CS/MATH 375, Fall 2024 — HOMEWORK # 12
Due : Nov. 22nd at 10:00pm on Canvas

Instructions

- **Report:** In general, your report needs to read coherently. That is, start off by answering question 1. Fully answer the question, and provide all the information needed to understand your answer. If Matlab code or output is part of the question, include that code or output (e.g., screenshot) alongside your narrative answer. If discussion is required for a question, include that. Overall, your report is your narrative explanation of what was done, your answers to the specific questions, and how you arrived at your answers. *Your report should include your Matlab scripts, code output, and any figures.*

- **What to hand in:** Submission must be one **single PDF** document, containing your entire report, submitted on Canvas.

- **Partners:** You are allowed to (even encouraged) to **work in pairs**. If you work with a partner, only one member of the group should need to submit a report . On Ca nvas, both partners should join a group (numbered 1 through 15). Then either member can upload the report for the entire group. Groups of more than 2 students are not allowed.

- **Typesetting:** If you write your answers by hand, then make sure that your handwriting is readable. Otherwise, I cannot grade it.

- **Plots:** All plots/figures in the report must be generated in Matlab or Python and not hand drawn (unless otherwise specified in the homework question).

  In general, make sure to (1) title figures, (2) label both axes, (3) make the curves nice and thick to be easily readable, and (4) include a legend for the plotted data sets. The font-size of all text in your figures must be large and easily readable.

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1. Accuracy of the trapezoid rule

   (a) Calculate how many points are needed to ensure the composite Trapezoid rule is accurate to 10^{-6} for
   $$I = \int_0^{\pi} x \sin(x)\, dx.$$

   (b) Write a MATLAB function `integral = comp_trap_int(f,a,b,n)` where `f` is the function to be integrated, `a`and `b` are the limits of integration, `n` is the number of intervals and `integral` is the approximation of the integral. This function approximates the definite integral $\int_a^b f(x)\,dx$ using `n` intervals for the composite trapezoid rule.

(c) Using your MATLAB function above, approximate the definite integral

$$\int_0^\pi x\sin(x)\,dx$$

using $n = 4, 8, 16, 32$ intervals. Tabulate your results by showing three columns: the value of the approximate integral computed by the composite Trapezoid rule, the error, and the observed order of convergence p for each value of n. Recall that the order of convergence p is computed as

$$p = \frac{\log(err^{(n)}/err^{(n-1)})}{\log(h^{(n)}/h^{(n-1)})}$$

where $err^{(n)}$ is the error for n intervals and $h^{(n)} = (b-a)/n$ is the sub-interval length.

Do you see the expected convergence? If not, try to explain your results.

(d) Approximate (by hand) the above integral using the **composite Simpson** rule with 2 intervals of equal length (take $n = 4$, which means 5 points). How many evaluations of the function $f(x)$ are required? How large is your error and how does this compare to using the composite trapezoid method?

2. Degree of precision (the other accuracy!).
We say that an integration rule has degree of precision p, if the rule can exactly integrate all polynomials of up to degree p exactly.

(a) Show that Simpson's rule

$$\int_a^b f(x)dx \approx \frac{b-a}{6}(f(a) + 4f((a+b)/2) + f(b))$$

is exact for $f(x) = 1$, $f(x) = x$, $f(x) = x^2$.

This can be shown directly, or via the fact that Simpson's rule is constructed using an interpolating polynomial for $f(x)$.

(b) Show that Simpson's rule is also exact for $f(x) = x^3$. This can be shown directly.

(c) Show that Simpson's rule is a linear operator. That is, let $S(f(x))$ represent one application of Simpson's rule to the function $f(x)$ in order to approximate the integral $\int_a^b f(x)dx$,, i.e.,

$$\int_a^b f(x)dx \approx S(f(x)) = \frac{b-a}{6}(f(a) + 4f((a+b)/2) + f(b)).$$

Show that $S(\cdot)$ is a linear operator, i.e., that

$$S(\alpha f(x) + \beta g(x)) = \alpha S(f(x)) + \beta S(g(x)),$$

for functions $f(x)$ and $g(x)$ and scalars α and β.

(d) Using the previous three facts, show that Simpson's rule is exact for all polynomials up to degree 3.

For your proof, you will likely want to use the linearity of Simpson's rule (proved above), and the well-known linearity of integration, i.e., that

$$\int_a^b \alpha f(x) + \beta g(x) dx = \alpha \int_a^b f(x) dx \; + \; \beta \int_a^b g(x) dx,$$

for integrable functions $f(x)$ and $g(x)$ and scalars α and β.