Задача. За кривата $C: \vec{r}\left(u, \frac{u^2}{2}, \frac{u^3}{3}\right)$. Намерете

а) ур-нието на нормалната равнина в т. $P\,(u=2)$.

Решение:

$$\vec{t} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}, \quad \nu = (\ni \text{т. } P, \perp \vec{t}) \text{ - нормална равнина.}$$

$$\vec{r}\left(u, \frac{u^2}{2}, \frac{u^3}{3}\right) \qquad \vec{r}_{P(u=2)}\left(2, 2, \frac{8}{3}\right)$$

$$\dot{\vec{r}}\left(1, u, u^2\right) \qquad \dot{\vec{r}}_{P(u=2)}\left(1, 2, 4\right),$$

$$\Rightarrow |\dot{\vec{r}}_{P(u=2)}| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}$$

$$\Rightarrow \vec{t} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} = \frac{1}{\sqrt{21}}(1, 2, 4)$$

$$\nu: \begin{cases} \ni \text{т. } P \equiv \vec{r}_{P(u=2)}\left(2, 2, \frac{8}{3}\right) \\ \perp \vec{t} \parallel \dot{\vec{r}}(1, 2, 4) \end{cases}$$

$$\nu: n_1(x - x_u) + n_2(y - y_u) + n_3(z - z_u) = 0$$

$$\nu: 1(x-2) + 2(y-2) + 4\left(z - \frac{8}{3}\right) = 0$$

$$\Rightarrow \nu: 3x + 6y + 9z - 50 = 0$$

б) оскулачната ѝ равнина, минаваща през точката $A(0,0,9)$.

Решение:

$$\mu = (\exists \text{т. } P, \perp \vec{b}) \quad , \quad \vec{b} = \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{|\dot{\vec{r}} \times \ddot{\vec{r}}|}$$

$$\vec{r}\left(u, \frac{u^2}{2}, \frac{u^3}{3}\right)$$

$$\dot{\vec{r}}(1, u, u^2)$$

$$\ddot{\vec{r}}(0, 1, 2u)$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \left(\begin{vmatrix} u & u^2 \\ 1 & 2u \end{vmatrix} ; - \begin{vmatrix} 1 & u^2 \\ 0 & 2u \end{vmatrix} ; \begin{vmatrix} 1 & u \\ 0 & 1 \end{vmatrix} \right) = (2u^2 - u^2, -2u, 1)$$

$$\Rightarrow \dot{\vec{r}} \times \ddot{\vec{r}} = (u^2, -2u, 1)$$

$$\mu : \begin{cases} \exists \text{т. } P \equiv \vec{r}\left(u, \frac{u^2}{2}, \frac{u^3}{3}\right) \\ \perp \vec{b} \parallel \dot{\vec{r}} \times \ddot{\vec{r}} \, (u^2, -2u, 1) \end{cases}$$

$$\mu : u^2(x-u) - 2u\left(y - \frac{u^2}{2}\right) + 1.\left(z - \frac{u^3}{3}\right) = 0$$

$$\mu : u^2x - u^3 - 2uy + \frac{2u^3}{2} + z - \frac{u^3}{3} = 0$$

$$\mu : u^2x - 2uy + z - \frac{u^3}{3} = 0.$$

Сега определяме стойността на параметъра u, така че A да лежи на равнината μ.

$A(\underset{x}{0}, \underset{y}{0}, \underset{z}{9})$ заместваме в ур-нието на μ $\Rightarrow$

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$\mathcal{M}: u^2 . 0 - 2u . 0 + 9 - \frac{u^3}{3} = 0$

$\Rightarrow \boxed{u = 3}$

$\Rightarrow \mathcal{M}: u^2 x - 2uy + z - \frac{u^3}{3} = 0$

$\mathcal{M}: 3^2 x - 2.3.y + 3 - \frac{3^3}{3} = 0$

$\mathcal{M}: 9x - 6y + z - 9 = 0$

Задача. Дадена е кривата $C: \vec{r}\left(\left(\frac{3}{2}\right)^{1/2} u^2, 2-u, u^3\right)$

а) да се намери ур-нието на бинормалата в произволна точка на C.

Решение:

$\vec{r}\left(\left(\frac{3}{2}\right)^{1/2} u^2, 2-u, u^3\right)$, бинормала – $b(\ni т. P, \parallel \vec{b})$

$$\vec{b} = \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{|\dot{\vec{r}} \times \ddot{\vec{r}}|}$$

$\dot{\vec{r}}\left(\sqrt{\frac{3}{2}}.2u, -1. 3u^2\right)$

$\ddot{\vec{r}}\left(2\sqrt{\frac{3}{2}}, 0, 6u\right)$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \left(\begin{vmatrix} -1 & 3u^2 \\ 0 & 6u \end{vmatrix}; -\begin{vmatrix} 2\sqrt{\frac{3}{2}}u & 3u^2 \\ 2\sqrt{\frac{3}{2}} & 6u \end{vmatrix}; \begin{vmatrix} 2\sqrt{\frac{3}{2}}u & -1 \\ 2\sqrt{\frac{3}{2}} & 0 \end{vmatrix}\right) =$$

$$= \left(-6u; -12\sqrt{\frac{3}{2}}u^2 + 6\sqrt{\frac{3}{2}}u^2; 2\sqrt{\frac{3}{2}}\right)$$

$$= \left(-6u; -6\sqrt{\frac{3}{2}}u^2; 2\sqrt{\frac{3}{2}}\right)$$

$$\vec{b} = \frac{1}{|\dot{\vec{r}} \times \ddot{\vec{r}}|}\left(-6u; -6\sqrt{\frac{3}{2}}u^2; 2\sqrt{\frac{3}{2}}\right)$$

$$b: \begin{cases} \ni т. P \equiv \vec{r}\left(\left(\frac{3}{2}\right)^{1/2} u^2, 2-u, u^3\right) \\ \parallel \vec{b}\left(-6u, -6\sqrt{\frac{3}{2}}u^2, 2\sqrt{\frac{3}{2}}\right) \end{cases}$$

$$\Rightarrow \quad b: \frac{x - \left(\frac{3}{2}\right)^{1/2} u^2}{-6u} = \frac{y-(2-u)}{-6\sqrt{\frac{3}{2}}u^2} = \frac{z-u^3}{2\sqrt{\frac{3}{2}}}$$

б) да се намери ур-нието на ректифициращата равнина в произволна точка на C.

Решение:

$$\pi : \begin{cases} \ni т. P \equiv \vec{r}\left(\left(\frac{3}{2}\right)^{1/2} u^2, 2-u, u^3\right) \\ \perp \vec{m} \end{cases}$$

$$\vec{m} = \vec{b} \times \vec{t} \quad , \quad \vec{b} = \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{|\dot{\vec{r}} \times \ddot{\vec{r}}|} \quad , \quad \vec{t} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$$

$$\vec{b} = \frac{1}{|\dot{\vec{r}} \times \ddot{\vec{r}}|}\left(-6u, -6\sqrt{\frac{3}{2}}u^2; 2\sqrt{\frac{3}{2}}\right)$$

$$\vec{t} = \frac{1}{|\dot{\vec{r}}|}\left(2\sqrt{\frac{3}{2}}u, -1, 3u^2\right)$$

$$\vec{b} \times \vec{t} = \left(\begin{vmatrix} -6\sqrt{\frac{3}{2}}u^2 & 2\sqrt{\frac{3}{2}} \\ -1 & 3u^2 \end{vmatrix}; -\begin{vmatrix} -6u & 2\sqrt{\frac{3}{2}} \\ 2\sqrt{\frac{3}{2}}u & 3u^2 \end{vmatrix}; \begin{vmatrix} -6u & -6\sqrt{\frac{3}{2}}u^2 \\ 2\sqrt{\frac{3}{2}}u & -1 \end{vmatrix}\right) =$$

$$\vec{m} = \left(-18\sqrt{\frac{3}{2}}u^4 + 2\sqrt{\frac{3}{2}}; 18u^3 + 6u; 6u + 27u^3\right)$$

$$\pi : \left(-18\sqrt{\frac{3}{2}}u^4 + 2\sqrt{\frac{3}{2}}\right)\left(x - \left(\frac{3}{2}\right)^{1/2}u^2\right) + (18u^3 + 6u)\left(y - (2-u)\right) + (6u + 27u^3)(z - u^3) = 0$$

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в) да се намери кривината и торзията за т. $P(u=0)$

$$\varkappa = \frac{|\dot{\vec r} \times \ddot{\vec r}|}{|\dot{\vec r}|^3}, \quad \tau = \frac{\dot{\vec r}\,\ddot{\vec r}\,\dddot{\vec r}}{(\dot{\vec r} \times \ddot{\vec r})^2}$$

$$\dot{\vec r} \times \ddot{\vec r} = \left(-6u, -6\sqrt{\tfrac{3}{2}}u^2, 2\sqrt{\tfrac{3}{2}}\right) \Rightarrow \dot{\vec r} \times \ddot{\vec r}_{u=0} = \left(0; 0, 2\sqrt{\tfrac{3}{2}}\right)$$

$$\Rightarrow |\dot{\vec r} \times \ddot{\vec r}|_{u=0} = \sqrt{0^2+0^2+\left(2\sqrt{\tfrac{3}{2}}\right)^2} = \sqrt{6}$$

$$\dot{\vec r}\left(2\sqrt{\tfrac{3}{2}}u, -1, 3u^2\right) \Rightarrow \dot{\vec r}_{u=0} = (0, -1, 0)$$

$$\Rightarrow |\dot{\vec r}|_{u=0} = \sqrt{0^2+(-1)^2+0^2} = 1$$

$$\varkappa_{u=0} = \frac{|\dot{\vec r} \times \ddot{\vec r}|_{u=0}}{|\dot{\vec r}|^3_{u=0}} = \frac{\sqrt{6}}{1^3} = \underline{\underline{\sqrt{6}}}$$

$$\dot{\vec r}\left(2\sqrt{\tfrac{3}{2}}u, -1, 3u^2\right)$$

$$\ddot{\vec r}\left(2\sqrt{\tfrac{3}{2}}, 0, 6u\right)$$

$$\dddot{\vec r}\left(0, 0, 6\right)$$

$$\Rightarrow \dot{\vec r}\,\ddot{\vec r}\,\dddot{\vec r} = (\dot{\vec r} \times \ddot{\vec r})\dddot{\vec r} = 0(-6u)+0\left(-6\sqrt{\tfrac{3}{2}}u^2\right)+6.2\sqrt{\tfrac{3}{2}}$$

$$\dot{\vec r} \times \ddot{\vec r} = \left(-6u, -6\sqrt{\tfrac{3}{2}}u^2, 2\sqrt{\tfrac{3}{2}}\right) \quad \Rightarrow \dot{\vec r}\,\ddot{\vec r}\,\dddot{\vec r} = 12\sqrt{\tfrac{3}{2}}$$

$$(\dot{\vec r} \times \ddot{\vec r})^2 = |\dot{\vec r} \times \ddot{\vec r}|^2 = 6$$

$$\Rightarrow \tau = \frac{12\sqrt{\tfrac{3}{2}}}{6} = \underline{\underline{2\sqrt{\tfrac{3}{2}}}}$$

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Задача. Да се намерят кривината и торзията:

б) $C: \vec{r}(3u, 3u^2, 2u^3)$ в произволна точка и за т. $M(u=0)$.

Решение:

$$\varkappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} \quad , \quad \tilde{\tau} = \frac{\dot{\vec{r}}\,\ddot{\vec{r}}\,\dddot{\vec{r}}}{(\dot{\vec{r}} \times \ddot{\vec{r}})^2}$$

$\dot{\vec{r}}(3, 6u, 6u^2)$

$\ddot{\vec{r}}(0, 6, 12u)$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \left(\begin{vmatrix} 6u & 6u^2 \\ 6 & 12u \end{vmatrix} ; -\begin{vmatrix} 3 & 6u^2 \\ 0 & 12u \end{vmatrix} ; \begin{vmatrix} 3 & 6u \\ 0 & 6 \end{vmatrix} \right) =$$

$$= (72u^2 - 36u^2 ; -(36u) ; 18)$$

$$= (36u^2 ; -36u ; 18)$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = \sqrt{(36u^2)^2 + (-36u)^2 + 18^2} = \sqrt{1296u^4 + 1296u^2 + 324} =$$

$$= \sqrt{324(4u^4 + 4u^2 + 1)} = 18\sqrt{(2u^2+1)^2} = 18(2u^2+1)$$

$$\Rightarrow \underline{|\dot{\vec{r}} \times \ddot{\vec{r}}| = 18(2u^2+1)}$$

$$|\dot{\vec{r}}| = \sqrt{3^2 + (6u)^2 + (6u^2)^2} = \sqrt{9 + 36u^2 + 36u^4} = \sqrt{9(1+4u^2+4u^4)} =$$

$$= 3\sqrt{(2u^2+1)^2} = 3(2u^2+1)$$

$$\Rightarrow \underline{|\dot{\vec{r}}| = 3(2u^2+1)}$$

$$\varkappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} = \frac{18(2u^2+1)}{[3(2u^2+1)]^3} = \frac{18(2u^2+1)}{27(2u^2+1)^3} = \frac{2}{3(2u^2+1)^2}$$

$$\Rightarrow \varkappa = \frac{2}{3(2u^2+1)^2}$$

$\varkappa = ?$ за т. $M(u=0)$

$$\varkappa_{u=0} = \frac{2}{3(2\cdot 0^2+1)^2} = \frac{2}{3} \Rightarrow \varkappa_{u=0} = \frac{2}{3}$$

$$\tau = \frac{\dot{\vec{r}}\,\ddot{\vec{r}}\,\dddot{\vec{r}}}{(\dot{\vec{r}} \times \ddot{\vec{r}})^2} = ?$$

$\ddot{\vec{r}}(0, 6, 12u)$, $\dot{\vec{r}} \times \ddot{\vec{r}} = (36u^2, -36u, 18)$

$\dddot{\vec{r}}(0, 0, 12)$

$$\dot{\vec{r}}\,\ddot{\vec{r}}\,\dddot{\vec{r}} = (\dot{\vec{r}} \times \ddot{\vec{r}})\dddot{\vec{r}} = 36u^2 \cdot 0 + (-36u)\cdot 0 + 18\cdot 12 = 216$$

$$(\dot{\vec{r}} \times \ddot{\vec{r}})^2 = [18(2u^2+1)^2]^2 = 324(2u^2+1)^4$$

$$\tau = \frac{\dot{\vec{r}}\,\ddot{\vec{r}}\,\dddot{\vec{r}}}{(\dot{\vec{r}} \times \ddot{\vec{r}})^2} = \frac{216}{324(2u^2+1)^4} = \frac{2}{3(2u^2+1)^4}$$

$$\Rightarrow \tau = \frac{2}{3(2u^2+1)^4}$$

$\tau = ?$ за т. $M(u=0)$

$$\tau_{u=0} = \frac{2}{3(2\cdot 0^2+1)^4} = \frac{2}{3} \Rightarrow \tau_{u=0} = \frac{2}{3}$$

Задача. Дадени са две криви, дефинирани в/у интервала $[0,\pi]$, които се съединяват в координатното начало

$\vec{f}(u)=(-\sin u, -1-\cos u, 0)$, $\vec{g}(v)=(\sin v, 0, 1-\cos v)$

Изследвайте за C^1-, C^2-, G^1-, G^2- и $\mathcal{H}$-непрекъснатост в точката на съединяване.

Решение:

$\vec{f}(\pi)=(-\sin\pi, -1-\cos\pi, 0)=(0,0,0)$
$\vec{g}(0)=(\sin 0, 0, 1-\cos 0)=(0,0,0)$
$\Big\}$ $\vec{f}(\pi)=\vec{g}(0) \Rightarrow$ имаме непрекъснатост в точката на съединяване. (т. $O(0,0,0)$)

$\vec{f}(0)=(-\sin 0, -1-\cos 0, 0)=(0,-2,0)$
$\vec{g}(\pi)=(\sin\pi, 0, 1-\cos\pi)=(0,0,2)$
$\Big\}$ $\vec{f}(0)\neq\vec{g}(\pi)$.

$\Rightarrow \vec{f}(u)$ и $\vec{g}(v)$ са C^0-непрекъснати (или G^0-непр) в т. O.

$\dot{\vec{f}}(u_0)=\dot{\vec{g}}(v_0) \Rightarrow ?$

$\dot{\vec{f}}(u)=(-\cos u, \sin u, 0) \Rightarrow \dot{\vec{f}}(\pi)=(-\cos\pi, \sin\pi, 0)=(1,0,0)$

$\dot{\vec{g}}(v)=(\cos v, 0, \sin v) \Rightarrow \dot{\vec{g}}(0)=(\cos 0, 0, \sin 0)=(1,0,0)$

$\Rightarrow \dot{\vec{f}}(\pi)=\dot{\vec{g}}(0) \Rightarrow \exists\ C^1$-непрекъснатост

$\dot{\vec{f}}(u_0)\uparrow\uparrow\dot{\vec{g}}(v_0) \Rightarrow \exists\ G^1$-непрекъснатост ?

$\dot{\vec{f}}(\pi)=\lambda\dot{\vec{g}}(0) \Rightarrow (1,0,0)=\lambda(1,0,0) \Rightarrow \lambda=1 \Rightarrow \dot{\vec{f}}(\pi)\uparrow\uparrow\dot{\vec{g}}(0)$

$\Rightarrow \exists\ G^1$-непрекъснатост в т. O

$\ddot{\vec{f}}(u_0) = \ddot{\vec{g}}(v_0) \Rightarrow \exists\ C^2$-непр. ?

$\dot{\vec{f}}(u) = (-\cos u, \sin u, 0)$

$\ddot{\vec{f}}(u) = (\sin u, \cos u, 0)$, $\ddot{\vec{f}}(\pi) = (\sin\pi, \cos\pi, 0) = (0,-1,0)$

$\dot{\vec{g}}(v) = (\cos v, 0, \sin v)$

$\ddot{\vec{g}}(v) = (-\sin v, 0, \cos v)$, $\ddot{\vec{g}}(0) = (0,0,1)$

$\Rightarrow \ddot{\vec{f}}(\pi) \neq \ddot{\vec{g}}(0) \Rightarrow \nexists\ C^2$-непрекъснатост в т. O

$\ddot{\vec{f}}(u_0) - \ddot{\vec{g}}(v_0) \parallel \dot{\vec{f}}(u_0) = \dot{\vec{g}}(v_0) \Rightarrow \exists\ G^2$-непр. ?

$\ddot{\vec{f}}(\pi) - \ddot{\vec{g}}(0) = (0,-1,0) - (0,0,1) = \underline{(0,-1,-1)}$

$\dot{\vec{f}}(\pi) = \dot{\vec{g}}(0) = \underline{(1,0,0)}$

$\ddot{\vec{f}}(\pi) - \ddot{\vec{g}}(0) \neq \lambda \dot{\vec{f}}(\pi) \Rightarrow \nexists\ G^2$-непр. в т. O

$\varkappa_{f}(u_0) = \varkappa_{g}(v_0) \Rightarrow \exists\ \varkappa$-непр. ?

$$\varkappa_{f(\pi)} = \frac{|\dot{\vec{f}}(\pi) \times \ddot{\vec{f}}(\pi)|}{|\dot{\vec{f}}(\pi)|^3} = \varkappa_{g(0)} = \frac{|\dot{\vec{g}}(0) \times \ddot{\vec{g}}(0)|}{|\dot{\vec{g}}(0)|^3}$$

$\dot{\vec{f}}(\pi) = (1,0,0)$, $|\dot{\vec{f}}(\pi)| = 1$

$\ddot{\vec{f}}(\pi) = (0,-1,0)$

$\dot{\vec{f}}(\pi) \times \ddot{\vec{f}}(\pi) = \left(\begin{vmatrix} 0 & 0 \\ -1 & 0 \end{vmatrix} ; -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} ; \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \right) = (0,0,-1)$, $|\dot{\vec{f}}(\pi) \times \ddot{\vec{f}}(\pi)| = 1$

$\dot{\vec{g}}(0) = (1,0,0)$ $\quad |\dot{\vec{g}}(0)| = 1$

$\ddot{\vec{g}}(0) = (0,0,1)$

$\dot{\vec{g}}(0) \times \ddot{\vec{g}}(0) = \left(\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} ; -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} ; \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \right) = (0,-1,0)$

$$|\dot{\vec{g}}(0) \times \ddot{\vec{g}}(0)| = 1.$$

$$\left.\begin{aligned} k_f(\pi) &= \frac{|\dot{\vec{f}}(\pi) \times \ddot{\vec{f}}(\pi)|}{|\dot{\vec{f}}(\pi)|^3} = \frac{1}{1^3} = 1 \\ k_g(0) &= \frac{|\dot{\vec{g}}(0) \times \ddot{\vec{g}}(0)|}{|\dot{\vec{g}}(0)|^3} = \frac{1}{1^3} = 1 \end{aligned}\right\} \ \exists\ k\text{-непр. в т. } 0$$

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