Задача. Намерете главните линии в точка $P(u=0, v=0)$, както и гаусовата и средна кривина на повърхнината.

а) $S: \vec{r}(a\cos u\cos v,\ a\cos u\sin v,\ a\sin u)$

Решение:

$K = \frac{h}{g}$ гаусова кривина

$H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2g}$ средна кривина.

$$\left.\begin{aligned} g_{11} &= \vec{r}_u^{\,2} \\ g_{12} &= \vec{r}_u \vec{r}_v \\ g_{22} &= \vec{r}_v^{\,2} \end{aligned}\right\} \text{коефициенти на първа осн. форма.}$$

$\vec{r}_u(-a\sin u\cos v,\ -a\sin u\sin v,\ a\cos u)$

$\vec{r}_v(-a\cos u\sin v,\ a\cos u\cos v,\ 0)$

$$g_{11} = \vec{r}_u^{\,2} = (-a\sin u\cos v)^2 + (-a\sin u\sin v)^2 + (a\cos u)^2 =$$
$$= a^2\sin^2 u\cos^2 v + a^2\sin^2 u\sin^2 v + a^2\cos^2 u =$$
$$= a^2\sin^2 u\underbrace{(\cos^2 v + \sin^2 v)}_{1} + a^2\cos^2 u =$$
$$= a^2\sin^2 u + a^2\cos^2 u = \underline{a^2}$$

$$g_{12} = \vec{r}_u\vec{r}_v = -a\sin u\cos v\cdot(-a\cos u\sin v) + (-a\sin u\sin v)a\cos u\cos v + a\cos u\cdot 0$$
$$= a^2\sin u\cos v\cos u\sin v - a^2\sin u\sin v\cos u\cos v \underline{\underline{=0}}$$

$$g_{22} = \vec{r}_v^{\,2} = (-a\cos u\sin v)^2 + (a\cos u\cos v)^2 + 0^2 =$$
$$= a^2\cos^2 u\sin^2 v + a^2\cos^2 u\cos^2 v = a^2\cos^2 u\underbrace{(\sin^2 v + \cos^2 v)}_{1} =$$
$$= \underline{a^2\cos^2 u}$$

$$g = g_{11} g_{22} - g_{12}^2 = a^2 \cdot a^2 \cos^2 u - 0^2 = a^4 \cos^2 u$$

$h_{11} = \vec{N}\vec{r}_{uu}$ $\quad h_{12} = \vec{N}\vec{r}_{uv}$ $\quad h_{22} = \vec{N}\vec{r}_{vv}$

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

$\vec{r}_u(-a\sin u\cos v, -a\sin u\sin v, a\cos u)$

$\vec{r}_v(-a\cos u\sin v, a\cos u\cos v, 0)$

$$\vec{r}_u \times \vec{r}_v = \left(\begin{vmatrix} -a\sin u\sin v & a\cos u \\ a\cos u\cos v & 0 \end{vmatrix}; -\begin{vmatrix} -a\sin u\cos v & a\cos u \\ -a\cos u\sin v & 0 \end{vmatrix}; \begin{vmatrix} -a\sin u\cos v & -a\sin u\sin v \\ -a\cos u\sin v & a\cos u\cos v \end{vmatrix} \right)$$

$$= \left(-a^2\cos^2 u\cos v; -a^2\cos^2 u\sin v; -a^2\cos u\sin u\right)$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\underline{a^4\cos^4} u\cos^2 v + \underline{a^4\cos^4} u\sin^2 v + a^4\cos^2 u\sin^2 u}$$

$$= \sqrt{a^4\cos^4 u + a^4\cos^2 u\sin^2 u} = \sqrt{a^4\cos^2 u} = a^2\cos u$$

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{1}{a^2\cos u}\left(-a^2\cos^2 u\cos v; -a^2\cos^2 u\sin v; -a^2\cos u\sin u\right)$$

$\vec{N} = (-\cos u\cos v; -\cos u\sin v; -\sin u)$

$\vec{r}_{uu}(-a\cos u\cos v; -a\cos u\sin v; -a\sin u)$

$\vec{r}_{uv}(a\sin u\sin v; -a\sin u\cos v; 0)$

$\vec{r}_{vv}(-a\cos u\cos v; -a\cos u\sin v; 0)$

$h_{11} = \vec{N}\vec{r}_{uu} = \underline{a\cos^2 u}\cos^2 v + \underline{a\cos^2 u}\sin^2 v + a\sin^2 u = a\cos^2 u + a\sin^2 u = a$

$h_{12} = \vec{N}\vec{r}_{uv} = -a\sin u\cos u\sin v\cos v + a\sin u\cos u\cos v\sin v + 0(-\sin u)$ $\quad 0$

$h_{22} = \vec{N}\vec{r}_{vv} = a\cos^2 u\cos^2 v + a\cos^2 u\sin^2 v + 0.(-s\sin u) = a\cos^2 u$

$h = h_{11} h_{22} - h_{12}^2 = a \cdot a\cos^2 u - 0^2 = a^2 \cos^2 u$

$$K = \frac{h}{g} = \frac{a^2 \cos^2 u}{a^4 \cos^2 u} = \frac{1}{a^2}$$

$$H = \frac{g_{11} h_{22} - 2 g_{12} h_{12} + g_{22} h_{11}}{2g} = \frac{a^2 \cdot a\cos^2 u + a^2 \cos^2 u \cdot a}{2a^4 \cos^2 u} =$$

$$= \frac{a^3 \cos^2 u + a^3 \cos^2 u}{2a^4 \cos^2 u} = \frac{2a^3 \cos^2 u}{2a^4 \cos^2 u} = \frac{1}{a}.$$

Ур-нието на главните линии се получава, като се реши диференциалното ур-ние:

$$\begin{vmatrix} g_{11} du + g_{12} dv & g_{12} du + g_{22} dv \\ h_{11} du + h_{12} dv & h_{12} du + h_{22} dv \end{vmatrix} = 0$$

Заместваме $g_{11} = a^2$, $g_{12} = 0$, $g_{22} = a^2 \cos^2 u$

$h_{11} = a$, $h_{12} = 0$, $h_{22} = a \cos^2 u$

$$\begin{vmatrix} a^2 du + 0 dv & 0 du + a^2 \cos^2 u \, dv \\ a du + 0 dv & 0 du + a\cos^2 u \, dv \end{vmatrix} = 0$$

$a^2 du \cdot a\cos^2 u \, dv - a^3 \cos^2 u \, du \, dv = 0$

$a^3 \cos^2 u \, du \, dv - a^3 \cos^2 u \, du \, dv = 0 \quad / : a^3 \cos^2 u$

$du\,dv - du\,dv = 0 \qquad \underline{0 = 0} \quad \Rightarrow$ няма главни линии

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б) $S: \vec{r}(u\cos v, u\sin v, bv)$ $b = const > 0$

Решение:

$$K = \frac{h}{g}, \quad H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2g}$$

$\vec{r}_u(\cos v, \sin v, 0)$

$\vec{r}_v(-u\sin v, u\cos v, b)$

$g_{11} = \vec{r}_u^{\,2} = \cos^2 v + \sin^2 v + 0^2 = 1$

$g_{12} = \vec{r}_u \vec{r}_v = -u\cos v\sin v + u\cos v\sin v + b\cdot 0 = 0$

$g_{22} = \vec{r}_v^{\,2} = (-u\sin v)^2 + (u\cos v)^2 + b^2 = u^2\sin^2 v + u^2\cos^2 v + b^2 = u^2 + b^2$

$h_{11} = \vec{N}\vec{r}_{uu}$ $\quad h_{12} = \vec{N}\vec{r}_{uv}$ $\quad h_{22} = \vec{N}\vec{r}_{vv}$

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = ?$$

$$\vec{r}_u \times \vec{r}_v = \left(\begin{vmatrix} \sin v & 0 \\ u\cos v & b \end{vmatrix}; -\begin{vmatrix} \cos v & 0 \\ -u\sin v & b \end{vmatrix}; \begin{vmatrix} \cos v & \sin v \\ -u\sin v & u\cos v \end{vmatrix} \right) =$$

$$= (b\sin v; -b\cos v; \underbrace{u\cos^2 v + u\sin^2 v}_{= u})$$

$\Rightarrow \vec{r}_u \times \vec{r}_v = (b\sin v; -b\cos v, u)$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{(b\sin v)^2 + (-b\cos v)^2 + u^2} = \sqrt{b^2\sin^2 v + b^2\cos^2 v + u^2} =$

$= \sqrt{b^2\underbrace{(\sin^2 v + \cos^2 v)}_{1} + u^2} = \sqrt{u^2 + b^2}$

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{1}{\sqrt{u^2+b^2}}(b\sin v; -b\cos v; u) = \left(\frac{b\sin v}{\sqrt{u^2+b^2}}, \frac{-b\cos v}{\sqrt{u^2+b^2}}, \frac{u}{\sqrt{u^2+b^2}}\right)$$

$\vec{r}_u(\cos v, \sin v, 0)$

$\vec{r}_v(-u\sin v, u\cos v, b)$

$\vec{N} = \left(\frac{b\sin v}{\sqrt{u^2+b^2}}, \frac{-b\cos v}{\sqrt{u^2+b^2}}, \frac{u}{\sqrt{u^2+b^2}}\right)$

$\vec{r}_{uu}(0, 0, 0)$

$\vec{r}_{uv}(-\sin v, \cos v, 0)$

$\vec{r}_{vv}(-u\cos v, -u\sin v, 0)$

$$h_{11} = \vec{N}\vec{r}_{uu} = \frac{b\sin v}{\sqrt{u^2+b^2}} \cdot 0 + \left(\frac{-b\cos v}{\sqrt{u^2+b^2}}\right) \cdot 0 + \frac{u}{\sqrt{u^2+b^2}} \cdot 0 = 0$$

$$h_{12} = \vec{N}\vec{r}_{uv} = \frac{b\sin v}{\sqrt{u^2+b^2}} \cdot (-\sin v) + \left(\frac{-b\cos v}{\sqrt{u^2+b^2}}\right)\cos v + \frac{u}{\sqrt{u^2+b^2}} \cdot 0 =$$

$$= \frac{-b\sin^2 v - b\cos^2 v}{\sqrt{u^2+b^2}} = \frac{-b}{\sqrt{u^2+b^2}}$$

$$h_{22} = \vec{N}\vec{r}_{vv} = \frac{b\sin v}{\sqrt{u^2+b^2}} \cdot (-u\cos v) + \left(\frac{-b\cos v}{\sqrt{u^2+b^2}}\right) \cdot (-u\sin v) + \frac{u}{\sqrt{u^2+b^2}} \cdot 0$$

$$= \frac{-bu\sin v\cos v + bu\cos v\sin v}{\sqrt{u^2+b^2}} = 0$$

$$\Rightarrow h_{11} = 0, \quad h_{12} = \frac{-b}{\sqrt{u^2+b^2}}, \quad h_{22} = 0$$

$$g = g_{11} g_{22} - g_{12}^2 = 1 \cdot (u^2+b^2) - 0^2 = u^2+b^2$$

$$h = h_{11} h_{22} - h_{12}^2 = 0 \cdot 0 - \left(\frac{-b}{\sqrt{u^2+b^2}}\right)^2 = -\frac{b^2}{u^2+b^2}$$

$$K = \frac{h}{g} = \frac{\frac{-b^2}{u^2+b^2}}{u^2+b^2} = \frac{-b^2}{(u^2+b^2)^2}$$

$$H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2g} = 0$$

Ур-нието на главните линии се получава от:

$$\begin{vmatrix} g_{11}du + g_{12}dv & g_{12}du + g_{22}dv \\ h_{11}du + h_{12}dv & h_{12}du + h_{22}dv \end{vmatrix} = 0$$

Замествание $g_{11} = 1$, $g_{12} = 0$, $g_{22} = u^2 + b^2$

$h_{11} = 0$, $h_{12} = \frac{-b}{\sqrt{u^2+b^2}}$, $h_{22} = 0$

$$\begin{vmatrix} 1.du & (u^2+b^2)dv \\ \frac{-b}{\sqrt{u^2+b^2}}dv & \frac{-b}{\sqrt{u^2+b^2}}du \end{vmatrix} = 0$$

$$\frac{-b}{\sqrt{u^2+b^2}}du^2 + \frac{b(u^2+b^2)}{\sqrt{u^2+b^2}}dv^2 = 0 \Big/ : -\frac{b}{\sqrt{u^2+b^2}}$$

$$du^2 - (u^2+b^2)dv^2 = 0$$

$$(du + \sqrt{u^2+b^2}\,dv)(du - \sqrt{u^2+b^2}\,dv) = 0$$

$du + \sqrt{u^2+b^2}\,dv = 0$ $\qquad\qquad$ $du - \sqrt{u^2+b^2}\,dv = 0$

$\frac{du}{\sqrt{u^2+b^2}} = -dv \Big/ \int$ $\qquad\qquad$ $\frac{du}{\sqrt{u^2+b^2}} = dv \Big/ \int$

$$\int \frac{du}{\sqrt{u^2+b^2}} = -\int dv$$

a_1: $\ln\left|u+\sqrt{u^2+b^2}\right| = -v + C_1$

$$\int \frac{du}{\sqrt{u^2+b^2}} = \int dv$$

a_2: $\ln\left|u+\sqrt{u^2+b^2}\right| = v + C_2$