Б-сплайн криви.

Задача. Изчислете сплайновата ф-я $N_{2,2}(u)$, определете интервала, в който тя е ненулева и скицирайте нейната графика при следния възлов вектор:

а) $U = \{0; 0,2; 0,5; 0,6; 0,8; 1\}$

б) $U = \{0; 0,2; 0,4; 0,5; 0,5; 1\}$

Решение: а) Посочените възли са подредени в таблица:

u_0	u_1	u_2	u_3	u_4	u_5
0	0,2	0,5	0,6	0,8	1

Всички възли са разпределени в последователни интервали

$[u_0, u_1)$	$[0; 0,2)$	$N_{0,0}$	$N_{0,1}$	$N_{0,2}$
$[u_1, u_2)$	$[0,2; 0,5)$	$N_{1,0}$	$N_{1,1}$	$N_{1,2}$
$[u_2, u_3)$	$[0,5; 0,6)$	$\boxed{N_{2,0}}$	$\boxed{N_{2,1}}$	$\boxed{N_{2,2}}$
$[u_3, u_4)$	$[0,6; 0,8)$	$\boxed{N_{3,0}}$	$\boxed{N_{3,1}}$	
$[u_4, u_5]$	$[0,8; 1]$	$\boxed{N_{4,0}}$		

$$\boxed{N_{i,0}(u) = \begin{cases} 1, \text{ако } u \in [u_i, u_{i+1}) \\ 0, \text{ако } u \notin [u_i, u_{i+1}) \end{cases}}$$

1. $N_{2,0}(u) = \begin{cases} 1, \text{ако } u \in [0,5; 0,6) \\ 0, \text{ако } u \notin [0,5; 0,6) \end{cases}$; $N_{3,0}(u) = \begin{cases} 1, \text{ако } u \in [0,6; 0,8) \\ 0, \text{ако } u \notin [0,6; 0,8) \end{cases}$

$$N_{4,0}(u) = \begin{cases} 1, \text{ ако } u \in [0,8;1] \\ 0, \text{ ако } u \notin [0,8,1]. \end{cases}$$

2. $N_{2,1}(u) = ?$ и $N_{3,1}(u) = ?$

$$\boxed{N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)}$$

$$\underset{i\ p}{N_{2,1}(u)} = \frac{u - u_2}{u_3 - u_2} N_{2,0}(u) + \frac{u_4 - u}{u_4 - u_3} N_{3,0}(u)$$

$$= \frac{u - 0,5}{0,6 - 0,5} N_{2,0}(u) + \frac{0,8 - u}{0,8 - 0,6} N_{3,0}(u)$$

$$= \frac{u - 0,5}{0,1} N_{2,0}(u) + \frac{0,8 - u}{0,2} N_{3,0}(u)$$

$$= 10.(u - 0,5) N_{2,0}(u) + 5.(0,8 - u) N_{3,0}(u)$$

$$= 5(2u - 1) N_{2,0}(u) + (4 - 5u) N_{3,0}(u)$$

$$\Rightarrow N_{2,1}(u) = \begin{cases} 5(2u-1), \text{ ако } u \in [0,5;0,6) \\ (4 - 5u), \text{ ако } u \in [0,6; 0,8) \\ 0, \text{ ако } u \notin [0,5; 0,8) \end{cases}$$

$$N_{3,1}(u) = \frac{u-u_3}{u_4-u_3} N_{3,0}(u) + \frac{u_5-u}{u_5-u_4} N_{4,0}(u)$$
ip

$$= \frac{u-0,6}{0,8-0,6} N_{3,0}(u) + \frac{1-u}{1-0,8} N_{4,0}(u)$$

$$= \frac{u-0,6}{0,2} N_{3,0}(u) + \frac{1-u}{0,2} N_{4,0}(u)$$

$$= 5(u-0,6) N_{3,0}(u) + 5(1-u) N_{4,0}(u)$$

$$= (5u-3) N_{3,0}(u) + (5-5u) N_{4,0}(u)$$

$$\Rightarrow N_{3,1}(u) = \begin{cases} (5u-3), \text{ ако } u \in [0,6; 0,8) \\ (5-5u), \text{ ако } u \in [0,8; 1] \\ 0, \text{ ако } u \notin [0,6; 1]. \end{cases}$$

3. $N_{2,2}(u) = ?$

$$\boxed{N_{i,p}(u) = \frac{u-u_i}{u_{i+p}-u_i} N_{i,p-1}(u) + \frac{u_{i+p+1}-u}{u_{i+p+1}-u_{i+1}} N_{i+1,p-1}(u)}$$

$$N_{2,2}(u) = \frac{u-u_2}{u_4-u_2} N_{2,1}(u) + \frac{u_5-u}{u_5-u_3} N_{3,1}(u)$$
ip

$$= \frac{u-0,5}{0,8-0,5} N_{2,1}(u) + \frac{1-u}{1-0,6} N_{3,1}(u)$$

$$= \frac{u-0{,}5}{0{,}3} N_{2,1}(u) + \frac{1-u}{0{,}4} N_{3,1}(u)$$

$$= \frac{10}{3}(u-0{,}5) N_{2,1}(u) + \frac{10}{4}(1-u) N_{3,1}(u)$$

$$\boxed{N_{2,2}(u) = \frac{5}{3}(2u-1) N_{2,1}(u) + \frac{5}{2}(1-u) N_{3,1}(u)} \quad (*)$$

$\Rightarrow$

1) за $u \in [0{,}5; 0{,}6)$ $N_{2,1}(u) = 5(2u-1)$ и $N_{3,1}(u) = 0$ $\Rightarrow$ заместване в (*)

$$N_{2,2}(u) = \frac{5}{3}(2u-1) \cdot 5(2u-1) = \frac{25}{3}(2u-1)^2 \quad \text{при } u \in [0{,}5; 0{,}6)$$

2) за $u \in [0{,}6; 0{,}8)$ $N_{2,1}(u) = 4-5u$ и $N_{3,1}(u) = 5u-3$ $\Rightarrow$ заместване в (*)

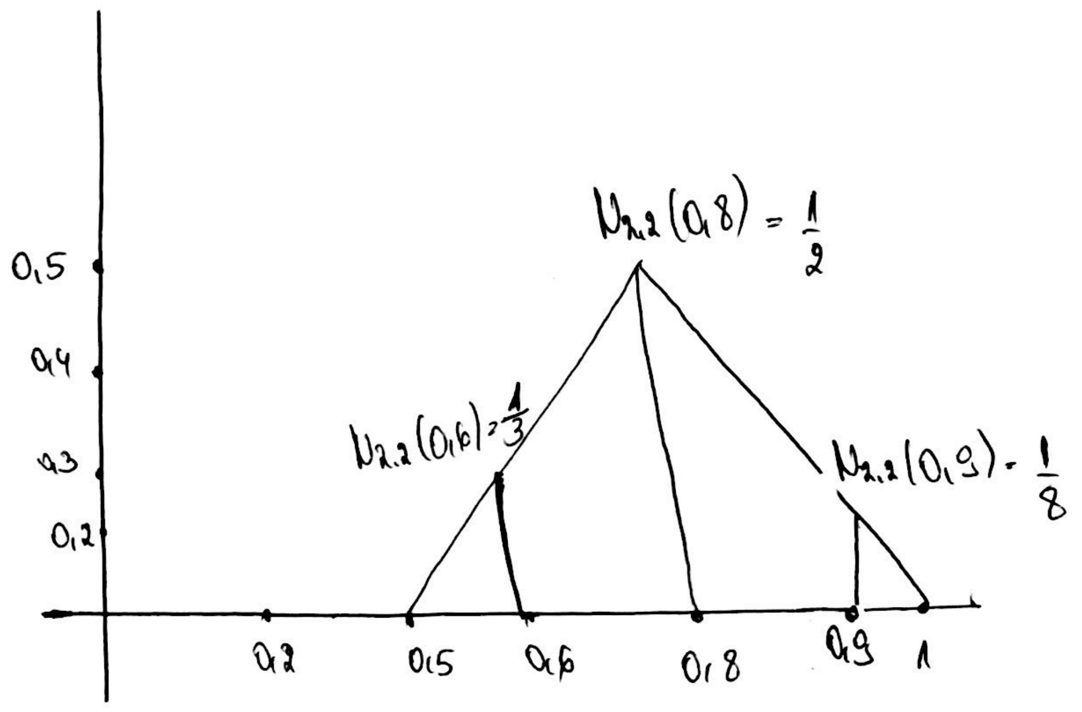
$$N_{2,2}(u) = \frac{5}{3}(2u-1) \cdot (4-5u) + \frac{5}{2}(1-u) \cdot (5u-3) =$$

$$N_{2,2}(u) = \frac{5}{6}(50u - 35u^2 - 17) \quad \text{при } u \in [0{,}6; 0{,}8)$$

3) за $u \in [0{,}8; 1]$, $N_{2,1}(u) = 0$, $N_{3,1}(u) = 5-5u$ $\rightarrow$ заместване в (*)

$$N_{2,2}(u) = \frac{5}{2}(1-u) N_{3,1}(u) = \frac{5}{2}(1-u)(5-5u) = \frac{25}{2}(1-u)^2 \quad \text{при } u \in [0{,}8; 1]$$

$$\Rightarrow N_{2,2}(u) = \begin{cases} \frac{25}{3}(2u-1)^2, & \text{ако } u \in [0{,}5; 0{,}6) \\ \frac{5}{6}(50u - 35u^2 - 17), & \text{ако } u \in [0{,}6; 0{,}8) \\ \frac{25}{2}(1-u)^2, & \text{ако } u \in [0{,}8; 1]. \end{cases}$$



$$N_{2,2}(0{,}6) = \frac{5}{6}\left(50.0{,}6 - 35.(0{,}6)^2 - 17\right) = \frac{1}{3}$$

$$N_{2,2}(0{,}8) = \frac{25}{2}(1-0{,}8)^2 = \frac{1}{2}$$

$$N_{2,2}(0{,}9) = \frac{25}{2}(1-0{,}9)^2 = \frac{1}{8}$$

б) $U = \{0; 0; 0{,}2; 0{,}4; 0{,}5; 0{,}5; 1\}$

Решение:

$u_0 = u_1$	u_2	u_3	$u_4 = u_5$	u_6
0	$0{,}2$	$0{,}4$	$0{,}5$	1

$[u_0, u_1)$	$[0; 0)$	$N_{0,0}$	$N_{0,1}$	$N_{0,2}$
$[u_1, u_2)$	$[0; 0{,}2)$	$N_{1,0}$	$N_{1,1}$	$N_{1,2}$
$[u_2, u_3)$	$[0{,}2; 0{,}4)$	$\boxed{N_{2,0}}$	$\boxed{N_{2,1}}$	$\boxed{N_{2,2}}$
$[u_3, u_4)$	$[0{,}4; 0{,}5)$	$\boxed{N_{3,0}}$	$\boxed{N_{3,1}}$	$N_{3,2}$
$[u_4, u_5)$	$[0{,}5; 0{,}5)$	$\boxed{N_{4,0}}$	$N_{4,1}$	
$[u_5, u_6]$	$[0{,}5; 1]$	$N_{5,0}$		

1. $N_{2,0}(u) = ?$, $N_{3,0}(u) = ?$, $N_{4,0}(u) = ?$

$$\boxed{N_{i,0}(u) = \begin{cases} 1, & \text{ако } u \in [u_i, u_{i+1}) \\ 0, & \text{ако } u \notin [u_i, u_{i+1}) \end{cases}}$$

$$N_{2,0}(u) = \begin{cases} 1, & \text{ако } u \in [0{,}2; 0{,}4) \\ 0, & \text{ако } u \notin [0{,}2; 0{,}4) \end{cases} ; \quad N_{3,0}(u) = \begin{cases} 1, & \text{ако } u \in [0{,}4; 0{,}5) \\ 0, & \text{ако } u \notin [0{,}4; 0{,}5) \end{cases}$$

$N_{4,0}(u) = 0$ за всяко $u \in [0; 1]$.

2. $N_{2,1}(u) = ?$, $N_{3,1}(u) = ?$

$$\boxed{N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)}$$

$$N_{\substack{2,1 \\ i\ p}}(u) = \frac{u - u_2}{u_3 - u_2} N_{2,0}(u) + \frac{u_4 - u}{u_4 - u_3} N_{3,0}(u)$$

$$= \frac{u - 0{,}2}{0{,}4 - 0{,}2} N_{2,0}(u) + \frac{0{,}5 - u}{0{,}5 - 0{,}4} N_{3,0}(u)$$

$$= \frac{u - 0{,}2}{0{,}2} N_{2,0}(u) + \frac{0{,}5 - u}{0{,}1} N_{3,0}(u)$$

$$= \frac{10}{2}(u - 0{,}2) N_{2,0}(u) + \frac{10}{1}(0{,}5 - u) N_{3,0}(u)$$

$$= (5u - 1) N_{2,0}(u) + (5 - 10u) N_{3,0}(u)$$

$$= (5u - 1) N_{2,0}(u) + 5(1 - 2u) N_{3,0}(u)$$

$$\Rightarrow N_{2,1}(u) = \begin{cases} 5u - 1, & \text{ако } u \in [0{,}2; 0{,}4) \\ 5(1 - 2u), & \text{ако } u \in [0{,}4; 0{,}5) \end{cases}$$

$$N_{\substack{3,1 \\ i\ p}}(u) = \frac{u - u_3}{u_4 - u_3} N_{3,0}(u) + \frac{u_5 - u}{u_5 - u_4} \underbrace{N_{4,0}(u)}_{0}$$

$$= \frac{u - 0{,}4}{0{,}5 - 0{,}4} N_{3,0}(u) = \frac{u - 0{,}4}{0{,}1} N_{3,0}(u) = \frac{10}{1}(u - 0{,}4) N_{3,0}(u)$$

$\Rightarrow N_{3,1}(u) = (10u - 4)\, N_{3,0}(u)$

$\Rightarrow N_{3,1}(u) = 10u - 4$, ако $u \in [0{,}4 ; 0{,}5)$

3. $N_{2,2}(u) = ?$

$$\boxed{N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)}$$

$$\underset{i\ p}{N_{2,2}(u)} = \frac{u - u_2}{u_4 - u_2} N_{2,1}(u) + \frac{u_5 - u}{u_5 - u_3} N_{3,1}(u)$$

$$= \frac{u - 0{,}2}{0{,}5 - 0{,}2} N_{2,1}(u) + \frac{0{,}5 - u}{0{,}5 - 0{,}4} N_{3,1}(u)$$

$$= \frac{u - 0{,}2}{0{,}3} N_{2,1}(u) + \frac{0{,}5 - u}{0{,}1} N_{3,1}(u)$$

$$= \frac{10}{3}(u - 0{,}2) N_{2,1}(u) + 10(0{,}5 - u) N_{3,1}(u)$$

$$= \left(\frac{10}{3}u - \frac{2}{3}\right) N_{2,1}(u) + (5 - 10u) N_{3,1}(u)$$

$$\boxed{N_{2,2}(u) = \frac{2}{3}(5u - 1) N_{2,1}(u) + 5(1 - 2u) N_{3,1}(u)} \quad (*)$$

$\Rightarrow$

1) за $u \in [0{,}2 ; 0{,}4)$, $N_{2,1}(u) = 5u - 1$, $N_{3,1}(u) = 0 \Rightarrow$ заместваме в $(*)$

$N_{2,2}(u) = \frac{2}{3}(5u-1).(5u-1) = \frac{2}{3}(5u-1)^2$ при $u \in [0,2;0,4)$

2) за $u \in [0,4;0,5)$, $N_{2,1}(u) = 5(1-2u)$, $N_{3,1}(u) = 10u-4$, заместваме в (*)

$$N_{2,2}(u) = \frac{2}{3}(5u-1).5.(1-2u) + 5(1-2u).(10u-4)$$

$$= 5(1-2u)\left(\frac{2}{3}(5u-1) + 10u-4\right)$$

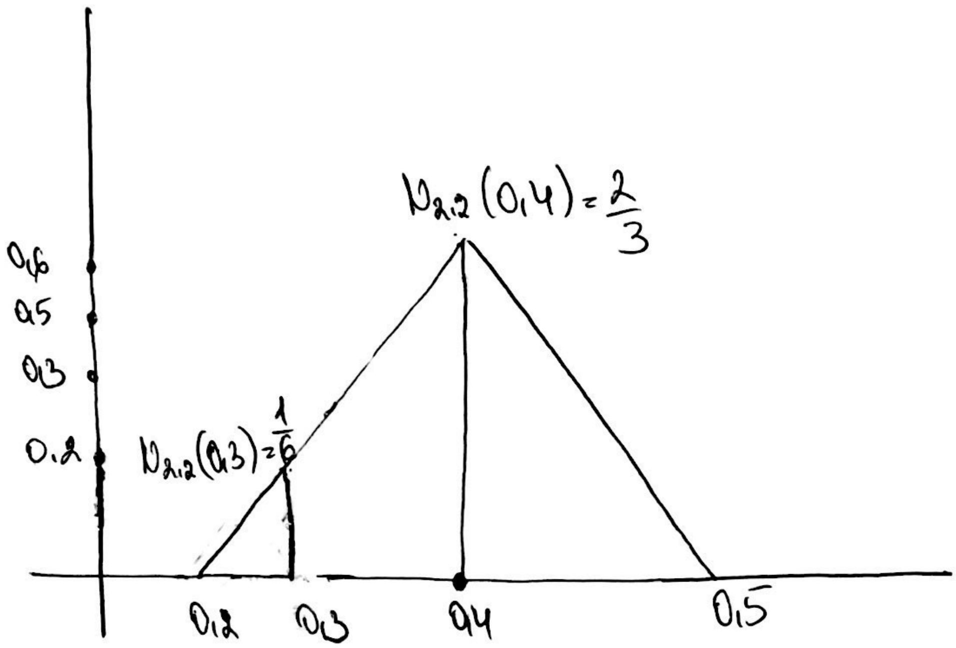
$$= 5(1-2u)\left(\frac{10}{3}u - \frac{2}{3} + 10u-4\right)$$

$$= 5(1-2u)\left(\frac{40u-14}{3}\right) = \frac{5}{3}\left(40u-14-80u^2+28u\right)$$

$$= \frac{5}{3}\left(-80u^2+68u-14\right) = \frac{10}{3}\left(-40u^2+34u-7\right)$$

$N_{2,2}(u) = \frac{10}{3}(-40u^2+34u-7)$ при $u \in [0,4;0,5)$

$$\Rightarrow N_{2,2}(u) = \begin{cases} \frac{2}{3}(5u-1)^2, \text{ ако } u \in [0,2;0,4) \\ \frac{10}{3}(-40u^2+34u-7), \text{ ако } u \in [0,4;0,5) \end{cases}$$



$$N_{2,2}(0,3) = \frac{2}{3}(5 \cdot 0,3 - 1)^2 = \frac{1}{6}$$

$$N_{2,2}(0,4) = \frac{10}{3}(-40(0,4)^2 + 34 \cdot 0,4 - 7) = \frac{2}{3}.$$