Криви на Безие.

Задача 1. Дадени са контролните точки $P_0(-1,0)$, $P_1(0,1)$, $P_2(2,0)$.

а) Запишете ур-нието на Безие кривата $C(u)$, дефинирана чрез дадените точки.

Решение:

$P_0(-1,0)$, $P_1(0,1)$, $P_2(2,0)$ $\quad n=2$ (степен на кривата на Безие).

$$\boxed{B_{n,i}(u) = \frac{n!}{i!\,(n-i)!}\,u^i(1-u)^{n-i}} \quad u \in [0,1].$$

$$B_{2,0}(u) = \frac{2!}{0!\,(2-0)!}\,u^0(1-u)^{2-0} = \underline{\underline{(1-u)^2}}$$

Забележка: $0! = 1$, $u^0 = 1$!

$$B_{2,1}(u) = \frac{2!}{1!\,(2-1)!}\,u^1(1-u)^{2-1} = \underline{\underline{2u(1-u)}}$$

$$B_{2,2}(u) = \frac{2!}{2!\,(2-2)!}\,u^2(1-u)^{2-2} = \underline{\underline{u^2}}$$

$$\boxed{C(u) = \sum_{i=0}^{n} B_{n,i}(u)\,P_i}$$

$$C(u) = B_{2,0}(u)P_0 + B_{2,1}(u)P_1 + B_{2,2}(u)P_2$$

$$C(u) = (1-u)^2(-1,0) + 2u(1-u)(0,1) + u^2(2,0)$$

$$= \left(-1(1-u)^2 + 2u^2,\ 2u(1-u)\right) = \left(-1(1-2u+u^2) + 2u^2,\ 2u(1-u)\right)$$

$$C(u) = \left(-1 + 2u + u^2,\ 2u(1-u)\right)$$

б) Запишете това ур-ние в еквивалентна параметрична форма.

$$C(u) = (1-u)^2(-1,0) + 2u(1-u)(0,1) + u^2(2,0) =$$

$$= \left(-1(1-u)^2 + 2u^2,\ 2u(1-u)\right) =$$

$$= \left(-(1-2u+u^2)+2u^2,\ 2u-2u^2\right) =$$

$$= \left(-1+2u-u^2+2u^2,\ 2u-2u^2\right) =$$

$$C(u) = \left(u^2+2u-1,\ 2u-2u^2\right).$$

в) Използвайте калкулатор и параметризацията на кривата, за да намерите точките от кривата, които съответстват на $u = 0{,}25;\ 0{,}4;\ 0{,}75$

Решение:

$$C(u) = \left(u^2+2u-1,\ 2u-2u^2\right)$$

при $u = 0{,}25 = \frac{25}{100} = \frac{1}{4}$

$$C\left(\frac{1}{4}\right) = \left(\left(\frac{1}{4}\right)^2 + 2\cdot\frac{1}{4} - 1,\ 2\cdot\frac{1}{4} - 2\cdot\left(\frac{1}{4}\right)^2\right)$$

$$= \left(\frac{1}{16} + \frac{2}{4} - 1,\ \frac{2}{4} - \frac{2}{16}\right) = \left(-\frac{7}{16}, \frac{6}{16}\right)$$

$$\Rightarrow C\left(\frac{1}{4}\right) = \left(-\frac{7}{16}, \frac{6}{16}\right)$$

при $u = 0{,}4 = \frac{4}{10} = \frac{2}{5}$

$$C\left(\frac{2}{5}\right) = \left(\left(\frac{2}{5}\right)^2 + 2\cdot\frac{2}{5} - 1,\ 2\cdot\frac{2}{5} - 2\cdot\left(\frac{2}{5}\right)^2\right)$$

$$= \left(\frac{4}{25} + \frac{4}{5} - 1,\ \frac{4}{5} - \frac{8}{25}\right) = \left(-\frac{1}{25}, \frac{12}{25}\right)$$

$$\Rightarrow C\left(\frac{2}{5}\right) = \left(-\frac{1}{25}, \frac{12}{25}\right)$$

при $u = 0{,}75 = \frac{75}{100} = \frac{3}{4}$

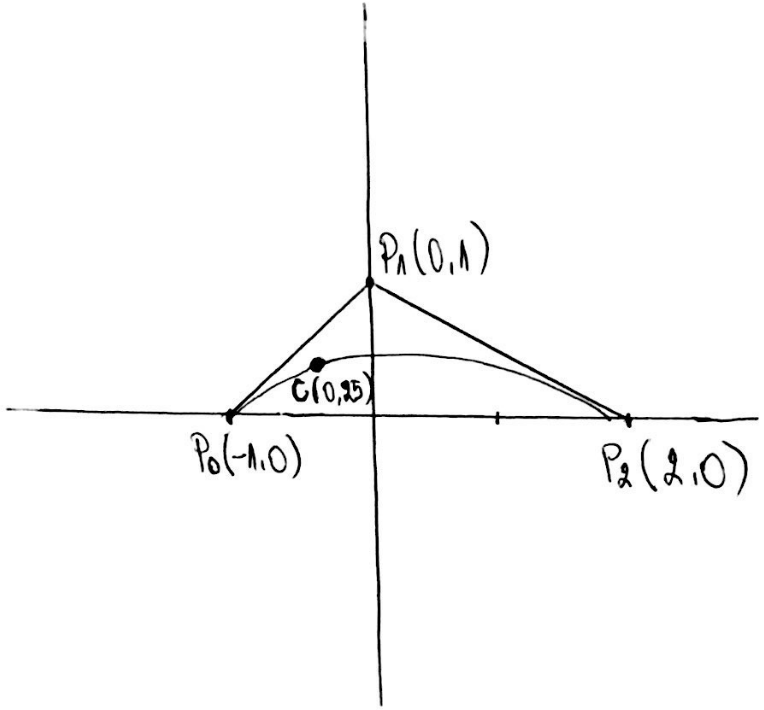
$$C\left(\frac{3}{4}\right) = \left(\left(\frac{3}{4}\right)^2 + 2\cdot\frac{3}{4} - 1,\ 2\cdot\frac{3}{4} - 2\cdot\left(\frac{3}{4}\right)^2\right) =$$

$$= \left(\frac{9}{16} + \frac{6}{4} - 1,\ \frac{6}{4} - \frac{18}{16}\right) = \left(\frac{17}{16}, \frac{6}{16}\right)$$

$$\Rightarrow C\left(\frac{3}{4}\right) = \left(\frac{17}{16}, \frac{6}{16}\right)$$

Отв: $C(0{,}25) = \left(-\frac{7}{16}, \frac{6}{16}\right)$; $C(0{,}4) = \left(\frac{-1}{25}, \frac{12}{25}\right)$; $C(0{,}75) = \left(\frac{17}{16}, \frac{6}{16}\right)$.

* чертеж на контролния полигон с $C(0{,}25)=\left(-\frac{7}{16}, \frac{6}{16}\right)$



б) намерете коефициентите на Безие при $u=0,4$ и $C(0,4)$ като използвате коеф. на Безие.

Решение:

$P_0(-1,0)$, $P_1(0,1)$, $P_2(2,0)$ $\quad n=2$ $\quad$, $u=0,4=\frac{4}{10}=\frac{2}{5}$

$$B_{n,i}(u)=\frac{n!}{i!(n-i)!}u^i(1-u)^{n-i}$$

$$B_{2,0}\left(\frac{2}{5}\right)=\frac{2!}{0!(2-0)!}\left(\frac{2}{5}\right)^0\left(1-\frac{2}{5}\right)^{2-0}=\left(\frac{5}{5}-\frac{2}{5}\right)^2=\left(\frac{3}{5}\right)^2=\frac{9}{25}$$

$$B_{2,1}\left(\frac{2}{5}\right)=\frac{2!}{1!(2-1)!}\left(\frac{2}{5}\right)^1\left(1-\frac{2}{5}\right)^{2-1}=2\cdot\frac{2}{5}\cdot\frac{3}{5}=\frac{12}{25}$$

$$B_{2,2}\left(\frac{2}{5}\right)=\frac{2!}{2!(2-2)!}\left(\frac{2}{5}\right)^2\left(1-\frac{2}{5}\right)^{2-2}=\left(\frac{2}{5}\right)^2=\frac{4}{25}$$

$$C(u)=\sum_{i=0}^{n}B_{n,i}(u)P_i$$

$$C(u)=B_{2,0}(u)P_0+B_{2,1}(u)P_1+B_{2,2}(u)P_2$$

$$C(0,4)=B_{2,0}(0,4)P_0+B_{2,1}(0,4)P_1+B_{2,2}(0,4)P_2$$

$$C(0,4)=\frac{9}{25}(-1,0)+\frac{12}{25}(0,1)+\frac{4}{25}(2,0)=$$

$$=\left(-\frac{9}{25}+\frac{8}{25};\ \frac{12}{25}\right)=\left(-\frac{1}{25},\ \frac{12}{25}\right)$$

$$\Rightarrow C(0,4)=\left(-\frac{1}{25},\ \frac{12}{25}\right)$$

г) Използвайте алгоритъма на дьо Кастелжо, за да намерите точките върху кривата съответни на $u = 0{,}25$; $u = 0{,}4$. $u = 0{,}75$.

Решение:

1) за $u = 0{,}25$

$$
\begin{tikzcd}[column sep=small]
P_0(-1,0) \arrow[dr] & & \\
 & P_{10}\left(-\frac{3}{4}, \frac{1}{4}\right) \arrow[dr] & \\
P_1(0,1) \arrow[ur] \arrow[dr] & & P_{20}\left(-\frac{7}{16}, \frac{6}{16}\right) = C(0{,}25) \\
 & P_{11}\left(\frac{2}{4}, \frac{3}{4}\right) \arrow[ur] & \\
P_2(2,0) \arrow[ur] & &
\end{tikzcd}
$$

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$$\boxed{\begin{array}{l} P_{1i} = (1-u_0)P_i + u_0 P_{i+1} \\ \searrow \cdot (1-u_0) \\ \nearrow \cdot u_0 \end{array}}$$

$u = 0{,}25 = \frac{25}{100} = \frac{1}{4}$, $1-u = 1-\frac{1}{4} = \frac{3}{4}$

$P_{10} = (1-u)P_0 + uP_1 = \frac{3}{4}(-1,0) + \frac{1}{4}(0,1) = \left(-\frac{3}{4}, \frac{1}{4}\right)$

$P_{11} = \frac{3}{4}P_1 + \frac{1}{4}P_2 = \frac{3}{4}(0,1) + \frac{1}{4}(2,0) = \left(\frac{2}{4}, \frac{3}{4}\right)$

$P_{20} = \frac{3}{4}P_{10} + \frac{1}{4}P_{11} = \frac{3}{4}\left(-\frac{3}{4}, \frac{1}{4}\right) + \frac{1}{4}\left(\frac{2}{4}, \frac{3}{4}\right) = \left(-\frac{7}{16}, \frac{6}{16}\right)$

2) $u = 0{,}4 \Rightarrow u = \frac{4}{10} = \frac{2}{5}$ $\quad 1-u = 1-\frac{2}{5} = \frac{5}{5} - \frac{2}{5} = \frac{3}{5}$

$$\begin{tikzcd}[column sep=small, row sep=tiny]
P_0(-1,0) \arrow[dr] & & \\
 & P_{10}\left(-\frac{3}{5}, \frac{2}{5}\right) \arrow[dr] & \\
P_1(0,1) \arrow[ur] \arrow[dr] & & P_{20}\left(\frac{-1}{25}, \frac{12}{25}\right) = C(0{,}4). \\
 & P_{11}\left(\frac{4}{5}, \frac{3}{5}\right) \arrow[ur] & \\
P_2(2,0) \arrow[ur] & &
\end{tikzcd}$$

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$P_{10} = (1-u)P_0 + uP_1 = \frac{3}{5}(-1,0) + \frac{2}{5}(0,1) = \left(-\frac{3}{5}, \frac{2}{5}\right)$

$P_{11} = (1-u)P_1 + uP_2 = \frac{3}{5}(0,1) + \frac{2}{5}(2,0) = \left(\frac{4}{5}, \frac{3}{5}\right)$

$P_{20} = (1-u)P_{10} + uP_{11} = \frac{3}{5}\left(-\frac{3}{5}, \frac{2}{5}\right) + \frac{2}{5}\left(\frac{4}{5}, \frac{3}{5}\right) = \left(-\frac{9}{25} + \frac{8}{25}, \frac{6}{25} + \frac{6}{25}\right)$

$= \left(\frac{-1}{25}, \frac{12}{25}\right)$

3) $u = 0{,}75 \Rightarrow u = \frac{75}{100} = \frac{3}{4}$, $1-u = 1-\frac{3}{4} = \frac{1}{4}$

$$\begin{tikzcd}[column sep=small, row sep=tiny]
P_0(-1,0) \arrow[dr] & & \\
 & P_{10}\left(-\frac{1}{4}, \frac{3}{4}\right) \arrow[dr] & \\
P_1(0,1) \arrow[ur] \arrow[dr] & & P_{20}\left(\frac{17}{16}, \frac{6}{16}\right) = C(0{,}75). \\
 & P_{11}\left(\frac{6}{4}, \frac{1}{4}\right) \arrow[ur] & \\
P_2(2,0) \arrow[ur] & &
\end{tikzcd}$$

$P_{10} = (1-u)P_0 + uP_1 = \frac{1}{4}(-1,0) + \frac{3}{4}(0,1) = \left(-\frac{1}{4}, \frac{3}{4}\right)$

$P_{11} = (1-u)P_1 + uP_2 = \frac{1}{4}(0,1) + \frac{3}{4}(2,0) = \left(\frac{6}{4}, \frac{1}{4}\right)$

$P_{20} = (1-u)P_{10} + uP_{11} = \frac{1}{4}\left(-\frac{1}{4}, \frac{3}{4}\right) + \frac{3}{4}\left(\frac{6}{4}, \frac{1}{4}\right) = \left(\frac{-1}{16} + \frac{18}{16}, \frac{3}{16} + \frac{3}{16}\right)$

$P_{20} = \left(\frac{17}{16}, \frac{6}{16}\right)$

д) Разделете кривата при $u = 0{,}4$ и подредете контролните точки на получените части на кривата.

Решение: От в) г) имаме

$$C_1(u):\ u \in [0;0{,}4] - P_0(-1,0),\ P_{10}\left(-\frac{3}{5}, \frac{2}{5}\right),\ P_{20}\left(-\frac{1}{25}, \frac{12}{25}\right)$$

$$C_2(u):\ u \in [0{,}4;1] - P_{20}\left(-\frac{1}{25}, \frac{12}{25}\right),\ P_{11}\left(\frac{4}{5}, \frac{3}{5}\right),\ P_2(2,0).$$

$$\begin{array}{ccccc}
P_0(-1,0) & \searrow & & & \\
 & & P_{10}\left(-\frac{3}{5}, \frac{2}{5}\right) & \searrow & \\
P_1(0,1) & \nearrow & & & P_{20}\left(-\frac{1}{25}, \frac{12}{25}\right) = C(0{,}4) \\
 & \searrow & P_{11}\left(\frac{4}{5}, \frac{3}{5}\right) & \nearrow & \\
P_2(2,0) & \nearrow & & &
\end{array}$$

е) Увеличете степента на тази крива на 3 и подредете новото множество от контролни точки Q_i. След това увеличете степента на 4 и запишете новото множество контролни точки R_i. Начертайте съответните контролни полигони заедно с дадения, за да покажете ефекта на "отрязване на върховете".

Решение:

$P_0(-1,0),\ P_1(0,1),\ P_2(2,0)$

$n=2 \longrightarrow 3$

$Q_0,\ Q_1,\ Q_2,\ Q_3$

$$\boxed{\begin{array}{l} Q_0 = P_0 \qquad , \quad Q_{n+1} = P_n \\ Q_i = \dfrac{i}{n+1} P_{i-1} + \left(1 - \dfrac{i}{n+1}\right) P_i \quad , \ i = 1, 2, \dots, n \end{array}}$$

$Q_0 = P_0 \Rightarrow Q_0(-1,0)$

$Q_3 = P_2 \Rightarrow Q_3(2,0)$

$$Q_1 = \frac{1}{3} P_0 + \left(1 - \frac{1}{3}\right) P_1 = \frac{1}{3}(-1,0) + \frac{2}{3}(0,1) = \left(-\frac{1}{3}, \frac{2}{3}\right)$$

$$Q_2 = \frac{2}{3} P_1 + \left(1 - \frac{2}{3}\right) P_2 = \frac{2}{3}(0,1) + \frac{1}{3}(2,0) = \left(\frac{2}{3}, \frac{2}{3}\right)$$

Отг: $D(u)$: $Q_0(-1,0)$, $Q_1\left(-\frac{1}{3}, \frac{2}{3}\right)$, $Q_2\left(\frac{2}{3}, \frac{2}{3}\right)$, $Q_3(2,0)$.

$D(u)$: $Q_0(-1,0)$, $Q_1\left(-\frac{1}{3}, \frac{2}{3}\right)$, $Q_2\left(\frac{2}{3}, \frac{2}{3}\right)$, $Q_3(2,0)$

$n = 3 \Rightarrow 4$

R_0, R_1, R_2, R_3, R_4

$R_0 = Q_0 \Rightarrow R_0(-1,0)$; $R_4 = Q_3 \to R_4(2,0)$

$$R_i = \frac{i}{n+1} Q_{i-1} + \left(1 - \frac{i}{n+1}\right) Q_i$$

$$R_1 = \frac{1}{4} Q_0 + \left(1 - \frac{1}{4}\right) Q_1 = \frac{1}{4}(-1,0) + \frac{3}{4}\left(-\frac{1}{3}, \frac{2}{3}\right) = \left(-\frac{1}{3} - \frac{3}{12}, \frac{6}{12}\right)$$

$$= \left(-\frac{7}{12}, \frac{6}{12}\right) = \left(-\frac{7}{12}, \frac{1}{2}\right)$$

$$R_2 = \frac{2}{4} Q_1 + \left(1 - \frac{2}{4}\right) Q_2 = \frac{2}{4}\left(-\frac{1}{3}, \frac{2}{3}\right) + \frac{2}{4}\left(\frac{2}{3}, \frac{2}{3}\right) = \left(-\frac{2}{12} + \frac{4}{12}, \frac{4}{12} + \frac{4}{12}\right)$$

$$= \left(\frac{2}{12}, \frac{8}{12}\right) = \left(\frac{1}{6}, \frac{2}{3}\right)$$

$$R_3 = \frac{3}{4} Q_2 + \left(1 - \frac{3}{4}\right) Q_3 = \frac{3}{4}\left(\frac{2}{3}, \frac{2}{3}\right) + \frac{1}{4}(2,0) = \left(\frac{6}{12} + \frac{2}{4}, \frac{6}{12}\right)$$

$$= \left(1, \frac{1}{2}\right)$$

Отг:

$F(u)$: $R_0(-1,0)$, $R_1\left(-\frac{7}{12}, \frac{1}{12}\right)$, $R_2\left(\frac{1}{6}, \frac{2}{3}\right)$, $R_3\left(1, \frac{1}{2}\right)$, $R_4(2,0)$