4.4 Асимптотични и главни линии на повърхнина.

I. **Асимптотични линии на повърхнината** $S(u, v)$

Дадена е повърхнината $S(u, v): \vec{r}(u, v) = \vec{r}\,(\,x(u, v),\; y(u, v),\; z(u, v)\,)$

Уравненията на асимптотичните линии се получават, като се приравни втората основна форма на нула, след което се решава диференциално уравнение относно параметрите u, v .

(4.59) $$II(du, dv) = h_{11}.\, d^2u + 2h_{12}.\, du.\, dv + h_{22}.\, d^2v \;=\; 0$$

Необходимо и достатъчно условие за съществуването на асимптотични линии върху $S(u, v)$ е да определим знака на $h = h_{11}.\, h_{22} - {h_{12}}^2$.

- Ако $h < 0 \;\Rightarrow\; \exists\; 2$ асимптотични линии.
- Ако $h = 0 \;\Rightarrow\; \exists\,!\; 1$ асимптотична линия.
- Ако $h > 0 \;\Rightarrow\; \nexists$ асимптотични линии.

<u>Пример 1</u> **4.12 е)** Намерете Асимптотичните Линии, ако съществуват за кос хеликоид

$$S\,(u,v):\ \ \vec{r}\,(ucosv, usinv,\ u+v)$$

$$II(du,dv)=\frac{(2dudv+u^2d^2v)}{\sqrt{1+2u^2}}\quad,\ h=-\frac{1}{2u^2+1}=-\frac{1}{g}<0\ \Rightarrow\ \exists\ 2\ \text{АЛ}$$

$$II(du,dv)=0\ \Rightarrow\ \frac{(2dudv+u^2d^2v)}{\sqrt{1+2u^2}}=0$$

$$\Rightarrow\ 2dudv+u^2(dv)^2=0\quad\Rightarrow\quad dv(2du+u^2dv)=0$$

1. Случай: $dv=0\ \Rightarrow\ v=const\ \Rightarrow\ a_1:\begin{cases}u=q\ -\text{параметър}\\ v=const\end{cases}$

2. Случай: $2du+u^2dv=0\ \Rightarrow\ \frac{2du}{u^2}=-dv\ \ |.\ \int$

$$\Rightarrow\ \int dv=-2.\int\frac{du}{u^2}\ \Rightarrow\ v=-2\int u^{-2}du\ \Rightarrow\ v=+2.\frac{1}{u}+C\ \Rightarrow\ a_2:\ v=\frac{2}{u}+C\,.$$

Задача 4.15 */ стр. 75* Намерете Асимптотичните Линии, ако съществуват за повърхнините S в произволна нейна точка и за точка P :

б) $S\,(u,v):\ \vec{r}\ (u^2, v^2,\ uv)\ ,\ P\,(u=v=1);$

в) $S\,(u,v):\ \vec{r}\,(chu.cosv, chu.sinv,\ u)\ ,\ P\,(u=2,\ v=1)\,.$

Решение: б) $S\,(u,v):\ \vec{r}\ (u^2, v^2,\ uv)\ ,\ P\,(u=v=1)$

$$\vec{r}\ (u^2, v^2,\ uv)\ \Rightarrow\ \vec{r}_u\,(2u,0,v)\ ,\ \vec{r}_v\,(0,2v,u)$$

$$\begin{matrix}\vec{r}_u\,(2u,0,v)\\ \vec{r}_v\,(0,2v,u)\end{matrix}\ \Rightarrow\ \vec{r}_u\times\vec{r}_v\left(\begin{vmatrix}0 & v\\ 2v & u\end{vmatrix}, -\begin{vmatrix}2u & v\\ 0 & u\end{vmatrix}, \begin{vmatrix}2u & 0\\ 0 & 2v\end{vmatrix}\right)\ \Rightarrow$$

$$\Rightarrow\ \vec{r}_u\times\vec{r}_v\,(-2v^2, -2u^2, 4uv\,)\ \Rightarrow$$

$$\Rightarrow\ |\vec{r}_u\times\vec{r}_v| = \sqrt{(-2v^2)^2+(-2u^2)^2+16u^2v^2} = 2\sqrt{v^4+u^4+4u^2v^2}$$

$$\Rightarrow\ \vec{N}=\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\ \Rightarrow\ \vec{N}\left(-\frac{v^2}{\sqrt{v^4+u^4+4u^2v^2}}, -\frac{u^2}{\sqrt{v^4+u^4+4u^2v^2}}, \frac{2uv}{\sqrt{v^4+u^4+4u^2v^2}}\right)\quad (1)$$

$$\begin{array}{l} \vec{r}_u\ (2u,\ 0, v) \\ \vec{r}_v\ (0, 2v, u) \end{array} \Rightarrow \begin{array}{l} \vec{r}_{uu}\ (2,\ 0,\ 0)\ ,\ \ \vec{r}_{uv}\ (0,0,1) \\ \vec{r}_{vv}\ (0,\ 2,\ 0) \end{array}$$

$$h_{11} = \overrightarrow{N}.\vec{r}_{uu} = -\frac{2v^2}{\sqrt{v^4+u^4+4u^2v^2}}\ ,\ \ h_{12} = \overrightarrow{N}.\vec{r}_{uv} = \frac{2uv}{\sqrt{v^4+u^4+4u^2v^2}}$$

$$h_{22} = \overrightarrow{N}.\vec{r}_{vv} = -\frac{2u^2}{\sqrt{v^4+u^4+4u^2v^2}}$$

$$\Rightarrow h = h_{11}.h_{22} - \ {h_{12}}^2 = \frac{2v^2}{\sqrt{v^4+u^4+4u^2v^2}}.\frac{2u^2}{\sqrt{v^4+u^4+4u^2v^2}} - \left(\frac{2uv}{\sqrt{v^4+u^4+4u^2v^2}}\right)^2 = 0$$

$$\Rightarrow\ II(du,dv) = -\frac{2v^2}{\sqrt{v^4+u^4+4u^2v^2}}.d^2u + 2.\frac{2uv}{\sqrt{v^4+u^4+4u^2v^2}}.du.dv - \frac{2u^2}{\sqrt{v^4+u^4+4u^2v^2}}.d^2v$$

$$\Rightarrow\ II(du,dv) = \frac{-2v^2}{\sqrt{v^4+u^4+4u^2v^2}}.d^2u + \frac{4uv}{\sqrt{v^4+u^4+4u^2v^2}}.du.dv + \frac{-2u^2}{\sqrt{v^4+u^4+4u^2v^2}}.d^2v$$

$$\Rightarrow\ II(du,dv) = \frac{-2}{\sqrt{v^4+u^4+4u^2v^2}}.(v^2d^2u - 2uvdudv + u^2d^2v)$$

$h = 0 \Rightarrow\ \ \exists\ !\ \ 1$ асимптотична линия.

$$\Rightarrow \; II(du, dv) = \frac{-2}{\sqrt{v^4+u^4+4u^2v^2}} . (v^2 d^2u - 2uvdudv + u^2 d^2 v) = 0$$

$$\Rightarrow \; (v^2 d^2u - 2uvdudv + u^2 d^2 v) = 0 \;\; \Rightarrow \;\; (v.du - u.dv)^2 = 0$$

$$\Rightarrow \;\; v.du = u.dv \;\; \Rightarrow \;\; \frac{du}{u} = \frac{dv}{v} \;\; |. \int \;\;\; \Rightarrow \;\; a_1 : \; ln\, u = ln\, v + C \, .$$

$$P\,(u = 1, \; v = 1) \; \Rightarrow \; {a_1}^P : \; ln\, 1 = ln\, 1 + C \Rightarrow \;\; C = 0$$

$$\Rightarrow {a_1}^P : \; ln\, u = ln\, v \;\; .$$

<u>*Решение*</u>: *задача 4.15в* $S\,(u, v) : \;\; \vec{r}\,(chu.cosv, chu.sinv, \; u) \;, \; P\,(u = 2, \; v = 1)$.

$\vec{r}\,(chu.cosv, chu.sinv, \; u)$ $\qquad (chu)' = shu \;, \;\; (shu)' = chu \;, \; 1 + sh^2u = ch^2u$

$\vec{r}_u\,(cosv.shu, sinv.shu, 1)$,

$\vec{r}_v\,(-chu.sinv, chu.cosv, 0)$

$$\Rightarrow \quad \vec{r}_u \times \vec{r}_v \left(\begin{vmatrix} sinv.shu & 1 \\ chu.cosv & 0 \end{vmatrix}, - \begin{vmatrix} cosv.shu & 1 \\ -chu.sinv & 0 \end{vmatrix}, \begin{vmatrix} cosv.shu & sinv.shu \\ -chu.sinv & chu.cosv \end{vmatrix} \right) \Rightarrow$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v \, (-chu.cosv, \; -chu.sinv, \; shu.chu) \quad \Rightarrow$$

$$\Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{(-chu.cosv)^2 + (-chu.sinv)^2 + (shu.chu)^2} =$$

$$\Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{(chu)^2 + (shu.chu)^2} = \sqrt{ch^2u(1 + (shu)^2)} = \sqrt{ch^4u} = ch^2u$$

$$\Rightarrow \vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \quad \Rightarrow \quad \vec{N} \left(-\frac{chu.cosv}{ch^2u}, -\frac{chu.sinv}{ch^2u}, \frac{shu.chu}{ch^2u} \right)$$

$$\Rightarrow \quad \vec{N} \left(-\frac{cosv}{chu}, -\frac{sinv}{chu}, \frac{shu}{chu} \right)$$

$$\begin{array}{l} \vec{r}_u \, (cosv.shu, sinv.shu, 1) \\ \vec{r}_v \, (-chu.sinv, chu.cosv, 0) \end{array} \quad \Rightarrow \quad \begin{array}{l} \vec{r}_{uu} \, (cosv.chu, \; sinv.chu, \; 0) \\ \vec{r}_{uv} \, (-shu.sinv, \; shu.cosv, \; 0) \\ \vec{r}_{vv} \, (-chu.cosv, -chu.sinv, \; 0) \end{array}$$

$$\vec{N} \left(-\frac{cosv}{chu}, -\frac{sinv}{chu}, \frac{shu}{chu} \right)$$

$$h_{11} = \overrightarrow{N}.\vec{r}_{uu} = -\frac{cos^2v.chu}{chu} - \frac{sin^2v.chu}{chu} + 0 = -1$$

$$h_{12} = \overrightarrow{N}.\vec{r}_{uv} = \frac{cosv}{chu}.shu.sinv - \frac{sinv}{chu}.\ shu.cosv + 0 = 0$$

$$h_{22} = \overrightarrow{N}.\vec{r}_{vv} = \frac{cos^2v.chu}{chu} + \frac{sin^2v.chu}{chu} = 1$$

$\Rightarrow \quad h = h_{11}.h_{22} - \ {h_{12}}^2 = (-1).1 - (0)^2 = -1 < 0 \ \Rightarrow \ \exists \ 2$ асимптотични линии.

$$\Rightarrow \quad II(du, dv) = -1.d^2u + 2.0.du.dv + 1.d^2v \ \Rightarrow \ -d^2u + d^2v = 0$$

$$\Rightarrow \quad \left(\frac{du}{dv}\right)^2 = 1$$

1. Случай: $\frac{du}{dv} = 1 \ \Rightarrow \ du = dv \ |. \int \quad \Rightarrow \int du = \int dv \Rightarrow a_1 : \ u = v + C_1$

2. Случай: $\frac{du}{dv} = -1 \ \Rightarrow \ du = -dv \ |. \int \quad \Rightarrow \int du = -\int dv \Rightarrow a_2 : \ u = -v + C_2$

$$P\ (u = 2,\ v = 1) \ \Rightarrow \ {a_1}^P : \ 2 = 1 + C_1 \Rightarrow \ C_1 = 1 \Rightarrow \ {a_1}^P : \ u = v + 1$$

$$P\ (u = 2,\ v = 1) \ \Rightarrow \ {a_2}^P : \ 2 = -1 + C_2 \Rightarrow \ C_2 = 3 \Rightarrow \ {a_2}^P : \ u = -v + 3$$