Интегриране чрез субституции
(полагане)

Нека Δ_x и Δ_t са интервали и $\varphi : \Delta_t \to \Delta_x$ е диференцируема ф-я в Δ_t, като $\varphi'(t)$ е положителна (отрицателна). Нека $f : \Delta_x \to \mathbb{R}$. Тогава, ако

$$\int f(\varphi(t)) . \varphi'(t)\, dt = F(t) + C,$$ то

$$\int f(x)\, dx = F(\psi(x)) + C,$$ където $t = \psi(x)$ е обратна на ф-та $x = \varphi(t)$

1. Субституция на Хорнер –
При интеграли от функции на квадратният тричлен $ax^2 + bx + c$ често е удобно да се използва субституцията $x = t - \frac{b}{2a}$, при която

$$ax^2 + bx + c = at^2 + \frac{4ac - b^2}{4a}$$

Интеграли от вида $\int \frac{Mx+N}{ax^2+bx+c}dx$ и

$\rightarrow \int \frac{Mx+N}{\sqrt{ax^2+bx+c}}dx$, при $a, b \neq 0$ и

привеждане към ТЪХ може да се пресмятат с помощта на Хорнеровата субституция

① $\int \frac{7-8x}{2x^2-3x+1}dx$

субст. Хорнер. $x = t - \frac{b}{2a}$

$x = t + \frac{3}{4}$ $\qquad$ $t = x - \frac{3}{4}$

$dx = d\underbrace{\left(t+\frac{3}{4}\right)} = 1 . dt \Rightarrow dx = dt$

$\frac{7-8x}{2x^2-3x+1} \overset{x=t+3/4}{=} \frac{7-8\left(t+\frac{3}{4}\right)}{2\left(t+\frac{3}{4}\right)^2-3\left(t+\frac{3}{4}\right)+1}$

$$= \frac{1-8t}{2\left(t^2+2\cdot t\cdot\frac{3}{4}+\frac{9}{16}\right)-3t-\frac{9}{4}+1} =$$

$$= \frac{1-8t}{2t^2+\cancel{3t}+\frac{9}{8}-\cancel{3t}-\frac{9}{4}+1} = \frac{1-8t}{\underbrace{\frac{2t^2}{1}-\frac{1}{8}}_{8}} =$$

$$= \frac{8(1-8t)}{16t^2-1}$$

$$\int \frac{7-8x}{2x^2-3x+1}\,dx \overset{x=t+\frac{3}{4}}{=\!=} \int \frac{8(1-8t)}{16t^2-1}\,dt =$$

$$= 8\int \frac{1-8t}{16t^2-1}\,dt = 8\int \frac{dt}{16t^2-1} - 8\int \frac{8t}{16t^2-1}\,dt$$

$$= \overset{2}{\frac{\cancel{8}}{\cancel{4}}}\int \frac{d(4t)}{(4t)^2-1^2} - \overset{2}{\frac{\cancel{64}}{\cancel{2\cdot 16}}}\int \frac{d(16t^2-1)}{16t^2-1} =$$

$$= \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot \ln\left|\frac{4t-1}{4t+1}\right| - 2\ln\left|16t^2-1\right| + C =$$

$$= \ln\left|\frac{4(x-\frac{3}{4})-1}{4(x-\frac{3}{4})+1}\right| + 2\ln\left|16\cdot(x-\tfrac{3}{4})^2-1\right| + C$$

$$= \ln\left|\frac{4x-4}{4x-2}\right| + 2\ln\left|16\left(x^2-2x\cdot\frac{3}{4}+\frac{9}{16}\right)\right| + C$$

$$= \ln\left|\frac{2x-2}{2x-1}\right| + 2\ln\left|16x^2-24+9\right| + C$$

② $\displaystyle\int \frac{3x-2}{\sqrt{x^2+6x+2}}\,dx$

субст. на Хорнер $x = t - \frac{b}{2a}$

$x = t - 3 \qquad t = x+3$

$dx = d(t-3) = dt$

$$\int \frac{3x-2}{\sqrt{x^2+6x+2}}\,dx \overset{x=t-3}{=\!=} \int \frac{3(t-3)-2}{\sqrt{(t-3)^2+6(t-3)+2}}\,dt$$

$$= \int \frac{3t-11}{\sqrt{t^2 - \cancel{6t} + 9 + \cancel{6t} - 18 + 2}}\, dt = \int \frac{3t-11}{\sqrt{t^2 - 7}}\, dt$$

$$= \int \frac{3t}{\sqrt{t^2-7}}\, dt - \int \frac{11}{\sqrt{t^2-7}}\, dt =$$

$$= \frac{3}{2} \int \frac{d(t^2-7)}{(t^2-7)^{\frac{1}{2}}} - 11 \int \frac{dt}{\sqrt{t^2-(\sqrt{7})^2}} =$$

$$= \frac{3}{2} \int (t^2-7)^{-\frac{1}{2}} d(t^2-7) - 11 \int \frac{dt}{\sqrt{t^2-(\sqrt{7})^2}} =$$

$$= \frac{3}{\cancel{2}} \cdot \frac{\sqrt{t^2-7}}{\frac{1}{\cancel{2}}} - 11 \cdot \ln\left| t + \sqrt{t^2-7} \right| + C$$

$$= 3 \cdot \sqrt{t^2-7} - 11 \cdot \ln\left| t + \sqrt{t^2-7} \right| + C$$

$$\overset{t = x+3}{=} 3 \cdot \sqrt{(x+3)^2 - 7} - 11 \cdot \ln\left| x+3 + \sqrt{(x+3)^2-7} \right| + C$$

$$= 3 \cdot \sqrt{x^2+6x+2} - 11 \cdot \ln\left| x+3+\sqrt{x^2+6x+2} \right| + C$$

③ $\int \frac{x}{x^4-4x^2+3}\,dx = \frac{1}{2}\int \frac{dx^2}{x^4-4x^2+3}$

Пол. $y = x^2 \Rightarrow$

$\frac{1}{2}\int \frac{dy}{y^2-4y+3} \overset{y=t+2}{=} \frac{1}{2}\int \frac{dt}{(t+2)^2-4(t+2)+3}$

$\left| \begin{array}{l} y = t+2 \qquad t = y-2 \\ dy = d(t+2) = dt \end{array} \right.$

$= \frac{1}{2}\int \frac{dt}{t^2+\cancel{4t}+4-\cancel{4t}-8+3} = \frac{1}{2}\int \frac{dt}{t^2-1^2}$

$= \frac{1}{2}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{4}\cdot\ln\left|\frac{t-1}{t+1}\right| + C$

$\overset{t=y-2}{=} \frac{1}{4}\cdot\ln\left|\frac{y-3}{y-1}\right| + C \overset{y=x^2}{=}$

$= \frac{1}{4}\cdot\ln\left|\frac{x^2-3}{x^2-1}\right| + C$

④ $\int \frac{x^{3=x\cdot x^2}}{3x^4-2x^2+1}\,dx = \frac{1}{2}\int \frac{x^2\,dx^2}{3x^4-2x^2+1} \overset{x^2=y}{=}$

Пол. $x^2 = y$

$= \frac{1}{2}\int \frac{y\,dy}{3y^2-2y+1} \to$ хорнер.

⑤ $\int \frac{\cos x}{\sin^2 x - 6\sin x + 12}\,dx =$

$= \int \frac{d(\sin x)}{\sin^2 x - 6\sin x + 12} \overset{\sin x = y}{=}$

Пол. $\sin x = y$

$= \int \frac{dy}{y^2-6y+12} \to$ хорнер

(7) $$\int \frac{e^x}{\sqrt{1+e^x+e^{2x}}}\,dx =$$

$$= \int \frac{d(e^x)}{\sqrt{1+e^x+(e^x)^2}} \overset{y=e^x}{=} \int \frac{dy}{\sqrt{y^2+y+1}}$$

$\to$ хорнер.

Пер. $y = e^x$

(8) $$\int \frac{1}{x} \cdot \frac{\ln x}{\sqrt{1-4\ln x-\ln^2 x}}\,dx =$$

$$= \int \frac{\ln x \, d(\ln x)}{\sqrt{1-4\ln x-\ln^2 x}} \overset{\ln x = y}{=}$$

Пер. $\ln x = y$

$$= \int \frac{y\,dy}{\sqrt{-y^2-4y+1}}$$ $\to$ хорнер

2. Реципрочна субституция

При решаването на интеграли от вида $\int \frac{dx}{(mx+n)\cdot\sqrt{ax^2+bx+c}}$ и приводими към тях може да се пресметнат с помощта на реципрочната субституция $mx+n=\frac{1}{t}$

(1) $\int \frac{dx}{(x+1)\sqrt{x^2+1}}$

$m=1 \quad n=1$

$x+1=\frac{1}{t} \qquad x=\frac{1}{t}-1$

$t(x+1)=1 \Rightarrow t=\frac{1}{x+1}$

$dx=d\left(\frac{1}{t}-1\right)$

$\left(\frac{1}{t}-1\right)'=\left(t^{-1}-1\right)'=-t^{-2}=-\frac{1}{t^2}$

$$\int \frac{-1}{\frac{1}{t}\cdot\sqrt{\left(\frac{1}{t}-1\right)^2+1}}\,\frac{1}{t^2}\,dt =$$

$$= -\int \frac{dt}{\frac{t^2}{t}\cdot\sqrt{\underbrace{\frac{1}{t^2}-\frac{2}{t}+\frac{2}{1}}_{t^2}}} =$$

$$= -\int \frac{dt}{t\cdot\sqrt{\frac{1-2t+2t^2}{t^2}}} =$$

$$= -\int \frac{dt}{t\cdot\frac{\sqrt{2t^2-2t+1}}{|t|}} =$$

$$= -\int \frac{dt}{\frac{t}{|t|}\cdot\sqrt{2t^2-2t+1}}$$

1 сл. $t > 0 \to -\int \frac{dt}{\sqrt{2t^2-2t+1}}$ → хор.

$t < 0$ $\qquad \int \frac{dt}{\sqrt{2t^2-2t+1}} \to$ Хорнер

Формула за интегриране по части

Ако $U = U(x)$ и $V = V(x)$ имат непрекъснати производни в интервала D, в сила е следната формула за интегриране по части:

$$\int u(x)\, d\, v(x) = u(x) . v(x) - \int v(x)\, d(u(x))$$

Някои интеграли от вида $\int f(x) . g(x)\, dx$, където f е алгебрична функция, а g е една от функциите: $\ln^n \varphi(x)$, $\arcsin^n \varphi(x)$, $\arccos^n \varphi(x)$, $\mathrm{arctg}^n \varphi(x)$ и $\mathrm{arccotg}^n \varphi(x)$, където

$n \in \mathbb{N}$, а φ е алгебрична функция се пресмятат с помощта на формулата за интегриране по части, като предварително f се внася под диференциала.

① $\int \ln x \, dx = x . \ln x - \int x . d \ln x =$

$= x . \ln x - \int \cancel{x} . \frac{1}{\cancel{x}} \, dx =$

$= x . \ln x - x + C$

② $\int \arccos x \, dx = x . \arccos x - \int x \, d \arccos x$

$= x . \arccos x - \int x . \left(- \frac{1}{\sqrt{1-x^2}} \right) dx =$

$= x . \arccos x + \int \frac{x}{\sqrt{1-x^2}} \, dx =$

$$= x \cdot \arccos x - \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}} d(-x^2+1) =$$

$$= x \cdot \arccos x - \frac{1}{\cancel{2}} \frac{\sqrt{1-x^2}}{\frac{1}{\cancel{2}}} + C =$$

$$= x \cdot \arccos x - \sqrt{1-x^2} + C$$

③ $$\int x^2 \ln x \, dx =$$

$$\left| \int x^2 dx = \frac{1}{3} x^3 + C \right.$$

$$= \frac{1}{3}\int \ln x \, d x^3 = \frac{1}{3} \cdot x^3 \cdot \ln x - \frac{1}{3}\int x^3 d(\ln x) =$$

$$= \frac{1}{3} x^3 \cdot \ln x - \frac{1}{3} \cdot \int x^{\cancel{3}\,2} \cdot \frac{1}{\cancel{x}} dx =$$

$$= \frac{1}{3} x^3 \cdot \ln x - \frac{1}{3}\int x^2 dx = \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C$$

④ $\int \frac{\arcsin x}{\sqrt{1+x}}\,dx = \int \frac{1}{\sqrt{1+x}} \cdot \arcsin x\, dx$

$\int \frac{1}{\sqrt{1+x}}\,dx = \int (1+x)^{-\frac{1}{2}}\, d(x+1) =$

$= 2 \cdot \sqrt{1+x} + C$

$= \int \arcsin x \; d\left(2\sqrt{1+x}\right) =$

$= 2 \int \arcsin x \; d\left(\sqrt{1+x}\right) =$

$= 2 \cdot \sqrt{1+x} \cdot \arcsin x - 2 \int \sqrt{1+x}\; d(\arcsin x) =$

$= 2 \cdot \sqrt{1+x} \cdot \arcsin x - 2 \int \frac{\sqrt{1+x}}{\sqrt{1-x^2}}\,dx =$

$= 2 \cdot \sqrt{1+x} \cdot \arcsin x - 2 \int \frac{\cancel{\sqrt{1+x}}}{\sqrt{1-x} \cdot \cancel{\sqrt{1+x}}}\,dx =$

$= 2 \cdot \sqrt{1+x} \cdot \arcsin x + 2 \int (1-x)^{-\frac{1}{2}}\, d(-x+1) =$

$$= 2\sqrt{1+x} \cdot \arcsin x + 4\sqrt{1-x} + C$$

(5) $$\int x \cdot \operatorname{arctg}^2 x \, dx = \frac{1}{2}\int \operatorname{arctg}^2 x \, dx^2 =$$

$$= \frac{1}{2} x^2 \cdot \operatorname{arctg}^2 x - \frac{1}{2}\int x^2 \, d\left(\operatorname{arctg}^2 x\right) =$$

$$= \frac{1}{2} x^2 \cdot \operatorname{arctg}^2 x - \frac{1}{2}\int x^2 \cdot 2 \operatorname{arctg} x \cdot \frac{1}{1+x^2} \, dx =$$

$$= \frac{1}{2} x^2 \cdot \operatorname{arctg}^2 x - \boxed{\int \frac{x^2}{1+x^2} \cdot \operatorname{arctg} x \, dx}$$

$$\int \frac{x^2}{1+x^2} \cdot \operatorname{arctg} x \, dx = \int \operatorname{arctg} x \, d(x - \operatorname{arctg} x)$$

$$\int \frac{x^2+1-1}{1+x^2} \, dx = \int \cancel{\frac{x^2+1}{1+x^2}} \, dx - \int \frac{dx}{1+x^2} =$$

$$= x - \operatorname{arctg} x$$

Интеграли от вида

$\int P(x)\sin\alpha x\,dx$, $\int P(x)\cos\alpha x\,dx$,

$\int P(x)e^{\alpha x}$, където $P(x)$ е полином

на x, се пресмятат с помощта на формулата за интегриране по части, като предварително се внасят $\sin\alpha x$, $\cos\alpha x$, $e^{\alpha x}$ под знака на диференциране.

① $\int x\cdot\cos x\,dx = \int x\,d\sin x =$

$= x\cdot\sin x - \int \sin x\,dx =$

$= x\cdot\sin x + \cos x + C$

② $\int (x^2-2x)\sin 3x\, dx =$

$\left| \frac{1}{3}\int \sin 3x \; d3x = -\frac{1}{3}\cos 3x \right.$

$= -\frac{1}{3}\int (x^2-2x)\, d(\cos 3x) =$

$= -\frac{1}{3}(x^2-2x) \cdot \cos 3x + \frac{1}{3}\int \cos 3x \, d(x^2-2x) =$

$= \underbrace{-\frac{1}{3}(x^2-2x) \cdot \cos 3x + \frac{1}{3}}\int (2x-2) \cdot \cos 3x \, dx =$

$\int (2x-2) \cdot \cos 3x \, dx =$

$\left| \frac{1}{3}\int \cos 3x \, d3x = \frac{1}{3}\sin 3x + C \right.$

$= \frac{1}{3}\int (2x-2)\, d \sin 3x = \frac{1}{3}(2x-2) \cdot \sin 3x - \frac{1}{3}\int \sin 3x \, d(2x)$

$$\frac{1}{3}(2x-2).\sin 3x-\frac{1}{3}\int \sin 3x\, d(2x-2)=$$

$$=\frac{1}{3}(2x-2).\sin 3x-\frac{1}{3.3}\int 2.\sin 3x\, d3x=$$

$$=\frac{1}{3}(2x-2)\sin 3x-\frac{2}{9}\int \sin 3x\, d3x=$$

$$=\frac{1}{3}(2x-2)\sin 3x+\frac{2}{9}\cos 3x+C$$

Отг:

$$-\frac{1}{3}(x^2-2x).\cos 3x+\frac{1}{9}(2x-2)\sin 3x+\frac{2}{27}\cos 3x+C$$

Пресмятане на интеграли, при които се получава линейно уравнение или линейна система относно търсения интеграл.

$$I = \int \sqrt{x^2+9}\,dx = x \cdot \sqrt{x^2+9} - \int x\, d\sqrt{x^2+9} =$$

$$\Big|\ \left(\sqrt{x^2+9}\right)' = \frac{1}{\cancel{2}\sqrt{x^2+9}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+9}}$$

$$= x \cdot \sqrt{x^2+9} - \int \frac{x^2+9-9}{\sqrt{x^2+9}}\,dx =$$

$$= x \cdot \sqrt{x^2+9} - \int \frac{(x^2+9)^{\cancel{1}\ \frac{1}{2}}}{(x^2+9)^{\frac{1}{2}}}\,dx + 9\int \frac{dx}{\sqrt{x^2+9}} =$$

$$= x \cdot \sqrt{x^2+9} - \underbrace{\int \sqrt{x^2+9}\,dx}_{I} + \ln\left|x+\sqrt{x^2+9}\right| + C$$

$$I = x \cdot \sqrt{x^2+9} - I + \ln\left|x+\sqrt{x^2+9}\right| + C$$

$$2I = x \cdot \sqrt{x^2+9} + \ln\left|x+\sqrt{x^2+9}\right| + C$$

$$I = \frac{x \cdot \sqrt{x^2+9}}{2} + \frac{\ln\left|x+\sqrt{x^2+9}\right|}{2} + C$$

С помощта на ф-ла за интегр. по части да се намерят рекурентни ф-ли за интегралите:

$$I_n = \int \ln^n x\, dx = x \cdot \ln^n x - \int x\, d(\ln^n x) =$$

$$= x \cdot \ln^n x - \int \cancel{x} \cdot n \cdot \ln^{n-1} x \cdot \frac{1}{\cancel{x}}\, dx =$$

$$= x \cdot \ln^n x - n \underbrace{\int \ln^{n-1} x\, dx}_{I_{n-1}} \qquad I_1 = \ldots$$

$$\boxed{I_n = x \cdot \ln^n x - n \cdot I_{n-1}}$$

$$I_2 = x \cdot \ln^2 x - 2 \cdot I_1$$

$$I_3 = x \cdot \ln^3 x - 3 \cdot (x \cdot \ln^2 x - 2 I_1) -$$