CS 561, HW 1

Ryan Scherbarth, University of New Mexico

1. Is $3^{n+1} = O(3^n)$? Is $3^{2n} = O(3^n)$?

$$(a)$$

Show that: $3^{n+1} \leq c * 3^n$ for all $n, c > 0$
Base Case, $n = 0$:

$$3^{(0)+1} \leq c * 3^{(0)}$$
$$3 \leq c * 1$$
$$c = 3$$

Inductive Hypothesis: $3^{j+1} \leq c * 3^j$ for all $j > 0$

$$3^{(j+1)+1} \leq c * 3^{(j+1)}$$
$$3 * 3^{j+1} \leq c * 3^{j+1}$$
$$3 \leq c$$

Therefore, with proof by induction we have shown that

$$\boxed{f(n) = 3^{n+1} \textbf{ is bounded by } O(3^n)\ \forall (n, c) > 0\textbf{.}}$$

$$(b)$$

Show that: $3^{2n} \leq c * 3^n$ for all $n, c > 0$

$$3^{2n} = (3^n)^2$$
$$(3^n)^2 \leq c * 3^n$$
$$3^n \leq c$$

Since C can not be simplified to a constant value,

$$\boxed{3^{2n} \neq O(3^n)}$$

2. Prove that $\log n! = \Theta(n \log n)$ and that $n! = \omega(2^n)$ and $n! = o(n^n)$

(a)

Show that $\log(n!) = O(n\log(n))$

$$\log(n!) = \sum_{i=1}^{n} \log(i)$$

$$\sum_{i=1}^{n} \log(i) \leq n\log(n), \forall i \leq n$$

$$\log(n!) = O(n\log(n))$$

Now, show that $\log(n!) = \Omega(n\log(n))$

$$\log(n!) = \sum_{i=1}^{n} \log(i)$$

$$= \int_{1}^{n} \log(x)dx$$

$$= n\log(n) - n - (1log(1) - 1)$$

$$\sum_{i=1}^{n} \log(i) \geq n\log(n) - n + 1$$

$$\log(n!) \geq n\log(n) - n + 1$$

$$\log(n!) = \Omega(n\log(n))$$

$$\boxed{\Omega(n\log(n)) = O(n\log(n)) \equiv \Theta(n\log(n))}$$

(b)

Prove that $n! = \omega(2^n)$
Show that For any positive c,
there exists a positive n_0 s.t. $0 \leq c * 2^n < n!$ for all $n \geq n_0$.
The inequality holds iff;

$$\frac{n!}{2^n} > 1$$

To find our n_0, we can set the inequality $\frac{n!}{2^n} > 1$.

$$\frac{n!}{2^n} = \Pi_{i=1}^{n} \frac{i}{2}$$

Therefore, the inequality is increasing for all $n \geq 2$, Therefore;

$$\boxed{\textbf{For } n_0 = 2\textbf{, } n! = \omega(2^n)}$$

(c)

Prove $n! = o(n^n)$
Show that for any positive c,
there exists a positive n_0 s.t. $0 \leq n! < c * n^n$ for all $n \geq n_0$.
The inequality holds iff;

$$\frac{n!}{n^n} < c * \frac{n^n}{n^n}$$

$$\frac{n!}{n^n} < 1$$

To find our n_0, we can set the inequality $\frac{n!}{n^n} > 1$.
Since $\frac{n!}{n^n} \leq 1$ for $n > 1$, we can see that n^n is increasing asymptotically quicker than $n!$ for any point greater than $n = 1$, therefore;

$$\boxed{\textbf{For } n_0 = 1\textbf{, } n! = o(n^n)}$$

3. Problem 3-3 (Ordering by Asymptotic Growth Rates)

$$2^{2^{n+1}} > 2^{2^n} > (n+1)! > n! > e^n > n*2^n > 2^n > (\frac{3}{2})^n > n^{lg(lg(n))} \geq ...$$

$$(lg(n))^{lg(n)} > (lg(n))! > n^3 > 4^{lg(n)} \geq n^2 > nlg(n) \geq 2^{lg(n)} > ...$$

$$(\sqrt{(2)})^{lg(n)} > 2^{\sqrt{2lg(n)}} > lg^2(n) > ln(n) > \sqrt{lg(n)} > ln(ln(n)) > ...$$

$$2^{lg*n} > lg*(lg(n)) \geq lg*n > lg(lg*n) > 1 \geq n^{\frac{1}{lg(n)}}$$

4. Assume you have functions f and g, such that $f(n)$ is $O(g(n))$. For each of the following statements, decide whether you think it is true or false and give either a proof or a counterexample

(a) $\log_2 f(n)$ is $O(\log_2 g(n))$
Prove that there exists some $c > 0$ s.t. $f(n) \leq c * g(n)$ for all $n \geq 0$.

$$f(n) \leq C * g(n)$$

$$log(f(n)) \leq log(C * g(n))$$

$$log(f(n)) \leq log(C) + log(g(n))$$

$$log(f(n)) \leq log(g(n))$$

Therefore;

$$\boxed{log(f(n)) = O(log(g(n))}$$

(b) $2^{f(n)}$ is $O(2^{g(n)})$
Let $f(n) = \log(n)$, and $g(n) = \sqrt{n}$;

$$2^{\log(n)} \leq 2^{c*\sqrt{n}}$$

$$2^{f(n)} = n$$

$$2^{g(n)} = 2^{\sqrt{n}}$$

Therefore;

$$\boxed{2^{f(n)} \neq O(2^{g(n)})}$$

(c) $f(n)^2$ is $O(g(n)^2)$

$$f(n)^2 \leq (c * g(n))^2$$

$$f(n)^2 \leq c^2 * g(n)^2$$

Therefore,

$$\boxed{f(n)^2 = O(g(n)^2)}$$