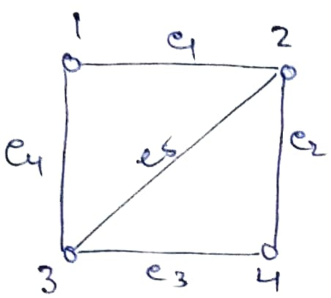
[Do all the question of standard book]

Graph Theory

[Imp. topic. So read it from notes and do questions]

A graph is a triple consisting of vertex set V[G], an edge set E[G], and the relation that associate with each edge, two vertices [not necessarily distinct] called its end points.



Cardinality

$V(G) = [1,2,3,4]$ $\quad |V(G)| = 4$

$E(G) = \{e_1, e_2, e_3, e_4, e_5\}$ $\quad |E(G)| = 5$

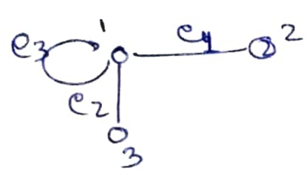
Ex → for undirected graph

$e_1 \to \{1,2\}$ → set of vertices

for directed graph

$e_1 \to (1,2)$ → order pair.

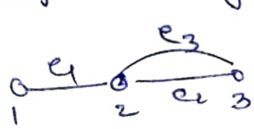
Loop → Edge with same starting and ending points.



$e_1 \to \{1,2\}$ $\quad e_3 = \{1,1\}$

$e_2 \to \{1,3\}$

Multiple edges having the same pair of end points.



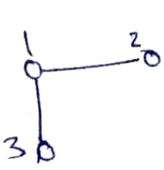
$G(V,E)$

$e_1 \to \{1,2\}$ $\quad e_2, e_3 \to \{2,3\}$

Graph with multiple edges called multi graph.

Graph with no loops and multi edges called simple graph.

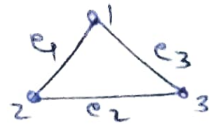
Two vertices joined through single edge called adjacent.



(1,2) and (1,3) are adjacent but (2,3) are not

Representation of Graph.

① we can list the vertices and edges.



$G(V,E)$

$V(G) = \{1,2,3\}$

$E(G) = \{e_1, e_2, e_3\}$

$e_1 \to \{1,2\}$

$e_2 \to \{2,3\}$

$e_3 \to \{1,3\}$

(2) Adjacency matrix $A(G)$:

$n \times n$ matrix in which entry a_{ij} is the number of edges in G with end points (v_i, v_j)

$$A(G) = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \Rightarrow \text{Symmetric for simple graph}$$

for every undirected simple graph –

① $A = A^T$

② Principal diagonal are all zeroes (bcoz No loop)

Degree of vertex → No. of ~~vertex~~ edges incident on the vertex.

Sum of all elements in row gives the degree of vertex represent by A.

$$A = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \to 2 \quad D(1) = 2$$

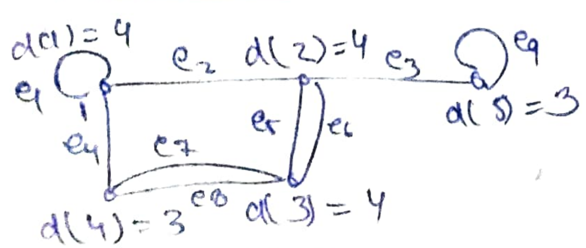
Hand Shaking Lemma → In any graph $G(V,E)$ the sum of degree of vertices is twice the number of edges.

$$\boxed{\sum_{v \in V} d(v) = 2\,|E(G)|}$$

[bcoz two odd combine to given even number]

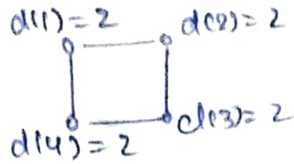
→ Number of odd degree vertices are even.

→ A loop is provide degree 2 to vertex.

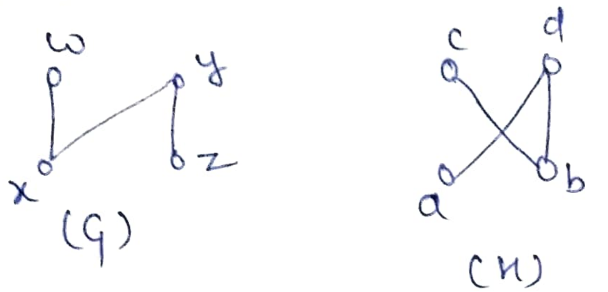


$4+4+4+3+3 = \underline{18}$

In a graph if all the vertices are having even degree called <u>even graph</u>.



<u>Isomorphism</u> :- An isomorphic from a simple graph G to simple graph H is a bisection $f: V(G) \rightarrow V(H)$ such that $uv \in E(G)$ iff $f(u)f(v) \in E(H)$.



$f: V(G) \rightarrow V(H)$

$w \rightarrow c$
$x \rightarrow b$
$y \rightarrow d$
$z \rightarrow a$

one-one onto

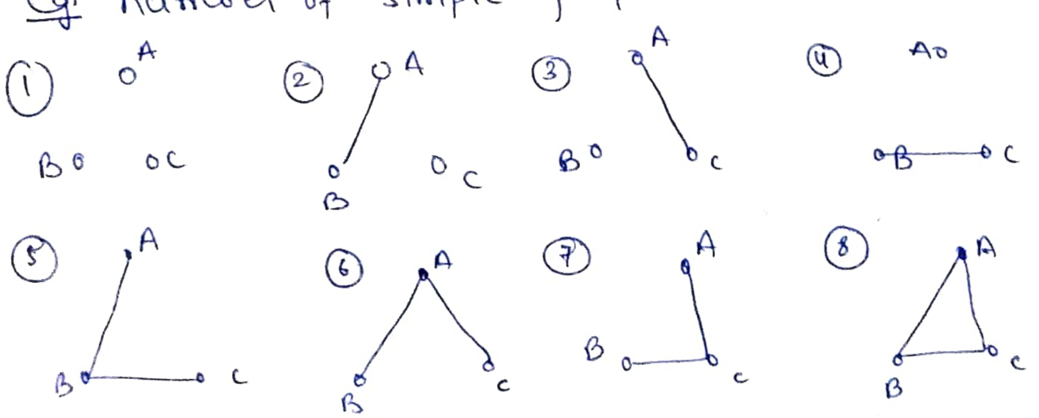
- Isomorphism in graph with same number of edges and vertices.
- Degree sequence must be same for both.
- Both have same adjacency matrix with different order of row.

eg
$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & a & b & c \\ 3 & & & \end{array} \qquad \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & a & b & c \\ 2 & & & \\ 3 & & & \end{array}$$
(G) (H)

we can acheive the same adjacency matrix with some row operations.

<u>Number of simple Graph</u> : The number of simple graph with n-vertices $= \boxed{2^{(^nC_2)}} \rightarrow$ max. no. of edges possible in simple graph

eg. number of simple-graph with 3-vertices $\rightarrow$ $\underline{8}$

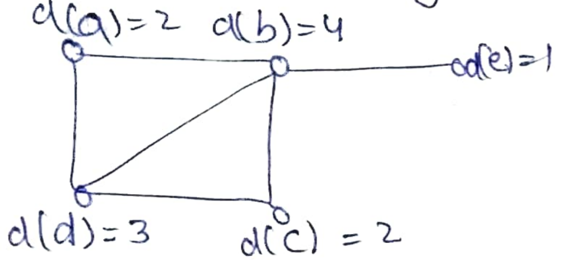


E_1	E_2	E_3
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

→ Here all the vertices are lebelled.

→ A complete graph conteins $\boxed{(^nC_2 \text{ edges})}$ ★★★

Degree sequence → The arrangement of degree in non-ascending or non-descending order.



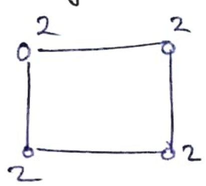
<4, 3, 2, 2, 1> ← Degree sequence

or

<1, 2, 2, 3, 4>

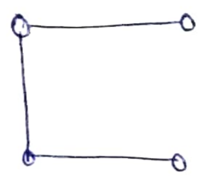
Ex:- find a graph for given degree sequence.

① Ex-1 : <2, 2, 2, 2>



→ ~~Graph of n vertices can have max. degree~~

Ex-2 < 3, 2, 1, 1, 0 >

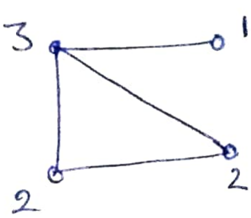


3+2+1+1 = 7
odd

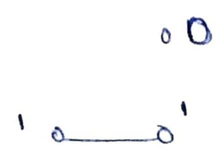
→ Since, sum of degree is odd so no graph can be formed by hand shaking lemma.

Imp → If sum of degree sequence is even then you can't confirmly say that graph is exist or not. for this purpose we have -

Havel-Hakimi Procedure [Do 5 question for practice on this procedure]



on deleting 3



→ On deleting a vertex decrease the degree the vertices adjacent to them, by one '1'.

Ex-3 $\langle 7, 6, 5, 4, 4, 3, 2, 1 \rangle \rightarrow S_1$
sum = 32

vertex with $d(v) = 7$ means it is adjacent to 7 more vertices
on delete 7 reduce the 1 degree from each vertex

$\langle 5, 4, 3, 3, 2, 1, 0 \rangle \rightarrow S_2$ Sum - 18

It is known that if graph is existing with degree sequence S_1 then it must be exist with S_2.

On deleting 5

$\langle 3, 2, 2, 1, 0, 0 \rangle \rightarrow S_3$ Sum = 8

On deleting 3

$\langle 1, 1, 0, 0, 0 \rangle \rightarrow S_4$ Sum = 2

On deleting 1

$\langle 0, 0, 0, 0 \rangle \rightarrow S_4$

so there can be graph with S_4.

✓ You can stop anywhere in between where you can assume that graph can exist.

imp. Min and Max degree [By heart this]
(δ) (Δ)

average degree

Relation: $\boxed{\delta \leq \frac{2|E|}{|V|} \leq \Delta}$

In a degree sequence $\langle 1, 2, 2, 3 \rangle$
$\swarrow (\delta)$ $\searrow (\Delta)$

Exam min and max. degree :

① G is a graph with 11 edges and $\delta = 3$. what is maximum no. of vertices

$$\boxed{\delta |V| \leq 2|E|}$$

$$|V| \leq \frac{22}{3} \approx 7 \ldots$$ take floor $\lfloor\ \rfloor$

$$\boxed{|V| = 7}_{max.}$$

② Graph with 12 vertices and $\Delta = 4$. find max. no. of edges.

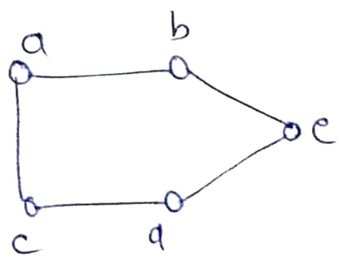
$$\boxed{2|E| \leq |V|\Delta}$$

$$|E| \leq \frac{48}{2}$$

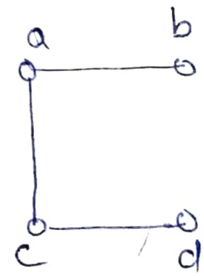
$$\boxed{|E| = 24}_{max.}$$

Note : Incase of decimal take floor. for max.

Sub-Graph : A sub-graph of graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, the assignments of end points to edges in H is same as in G.



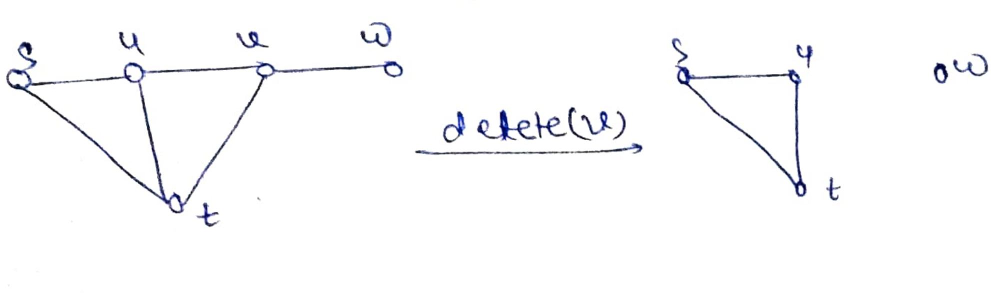
(G)



(H)

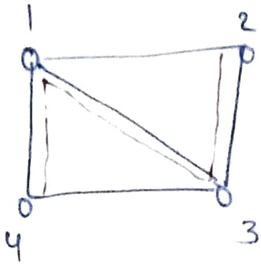
→ A graph is also the sub-graph of its own.

Induced sub-graph → It is a subgraph obtained by deleting vertices.

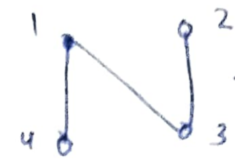


Path : 'A path is a simple graph' whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.

$P_n \rightarrow$ Path with n- vertices.

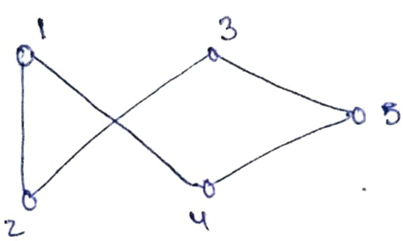


$P_4 = < 4, 1, 3, 2 \}$



$\rightarrow$ Sub graph

Cycle $\rightarrow$ Cycle is a graph 'with equal vertices and edges', whose vertices can be placed around a circle so that two vertices are adjacent if and only if 'they appear consequetively along circle'



$C_5 = \{ 1, 4, 5, 3, 2, 1 \}$

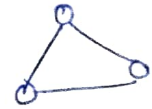
$\rightarrow$ Degree of each vertex is 2.

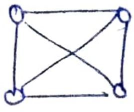
$\rightarrow$ If there is a cycle then we can acheive a path by just removing an edge, but we can't acheive a cycle from path (without including anything from outside).

Complete Graph (K_n) :- Graph in which an edge connected to each other vertex, adjacent to it.

K_1

K_2 

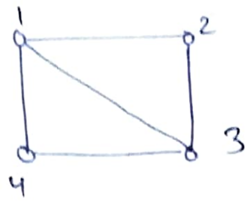
K_3 

K_4 

* Each vertex have degree 'n-1'

* Number of edges is $^nC_2$.

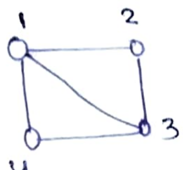
Clique → set of pairwise adjacent vertices.



{1,2} {3,4}
{1,3} {2,3}
{1,4}
] Clique with two vertices

→ $\underbrace{\{1,2,3\} \text{ and } \{1,3,4\}}_{\text{Also max. cliques (here).}}$ → Clique with 3 vertices.

Independent-Set: Set of pairwise non-adjacent.

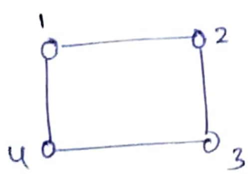


{2,4} is an independent set.

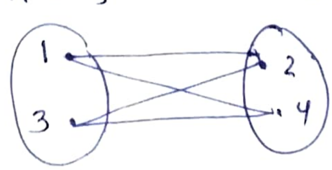
→ There is no Independent set in a complete graph.

Bipartite-Graph: A graph G whose vertices can be divided into two disjoint sets U and V such that every edge connects vertex in U and to one in V. [U and V are independent sets].

Ex)



U {1,3} V {2,4}

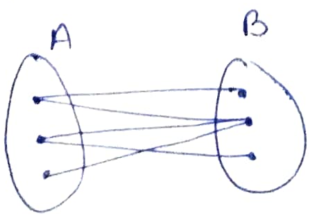


[1, 3 must have relation with both 2, 4 but not with each other] → for complete bipartite

$K_{2,2}$ → Complete bipartite with 2 on left side and 2 on RHS.

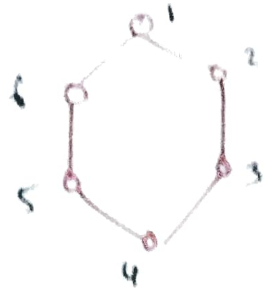
→ A complete bipartite graph $K_{m,n}$ have $m \times n$ edges and $(m+n)$ vertices.

→



→ Bipartite but not complete bipartite.

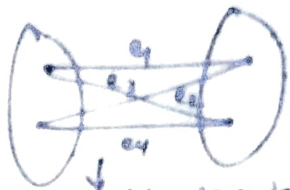
Ex →



$\{1, 3, 5\}$, $\{2, 4, 6\}$

→ A graph is bipartite if and only if it has no odd cycles

(odd no of edges in a cycle)



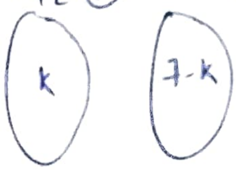
↓ 4 edges in cycle so bipartite

gmp

The maximum number of edges in bipartite graph with n-vertices -

Ex: $n=7$

$n/2 = 3$ $\quad 7-k = 4$



→ Only for a complete bipartite graph.

Total no. of edges $= k(n-k) = kn - k^2$

To get max. edges $\frac{d}{dk} f(k) = 0$

$$\Rightarrow n - 2k = 0$$

$$\Rightarrow \boxed{k = n/2} \rightarrow \text{critical point}$$

$$\frac{d^2 f(k)}{dk} = -2 < 0$$

So $f(k)$ is max. at $k = n/2$

gmb

Max. edges $= \frac{n}{2} \times \frac{n}{2} = \boxed{\frac{n^2}{4}}$

for $n=7 = \frac{49}{4} \approx 12$

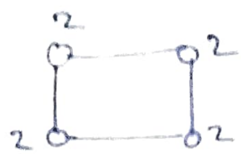
or/ $k(7-k)$

$3(4) = 12$

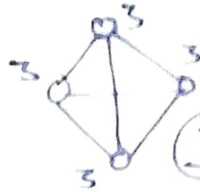
Regular Graph :- Graph in which every vertex have same degree.

n-regular graph → each vertex with degree n.

Ex



2-regular graph.



3-regular graph
↳ 4 vertices

→ If degree (n) of graph is odd then no of vertices should be even for a regular graph

→ A complete graph with 'n' vertices is (n-1) regular.

→ Every cycle graph is 2-regular.

→ No. of edges in k-regular graph with n vertices is, $\boxed{\frac{n*k}{2}}$

Imp.

Hyber-Cube or k-dimensional Cube (Q_k)

A simple graph whose vertices are the k-tuple with entries in {0, 1} and whose edges are the pairs of k-tuples that differ in exactly one position.

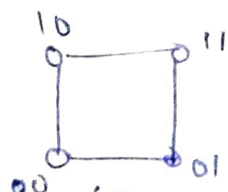
Q_1 → 1 bit

Q_2 → 2 bit

⋮

Q_k → k bit

(Q_1)



(Q_2)

→

→ In Q_k → 2^k vertices.

→ In Q_k → $k2^{k-1}$ edges.

→ In Q_k → $d(v) = k$

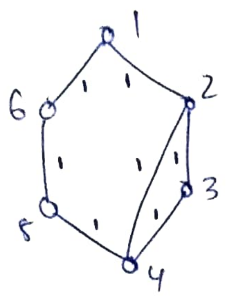
We can make a Q_k using two Q_{k-1} cubes

0 0
0 1
1 0
1 1

Edge exist between number that have difference in one bit

Diameter, radius & Eccentricity

→ If G is a uv-path then distance from 'u' to 'v' is the least length of uv path.



$d(1,2) = 1$ $d(1,6) = 1$ $d(2,6) = 2$

$d(1,3) = 2$ $d(2,3) = 1$ $d(3,4) = 1$

$d(1,4) = 2$ $d(2,4) = 1$ $d(3,5) = 2$

$d(1,5) = 2$ $d(2,5) = 2$ $d(3,6) = 3$

$d(4,5) = 1$ $d(4,6) = 2$ $d(5,6) = 1$

Consider, $d(n,k) = d(k,n)$

→ If graph have no path between u, v then $\boxed{d(u,v) = \infty}$

① **Diameter** → $\boxed{\max_{u,v \in G} d(u,v)}$

Diameter = 3 [for above One]

② **Eccentricity** [of a vertex]: $E(u) = \max_{v \in V(G)} d(u,v)$

→ from vertex 'u' max. distance to any other vertex.

$E(1) = 2$
$E(2) = 2$
$E(3) = 3$
$E(4) = 2$
$E(5) = 2$
$E(6) = 3$ → from $d(3,6) = 3$

→ Diameter is a property of graph but eccentricity is a property of vertex.

③ **Radius** → Minimum of eccentricity.

$\boxed{\min_{u \in V(G)} E(u)}$

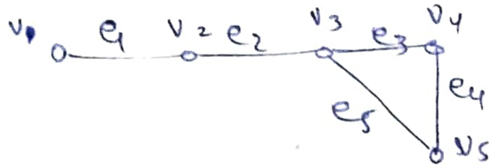
Radius = 2

Isolated → Vertex of degree 0.

Pendant → Vertex of degree ~~atleast~~ only 1.

Walk/Trail :-

walk: It is a list $v_0 e_1 v_1 \ldots e_k v_k$ of vertices and edges such that, for $1 \le i \le k$, the edge e_i has end points v_{i-1} and v_i.



→ $v_1 e_1 v_2 e_2 v_3$ is walk from v_1 to v_3

→ walk can be from any vertex to any vertex.

Trail: Walk with no repeated edge

(v_1-v_3 walk) $v_1 e_1 v_2 e_2 v_3$ → walk and trial both

(v_1-v_3 walk) $v_1 e_1 v_2 e_2 v_3 e_5 v_5 e_4 v_4 e_3 v_3$ → walk ~~and~~ trial

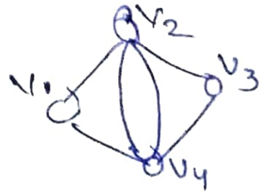
→ A u-v walk, u-v trial } u → starting vertex, v → end vertex

→ There can be more than 1 trial to a graph. as -

v_1-v_5 ~~from~~ via v_4] 2 trials.
v_1-v_4 via v_5

Eulerian Graph :-

Graph has a 'closed trial' connecting all edges.



→ Closed trial also called as circuit.

→ An Eulerian circuit in a graph is circuit connecting all edges.

→ If every vertex of a graph G has degree at least 2 then G has a cycle.

→ A graph G is Eulerian if and only if it has atmost one non-trivial component and all vertices have even degrees.

Let G be a undirected complete graph K6. If vertices of G are labelled then the number of distinct cycles of length 4 in G is ______.

→ no. of vertices → 6

Cycle 1 — 2 — 3 — 4 — 1

We need 4 vertices to form cycle.

- So, $\left(6C_4\right)\left(\frac{3!}{2}\right)$ 1 (2 3 4) 1

↓ to choose any four from 6   ↓ these 3 can be interchange with each other so 3!

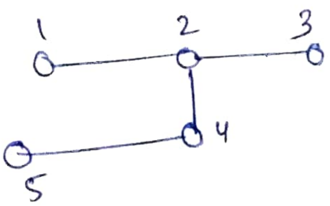
and
1 2 3 4 1
1 4 3 2 1
] both are same so no need to include twice

So, no. of Cycles = $\left(\frac{6C_4 \times 3!}{2}\right)$

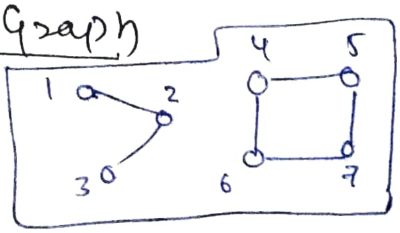
In General for kn with m vertices cycle

$$\boxed{\frac{nC_m \, (m-1)!}{2}}$$

Connected graph → You can reach any vertex from another vertex.

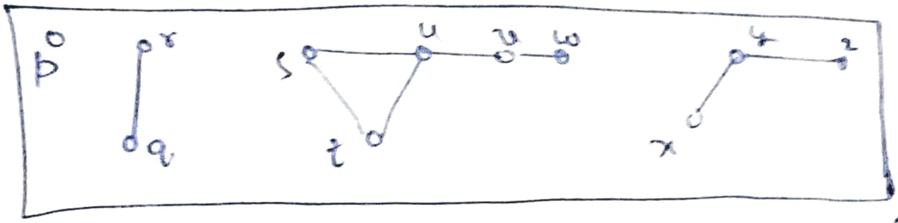


Disconnected Graph



No path to reach (4, 5, 6, 7) from any one from (1, 2, 3)

Components: The component of a graph is a sub-graph in which any two vertices are connected to each other by paths, and which is connected to no addition vertices in graph



All the components are pairwise disjoint] → no two component have connecting path

Components: {p}, {r, q}, {x, y, z} {t, s, u, v, w}
Isolated vertex.

[Adding an edge between two components reduce the components by 1 and edge with same component reduce the component by 0.

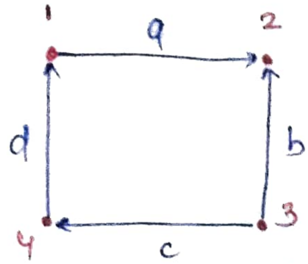
[Deleting an edge increases the number of component by 0 or 1.

→ Every graph with n vertices and k edges has atleast (n−k) components.

Directed Graph :- A directed graph or digraph G is a 'triple' consisting of a vertex set 'V(G)', an edge set 'E(G)' and 'function' assigning each edge an ordered pair of vertices.

* The first vertex of the ordered pair is the tail of edge and second is the head.

(u, v) : u ⟶ v



G(V, E)

V(G) = {1, 2, 3, 4}

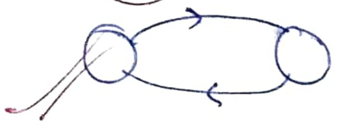
E(G) = {a, b, c, d}

a → (1, 2) b (3, 2) c → (3, 4)

d → (4, 1)

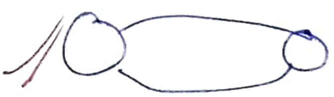
Directed Graph

→ Digraph can be simple or multiple graph (Multi).



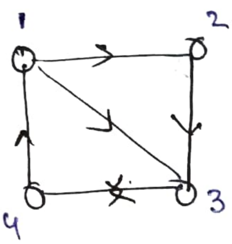
simple graph bcoz it has definite direction

but



→ not a simple graph

<u>Path</u> →

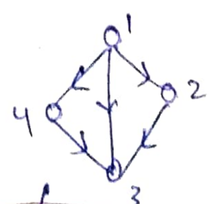


(Path)

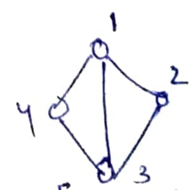
You can go from one vertex to another if tail of one connected to head of another vertex.

→ Vertex in path should be linearly ordered.

<u>Underlying Graph</u> :- The underlying graph of a digraph D is graph G obtained by treating the edge of D as unordered pairs.



→



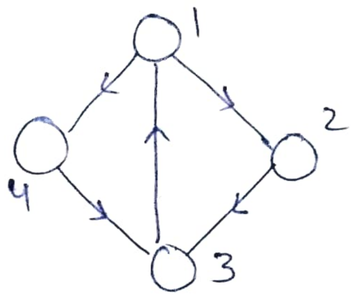
[Underlying Graph]

Not connected bcoz no path to go 4 from 2.

But it connected bcoz there is no hinderence of directions.

* A diagraph is weakly connected if its underlying graph is connected.

* A diagraph is strongly connected if for each ordered u, v vertices, there is a path from 'u' to 'v'



→ Strongly connected with a modification in previous graph.

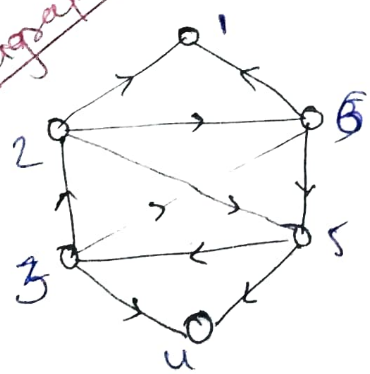
* weakly connected ⊆ Strongly connected.

weakly connected —Not Also→ Strong connected

Strongly connected —Also→ weak connected

Strong Components → The strong components of a digraph are its maximal strong subgraphs.

Strongly connected part of digraph.



{2, 3, 6, 5} → sub-graph that is strongly connected. because if you choose any two vertex from set then there is a path between u → v and v → u either direct or indirect.

Vertex, degrees :

$d^+(v)$ → out degree (tail towards v)

$d^-(v)$ → in degree (head towards (v))

$$\sum_{v\in V(G)} d^+(v) = |E(G)| = \sum_{v\in V(G)} d^-(v)$$

out-degree = no. of edges = in-degree

$\delta^-(G)$: min. in-degree

$\Delta^-(G)$: max. in-degree

$\delta^+(G)$: min. out-degree

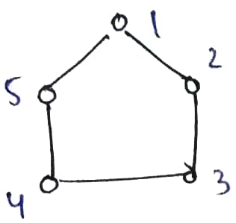
$\Delta^+(G)$: max. out degree

Average $= \frac{|E|}{|V|}$

① $\delta^-(G) \leq \frac{|E|}{|V|} \leq \Delta^-(G)$ ② $\delta^+(G) \leq \frac{|E|}{|V|} \leq \Delta^+(G)$

<u>Matching</u>:- A matching in a graph G is a set of non-loop edges with no shared end points.

<u>Ex</u> →



① $\boxed{(1,2)\ (3,4)}$ ② $\boxed{(1,5)\ (4,3)}$ ③ $\boxed{(5,4)\ (2,3)}$

and many more are matchings here

① $\boxed{(1,2)\ (3,4)}$ → Saturated and ⑤ → Unsaturated.

The vertex incident to the edges of a matching M are saturated by M and all other are unsaturated.

② A perfect matching in a graph is a matching that saturate every vertex in a graph.

Number of perfect matching in complete bipartite graph ($K_{n,n}$):

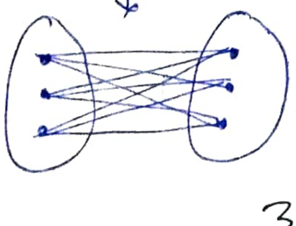
possible combinations of

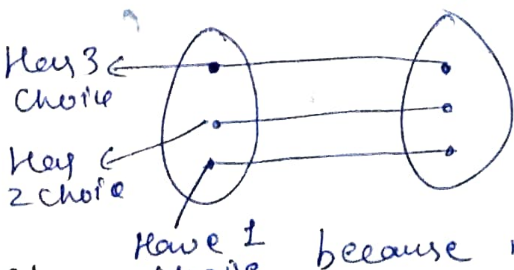
$\boxed{\text{No. of perfect matching} = n!}$

Perfect match → one-one

→ for perfect matching no. of vertices should be even.

<u>$K_{3,3}$</u>





because no adjacent vertex is common in matching.

$3 \times 2 \times 1 \rightarrow 3!$

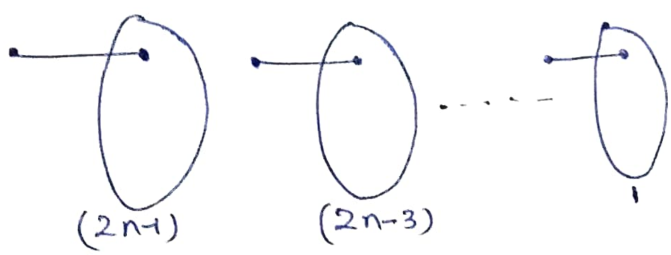
Number of perfect matching in complete graph.

⇒ K_{2n+1} has no perfect matching. (odd no. of vertices)

⇒ The number ~~of the~~ of perfect matchings in K_{2n} is the number of ways to pair up 2n distinct peoples.

→ Pair up 2n peoples to get 'n' pairs.

first person can choose from (2n-1) to form a pair, means (2n-2) get excluded from 2n after first pair, then Second will chose from (2n-3) and third from (2n-5) and so on.



so, $(2n-1)\times(2n-3)\cdots 1$ is total no of perfect matching in K_{2n}

No. of perfect matching $= \boxed{(2n-1)\times(2n-3)\times\cdots\times 1}$

OR $\boxed{\dfrac{(2n)!}{n!\,2^n}}$ → Remember both

with (2n) vertices (2n)! combinations are possible like

(1 2)(3 4)(5 6)(7 8)⋯(2n-1 2n) or

1 (2 3)(4 5)(6 7)(8 9)(10 ⋯ 2n-1) 2n

and this way you can choose 2n combinations by just changing the position of bracket

But we fix the position of brackets to form only 'n' combination as -

(1 2)(3 4)(5,6) ⋯ (2n-1, 2n) by chosing two successive terms

and these order pair need not be occur in same order so total (n!) orders are possible by changing the position of order pairs.

Similarly (1 2) and (2 1) both can consider as 2 different pair so each from the 'n' pairs contribute 2 different pairs to total,

(1.2) (3,4) ⋯ (2n-1, 2n)

$2\times 2\times 2\times 2 \cdots = 2^n$ pairs.

That is why total no. of perfect matching in K_n is

$$\Rightarrow \frac{(2n)!}{n!\,2^n} = (2n-1)\times(2n-3)\times \cdots \times 1$$

Maximal and ~~minimal~~ maximum matching [α']

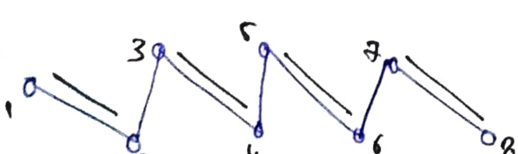
A matching contains sets of edges, so size of matching is equals to no. of edges in matching.

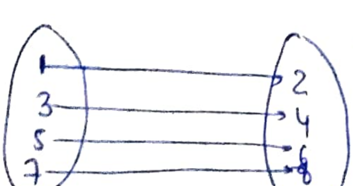
→ A maximal matching in a graph is a matching that can't be enlarged by adding an edge. It means if you can't add an edge in a matching without disturbing its property then matching is maximal.

→ A maximum matching is a matching of maximum size among all matchings in the graph.

→ Every maximum matching is maximal but each maximal is not maximum.

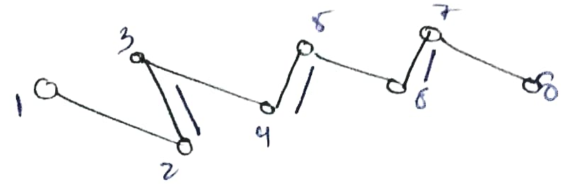
Ex →

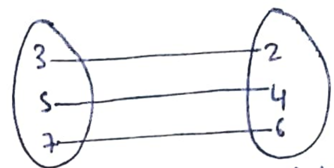




Maximal with 4 edges.

perfect matching





Maximal with 3 edges.

Both are maximal → bcoz we can't add even a single edge more.

But the maximum matching is 4 among both.

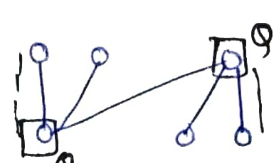
So, both are maximal but maximum is just one.

Vertex Cover :- A vertex cover of a graph is a set of $Q \subseteq V(G)$ that contains at least one end point of every edge. The vertex in Q cover $E(G)$.

→ Since no vertex can cover two edges of matching the size vertex cover is always greater than or equals to matching size.

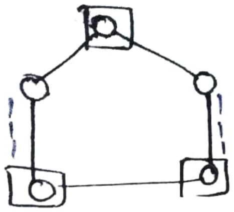
|Vertex cover| ≥ |matching size|

Ex

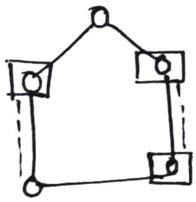


$Q \to$ vertex cover $= 2$

$| \to$ matching $= 2$



matching = 2
vertex cover
Q (= 3)

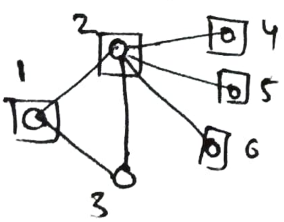


matching = 2
Q = 3

→ In vertex cover go through or involve each edge.

Minimum vertex cover [β]

Minimal vertex cover → The vertex cover from which you can't remove any vertex

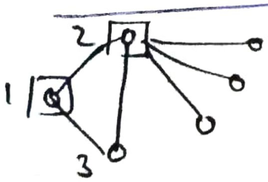


Q = {1, 2, 4, 5, 6}

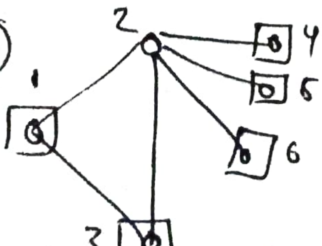
Q is not a minimal vertex cover.

Minimal vertex cover is :

Q = {1, 2} only



(2)



Q = {1, 3, 4, 5, 6}

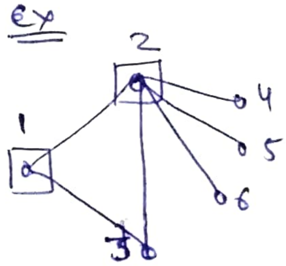
Here 2 is not present
So its necessary to involve 4, 5, 6 all
so, given Q is minimal vertex covers.

Trick → If you want to check that the vertex cover is minimal or not then one-by one delete each vertex or remove vertex from Q set and verify each edge should be cover. if any edge is remaining uncovered then that vetex should be there.

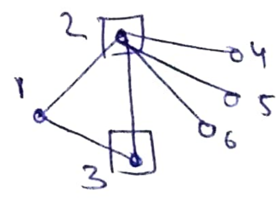
Minimum Vertex cover: It is minimal vertex cover with least no. of vertex.

→ There can be more than two minimum vertex covers.

ex



$Q_1 = \{1, 2\}$
$|Q_1| = 2$



$Q_2 = \{2, 3\}$
$|Q_2| = 2$

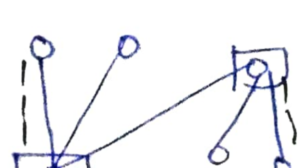
Both Q_1 and Q_2 are minimum vertex cover with cardinality 2.

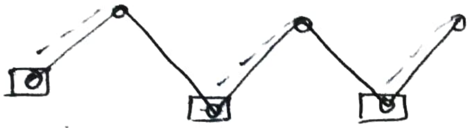
→ Find the minimum vertex cover is 'NP- Complete problem'.

⇒ Obtaining a matching and a vertex cover of same size proves that each is optimal

Max. matching → M (say)
Mini mum. vertex cover → Q_m (say)

$\boxed{Q_m = M = 2}$





$\boxed{Q_m = M = 3}$

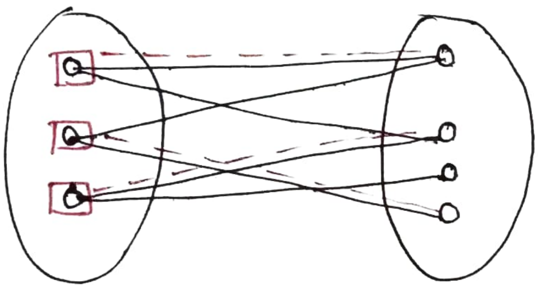
So, if Ma → Maximal matching

$$\boxed{|M_a| \le |Q_m| \le 2 * |M_a|}$$

Just Remember

No need to by heart

→ If G is a bipartite graph then max. size of a matching in G is equal to minimum size of vertex cover in graph.



|max No. of matching| = |min. vertex cover| for (Kmn)

Note → Imp

$\boxed{\text{min. vertex cover in } K_{mn} = \min(m,n)}$

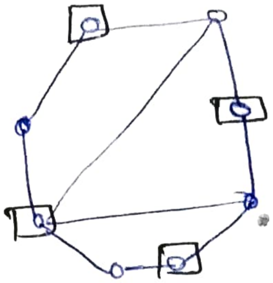
↳ bcoz it will cover every edge if you choose all vertex in any of the single set.

Independent sets and covers

→ An independent set in a graph is a set of pairwise non-adjacent vertices.

⇒ The independence number of a graph is the max. size of an independent set of vertices.

Ex →

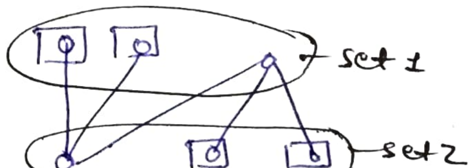


Independence number = 4

→ There can be more than one independent sets possible.

→ In case of bipartite graph $K_{m,n}$ the max. of independent set equals to 'max (m,n)'

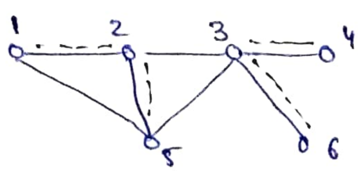
→ The independence no. of a bipartite set doesn't 'always' have the size of a partite set.



Size of partite set = 3 but independence no = 4.

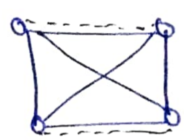
→ An edge cover of G is a set 'L' of edges such that every vertex of G is incident in 'L'.

G_1 :



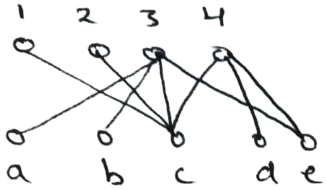
It needs not be unique.

G_2 :



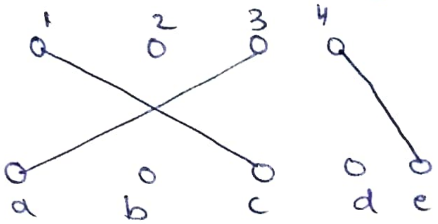
→ Minimal edge cover.

A perfect matching forms an edge cover with $\frac{|V|}{2}$.



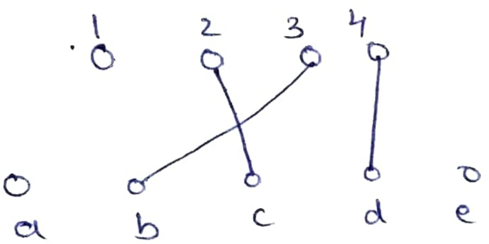
is a graph then

We can obtain an edge cover by adding edges to maximum matching, as—



(max. matching)

+



(additional edges)

Relation between Edge Cover [β'], vertex cover [β], Independent set [α] & matching [α']

Notations → Same as given below

max. size of	independent set	$\alpha(G)$
	matching	$(\alpha' G)$
minimum	vertex cover	$\beta(G)$
	edge cover	$\beta'(G)$

then,

① for every bipartite graph–
$\alpha'(G) = \beta(G)$.

② for every bipartite graph with no isolated vertices
$\alpha(G) = \beta'(G)$

③ In a graph G,
$\alpha(G) + \beta(G) = n(G)$
↳ no. of vertices

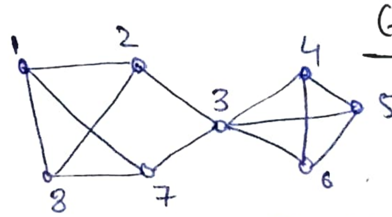
④ If G is a graph without isolated vertices then,
$\alpha'(G) + \beta'(G) = n(G)$

Cuts and Connectivities

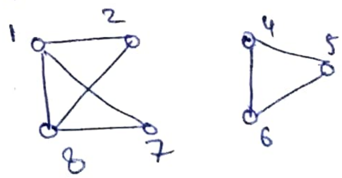
A separating set or vertex cut of a graph G is set $S \subseteq V(G)$ such that ~~graph~~ $(G-S)$ – {G minus S} has more than one component

Ex

Connectivity = 1



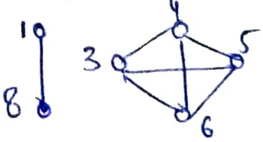
If we remove ③ then



→ 2 components

So, $S = \{3\}$

→ Vertex cut is not always unique as. If $S = \{2, 7\}$ then,



And many more

→ There can be more than one element in S.

The connectivity of $G(\kappa(G))$ is the minimum size of a vertex set S such that $(G-S)$ is disconnected or has only one vertex. (complete graph)

It means we can get two or more components either by removing one vertex only or in case of complete graph we have two remove all $(n-1)$ vertex.

→ In a cycle connectivity of graph is 2, and vertex removed should be non-adjacent.

→ Connectivity of complete graph is $(n-1)$.

→ A graph is k-connected if its connectivity is atleast k.

imp → A clique has no separating set.

Separating set: To read from Internet.

Properties of cuts and Connectivity

① $\kappa(K_n) = n-1 = \delta(G)$ $G \rightarrow$ complete graph.

$\kappa \rightarrow$ Kappa denotes connectivity.

If G is not a complete graph then,

② $\kappa(G) \le n(G) - 2$

③ $\kappa(K_{m,n}) = \min(m,n)$

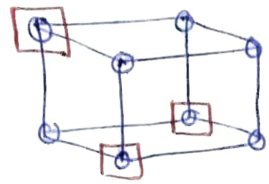
↳ for a bipartite graph.

$\kappa(K_{3,2}) = \min(3,2) = 2$

In a hypercube Q_k, for $k \geq 2$ the neighbours of one vertex in Q_k for a separating set,

So,

imp $\boxed{\kappa(Q_k) \le k}$



→ Deleting the neighbours of a vertex disconnects a graph So,

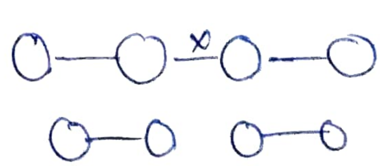
imp $\boxed{\kappa(G) \le \delta(G)}$

Edge Connectivity

A disconnecting set of edges is a set $F \subseteq E(G)$ such that $(G-F)$ have more than one components.

The edge connectivity of $G(\kappa'(G))$ is the minimum size of disconnecting sets.

We can delete all the edges but we prefer to delete min. no. of edges.



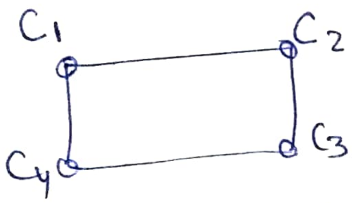
edge connectivity = 1

$\boxed{K'(G) \leq \delta(G)}$ → min. degree of graph.

→ A graph is k-edge connected if every disconnecting set has atleast k edges.

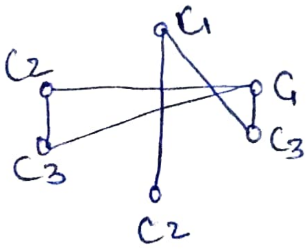
Coloring:- (Vertex Coloring)

→ A k-coloring of a graph (G) is a labeling [coloring] $f: V(G) \to S$, where S is a set of label or colors and $|S| = k$.



4 color graph

→ A k-coloring is proper if adjacent vertices have different colors.



3-color graph

→ A graph is k-colorable if it has a proper k-coloring.

→ The above graph is 3-colorable but not 2-colorable.

Chromatic Number:

We want to color the graph with least no. of colors. If graph G is k-colorable then it needs at least k-colors.

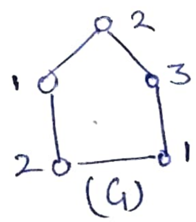
→ The chromatic no of bipartite graph is 2.

→ The chromatic no. of odd length cycle is 3 and even length cycle is 2.

Chromatic number lower bounds:-

say, $\chi(G)$ → Chromatic no. of G.

→ If $\chi(H) < [\chi(G) = k]$ for every proper subgraph H of G, then G is k-critical

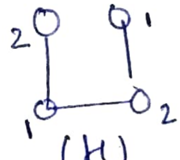


(G)

3-Colorable

$\chi(G) = 3$

then



(H)

2-Colorable

$\chi(H) = 2$

→ The chromatic no. of a complete graph K_n is 'n'.

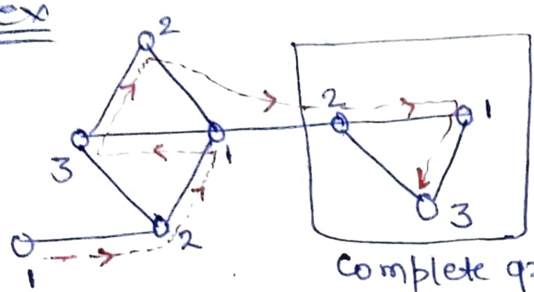
Greedy coloring Algo.

→ Algo. doesn't give optimal Solution in all the cases.

Cont-

A greedy coloring to a vertex ordering $v_1 v_2 \ldots v_n$ of $V(G)$ is obtained by coloring vertices in order $v_1 v_2 --- v_n$ assigning to v_i the smallest indexed color not used on its lower-indexed neighbours.

Ex



complete graph (K_3)

→ So it will hold min. 3 colors

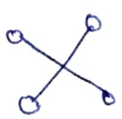
→ But in some cases we might get less than 3 colors if we dont have complete graph.

⇒ $\boxed{\chi(G) \le \Delta(G)+1}$

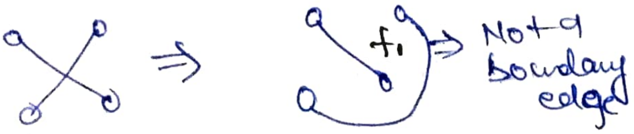
↳ In worst case

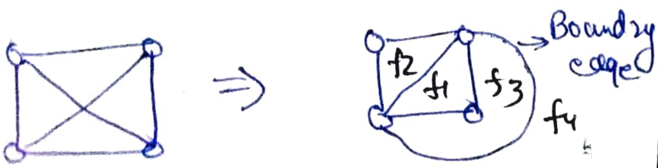
Planarity

Cross over:



If we can draw the graph on plane without cross-over called planar representation.





face and region

→ 2 faces

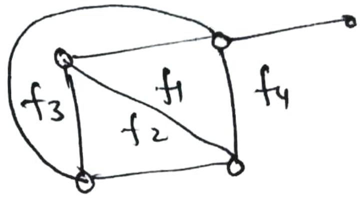
The planar region can divided into planar and faces.

Degree of interior region

Number of edges enclosing the region / face.

Degree of exterior region

Number of edges exposed to the region / face.



$deg(f_1) = 3$

$deg(f_2) = deg(f_3) = 3$

$deg(f_4) = 5$.

→ for any planar graph with n vertices

$$\boxed{\sum_{i=1}^{n} deg(f_i) = 2*e}$$

e → no. of edges

$3+3+3+5 = 2*7 = 14$

→ In a planar graph

① $k*f = 2e$ if $deg(f) = k$

② $k*f \le 2e$ if $deg(f) \ge k$

③ $k*f \ge 2e$ if $deg(f) \le k$

↳ k → least value or highest values among all the faces.

means replace all degrees with either max or min degree values.