Лице на равнинна фигура зададена в декартови координати

Нека функциите $f(x)$ и $g(x)$ са непрекъснати и удовлетворяват неравенството $f(x) \le g(x)$ за $\forall x \in [a, b]$. Множеството T на всички точки (x, y) от равнината, за които са изпълнени неравенствата

$$\begin{cases} a \le x \le b \\ f(x) \le y \le g(x) \end{cases} \qquad \left(\begin{array}{c} c \le y \le d \\ f(y) \le x \le g(y) \end{array} \right)$$

се нарича криволинеен трапец с вертикални (хоризонтални) основи. Неговото лице се определя по формулата

$$\mu(T) = \int_a^b [g(x) - f(x)]\,dx,$$

$$\left(\mu(T) = \int_c^d [g(y) - f(y)]\,dy \right)$$

① Намерете лицето на фигурата, заградена от параболата $y = 2x^2 - 5x + 2$ и правата $y = 4 - 2x$

Р-е: Нам. пресечните точки на двете криви

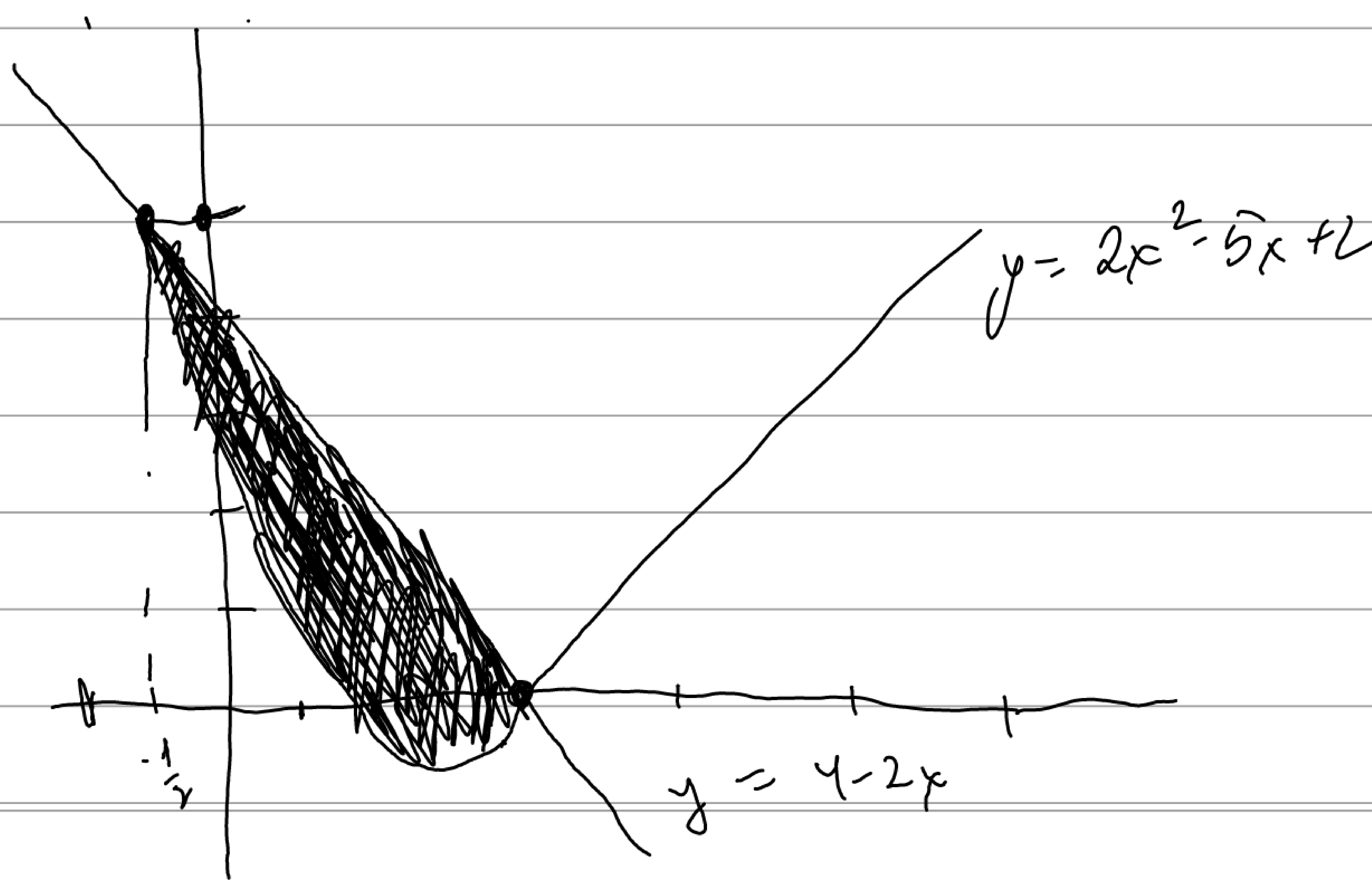
$$2x^2 - 5x + 2 = 4 - 2x$$

$$2x^2 - 3x - 2 = 0$$

$$x_{1,2} = \frac{3 \pm 5}{4} \quad x_1 = 2, \quad x_2 = -\frac{1}{2}$$

$$D = 9 + 16 = 25$$

$$(2 ; 0) \qquad \left(-\frac{1}{2} ; 5\right)$$



$$\mu(T) = \int_{-\frac{1}{2}}^{2} \left[4-2x-(2x^2-5x+2)\right] dx =$$

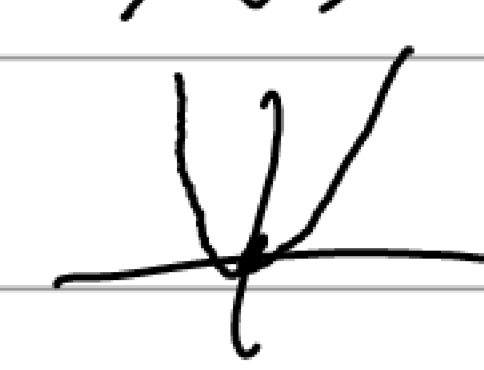
$$= \frac{125}{24}$$

$$x^3-a^3 = \boxed{(x-a)}\underbrace{(x^2+ax+a^2)}_{>0}$$

② Да се намери лицето на фигурата заградено от графиките на кривите

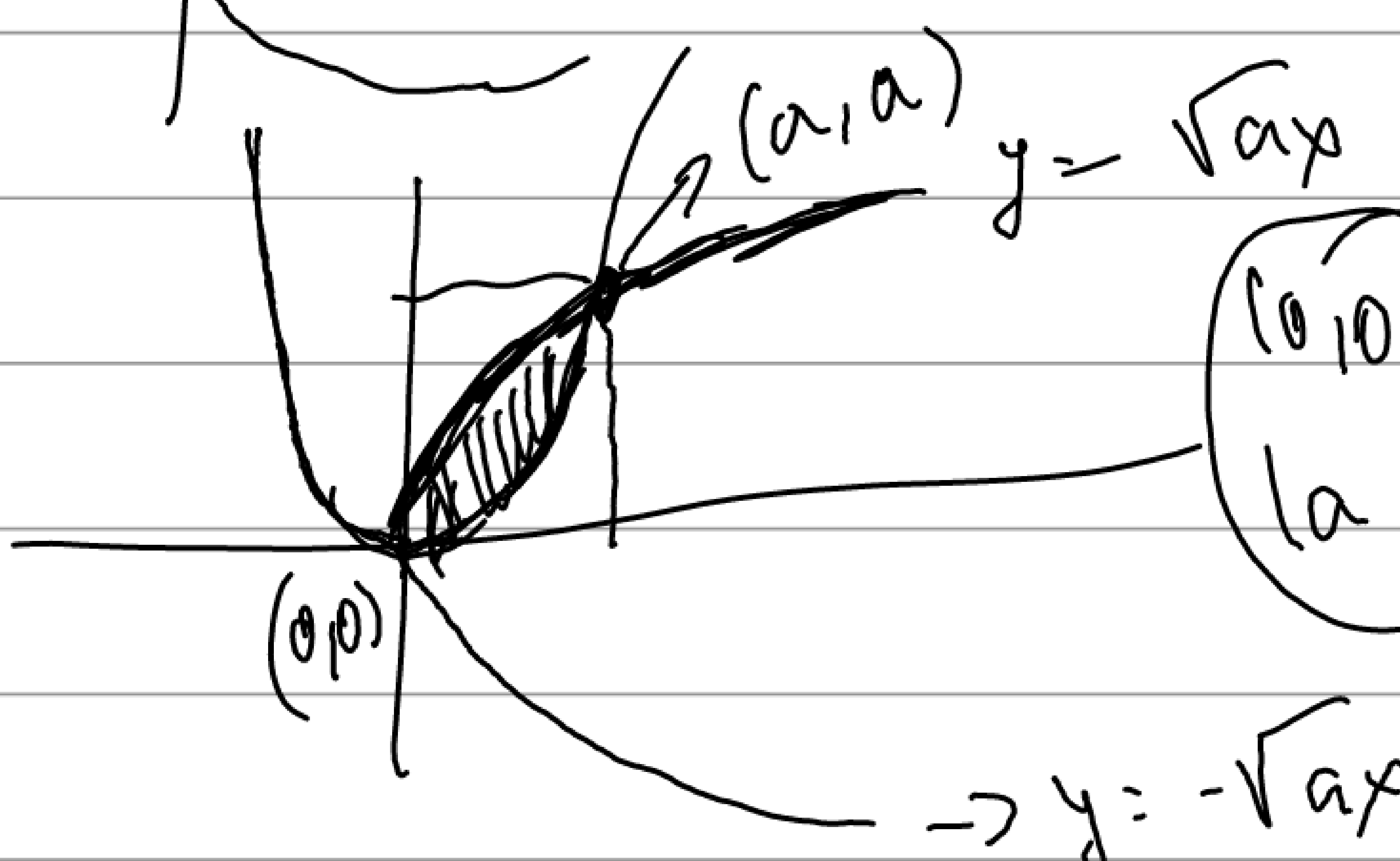
$ax = y^2$, $ay = x^2$, $(a>0)$

$x = \frac{1}{a}y^2$ $y = \frac{1}{a}x^2$



$y = \pm\sqrt{ax}$

$\sqrt{ax} = \frac{x^2}{a}$ $/^2$



(0,0)
(a,a)

$\frac{ax}{1} = \frac{x^4}{a^2}$

$a^3x = x^4$

$x^4 - a^3x = 0$

$x(x^3-a^3) = 0$

$x_1 = 0 \vee x = a$

$$\int_0^a \left(\sqrt{ax} - \frac{1}{a}x^2\right) dx = \frac{a^2}{3}$$

① $$\int_0^{+\infty} e^{-x}\, dx = \lim_{p \to +\infty} \int_0^p e^{-x}\, dx =$$

$$\boxed{-\int e^{-x}\, d(-x) = -e^{-x}\, dx}$$

$$= -\lim_{p \to +\infty} e^{-x} \Big|_0^p =$$

$$= -\lim_{p \to +\infty} e^{-p} = 1$$ -> сходится

② $$\int_0^{+\infty} \sin x\, dx = \lim_{p \to +\infty} \int_0^p \sin x\, dx =$$

$$= -\lim_{p \to +\infty} \left(\cos p - \underbrace{\cos 0}_{1}\right) = 1 - \lim_{p \to +\infty} \cos p$$

$\lim\limits_{p \to +\infty} \cos p \ \nexists \Rightarrow \int\limits_0^{+\infty} \sin x \, dx$ е разходящ

(B) $\int\limits_{-\infty}^{0} \frac{\operatorname{arctg} x}{1+x^2} dx = \lim\limits_{p \to -\infty} \int\limits_p^0 \frac{\operatorname{arctg} x}{1+x^2} dx =$

$= \lim\limits_{p \to -\infty} \int\limits_p^0 \operatorname{arctg} x \, d(\operatorname{arctg} x) =$

$= \lim\limits_{p \to -\infty} \frac{\operatorname{arctg}^2 x}{2} \Big|_p^0 =$

$= \frac{1}{2} \lim\limits_{p \to -\infty} \left(\operatorname{arctg}^2 0 - \operatorname{arctg}^2 p \right) =$

$= -\frac{1}{2} \lim\limits_{p \to -\infty} \operatorname{arctg}^2 p = -\frac{\pi^2}{8}$ → сход.

$$\int_0^1 \frac{dx}{1-x^3}$$

$$\frac{1}{1-x^3} = \frac{1}{(1-x)(1+x+x^2)} = \underbrace{\frac{\overset{x^2+x+1}{A}}{1-x} + \frac{\overset{1-x}{Mx+N}}{x^2+x+1}}_{(1-x)(x^2+x+1)} =$$

$$= \frac{Ax^2 + Ax + A + Mx - Mx^2 + N - Nx}{(1-x)(x^2+x+1)} =$$

$$= \frac{(A-M)x^2 + (A+M-N)x + (A+N)}{(1-x)(x^2+x+1)} =$$

$$= \frac{1 + 0\cdot x + 0\cdot x^2}{1-x^3}$$

$$\begin{cases} A+N = 1 \\ A+M-N = 0 \\ A-M = 0 \end{cases} \qquad \begin{aligned} A &= \tfrac{1}{3} \\ M &= \tfrac{1}{3} \\ N &= \tfrac{2}{3} \end{aligned}$$

$$\frac{1}{3}\cdot\frac{1}{1-x}+\frac{1}{3}\frac{x+2}{x^2+x+1}=\frac{1}{3}\left(\frac{1}{1-x}+\frac{x+2}{x^2+x+1}\right)$$

$$\frac{1}{3}\int_0^1\frac{1}{1-x}dx+\frac{1}{3}\int_0^1\frac{x+2}{x^2+x+1}$$ → собствен

$$\int_0^{1}\frac{1}{1-x}=\lim_{p\to 1-0}\int_0^p\frac{dx}{1-x}=+\infty$$

разходящ → $\int\frac{1}{1-x^3}dx$ е разходящ

$$\int h(x)=\int f(x)+\int g(x)$$

f – непрек. $[a,b)$ и F – прим.

$$\Rightarrow \int_a^b f(x)dx=\begin{cases}F(b-0)-F(a), & \text{ако } b\in\mathbb{R}\\ F(+\infty)-F(a), & \text{ако } b=+\infty\end{cases}$$

$F(b-0)=\lim_{x\to b-0}F(x)$ и $F(+\infty)=\lim_{x\to+\infty}F(x)$

$$\int_a^b f(x)\,dx \to$$ абсолютно сход. ако

$$\int_a^b |f(x)|\,dx \to$$ сход.

$$\int_a^b f(x)\,dx \to$$ условно сход, ако

$\int_a^b f(x)\,dx$ е сход, но

$$\int_a^b |f(x)|\,dx \to$$ разход.