

# CONVEX ANALYSIS WORKSHOP

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## 1. CONVEX FUNCTIONS

I made this material referring to [1].

### 1.1. Definitions.

**Definition 1.1.1** (Convex function): A function  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  is *convex* if  $\mathbf{dom} f$  is a convex set and

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{dom} f, \forall t \in [0, 1], f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y}) \quad (1)$$

where  $\mathbf{dom} f$  is the effective domain of  $f$ :

$$\mathbf{dom} f := \{\mathbf{x} \mid f(\mathbf{x}) < \infty\}. \quad (2)$$

**Definition 1.1.2** (Concave function): A function  $f$  is said to be *concave* if  $-f$  is convex.

**Definition 1.1.3** (Non-decreasing and non-increasing): A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is called *non-decreasing* if

$$\forall a, b \in \mathbb{R}, a \leq b \implies f(a) \leq f(b). \quad (3)$$

Likewise, a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is called *non-increasing* if

$$\forall a, b \in \mathbb{R}, a \leq b \implies f(a) \geq f(b). \quad (4)$$

### 1.2. Exercise.

**Proposition 1.2.1** (Scalar composition): For  $h : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ ,  $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ , define  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  by

$$f(\mathbf{x}) := h(g(\mathbf{x})) \quad (5)$$

with

$$\mathbf{dom} f = \{\mathbf{x} \in \mathbf{dom} g \mid g(\mathbf{x}) \in \mathbf{dom} h\}. \quad (6)$$

Then  $f$  is convex if

- $h$  is convex and nondecreasing and  $g$  is convex, or
- $h$  is convex and nonincreasing and  $g$  is concave.

*Proof:* First, we prove that  $f$  is convex if  $h$  is convex and nondecreasing and  $g$  is convex.

Let  $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$ , and  $t \in [0, 1]$ . Since  $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$ , we have  $\mathbf{x}, \mathbf{y} \in \mathbf{dom} g$

and  $g(\mathbf{x}), g(\mathbf{y}) \in \mathbf{dom} h$ . From the convexity of  $\mathbf{dom} g$ ,  $t\mathbf{x} + (1-t)\mathbf{y} \in \mathbf{dom} g$ . Then, since  $g$  is convex, we have

$$g(t\mathbf{x} + (1-t)\mathbf{y}) \leq tg(\mathbf{x}) + (1-t)g(\mathbf{y}). \quad (7)$$

Since  $g(\mathbf{x}), g(\mathbf{y}) \in \mathbf{dom} h$  and  $\mathbf{dom} h$  is convex, we have  $tg(\mathbf{x}) + (1-t)g(\mathbf{y}) \in \mathbf{dom} h$ . Then,

$$h(tg(\mathbf{x}) + (1-t)g(\mathbf{y})) < \infty. \quad (8)$$

Using the assumption that  $h$  is nondecreasing and (7), it follows that

$$h(g(t\mathbf{x} + (1-t)\mathbf{y})) \leq h(tg(\mathbf{x}) + (1-t)g(\mathbf{y})). \quad (9)$$

From (9) and (8), we get

$$h(g(t\mathbf{x} + (1-t)\mathbf{y})) < \infty, \quad (10)$$

which means  $g(t\mathbf{x} + (1-t)\mathbf{y}) \in \mathbf{dom} h$ . Since  $t\mathbf{x} + (1-t)\mathbf{y} \in \mathbf{dom} g$  and  $g(t\mathbf{x} + (1-t)\mathbf{y}) \in \mathbf{dom} h$ , we get  $t\mathbf{x} + (1-t)\mathbf{y} \in \mathbf{dom} f$ . Therefore,  $\mathbf{dom} f$  is convex set. From the convexity of  $h$ , we have

$$h(tg(\mathbf{x}) + (1-t)g(\mathbf{y})) \leq th(g(\mathbf{x})) + (1-t)h(g(\mathbf{y})). \quad (11)$$

From (9) and (11), we get

$$h(g(t\mathbf{x} + (1-t)\mathbf{y})) \leq th(g(\mathbf{x})) + (1-t)h(g(\mathbf{y})). \quad (12)$$

That is

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y}). \quad (13)$$

Then, we have shown that  $f$  is convex if  $h$  is convex and nondecreasing and  $g$  is convex.

Next, we prove that  $f$  is convex if  $h$  is convex and nonincreasing and  $g$  is concave.

$$h(g(t\mathbf{x} + (1-t)\mathbf{y})) \leq th(g(\mathbf{x})) + (1-t)h(g(\mathbf{y})). \quad (14)$$

That is

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y}). \quad (15)$$

Then, we have shown that  $f$  is convex if  $h$  is convex and nonincreasing and  $g$  is concave.

□

## REFERENCES

1. Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press (2004)

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