Asymptotic flocking for the three-zone model

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Abstract

We prove the asymptotic flocking behavior for a general system of swarming agents. The model describes particles interacting with three types of behavior: repulsion, alignment, and attraction. We refer to this dynamics as the three-zone model. Our result expands the analysis of the so-called Cucker-Smale model, where only the alignment rule is taken into account. Whereas in the Cucker-Smale model, the alignment should be strong enough at long distances to ensure flocking behavior, here we only require that the attraction is described by a *confinement* potential. The key for the proof is to use that the dynamics is dissipative due to the alignment term which plays the role of a friction term. Several numerical examples illustrate the result and we also extend the proof for the kinetic equation associated with the three-zone dynamics.

1 Introduction

Flocking behavior is an intriguing phenomenon observed in nature. It remains an open question how birds or fish are able to organize on several scales to form a coherent motion. Modeling has proved to be crucial in highlighting how such complex behaviors can be generated from simple interaction rules. Among the different models proposed, the three-zone model has been particularly popular in biology [2, 14, 27, 30, 33]. In the three-zone model, agents representing birds or fish engage in three types of interactions: repulsion, alignment, and attraction, depending on whether their neighbor is at a short, intermediate, or long distance, respectively. The goal of this manuscript is to provide sufficient conditions for the convergence of such a system towards a *flock*, which occurs when all agents approach a common velocity.

Analytical studies of flocking dynamics have mainly been inspired by the seminal work of Cucker and Smale [15, 16]. In their research, they studied a simplified version of the so-called Vicsek model [36], where only the alignment force between agents is considered. They proved rigorously the convergence of the dynamics to a flock, given the condition that the alignment force is sufficiently *strong*. This work has been followed by many generalizations [1, 31, 32] and improvements [10, 24, 25].

One key element to prove the convergence of the Cucker-Smale model to a flock is the decay of the *kinetic energy* of the system due to the alignment rule, which acts as a source of friction. In the present manuscript, the dynamics combine both attraction-repulsion and alignment, therefore we have to define the energy of the system as the sum of its *kinetic energy* and *potential energy*. The attraction-repulsion term does not modify the total energy since it is Hamiltonian dynamics. However, the alignment term causes the kinetic energy and consequently the total energy of the system to decay with respect to time. As a result, if the attraction-repulsion force is such that the particle configuration remains spatially bounded, the influence of the alignment term will guarantee the system converges to a flock. Notice that we cannot draw any conclusions about the spatial organization of the flock, but many analytic and numerical studies have been conducted on this problem [9, 12, 38].

Since our proof is mainly based on an energy functional, it is possible to extend the method for the kinetic equation associated with the three-zone model. However, we have to define a weaker notion of flocking for the kinetic equation. There is additional difficulty in dealing with a kinetic equation: since we are working with a continuum distribution, there is no longer a maximum distance between two agents. Despite these obstacles, we manage to prove a L^1 type convergence result by applying stochastic process theory.

So far, the sufficient condition for flocking requires that the attraction term is given by a confinement potential. However, this condition can be weakened by incorporating the effect of the alignment in the non-spatial dispersion of the agents. Here, the effects of attraction and alignment are treated separately. Using commutator techniques, it might be possible to improve the sufficient condition for flocking and/or to find a joint condition on the strength of attraction and alignment at large distances. Other perspectives will be to extend the proof for other types of interactions, such as non-symmetric interactions [28, 31, 32], or using the so-called topological distance [5, 6, 26].

The paper is organized as followed: in section 2, we introduce the three-zone model for agent-based dynamics. We prove the main result by finding a sufficient condition for the emergence of a flock and give several numerical illustrations. In section 3, we extend the result for the kinetic equation associated with the agent-based model. Finally, we draw a conclusions in section 4.

2 Agent-based models

2.1 Three zones model

We consider the three-zone model, which describes agents moving according to three rules of interaction: repulsion (at short distance), alignment and attraction (long distance). A schematic representation of the model is given in figure 1. Each agent i is represented by a vector position $\mathbf{x}_i$ and a velocity $\mathbf{v}_i$ both belonging to $\mathbb{R}^d$ (with $d = 2$ or 3). The evolution of the N agents is governed by the following system:

$$\dot{\mathbf{x}}_i = \mathbf{v}_i \tag{1}$$

$$\dot{\mathbf{v}}_i = \frac{1}{N}\sum_{j=1}^{N} \phi_{ij}(\mathbf{v}_j - \mathbf{v}_i) - \frac{1}{N}\sum_{j\neq i} \nabla_{\mathbf{x}_i} V(|\mathbf{x}_j - \mathbf{x}_i|). \tag{2}$$

Here, $\phi_{ij} = \phi(|\mathbf{x}_j - \mathbf{x}_i|)$ represents the strength of the alignment between agents i and j. We suppose that the function ϕ is strictly positive. Similarly,

$\nabla_{\mathbf{x}_i} V(|\mathbf{x}_j - \mathbf{x}_i|)$ represents the attraction or repulsion of agent j on i. Indeed, developing the gradient gives:

$$-\nabla_{\mathbf{x}_i} V(|\mathbf{x}_j - \mathbf{x}_i|) = V'(|\mathbf{x}_j - \mathbf{x}_i|)\frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|}.$$

Thus, agent i is attracted to agent j if $V' > 0$ and repulsed if $V' < 0$. In fig. 1, we illustrate two possible choices for ϕ and V.

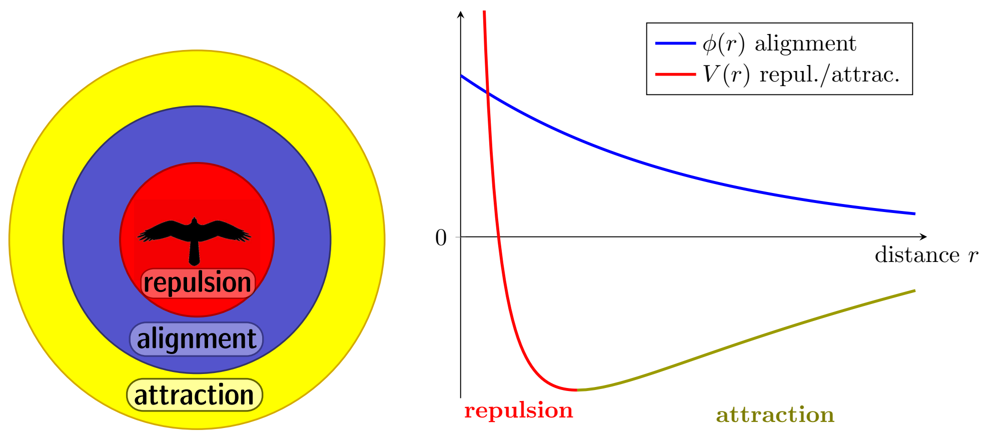

Figure 1: Left: Illustration of the three-zone model. The model includes three type of behavior: attraction/alignment/repulsion. **Right:** attraction and repulsion are represented through the function V, alignment is described via ϕ.

2.2 Flocking: rigorous results

The goal of this section is to prove conditions guaranteeing that the three-zone model (1)(2) converges to a *flock*.

Definition 1 *We say that a configuration $\{\mathbf{x}_i, \mathbf{v}_i\}_i$ converges to a flock if the following are satisfied:*

1) *There exists $\mathbf{v}_\infty$ such that $\mathbf{v}_i \xrightarrow{t\to\infty} \mathbf{v}_\infty$ for all $i = 1, \ldots, N$.*

2) *There exists M such that $|\mathbf{x}_j - \mathbf{x}_i| \leq M$ for all $i, j = 1, \ldots, N$ and for all $t \geq 0$.*

In other words, in order to achieve a flock, agents should converge to a common velocity $\mathbf{v}_\infty$ and the distance between agents should remain (uniformly) bounded in time.

The key quantity for studying the emergence of a flock is the energy function, defined below:

$$\mathcal{E}(\{\mathbf{x}_i, \mathbf{v}_i\}_i) = \frac{1}{2N}\sum_{i=1}^{N}|\mathbf{v}_i|^2 + \frac{1}{2N^2}\sum_{i,j,i\neq j}^{N} V(|\mathbf{x}_j - \mathbf{x}_i|). \tag{3}$$

We can interpret this value as a sum of the kinetic and potential energy of the system.

By itself, the attraction-repulsion term (1)-(2) describes a Hamiltonian system and therefore preserves the total energy $\mathcal{E}$. However, the alignment term causes the total energy to decay with respect to time. That is, it plays the role of a 'friction term', making the system dissipative. More precisely, we can estimate the decay rate of the energy $\mathcal{E}$.

Lemma 2.1 *Let $\{\mathbf{x}_i, \mathbf{v}_i\}_i$ be the solution of the N-bird system* (1)(2)*. Then the energy $\mathcal{E}$* (3) *satisfies:*

$$\frac{d}{dt}\mathcal{E}(\{\mathbf{x}_i, \mathbf{v}_i\}_i) = -\frac{1}{2N^2}\sum_{i,j=1}^{N}\phi_{ij}|\mathbf{v}_j - \mathbf{v}_i|^2. \tag{4}$$

Since ϕ is a positive function, the energy $\mathcal{E}$ is decaying along the solution trajectory.

Proof. Taking the derivative in time of the energy leads to:

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(\{\mathbf{x}_i, \mathbf{v}_i\}_i) &= \frac{1}{N}\sum_{i=1}^{N}\dot{\mathbf{v}}_i\cdot\mathbf{v}_i + \frac{1}{2N^2}\sum_{i,j,i\neq j}^{N}\nabla_{\mathbf{x}_i}V(|\mathbf{x}_j - \mathbf{x}_i|)\cdot(\mathbf{v}_j - \mathbf{v}_i) \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j,j\neq i}^{N}\Big(\nabla_{\mathbf{x}_i}V(|\mathbf{x}_j - \mathbf{x}_i|)\cdot\mathbf{v}_i + \phi_{ij}(\mathbf{v}_j - \mathbf{v}_i)\cdot\mathbf{v}_i\Big) \\
&\quad + \frac{1}{2N^2}\sum_{i,j,i\neq j}^{N}\nabla_{\mathbf{x}_i}V(|\mathbf{x}_j - \mathbf{x}_i|)\cdot(\mathbf{v}_j - \mathbf{v}_i).
\end{aligned}$$

By an argument of symmetry, we find:

$$\begin{aligned}\sum_{i,j,i\neq j}^{N} \nabla_{\mathbf{x}_i} V(|\mathbf{x}_j-\mathbf{x}_i|)\cdot \mathbf{v}_i &= \sum_{i,j,i\neq j}^{N} V'(|\mathbf{x}_j-\mathbf{x}_i|)\frac{\mathbf{x}_j-\mathbf{x}_i}{|\mathbf{x}_j-\mathbf{x}_i|}\cdot \mathbf{v}_i \\ &= -\sum_{i,j,i\neq j}^{N} V'(|\mathbf{x}_j-\mathbf{x}_i|)\frac{\mathbf{x}_j-\mathbf{x}_i}{|\mathbf{x}_j-\mathbf{x}_i|}\cdot \mathbf{v}_j \\ &= \frac{1}{2}\sum_{i,j,i\neq j}^{N} V'(|\mathbf{x}_j-\mathbf{x}_i|)\frac{\mathbf{x}_j-\mathbf{x}_i}{|\mathbf{x}_j-\mathbf{x}_i|}\cdot (\mathbf{v}_i-\mathbf{v}_j).\end{aligned}$$

Therefore, we can simplify:

$$\frac{d}{dt}\mathcal{E}(\{\mathbf{x}_i,\mathbf{v}_i\}_i) = \frac{1}{N^2}\sum_{i\neq j}\phi_{ij}(\mathbf{v}_j-\mathbf{v}_i)\cdot \mathbf{v}_i.$$

Using now the symmetry $\phi_{ij}=\phi_{ji}$, we conclude

$$\frac{d}{dt}\mathcal{E} = \frac{1}{2N^2}\sum_{i,j=1}^{N}\phi_{ij}(\mathbf{v}_j-\mathbf{v}_i)\cdot(\mathbf{v}_i-\mathbf{v}_j) = -\frac{1}{2N^2}\sum_{i,j=1}^{N}\phi_{ij}|\mathbf{v}_j-\mathbf{v}_i|^2.$$

□

Since the energy $\mathcal{E}$ is decaying, we deduce that the *potential energy* is bounded uniformly.

Lemma 2.2 *Take $\{\mathbf{x}_i,\mathbf{v}_i\}_i$ to be a solution of the three-zone model* (1)(2)*. There exists C such that for any time $t\geq 0$ and i, j:*

$$\sum_{i,j,i\neq j}^{N} V(|\mathbf{x}_j(t)-\mathbf{x}_i(t)|) \quad \leq \quad C. \tag{5}$$

Proof. Take $C_0=\mathcal{E}(\{\mathbf{x}_i(0),\mathbf{v}_i(0)\}_i)$. Since $\mathcal{E}$ is decaying along the solution trajectory, we deduce that for all t:

$$\frac{1}{2N}\sum_{i=1}^{N}|\mathbf{v}_i(t)|^2 + \frac{1}{2N^2}\sum_{i,j,i\neq j}^{N} V(|\mathbf{x}_j(t)-\mathbf{x}_i(t)|) \quad \leq \quad C_0,$$

Since the kinetic energy $\frac{1}{2}\sum_{i=1}^{N}|\mathbf{v}_i|^2$ is always positive, we deduce:

$$\sum_{i,j,i\neq j}^{N} V(|\mathbf{x}_j(t)-\mathbf{x}_i(t)|) \quad \leq \quad 2C_0N^2.$$

Taking $C = 2C_0N^2$ yields the result. $\square$

To take advantage of the lemma 2.2, we suppose that V is a *confinement* potential [28]:

$$V(r) \stackrel{r\to+\infty}{\longrightarrow} +\infty. \tag{6}$$

Roughly speaking, agents still experience an attraction force at long distances.

Under this assumption, we can prove the first part of flocking behavior, namely that the distances between agents are bounded.

Lemma 2.3 *Suppose V satisfies* (6)*. Then there exists r_M such that:*

$$|\mathbf{x}_j(t)-\mathbf{x}_i(t)| \leq r_M \qquad \textit{for any } i,j, \textit{ and } t \geq 0. \tag{7}$$

Proof. From lemma 2.2, we know that the potential energy is bounded, in particular:

$$V(|\mathbf{x}_j(t)-\mathbf{x}_i(t)|) \quad \leq \quad C,$$

Since V satisfies (6), we deduce that there exists r_M such that:

$$V(r) > C \qquad \text{if} \quad r > r_M.$$

Since $V(|\mathbf{x}_j(t)-\mathbf{x}_i(t)|)$ is bounded by C, we conclude that $|\mathbf{x}_j(t)-\mathbf{x}_i(t)|$ is bounded by r_M. $\square$

Remark 2.4 *Similarly, if we suppose that V diverges at $r=0$, then there exists a minimal distance r_m between agents:*

$$|\mathbf{x}_j(t)-\mathbf{x}_i(t)| \geq r_m.$$

We can now conclude by deriving sufficient conditions to guarantee flocking behavior for the three-zone model.

Theorem 1 *Suppose V satisfies* (6) *and ϕ is strictly positive. Then the three-zone model converges to a flock.*

Proof. Using lemma 2.3, we know that the distance between agents remains bounded: $|\mathbf{x}_j(t) - \mathbf{x}_i(t)| \leq r_M$. Since r_M is finite, we can take the minimum of ϕ on this interval:

$$m = \min_{s \in [0, r_M]} \phi(s).$$

Since ϕ is strictly positive, we deduce that $m > 0$. Therefore,

$$\phi_{ij} = \phi(|\mathbf{x}_j(t) - \mathbf{x}_i(t)|) \geq m > 0.$$

Since the energy $\mathcal{E}(\{\mathbf{x}_i, \mathbf{v}_i\}_i)$ is decaying and bounded from below we deduce that $\frac{d}{dt}\mathcal{E}(\{\mathbf{x}_i, \mathbf{v}_i\}_i) \stackrel{t\to\infty}{\longrightarrow} 0$ ($\frac{d}{dt}\mathcal{E}$ being uniformly continuous). Therefore, using lemma 2.1,

$$\phi_{ij}|\mathbf{v}_j - \mathbf{v}_i|^2 \stackrel{t\to+\infty}{\longrightarrow} 0.$$

Since $\phi_{ij} \geq m > 0$, we conclude that $|\mathbf{v}_j - \mathbf{v}_i| \stackrel{t\to\infty}{\longrightarrow} 0$.

Moreover, the mean velocity, $\bar{\mathbf{v}} = \frac{1}{N}\sum_i \mathbf{v}_i$, is preserved by the dynamics (by symmetry), thus

$$\mathbf{v}_i(t) \stackrel{t\to\infty}{\longrightarrow} \bar{\mathbf{v}} \qquad \text{for all } i$$

which concludes the proof. □

2.3 Numerical investigation

To illustrate the theorem 1, we perform numerical experiments for various choices of attraction-repulsion potentials. The theorem 1 guarantees that the velocities of the agents will converge to a single value, but there is no information about their position and in particular their relative distance. It is an open question to predict what *shape* the flock will have. All we know is that the distance between agents is bounded uniformly in time, which leaves the door open for many possible scenarios. Several studies have examined the stability of particular equilibrium solutions [9, 11, 12, 22, 29]. Models feaaturing self-propelled particles with an attraction-repulsion influence function have also been extensively studied [13, 21], and several patterns observed (e.g. flock, mill formation). In our settings, however, mill formation cannot occur as the agents' velocities will converge to same value. Of particular interest is the shape of the swarm as the number of individuals N increases.

To first illustrate the theorem 1, we choose the following functions for the three-zone model (see Fig. 2-left):

$$V(r) = r\big(\ln r - 1\big) \quad , \quad \phi(r) = \frac{1}{1+r^2}. \tag{8}$$

The potential $V(r)$ diverges at $+\infty$ (i.e. satisfies (6)) and the alignment function is strictly positive (i.e. $\phi(r) > 0$), thus we can apply the theorem 1 and deduces that the agents will always converge to a flock. Notice that the alignment function $\phi(r)$ is integrable, thus without the attraction/repulsion term $V(r)$ there is no guarantee that a flock will occur. In others words, since the three-zones model (1)-(2) reduces to the Cucker-Smale model when $V' = 0$, having ϕ integrable is not a sufficient condition to guarantee flocking.

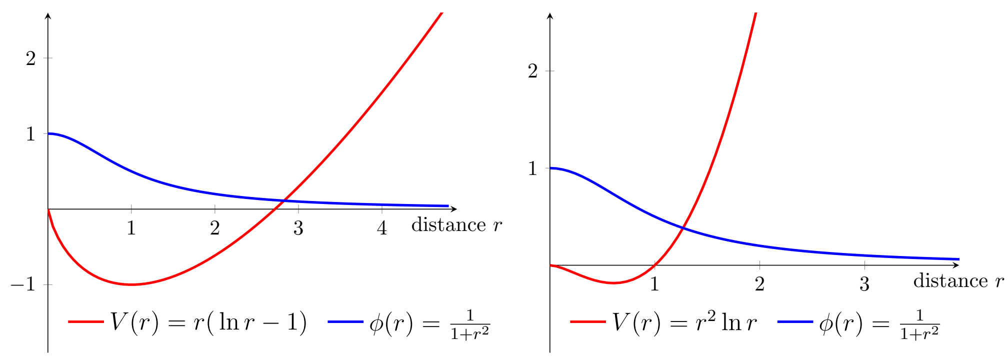

Figure 2: Attraction-repulsion V and alignment ϕ used for the simulations. In both cases, V diverges at infinity (i.e. satisfies (6)).

We use as initial condition a uniform distribution of agents on a square of size $\sqrt{N}$. Their velocity is taken from a normal distribution. In the figure 3, we plot the distribution of agents after $t = 200$ time units for four different group size: $N = 20$, 50, 100 and 1000. For each group size, the agents regroup on a disc of radius close to 2 space units and the distribution is uniform on the disc. As the number of agents N increases, the radius remains constant and therefore the average distance between particles decreases. This type of pattern has been called *catastrophic* [21, 34], since the density will eventually become singular as $N \longrightarrow +\infty$ and thus there is no *thermodynamic* limit. In other words, the repulsion is not strong enough to push back nearby agents.

In our second illustration, we reduce even further the repulsion force among agents using the following potential $V(r)$ (see Fig. 2-right):

$$V(r) = r^2 \ln r \quad , \quad \phi(r) = \frac{1}{1+r^2}. \tag{9}$$

There are now two possible equilibrium points for attraction/repulsion at the distances $r = 1$ and $r = 0$. In the figure 4, we plot the distribution of agents at time $t = 100$ for two group sizes ($N = 50$ and $N = 200$). We observe that the agents are now aggregating on a circle. This indicates that the equilibrium distribution might be a domain of dimension one [4].

The evolution of the total energy $\mathcal{E}$ (3) for all cases is given in figure 5. As predicted by lemma 2.1, the energy is always strictly decaying. Moreover, we observe oscillatory behavior between fast and slow decay. The reason for this behavior is the repreated *contraction-expansion* of the spatial configuration as the agents approach equilibrium. The energy is decaying faster when agents are closer since the alignment function ϕ is a decaying function. These types of oscillation are also observed for the convergence of Boltzmann equation toward global equilibrium [20, 23].

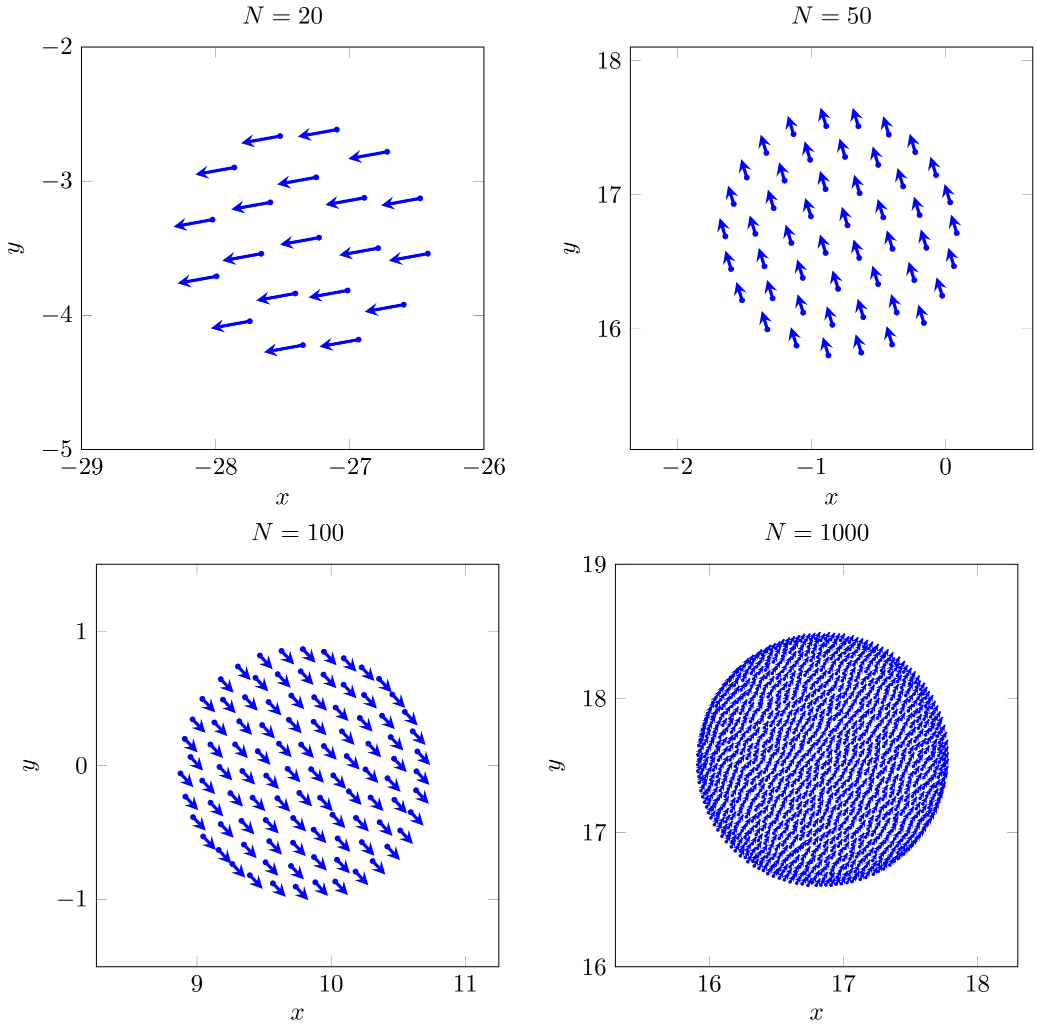

Figure 3: Simulation of the three-zone model (1)-(2) with potential V and alignment function ϕ given by (8). Agents regroup on a disc of size $R \approx 1.8$ for any group sizes. Parameters: $\Delta t = .05$, total time $T = 100$ unit time.

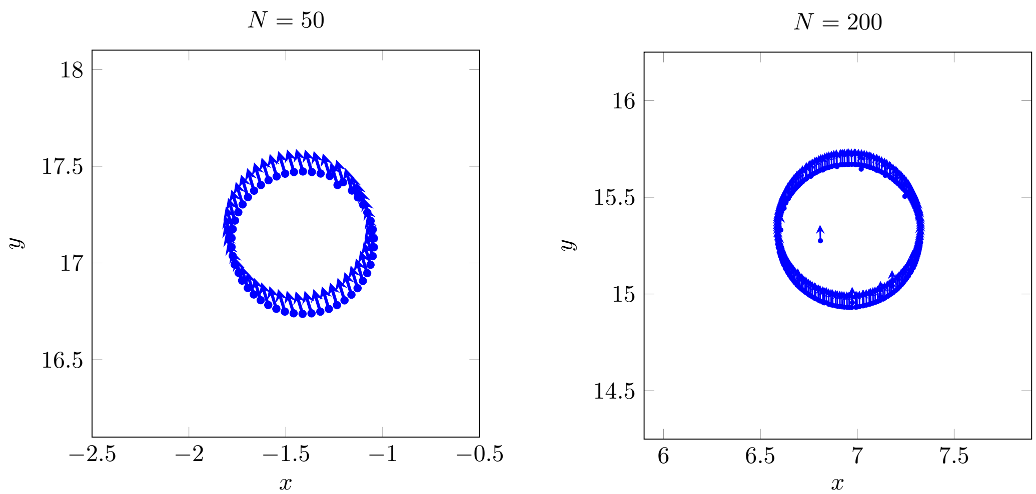

Figure 4: Simulation of the three-zone model (1)-(2) with potential V and alignment function ϕ given by (9). Agents regroup on a circle of size $R \approx .5$. Parameters: $\Delta t = .05$, total time $T = 100$ unit time.

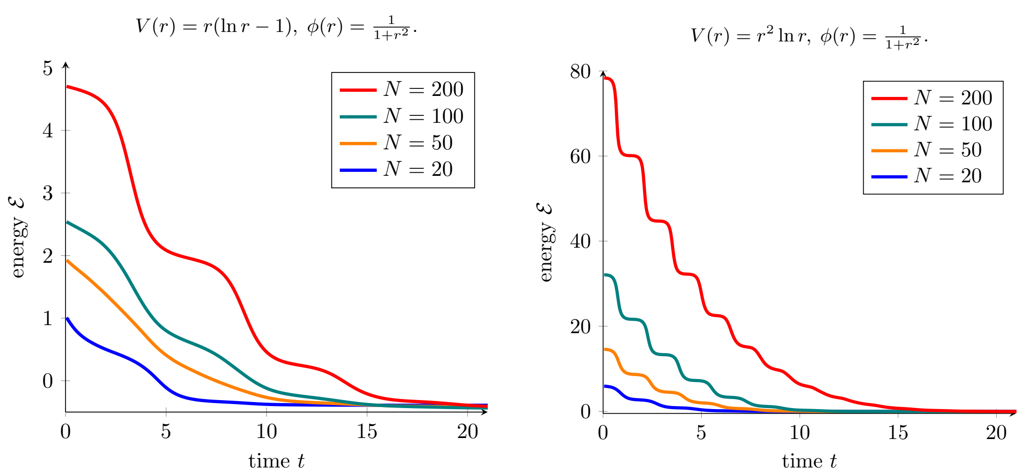

Figure 5: Evolution of the energy $\mathcal{E}$ for the solutions depicted in figures 3 and 4 (left and right figure respectively). The energy is always decaying but also oscillates between fast and slow decays. These oscillations can be explained by the successive *contraction-expansion* of the spatial configuration. The decay of the energy is faster when agents are closer to each other.

3 Kinetic equation

3.1 Formal derivation

We would like to investigate the flocking behavior of dynamics in the limit of infinitely many agents, i.e. $N \longrightarrow \infty$. With this aim, we introduce the so-called *kinetic equation* associated to *the particle dynamics* (1)-(2). To derive the kinetic equation, one can introduce the empirical distribution [7, 8, 8, 17–19]:

$$f_N(\mathbf{x}, \mathbf{v}, t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\mathbf{x}_i(t)}(\mathbf{x}) \otimes \delta_{\mathbf{v}_i(t)}(\mathbf{v}), \tag{10}$$

where $\{\mathbf{x}_i(t), \mathbf{v}_i(t)\}$ is the solution of the system (1)-(2). By integrating the empirical distribution f_N against a test function, we can show that f_N satisfies in a weak-sense the following kinetic equation:

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \Big(F[f] f \Big) = 0, \tag{11}$$

with

$$\begin{aligned} F[f](\mathbf{x}, \mathbf{v}) \;\; = \;\; & - \int_{\mathbf{y} \in \mathbb{R}^n} \nabla_{\mathbf{x}} V(|\mathbf{y} - \mathbf{x}|) \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y} \qquad (12) \\ & + \int_{(\mathbf{y}, \mathbf{w}) \in \mathbb{R}^n \times \mathbb{R}^n} \phi(|\mathbf{y} - \mathbf{x}|)(\mathbf{w} - \mathbf{v}) f(\mathbf{y}, \mathbf{w}) \, \mathrm{d}\mathbf{y} \mathrm{d}\mathbf{w}, \end{aligned}$$

and

$$\rho(\mathbf{x}) = \int_{\mathbf{v} \in \mathbb{R}^n} f(\mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v} \tag{13}$$

the spatial distribution of particles.

The rigorous convergence of the particle dynamics (1)-(2) toward the kinetic equation (11) is out of the scope of the present paper. But following the methods developed in [7,8,35], one would expect to have an error estimation between the empirical distribution f_N (10) and a 'classic' solution f to the kinetic equation (11)-(12). More precisely, for any time T, there exists a constant c such that the Wasserstein distance between the two distributions satisfies:

$$\mathcal{W}(f(T), f_N(T)) \leq \mathcal{W}(f(0), f_N(0)) \mathrm{e}^{c \cdot T}.$$

Notice that this result cannot be used to study the long time behavior of the solution of the kinetic equation $f(t)$ since the error-bound is not uniform in time.

3.2 Flocking behavior

To analyze the long-term behavior of the solution to the kinetic equation (11), we introduce the following energy function:

$$\mathcal{E} = \frac{1}{2}\int_{(\mathbf{x},\mathbf{v})\in\mathbb{R}^n\times\mathbb{R}^n} |\mathbf{v}|^2 f(\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} + \frac{1}{2}\int_{(\mathbf{x},\mathbf{y})\in\mathbb{R}^n\times\mathbb{R}^n} V(|\mathbf{y}-\mathbf{x}|)\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}. \tag{14}$$

Symmetry argument shows that the energy $\mathcal{E}$ is decaying (i.e. the system is dissipative).

Lemma 3.1 *The functional $\mathcal{E}$ satisfies:*

$$\frac{d}{dt}\mathcal{E} = -\frac{1}{2}\int_{(\mathbf{x},\mathbf{v}),(\mathbf{y},\mathbf{w})} \phi(|\mathbf{y}-\mathbf{x}|)|\mathbf{w}-\mathbf{v}|^2 f(\mathbf{x},\mathbf{v})f(\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{w}. \tag{15}$$

Proof. Taking the derivative of the energy with respect to time leads to:

$$\begin{aligned}
\frac{d}{dt}\mathcal{E} &= \frac{1}{2}\int_{\mathbf{x},\mathbf{v}} |\mathbf{v}|^2\partial_t f(\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\
&\qquad + \frac{1}{2}\int_{\mathbf{x},\mathbf{y}} V(|\mathbf{y}-\mathbf{x}|)\Big(\partial_t\rho(\mathbf{x})\rho(\mathbf{y}) + \rho(\mathbf{x})\partial_t\rho(\mathbf{y})\Big)\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&=: A + B.
\end{aligned}$$

Then,

$$\begin{aligned}
A &= -\frac{1}{2}\int_{\mathbf{x},\mathbf{v}} |\mathbf{v}|^2\nabla_{\mathbf{x}}\cdot\big(\mathbf{v}f\big)\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} - \frac{1}{2}\int_{\mathbf{x},\mathbf{v}} |\mathbf{v}|^2\nabla_{\mathbf{v}}\cdot\big(F[f]f\big)\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\
&= 0 + \int_{\mathbf{x},\mathbf{v}} \mathbf{v}\cdot F[f]f\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\
&= \int_{\mathbf{x},\mathbf{v}}\int_{\mathbf{y},\mathbf{w}} \mathbf{v}\cdot\phi(|\mathbf{y}-\mathbf{x}|)(\mathbf{w}-\mathbf{v})f(\mathbf{x},\mathbf{v})f(\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w} \\
&\qquad - \int_{\mathbf{x},\mathbf{v}}\int_{\mathbf{v}} \mathbf{v}\cdot\nabla_{\mathbf{x}}V(|\mathbf{y}-\mathbf{x}|)\rho(\mathbf{y})f(\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v} \\
&= \int_{\mathbf{x},\mathbf{v}}\int_{\mathbf{y},\mathbf{w}} \mathbf{v}\cdot\phi(|\mathbf{y}-\mathbf{x}|)(\mathbf{w}-\mathbf{v})f(\mathbf{x},\mathbf{v})f(\mathbf{y},\mathbf{w})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w} \\
&\qquad - \int_{\mathbf{x},\mathbf{y}} \nabla_{\mathbf{x}}V(|\mathbf{y}-\mathbf{x}|)\cdot\rho(\mathbf{x})\rho(\mathbf{y})u(\mathbf{x})\,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}.
\end{aligned}$$

To compute the term B, we notice that the density distribution ρ satisfies the continuity equation:

$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho u) = 0 \quad \text{with} \;\; \rho(\mathbf{x})u(\mathbf{x}) = \int_{\mathbf{v}} \mathbf{v} f(\mathbf{x},\mathbf{v})\, \mathrm{d}\mathbf{v}. \tag{16}$$

We deduce:

$$\begin{aligned}
B &= \frac{1}{2}\int_{\mathbf{x},\mathbf{y}} V(|\mathbf{y}-\mathbf{x}|)\Big(-\nabla_{\mathbf{x}} \cdot \big[\rho(\mathbf{x})u(\mathbf{x})\big]\; \rho(\mathbf{y}) - \rho(\mathbf{x})\; \nabla_{\mathbf{y}} \cdot \big[\rho(\mathbf{y})u(\mathbf{y})\big]\Big)\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&= \frac{1}{2}\int_{\mathbf{x},\mathbf{y}} \nabla_{\mathbf{x}}\big[V(|\mathbf{y}-\mathbf{x}|)\rho(\mathbf{y})\big] \cdot \rho(\mathbf{x})u(\mathbf{x}) + \nabla_{\mathbf{y}}\big[V(|\mathbf{y}-\mathbf{x}|)\rho(\mathbf{x})\big] \cdot \rho(\mathbf{y})u(\mathbf{y})\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&= \int_{\mathbf{x},\mathbf{y}} \nabla_{\mathbf{x}} V(|\mathbf{y}-\mathbf{x}|) \cdot \rho(\mathbf{x})\rho(\mathbf{y})u(\mathbf{x})\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y},
\end{aligned}$$

using the change of variable in the second term $\mathbf{x} \leftrightarrow \mathbf{y}$. Therefore,

$$\begin{aligned}
A + B &= \int_{\mathbf{x},\mathbf{v}} \int_{\mathbf{y}} \mathbf{v} \cdot \phi(|\mathbf{y}-\mathbf{x}|)(\mathbf{w}-\mathbf{v}) f(\mathbf{x},\mathbf{v}) f(\mathbf{y},\mathbf{w})\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w} \\
&= -\frac{1}{2}\int_{\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w}} \phi(|\mathbf{y}-\mathbf{x}|)|\mathbf{w}-\mathbf{v}|^2 f(\mathbf{x},\mathbf{v}) f(\mathbf{y},\mathbf{w})\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w}.
\end{aligned}$$

□

The decay of the energy $\mathcal{E}$ is the corner-store to prove the flocking of the dynamics. However, in the context of the kinetic equation (11), we deal with a continuum of agents and therefore it is more delicate to prove that the velocity of *all* agents converge to a common value. We prove a L^1 type estimate for the decay of the velocity toward its average value. The method relies mainly on stochastic process theory.

We denote by $(\mathbf{X}_t, \mathbf{V}_t)$ and $(\mathbf{Y}_t, \mathbf{W}_t)$ two independent stochastic process with probability density function $f(\cdot, t)$ solution to (11). The energy $\mathcal{E}$ (14) can be written as:

$$\mathcal{E} = \frac{1}{2}\mathbb{E}[|\mathbf{V}|^2] + \frac{1}{2}\mathbb{E}[V(|\mathbf{X}-\mathbf{Y}|)], \tag{17}$$

and its decay as:

$$\frac{d}{dt}\mathcal{E} = -\frac{1}{2}\mathbb{E}[\phi(|\mathbf{X}-\mathbf{Y}|)|\mathbf{V}-\mathbf{W}|^2]. \tag{18}$$

We first recall an elementary lemma in stochastic process theory that will be useful later. For the sake of completeness of the manuscript, we also give the proof.

Lemma 3.2 *Suppose* $\mathbf{X}_t$ *bounded uniformly in* L^2 *and that* $\mathbf{X}_t$ *converges in probability to* 0 *(i.e.* $\mathbf{X}_t \xrightarrow{P} 0$*). Then:* $\mathbf{X}_t \overset{t\to+\infty}{\longrightarrow} 0$ *in* L^1.

Proof. First, we show that $\mathbf{X}_t$ bounded in L^2 implies that $\mathbf{X}_t$ is uniformly integrable. Denote C the constant such that $\mathbb{E}[|\mathbf{X}_t|^2] \leq C$ for all t. We have:

$$\begin{aligned}
\mathbb{E}[|\mathbf{X}_t|\, \mathbb{1}_{\{|\mathbf{X}_t|\geq k\}}] &\leq \left(\mathbb{E}[\mathbf{X}_t^2]\, \mathbb{E}[\mathbb{1}^2_{\{|\mathbf{X}_t|\geq k\}}]\right)^{1/2} \leq C^{1/2}\mathbb{P}(|\mathbf{X}_t| \geq k)^{1/2} \\
&\leq C^{1/2}\left(\frac{1}{k^2}\mathbb{E}[\mathbf{X}_t^2]\right)^{1/2} \leq \frac{C}{k} \quad \overset{k\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

using (resp.) Cauchy-Schwarz and Markov inequalities.

We now use that $\mathbf{X}_t \xrightarrow{P} 0$ to conclude. Fix $\varepsilon > 0$:

$$\mathbb{E}[|\mathbf{X}_t|] = \mathbb{E}[|\mathbf{X}_t|\, \mathbb{1}_{\{|\mathbf{X}_t|\leq\varepsilon/2\}}] + \mathbb{E}[|\mathbf{X}_t|\mathbb{1}_{\{|\mathbf{X}_t|>\varepsilon/2\}}] \leq \frac{\varepsilon}{2} + \mathbb{E}[|\mathbf{X}_t|\mathbb{1}_{\{|\mathbf{X}_t|>\varepsilon/2\}}].$$

By uniform integrability, there exists $\delta > 0$ such that:

$$\mathbb{E}[|\mathbf{X}_t|\mathbb{1}_A] \leq \varepsilon/2 \qquad \text{if} \quad \mathbb{P}(A) \leq \delta. \tag{19}$$

Since $\mathbf{X}_t \xrightarrow{P} 0$, there exists t_* such that $\mathbb{P}(|\mathbf{X}_t| > \varepsilon/2) \leq \delta$ for $t \geq t_*$. Combined with (19), we deduce $\mathbb{E}[|\mathbf{X}_t|\mathbb{1}_{\{|\mathbf{X}_t|>\varepsilon/2\}}] \leq \varepsilon/2$. Therefore, $\mathbb{E}[|\mathbf{X}_t|] \leq \varepsilon$ for $t \geq t_*$. □

We now can prove our main theorem.

Theorem 2 *Suppose* V *satisfies* (6) *and* ϕ *is strictly positive. Then the solution* f *of* (11) *satisfies:*

$$\int_{\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w}} |\mathbf{v}-\mathbf{w}| f(\mathbf{x},\mathbf{v},t) f(\mathbf{y},\mathbf{w},t)\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{w} \overset{t\to+\infty}{\longrightarrow} 0 \tag{20}$$

Proof. Denote $(\mathbf{X}_t, \mathbf{V}_t)$ and $(\mathbf{Y}_t, \mathbf{W}_t)$ two independent stochastic processes with density distribution f (11). We first show that $\mathbf{V}_t - \mathbf{W}_t \xrightarrow{P} 0$. Fix $\delta > 0$ and $\varepsilon > 0$. We have to show that: $\mathbb{P}(|\mathbf{V}_t - \mathbf{W}_t| > \delta) < \varepsilon$ for a sufficiently large value of t.

Since the energy $\mathcal{E}$ is uniformly bounded, there exists C such that $\mathbb{E}[V(|\mathbf{X}_t - \mathbf{Y}_t|)] \leq C$ for all time t. Since the potential V satisfies (6), there exists L such that $V(r) \geq 2C/\varepsilon$ for $r > L$. Now we split our estimation:

$$\begin{aligned}
\mathbb{P}(|\mathbf{V}_t - \mathbf{W}_t| > \delta) &= \mathbb{P}(|\mathbf{V}_t - \mathbf{W}_t| > \delta,\, |\mathbf{X}_t - \mathbf{Y}_t| \leq L) \\
&\qquad + \mathbb{P}(|\mathbf{V}_t - \mathbf{W}_t| > \delta,\, |\mathbf{X}_t - \mathbf{Y}_t| > L) \\
&=: A + B.
\end{aligned}$$

First, we find an upper-bound for B:

$$\begin{aligned} B &\leq \mathbb{P}(|\mathbf{X}_t - \mathbf{Y}_t| > L) \;\leq\; \mathbb{P}(V(|\mathbf{X}_t - \mathbf{Y}_t|) > 2C/\varepsilon) \\ &\leq \frac{\varepsilon}{2C}\mathbb{E}[V(|\mathbf{X}_t - \mathbf{Y}_t|)] \;\leq\; \frac{\varepsilon}{2}. \end{aligned}$$

Then, we investigate A:

$$\begin{aligned} A &\leq \mathbb{P}\Big(|\mathbf{V}_t - \mathbf{W}_t| > \delta \,\Big|\, |\mathbf{X}_t - \mathbf{Y}_t| \leq L\Big) \\ &\leq \frac{1}{\delta^2}\mathbb{E}\Big[|\mathbf{V}_t - \mathbf{W}_t|^2 \,\Big|\, |\mathbf{X}_t - \mathbf{Y}_t| \leq L\Big] \end{aligned}$$

Consider $m = \inf_{r \leq L} \phi(r) > 0$. We have:

$$\begin{aligned} A &\leq \frac{1}{m\delta^2}\mathbb{E}\Big[\phi(|\mathbf{X}_t - \mathbf{Y}_t|)|\mathbf{V}_t - \mathbf{W}_t|^2 \,\Big|\, |\mathbf{X}_t - \mathbf{Y}_t| \leq L\Big] \\ &\leq \frac{1}{m\delta^2}\frac{d\mathcal{E}}{dt}. \end{aligned}$$

Since $\frac{d\mathcal{E}}{dt} \stackrel{t\to+\infty}{\longrightarrow} 0$, there exists t_* such that $A \leq \varepsilon/2$ for $t \geq t_*$. Therefore, we conclude:

$$\mathbb{P}(|\mathbf{V}_t - \mathbf{W}_t| > \delta) = A + B \leq \varepsilon$$

for $t \geq t_*$. Hence, $\mathbf{V}_t - \mathbf{W}_t \stackrel{P}{\longrightarrow} 0$.

Now, since the energy $\mathcal{E}$ remains uniformly bounded, we deduce that $\mathbf{V}_t - \mathbf{W}_t$ is uniformly bounded in L^2:

$$\mathbb{E}[|\mathbf{V}_t - \mathbf{W}_t|^2] \leq C.$$

Using lemma 3.2, we conclude that: $\mathbf{V}_t - \mathbf{W}_t \stackrel{t\to+\infty}{\longrightarrow} 0$ in L^1 leading to the result (20). □

4 Conclusion

In this study, we have derived sufficient conditions for the emergence of flock in a system of particles which includes attraction-repulsion and alignment interactions. The result sides with previous work on the Cucker-Smale model, but there is additional difficulty in the dynamics considered in this study, since energy estimates are insufficient to prove convergence. In particular, we do not have exponential decay towards equilibrium. However, it might be possible to obtain a stronger result, for instance by using commutator techniques [3, 37] to compensate the lack of Gronwall-type inequality.

It would also be interesting to adapt this model to other types of collective behavior, such as milling. Of course, this would require several adjustments to our starting assumptions: one could suppose that alignment only occurs at close distances (i.e. the function ϕ has a compact support). Another extension would be to consider non-metric interactions such as topological distance or non-symmetric interactions (e.g. presence of leaders).

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