CS/MATH 375, Fall 2024 — HOMEWORK # 2
Due : Friday, Sept. 6 at 10:00pm on Canvas

Instructions

- **Report:** In general, your report needs to read coherently. That is, start off by answering question 1. Fully answer the question, and provide all the information needed to understand your answer. If Matlab code or output is part of the question, include that code or output (e.g., screenshot) alongside your narrative answer. If discussion is required for a question, include that. Overall, your report is your narrative explanation of what was done, your answers to the specific questions, and how you arrived at your answers. *Your report should include your Matlab scripts, code output, and any figures.*
- **What to hand in:** Submission must be one **single PDF** document, containing your entire report, submitted on Canvas.
- **Partners:** You are allowed to (even encouraged) to **work in pairs**. If you work with a partner, only one member of the group should need to submit a report. On Canvas, both partners should join a group (numbered 1 through 15). Then either member can upload the report for the entire group. Groups of more than 2 students are not allowed.
- **Typesetting:** If you write your answers by hand, then make sure that your handwriting is readable. Otherwise, I cannot grade it.
- **Plots:** All plots/figures in the report must be generated in Matlab or Python and not hand drawn (unless otherwise specified in the homework question).

  In general, make sure to (1) title figures, (2) label both axes, (3) make the curves nice and thick to be easily readable, and (4) include a legend for the plotted data sets. The font-size of all text in your figures must be large and easily readable.

Reading: Read Chapter 0 from the book.

Problems:

1. *Binary and Floating Point numbers*

   (a) Find the binary representation of the number -26.1. Explain all of your work.

   (b) Using the previous part, find the double precision machine number (written as a binary string of 1's and 0's) which represents -26.1. Explain all of your work. (I know I said in class that I wouldn't make you do this in double precision, but because this is a homework and you have a week I feel justified. I definitely won't make you do this in an exam).

(c) Convert your answer from part b) back to base 10 and call it $fl(-26.1)$. You can use Matlab or a calculator to do this and do not need to show your work. Verify by hand that

$$\left|\frac{fl(-26.1) - -26.1}{-26.1}\right| < \frac{\epsilon_m}{2}.$$

2. *Cancellation, Precision and Loss of Precision.*

   (a) Write a very naive code to evaluate (as written) the function

   $$f(x) = \frac{1-(1-x)^3}{x}$$

   (b) Evaluate your function for $x = 10^{-1}, 10^{-2}, \ldots 10^{-14}$. (Note: I would recommend doing your calculations in Matlab with `format long e`.)

   (c) Use the Loss of Precision Theorem (with base 10) to determine the number of accurate digits in your answers. Recall that double precision numbers "begin" with about 15 or 16 decimal digits of precision.

   (d) Rearrange the given function $f(x)$ into a new expression which avoids catastrophic cancellation for small x and repeat part a).

   (e) Repeat part b) with your new function.

   (f) Assuming that the results of b) are the "approximate" values and the results of e) are the "true" values, compile a table with 14 rows showing your approximations, the true values, and the relative error of each. Does this match your predictions from part c)?

3. *Taylor Series Errors*

   In this problem, we approximate $f(x) = \cos(x)$ using Taylor series.

   (a) How many terms in the Taylor series approximation to $f(x) = \cos(x)$ do I need to make sure that my error is less than 2×10^{-8} for $x \in [0, \pi/2]$? Use the Taylor approximation error of

   $$E_n(x) = \frac{(x-c)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

   to derive your answer, or another another theorem from calculus. If you expand $\cos(x)$ around $c = 0$, then the Taylor series will have some zero terms. You should count these terms towards the n in your answer.

   (b) Write a function called `my_cos` which takes in two arguments: a number x and a positive integer n. Your function should approximate $\cos(x)$ using a Taylor series approximation of order n. The function should return the approximate value.

(c) Now we experimentally verify the bound from part (a) in Matlab using your function from part (b). Define a vector $\mathbf{x}$ of 100 equally spaced points between $[0, \pi/2]$. For each of these points, compute the error, $|\cos(\mathbf{x}) - \mathrm{my\_cos}(\mathbf{x}, \mathrm{n})|$, using two different values of n:

- $n = 3$
- Value of n that you compute in part (a)

Plot the errors (I recommend a logarithmic vertical axis) for both these values of n against the vector $\mathbf{x}$. Is your error below the bound in part (a) at all these points for the two values of n?

4. *Taylor Series and Loss of Precision*
Bessel functions are an important class of functions which often arise in PDEs. They cannot be written using any finite collection elementary functions (powers, radicals, exponentials, logarithms, etc.), but they are still very useful functions and we *can* write their Taylor Series. The zero'th order Bessel function of the first kind has a Taylor Series (centered at $c = 0$)

$$J_0(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k!)^2} \left(\frac{x}{2}\right)^{2k} = 1 - \frac{x}{2} + \frac{1}{4}\frac{x^4}{2^4} - \frac{1}{36}\frac{x^6}{2^6} + \dots$$

Matlab has a built-in way to evaluate this function using `besselj(0,x)`.

(a) Write a Matlab function named `ApproxBesselJ0(x,N)` that, for inputs `x` and `N` evaluates the Taylor Polynomial

$$\sum_{k=0}^{N} (-1)^k \frac{1}{(k!)^2} \left(\frac{x}{2}\right)^{2k}.$$

(b) Using Matlab's built in function as the "true" value, using your function to evaluate J_0 for $x = 20$. Do this for $N = 1, 2, \dots M$, where M is the integer such that the last term of your polynomial is less than 10^{-8}. That is,

$$\frac{1}{(M!)^2} \left(\frac{x}{2}\right)^{2M} < 10^{-8} \text{ for } x = 20.$$

Plot the absolute error in your approximation as a function of N on a `semilogy` plot and comment on your results.

(c) Repeat the previous part for $x = 40$. What is going wrong?