Задача. Дадена е повърхнина S, зададена чрез следните параметрични ур-ния $x = u\cos v$, $y = u\sin v$, $z = \ln u$. Намерете:

а) първа и втора основна форма

Решение:

$$S: \vec{r}(u\cos v, u\sin v, \ln u)$$

$$\boxed{I(du, dv) = g_{11} du^2 + 2g_{12} du dv + g_{22} dv^2} \quad (1)$$

$g_{11} = \vec{r_u}^2$, $g_{12} = \vec{r_u}\vec{r_v}$, $g_{22} = \vec{r_v}^2$

$\vec{r_u}(\cos v, \sin v, \frac{1}{u})$

$\vec{r_v}(-u\sin v, u\cos v, 0)$

$$g_{11} = \vec{r_u}^2 = \underbrace{\cos^2 v + \sin^2 v}_{1} + \left(\frac{1}{u}\right)^2 = 1 + \frac{1}{u^2} = \frac{u^2+1}{u^2}$$

$$g_{12} = \vec{r_u}\vec{r_v} = \cos v.(-u\sin v) + \sin v.u\cos v + \frac{1}{u}.0 = 0$$

$$g_{22} = \vec{r_v}^2 = (-u\sin v)^2 + (u\cos v)^2 = u^2\sin^2 v + u^2\cos^2 v = u^2$$

заместваме в (1) =>

$$I(du, dv) = \left(\frac{u^2+1}{u^2}\right)du^2 + 2.0.du.dv + u^2 dv^2$$

$I(du, dv) = \left(\frac{u^2+1}{u^2}\right)du^2 + u^2 dv^2$ - първа осн. форма.

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$$\mathrm{II}(du, dv) = h_{11} du^2 + 2 h_{12} du dv + h_{22} dv^2 \quad (2)$$ втора осн форма

$$\left.\begin{aligned} h_{11} &= \vec{N}\vec{r}_{uu} \\ h_{12} &= \vec{N}\vec{r}_{uv} \\ h_{22} &= \vec{N}\vec{r}_{vv} \end{aligned}\right\}$$ коефициенти на втора осн. форма

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = ?$$ – нормален вектор

$$\vec{r}_u\left(\cos v, \sin v, \frac{1}{u}\right)$$

$$\vec{r}_v(-u\sin v, u\cos v, 0)$$

$$\vec{r}_u \times \vec{r}_v = \left(\begin{vmatrix} \sin v & 1/u \\ u\cos v & 0 \end{vmatrix}; -\begin{vmatrix} \cos v & 1/u \\ -u\sin v & 0 \end{vmatrix}; \begin{vmatrix} \cos v & \sin v \\ -u\sin v & u\cos v \end{vmatrix} \right) =$$

$$= \left(-\frac{1}{u}\cdot u\cos v; -\frac{1}{u}\cdot u\sin v; u\cos^2 v + u\sin^2 v\right)$$

$$= (-\cos v; -\sin v; u)$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = (-\cos v; -\sin v; u)$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(-\cos v)^2 + (-\sin v)^2 + u^2} = \sqrt{1+u^2}$$

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{1}{\sqrt{1+u^2}}(-\cos v; -\sin v; u) = \left(\frac{-\cos v}{\sqrt{1+u^2}}; \frac{-\sin v}{\sqrt{1+u^2}}; \frac{u}{\sqrt{1+u^2}}\right)$$

$\vec{r}_u(\cos v; \sin v; \frac{1}{u})$

$\vec{r}_v(-u\sin v, u\cos v, 0)$

$\vec{N} = \left(\frac{-\cos v}{\sqrt{1+u^2}}, \frac{-\sin v}{\sqrt{1+u^2}}, \frac{u}{\sqrt{1+u^2}}\right)$

$\vec{r}_{uu}\left(0, 0, -\frac{1}{u^2}\right)$

$\vec{r}_{uv}(-\sin v, \cos v; 0)$

$\vec{r}_{vv}(-u\cos v; -u\sin v; 0)$

$$h_{11} = \vec{N}\vec{r}_{uu} = \frac{-\cos v}{\sqrt{1+u^2}} \cdot 0 + \left(\frac{-\sin v}{\sqrt{1+u^2}}\right) \cdot 0 + \frac{u}{\sqrt{1+u^2}} \cdot \left(-\frac{1}{u^2}\right) = \frac{-1}{u\sqrt{1+u^2}}$$

$$h_{12} = \vec{N}\vec{r}_{uv} = \left(\frac{-\cos v}{\sqrt{1+u^2}}\right) \cdot (-\sin v) + \left(\frac{-\sin v}{\sqrt{1+u^2}}\right) \cdot \cos v + \frac{u}{\sqrt{1+u^2}} \cdot 0 = 0$$

$$h_{22} = \vec{N}\vec{r}_{vv} = \left(-\frac{\cos v}{\sqrt{1+u^2}}\right) \cdot (-u\cos v) + \left(\frac{-\sin v}{\sqrt{1+u^2}}\right) \cdot (-u\sin v) + \frac{u}{\sqrt{1+u^2}} \cdot 0 =$$

$$= \frac{u\cos^2 v + u\sin^2 v}{\sqrt{1+u^2}} = \frac{u}{\sqrt{1+u^2}}$$

$\Rightarrow h_{11} = -\frac{1}{u^2\sqrt{1+u^2}}, \quad h_{12} = 0, \quad h_{22} = \frac{u}{\sqrt{1+u^2}}$

заместване в (2) $\Rightarrow$

$$\text{II}(du, dv) = -\frac{1}{u\sqrt{1+u^2}} du^2 + 2 \cdot 0 \cdot du \cdot dv + \frac{u}{\sqrt{1+u^2}} dv^2$$

$$\text{II}(du, dv) = -\frac{1}{u\sqrt{1+u^2}} du^2 + \frac{u}{\sqrt{1+u^2}} dv^2.$$

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б) Нормалната кривина $\nu = ?$ в т. $M(u=1, v=\frac{\pi}{4})$ по допирателното направление на кривата $C: u = tg\, v$ върху S.

Решение:

Нормална кривина: $\nu = \dfrac{II(du, dv)_M}{I(du, dv)_M}$

От а) получихме:

$$I(du, dv) = \left(\frac{u^2+1}{u^2}\right) du^2 + u^2 dv^2$$

$$II(du, dv) = -\frac{1}{u\sqrt{1+u^2}} du^2 + \frac{u}{\sqrt{1+u^2}} dv^2$$

т. $M(u=1, v=\frac{\pi}{4})$, остава да намерим допирателното направление (du, dv) на C.

От $C: u = tg\, v \;|\, d$

$$du = d\, tg\, v$$

$$du = \frac{1}{\cos^2 v} dv \Rightarrow \cos^2 v\, du = dv$$

$$\frac{du}{dv} = \frac{1}{\cos^2 v} \Rightarrow (du, dv) \uparrow\uparrow (1, \cos^2 v)$$

$$(du, dv)_M \uparrow\uparrow (1, \cos^2 \tfrac{\pi}{4}) \Rightarrow (du, dv)_M \uparrow\uparrow (1, \tfrac{1}{2}) \uparrow\uparrow (2, 1)$$

Сега заместваме $u=1$, $v=\frac{\pi}{4}$ и $(du, dv) \uparrow\uparrow (2,1)$ в получените резултати за първа и втора основна форма:

$$I(du,dv)_u = \left(\frac{u^2+1}{u^2}\right) du^2 + u^2 dv^2 = \left(\frac{1^2+1}{1^2}\right) \cdot 2^2 + 1^2 \cdot 1^2 = 9$$

$$II(du,dv)_u = \frac{-1}{u\sqrt{1+u^2}} du^2 + \frac{u}{\sqrt{1+u^2}} dv^2 = -\frac{1}{1\sqrt{1+1^2}} \cdot 2^2 + \frac{1}{\sqrt{1+1^2}} \cdot 1^2 = -\frac{3}{\sqrt{2}}$$

По повод за нормалната кривина получаваме:

$$\nu = \frac{II(du,dv)_u}{I(du,dv)_u} = \frac{-\frac{3}{\sqrt{2}}}{9} = -\frac{3}{9\sqrt{2}} = -\frac{1}{3\sqrt{2}} = -\frac{\sqrt{2}}{6}$$

$$\Rightarrow \nu = -\frac{\sqrt{2}}{6}$$

в) асимптотичните линии в произволна точка на S и през т. M. $(u=1, v=1)$

Решение:

Асимптотичните линии се получават като приравним втора основна форма $=0$ и решим диференциалното ур-ние:

$$\text{II}(du,dv) = h_{11}\,du^2 + 2h_{12}\,du\,dv + h_{22}\,dv^2 = 0$$

$$h = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} = h_{11}h_{22} - h_{12}^2$$

1) $h < 0 \Rightarrow \exists$ 2 асимп. линии

2) $h = 0 \Rightarrow \exists$ 1 асимп. линия

3) $h > 0 \Rightarrow \nexists$ асимп. линии.

От а) получихме:

$$h_{11} = -\frac{1}{u\sqrt{1+u^2}}, \quad h_{12} = 0, \quad h_{22} = \frac{u}{\sqrt{1+u^2}}$$

$$h = h_{11}.h_{22} - h_{12}^2 = -\frac{1}{u\sqrt{1+u^2}} \cdot \frac{u}{\sqrt{1+u^2}} - 0^2 = -\frac{1}{(1+u^2)} < 0$$

$\Rightarrow h < 0 \Rightarrow \exists$ 2 асимптотични линии !

За да намерим асимп. линии приравняваме $\mathrm{II}(du,dv)=0$

$$-\frac{1}{u\sqrt{1+u^2}}du^2+\frac{u}{\sqrt{1+u^2}}dv^2=0 \quad / \cdot \left(-\sqrt{1+u^2}\right)$$

$$\frac{du^2}{u}-u\,dv^2=0$$

$$du^2-u^2dv^2=0$$

$$(du+u\,dv)(du-u\,dv)=0$$

a_1: $du+u\,dv=0$

$$du=-u\,dv$$

$$\frac{du}{u}=-dv \quad / \int$$

$$\int\frac{du}{u}=-\int dv$$

a_1: $\ln u=-v+C_1$

a_2: $du-u\,dv=0$

$$du=u\,dv$$

$$\frac{du}{u}=dv \quad / \int$$

$$\int\frac{du}{u}=\int dv$$

a_2: $\ln u=v+C_2$

В т. $M(u=v=1)$

a_1: $\ln 1=-1+C_1$

$$\Rightarrow C_1=1$$

a_2: $\ln 1=1+C_2$

$$\Rightarrow C_2=-1$$

a_1: $\ln u=-v+1$

a_2: $\ln u=-v-1$