Задача. Нека повърхнина S е зададена чрез параметричните уравнения: $x = u^2 + v^2$, $y = u^2 - v^2$, $z = uv$, $u, v > 0$.

Намерете:

а) Първата и втората основна форма.

$S: \vec{r}\,(u^2+v^2, u^2-v^2, uv)$

$\mathrm{I}(du, dv) = g_{11} du^2 + 2g_{12} du dv + g_{22} dv^2$ – първа основна форма.

$g_{11} = \vec{r}_u^{\,2}$, $g_{12} = \vec{r}_u \vec{r}_v$, $g_{22} = \vec{r}_v^{\,2}$

$\vec{r}\,(u^2+v^2, u^2-v^2, uv)$

$\vec{r}_u\,(2u, 2u, v)$

$\vec{r}_v\,(2v, -2v, u)$

$g_{11} = \vec{r}_u^{\,2} = (2u)^2 + (2u)^2 + (v)^2 = 4u^2 + 4u^2 + v^2 = \underline{8u^2 + v^2}$

$g_{12} = \vec{r}_u \vec{r}_v = 2u \cdot 2v + 2u(-2v) + v \cdot u = \underline{uv}$

$g_{22} = \vec{r}_v^{\,2} = (2v)^2 + (-2v)^2 + (u)^2 = 4v^2 + 4v^2 + u^2 = \underline{8v^2 + u^2}$

$$\boxed{\mathrm{I}(du, dv) = (8u^2+v^2)du^2 + 2uv\, du dv + (8v^2+u^2) dv^2}$$

$\mathrm{II}(du, dv) = h_{11} du^2 + 2h_{12} du dv + h_{22} dv^2$ – втора основна форма

$h_{11} = \vec{N}\vec{r}_{uu}$, $h_{12} = \vec{N}\vec{r}_{uv}$, $h_{22} = \vec{N}\vec{r}_{vv}$

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = ?$$

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$\vec{r}_u(2u, 2u, v)$

$\vec{r}_v(2v, -2v, u)$

$$\vec{r}_u \times \vec{r}_v = \left(\begin{vmatrix} 2u & v \\ -2v & u \end{vmatrix}; -\begin{vmatrix} 2u & v \\ 2v & u \end{vmatrix}; \begin{vmatrix} 2u & 2u \\ 2v & -2v \end{vmatrix} \right) =$$

$$= (2u^2 + 2v^2; -2u^2 + 2v^2; -4uv - 4uv)$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = (2u^2 + 2v^2, -2u^2 + 2v^2, -8uv)$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(2u^2 + 2v^2)^2 + (2v^2 - 2u^2)^2 + (-8uv)^2} =$$

$$= \sqrt{4u^4 + \cancel{8u^2v^2} + 4v^4 + 4v^4 - \cancel{8u^2v^2} + 4u^4 + 64u^2v^2}$$

$$= \sqrt{8u^4 + 8v^4 + 64u^2v^2} = \sqrt{8(u^4 + v^4 + 8u^2v^2)}$$

$$\vec{N} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{1}{\sqrt{8(u^4 + v^4 + 8u^2v^2)}} (2u^2 + 2v^2, -2u^2 + 2v^2, -8uv)$$

$$\vec{N} = \left(\frac{2u^2 + 2v^2}{\sqrt{8(u^4 + v^4 + 8u^2v^2)}}, \frac{-2u^2 + 2v^2}{\sqrt{8(u^4 + v^4 + 8u^2v^2)}}, \frac{-8uv}{\sqrt{8(u^4 + v^4 + 8u^2v^2)}} \right)$$

$\vec{r}_u(2u, 2u, v)$

$\vec{r}_v(2v, -2v, u)$

$\vec{r}_{uu}(2, 2, 0)$

$\vec{r}_{uv}(0, 0, 1)$

$\vec{r}_{vv}(2, -2, 0)$

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$$h_{11} = \vec{N}\vec{r}_{uu} = 2\left(\frac{2u^2+2v^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}\right) + 2\left(\frac{-2u^2+2v^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}\right) =$$

$$= \underbrace{\frac{8v^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}}$$

$$h_{12} = \vec{N}\vec{r}_{uv} = 1.\left(\frac{-8uv}{\sqrt{8(u^4+v^4+8u^2v^2)}}\right) = \underbrace{\frac{-8uv}{\sqrt{8(u^4+v^4+8u^2v^2)}}}$$

$$h_{22} = \vec{N}\vec{r}_{vv} = 2\left(\frac{2u^2+2v^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}\right) - 2\left(\frac{-2u^2+2v^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}\right) = \underbrace{\frac{8u^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}}$$

$$\mathrm{II}(du,dv) = \frac{8v^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}du^2 + 2\left(\frac{-8uv}{\sqrt{8(u^4+v^4+8u^2v^2)}}\right)du\,dv + \frac{8u^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}dv^2$$

$$\mathrm{II}(du,dv) = \frac{8v^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}du^2 - \frac{16uv}{\sqrt{8(u^4+v^4+8u^2v^2)}}du\,dv + \frac{8u^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}dv^2$$

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б) Гаусовата и средната кривина на S.

$$K = \frac{h}{g} \text{ - гаусова кривина} \qquad H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2g}.$$

$$g = g_{11}g_{22} - g_{12}^2 = (8u^2+v^2)(8v^2+u^2) - (uv)^2 = 8(u^4+v^4+8u^2 v^2)$$

$$h = h_{11}h_{22} - h_{12}^2 = \frac{8v^2}{\sqrt{8(u^4+v^4+8u^2v^2)}} \cdot \frac{8u^2}{\sqrt{8(u^4+v^4+8u^2v^2)}} - \left(\frac{-8uv}{\sqrt{8(u^4+v^4+8u^2v^2)}}\right)^2 = 0$$

$$K = \frac{h}{g} = \frac{0}{8(u^4+v^4+8u^2v^2)} = 0 \,//$$

$$H = \frac{2u^4+2v^4+u^2v^2}{\sqrt{2}\left(u^4+v^4+8u^2v^2\right)^{3/2}} \,//$$

б) нормалната кривина на S в т. M(u = v = 1) по допирателното направление C: $2v = u^2$ върху S.

Решение:

$$\nu_M = \frac{\mathrm{II}(du,dv)_M}{\mathrm{I}(du,dv)_M}$$ - нормална кривина на S.

От C: $2v = u^2 \;/d$

$2dv = 2u\,du$ (т. M(u = v = 1))

$2dv = 2.1\,du$

$2dv = 2du$

$\frac{du}{dv} = \frac{2}{2} \Rightarrow \underline{(du,dv) \parallel (2,2)}$

От а) намираме I и II в доп. направление на C и т. M.

$\mathrm{I}(du,dv) = (8u^2+v^2)du^2 + 2uv\,du\,dv + (8v^2+u^2)dv^2$

$\mathrm{I}(2,2) = (8.1^2+1^2).2^2 + 2.1.1.2.2 + (8.1^2+1^2).2^2 = 80$

$$\mathrm{II}(du,dv) = \frac{8v^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}du^2 - \frac{16uv}{\sqrt{8(u^4+v^4+8u^2v^2)}}du\,dv + \frac{8v^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}dv^2$$

$$\mathrm{II}(2,2) = \frac{8.1^2}{\sqrt{8(1^4+1^4+8.1^2.1^2)}}.2^2 - \frac{16.1.1}{\sqrt{8(1^4+1^4+8.1^2.1^2)}}.2.2 + \frac{8.1^2}{\sqrt{8(1^4+1^4+8.1^2.1^2)}}.2^2 = 0$$

$$\nu_M = \frac{\mathrm{II}(du,dv)_M}{\mathrm{I}(du,dv)_M} = \frac{0}{80} = \underline{\underline{0}}$$

в) Асимптотичните линии през произволна точка на S и през точка M.

Решение:

Асимптотичните линии на S се получават като приравним втора основна форма на S = 0 и решим диференциалното ур-ние:

$$\mathrm{II}(du,dv) = h_{11}du^2 + 2h_{12}du\,dv + h_{22}dv^2 = 0$$

1) Определяме броя на асимп. линии:

$$h = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} = h_{11}h_{22} - h_{12}^2$$

$h<0 \Rightarrow \exists$ 2 асим. линии

$h=0 \Rightarrow \exists$ 1 асим. линия

$h>0 \Rightarrow \nexists$ асим. линии.

От а) получихме

$$h_{11} = \frac{8v^2}{\sqrt{8(u^4+v^4+8u^2v^2)}} ; \quad h_{12} = \frac{-8uv}{\sqrt{8(u^4+v^4+8u^2v^2)}} ; \quad h_{22} = \frac{8u^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}$$

$$\Rightarrow h = h_{11}h_{22} - h_{12}^2 = \frac{8v^2 \cdot 8u^2}{8(u^4+v^4+8u^2v^2)} - \left(-\frac{8uv}{\sqrt{8(u^4+v^4+8u^2v^2)}}\right)^2$$

$$= \frac{64v^2u^2}{8(u^4+v^4+8u^2v^2)} - \frac{64u^2v^2}{8(u^4+v^4+8u^2v^2)} = 0$$

$\Rightarrow h=0 \Rightarrow \exists$ 1 асим. линия

$$\mathrm{II}(du,dv) = \frac{8v^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}du^2 - \frac{16uv}{\sqrt{8(u^4+v^4+8u^2v^2)}}du\,dv + \frac{8u^2}{\sqrt{8(u^4+v^4+8u^2v^2)}}dv^2 = 0$$

$$8v^2du^2 - 16uv\,du\,dv + 8u^2dv^2 = 0 \quad /:8$$

$$v^2du^2 - 2uv\,du\,dv + u^2dv^2 = 0$$

$$(v\,du - u\,dv) = 0$$

$$v\,du - u\,dv = 0$$

$$v\,du = u\,dv$$

$$\frac{du}{u} = \frac{dv}{v} \quad \Big/ \int$$

$$\int \frac{du}{u} = \int \frac{dv}{v}$$

$$\ln u = \ln v + \ln c$$

$$\ln u = \ln cv$$

Можем да се освободим от логаритмите

$\Rightarrow \underline{u = cv}$ – ур-ние на асимптотична линия в произволна точка на S

Намираме асимптотичната линия през т. $M(u=1, v=1)$, като заместим нейните координати в ур-нието и по този начин намерим ст-ста на интеграционната константа c:

$$u = cv \Rightarrow 1 = c \cdot 1 \Rightarrow c = \underline{1}$$

$$\Rightarrow \quad \underline{u = v}$$

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