CS/MATH 375, Fall 2024 — HOMEWORK # 8
Due : Oct 25th at 10:00pm on Canvas

Instructions

– **Report:** In general, your report needs to read coherently. That is, start off by answering question 1. Fully answer the question, and provide all the information needed to understand your answer. If Matlab code or output is part of the question, include that code or output (e.g., screenshot) alongside your narrative answer. If discussion is required for a question, include that. Overall, your report is your narrative explanation of what was done, your answers to the specific questions, and how you arrived at your answers. *Your report should include your Matlab scripts, code output, and any figures.*

– **What to hand in:** Submission must be one **single PDF** document, containing your entire report, submitted on Canvas.

– **Partners:** You are allowed to (even encouraged) to **work in pairs**. If you work with a partner, only one member of the group should need to submit a report. On Canvas, both partners should join a group (numbered 1 through 15). Then either member can upload the report for the entire group. Groups of more than 2 students are not allowed.

– **Typesetting:** If you write your answers by hand, then make sure that your handwriting is readable. Otherwise, I cannot grade it.

– **Plots:** All plots/figures in the report must be generated in Matlab or Python and not hand drawn (unless otherwise specified in the homework question).

In general, make sure to (1) title figures, (2) label both axes, (3) make the curves nice and thick to be easily readable, and (4) include a legend for the plotted data sets. The font-size of all text in your figures must be large and easily readable.

Reading: Read chapters 3.1 - 3.3 in the book.

Problems: In this homework, n refers to the degree of the interpolating polynomial. Generally, this means there will be $n+1$ points of data.

1. Can you find a degree d polynomial that passes through $(-1,3),(1,1),(2,3),(3,7)$ for for $d=2,3,6$? If you can find such a polynomial, write it down. If no such polynomial exists, explain why. *(Hint):* The Newton polynomial basis makes this easy.

2. Consider the following points of data (n = 3):

$$(-1, 1), \quad (1, 1), \quad (2, 1), \quad (3, 9).$$

   (a) Find the monomial polynomial interpolating these data points by constructing the Vandermonde matrix and inverting it (you can use Matlab to do the inversion if you wish).

   (b) Find the Lagrange polynomial interpolating these data points.

   (c) Find the Newton polynomial interpolating these data points.

   (d) Show that your answers to a), b), and c) are equivalent.

3. Consider the data points $(1, 0)$, $(2, \ln 2)$, and $(4, \ln 4)$ which came from sampling the function $f(x) = \ln(x)$.

   (a) Using Lagrange or Newton form, find the quadratic polynomial interpolation of this data.

   (b) Use the error formula from lecture to find an upper bound on error if you evaluate your polynomial at $c = 3$ (i.e. $|f(3) - p(3)|$).

   (c) Evaluate your polynomial at 3, calculate $|p(3) - \ln(3)|$ and compare with your answer from part b).

4. In this problem you will write (and use) your own code to compute the monomial-based interpolating polynomial for arbitrary data: $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ for $(x_j, y_j) = (x_j, f(x_j)), j = 0, \ldots, n$.

   (a) Write a MATLAB function `function a=interp_monomials(x,y)` that returns the vector of coefficients $a = [a_0, \cdots, a_n]$ given input vectors

$$x = [x_0, x_1, \ldots, x_n]$$
$$y = [y_0, y_1, \ldots, y_n]$$

   (b) Apply your function from (b) to find the polynomial (i.e. the coefficients a) that interpolates

$$f(x) = \frac{1}{1 + 25x^2},$$

   for the points $x_i = -1 + ih$, $h = \frac{2}{n}$, $i = 0, \ldots, n$. Plot the data points $(x_j, f(x_j))$ as circles (use `plot(x,f,'o')`) and, in the same plot, plot the polynomial and $f(x)$ on a finer grid (still on $x \in [-1, 1]$), by using $n_{fine} = n * 100$ total points for plotting. Show what happens when you increase n. Try $n = 2, 4, \ldots, 20$. Comment. To evaluate the polynomial at some given values of x you can use `polyval` (note that `polyval` requires your coefficients to be in a specific order. Do `help polyval` for more information).

(c) Keep increasing n until the polynomial does not even interpolate $f(x)$. For what n does this happen? (You can find this by evaluating your polynomial at the original interpolating points x_i and then subtracting from y_i). What is the condition number of the matrix V for this n? Can you explain what is happening?

5. Solving the interpolation problem using the monomial basis (as above) is notoriously ill-conditioned, in fact it is possible to show

$$cond(V) \sim \pi^{-1} e^{\pi/4} (3.1)^n,$$

where V is the Vandermonde matrix. Perhaps Lagrage polynomials will help!

I have provided (check Canvas) a MATLAB function `my_lagrange` which can be used to evaluate the Lagrange interpolating polynomial at an arbitrary set of points. Its syntax is:

```
y = my_lagrange(x,pointx,pointy),
```

where `pointx` and `pointy` are arrays of the x and y values of your interpolating data respectively, and `x` is an array of locations where you would like to evaluate the polynomial.

(a) Show with some plots that the Lagrange polynomial still suffers from Runge phenomenon near the end points -1 and 1.

(b) Find the Lagrange polynomial for a value of n which failed to interpolate the data in Problem 3. Does this polynomial correctly interpolate the data now? What changed from Problem 3?