** All plots/figures in the report must be generated in Matlab or Python and not hand drawn (unless otherwise specified in the homework question).

  In general, make sure to (1) title figures, (2) label both axes, (3) make the curves nice and thick to be easily readable, and (4) include a legend for the plotted data sets. The font-size of all text in your figures must be large and easily readable.

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1. We saw last homework the problems inherent in interpolating the Runge function

$$f(x) = \frac{1}{1+25x^2}, \quad -1 \le x \le 1$$

with evenly spaced points. It is much better to interpolate on a grid made up of points that are clustered towards the endpoints. Interpolate $f(x)$ using the Chebyshev points

$$x_i = \cos\frac{(2i+1)\pi}{2n+2}, \;\; i = 0, \dots, n,$$

and Lagrange interpolation. Use the Chebyshev points with $n = 150$. Do you see any of the issues that were illustrated in the previous homework (either Runge phenomena or the interpolant failing to match the given data)? Justify your answer with suitable plots.

2. For the functions

$$f_1(x) = \sin(x), \quad f_2(x) = |x|,$$

on the interval $[-1, 1]$ tabulate the maximum error,

$$\max_{x \in [-1,1]} |f(x) - p(x)|,$$

as a function of n (make sure you compute the error by sampling the error on a fine grid). Take values of n from 1 to 16 and use Chebyshev points with Lagrange interpolation. Is the interpolating polynomial converging for both functions? If not, what could be going wrong?

3. We saw on previous homework that the polynomial interpolation behaves badly near the endpoints. In problem 1, we tackled this problem by using Chebyshev points instead of equispaced points. For this problem, we take a different approach and use natural cubic splines to interpolate the function. That is, we will find a natural cubic spline $S(x)$ on $[-1, 1]$ that interpolates the equispaced knots $-1 = x_0, x_1, \ldots, x_n = 1$ and function values $y_0 = f(x_0), y_1 = f(x_1), \ldots, y_n = f(x_n)$.

   (a) **Matlab problem** Write a MATLAB function `coeffs = spline3_coeff(t,y)` that takes as input the knot vector $t = [x_0, x_1, \ldots, x_n]$ and the function values $y = [y_0, y_1, \ldots, y_n]$ and return the vector $z = [0 = z_0, z_1, z_2, \ldots, z_{n-1}, z_n = 0]$. The z vector is computed by setting up and solving a tridiagonal system as discussed during the lecture. You can use the MATLAB backslash for solving the tridiagonal system.

   (b) **Matlab problem** Compute the natural cubic spline, $S(x)$, using the function you just wrote for $t = [-1 = x_0, x_1, \ldots, x_n = 1]$ and $y = [y_0 = f(x_0), y_1 = f(x_1), \ldots, y_n = f(x_n)]$ for $n = 11$ and plot $S(x)$ as well as $f(x)$ on $[-1, 1]$. How does the natural cubic spline behave relative to the polynomial interpolation in from last homework?

   You may use the function `eval_spine.m`, provided with this assignment, to evaluate $S(x)$.