4.3 Втора основна форма на повърхнина. Гаусова и средна кривина

I. **Втора основна форма на повърхнината** $S(u,v)$

Дадена е повърхнината $S(u,v) : \vec{r}(u,v) = \vec{r}\,(\,x(u,v),\; y(u,v),\; z(u,v)\,)$

Коефициенти на *втора основна форма*, които се означават с $h_{ij} = h_{ij}(u,v)$, $i,j = 1,2$, се дефинират чрез равенствата:

(4.40)
$$\begin{aligned} h_{11} &= \overrightarrow{N}.\vec{r}_{uu} \quad , \; h_{22} = \overrightarrow{N}.\vec{r}_{vv} \\ h_{12} &= h_{21} = \overrightarrow{N}.\vec{r}_{uv} = \overrightarrow{N}.\vec{r}_{vu} \end{aligned} \quad \Rightarrow \quad h = h_{11}.h_{22} - {h_{12}}^2 \; ,$$

където $\overrightarrow{N}$ е нормалния вектор на повърхнината S , а векторите $\vec{r}_{uu}$,$\vec{r}_{uv}$,$\vec{r}_{vv}$ са частните производни от втори ред, които имат вида:

(4.38)
$$\vec{r}_{uu} = (\,\vec{r}_u\,)'_u \;\; , \; \vec{r}_{uv} = \;(\,\vec{r}_u\,)'_v = (\,\vec{r}_v\,)'_u = \vec{r}_{vu} \;\; , \; \vec{r}_{vv} = (\,\vec{r}_v\,)'_v$$

Тогава, втора основна форма по допирателното направление (du,dv) има вида:

(4.41)
$$II(du,dv) = h_{11}.d^2u + 2h_{12}.du.dv + h_{22}.d^2v$$

II. Гаусова и средна кривина на повърхнината $S(u,v)$

Гаусова кривина K и средна кривина H са *инвариантни* величини за повърхнината $S(u,v)$ и се изчисляват съответно чрез равенства:

(4.43) $$K = \frac{h}{g} = \frac{h_{11}.h_{22} - {h_{12}}^2}{g_{11}.g_{22} - {g_{12}}^2} \quad , \quad H = \frac{1}{2g}\left(g_{11}.h_{22} - 2g_{12}.h_{12} + g_{22}.h_{11}\right)$$

Задача 4.12 */ стр. 70* Намерете втора основна форма, гаусова кривина K и средна кривина H на следните повърхнини S :

д) цилиндър $S\,(u,v):\ \vec{r}\ (\,acosu, asinu,\ v)\,,\ a = const > 0$;

е) кос хеликоид $S\,(u,v):\ \vec{r}\,(ucosv, usinv,\ u+v)$.

<u>Решение</u>: **4.12 д)** цилиндър $S\,(u,v):\ \vec{r}\ (\,acosu, asinu,\ v)$

$(\,?\,)\ II(du,dv) =\ ?\ ,\ K = ?\ ,\ H =\ ?$

$\vec{r}\ (\,acosu, asinu,\ v)$

$\vec{r}_u(u,v) = \vec{r}_u\ (-asinu, acosu, 0) \Rightarrow\quad {\vec{r}_u}^{\,2} =\ a^2 = g_{11}$

$\vec{r}_v(u,v) = \vec{r}_v\ (0\,,0\,,1) \Rightarrow\ {\vec{r}_v}^{\,2} =\ 1^2 = 1 = g_{22}$

$g_{12} = \vec{r}_u.\vec{r}_v = 0\ \ \Rightarrow\quad g = g_{11}.g_{22} - {g_{12}}^2 =\ 1.a^2 - 0^2 = a^2 =\ g$

$$\vec{r}_u\ \times\ \vec{r}_v\ \left(\begin{vmatrix} acosu & 0 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} -asinu & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} -asinu & acosu \\ 0 & 0 \end{vmatrix}\right)\ \Rightarrow$$

$$\Rightarrow\ \vec{r}_u\ \times\ \vec{r}_v\ (acosu, asinu, 0\,)\quad\Rightarrow\quad |\vec{r}_u\ \times\ \vec{r}_v| = \sqrt{a^2+0} = a$$

$$\Rightarrow\ \vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \left(\frac{acosu}{a}\ ,\frac{asinu}{a}, \frac{0}{a}\right)\ \Rightarrow\quad \vec{N}\,(cosu,\ sinu\ ,0)\quad (1)$$

$$\vec{r}_u\,(-asinu, acosu, 0) \;\Rightarrow\; \vec{r}_{uu}\,(-acosu, -asinu, 0\,)\,, \vec{r}_{uv}\,(0,0,0)$$

$$\vec{r}_v\,(0\,,0\,,1) \Rightarrow\; \vec{r}_{vv}\,(0,0,0)$$

$$\vec{N}\,(cosu,\; sinu\,, 0) \Rightarrow \begin{matrix} h_{12} = \overrightarrow{N}.\vec{r}_{uv} = 0 \quad , \; h_{22} = \overrightarrow{N}.\vec{r}_{vv} = 0 \\ h_{11} = \overrightarrow{N}.\vec{r}_{uu} = -acos^2u - asin^2u = -a \end{matrix} \Rightarrow \quad h = 0$$

$$\Rightarrow\; II(du, dv) = -a.\,d^2u + 2.0.\,du.\,dv + 0.\,d^2v = -a.\,d^2u$$

$$\Rightarrow K = \frac{h}{g} = \frac{0}{a^2} = 0 \quad ,$$

$$H = \frac{1}{2g}\,(g_{11}.h_{22} - 2g_{12}.h_{12} + g_{22}.h_{11}) = \frac{1}{2g}\,(a^2.\,0 - 2.0.0 - 1.\,a) = \frac{-a}{2a^2} = -\frac{1}{2a}$$

Наричаме цилиндъра *параболична повърхнина*, защото *гаусовата кривина* $K = 0$.

Решение: **4.12 е)** кос хеликоид $S\,(u,v):\ \vec{r}\,(ucosv, usinv,\ u+v)$

$(\,?\,)\ \ II(du,dv) = \ ?\ ,\ K = ?\ ,\ H = \ ?$

$\vec{r}\,(ucosv, usinv,\ u+v)$

$\vec{r}_u(u,v) = \vec{r}_u\,(cosv, sinv, 1) \Rightarrow \quad \vec{r}_u^{\;2} = \ 2 = g_{11}$

$\vec{r}_v(u,v) = \vec{r}_v\,(-usinv\,,\ ucosv\,, 1) \Rightarrow \ \vec{r}_v^{\;2} = \ u^2 + 1 = g_{22}$

$g_{12} = \vec{r}_u.\vec{r}_v = 1 \quad \Rightarrow \quad g = g_{11}.g_{22} - g_{12}{}^2 = \ 2.(u^2+1) - 1^2 = 2u^2 + 1 = \ g$

$\vec{r}_u \times \vec{r}_v \left(\begin{vmatrix} sinv & 1 \\ ucosv & 1 \end{vmatrix}, -\begin{vmatrix} cosv & 1 \\ -usinv & 1 \end{vmatrix}, \begin{vmatrix} cosv & sinv \\ -usinv & ucosv \end{vmatrix} \right) \Rightarrow$

$\Rightarrow \ \vec{r}_u \times \vec{r}_v\,(sinv - ucosv, -(cosv + usinv), u\,) \quad \Rightarrow$

$\Rightarrow \ |\vec{r}_u \times \vec{r}_v| = \sqrt{(sinv - ucosv)^2 + (cosv + usinv)^2 + u^2} = \sqrt{1 + 2u^2}$

$\Rightarrow \ \vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \ \Rightarrow \quad \vec{N}\,\left(\frac{sinv - ucosv}{\sqrt{1+2u^2}}, -\frac{cosv + usinv}{\sqrt{1+2u^2}}\,, \frac{u}{\sqrt{1+2u^2}}\right) \quad (1)$

$$\vec{r}_u\ (cosv, sinv, 1)\ \ \Rightarrow\ \vec{r}_{uu}\ (0,0,0\ )\ , \vec{r}_{uv}\ (-sinv, cosv, 0)$$

$$\vec{r}_v\ (-usinv\ ,\ ucosv\ , 1) \Rightarrow\ \vec{r}_{vv}\ (-ucosv\ , -usinv\ , 0)$$

$$\vec{N}\ \left(\frac{sinv-ucosv}{\sqrt{1+2u^2}}, -\frac{cosv+usinv}{\sqrt{1+2u^2}}\ , \frac{u}{\sqrt{1+2u^2}}\right) \Rightarrow$$

$$h_{11} = \overrightarrow{N}.\vec{r}_{uu} = 0\ ,\ h_{12} = \overrightarrow{N}.\vec{r}_{uv} = \frac{-sinv(sinv-ucosv)}{\sqrt{1+2u^2}} - \frac{(cosv+usinv)cosv}{\sqrt{1+2u^2}} + 0 = -\frac{1}{\sqrt{1+2u^2}}$$

$$h_{22} = \overrightarrow{N}.\vec{r}_{vv} = \frac{-ucosv(sinv-ucosv)}{\sqrt{1+2u^2}} + \frac{(cosv+usinv)usinv}{\sqrt{1+2u^2}} + 0 = \frac{u^2}{\sqrt{1+2u^2}}$$

$$\Rightarrow h = h_{11}.h_{22} - \ {h_{12}}^2 = 0 - \left(-\frac{1}{\sqrt{1+2u^2}}\right)^2 = -\frac{1}{2u^2+1} = -\frac{1}{g}$$

$$\Rightarrow\ II(du, dv) = 0.d^2u - 2.\frac{1}{\sqrt{1+2u^2}}.du.dv + \frac{u^2}{\sqrt{1+2u^2}}.d^2v = \frac{(2dudv+u^2d^2v)}{\sqrt{1+2u^2}}$$

$$\Rightarrow K = \frac{h}{g} = \ -\frac{1}{g^2} = -\frac{1}{(\ 1+2u^2)^2}\quad ; \qquad H = \frac{1}{2g}(g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}) = ?$$

$$H = \frac{1}{2g}\left(\frac{2u^2}{\sqrt{1+2u^2}} + 2.1.\frac{1}{\sqrt{1+2u^2}} - g_{22}.0\right) = \frac{u^2+1}{(1+2u^2)^{3/2}}\ -\ \textit{хиперболична повърхнина}$$