CONVEX ANALYSIS WORKSHOP

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1. Convex functions

I made this material referring to [1].

1.1. Definitions.

Definition 1.1.1 (Convex function): A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is *convex* if $\mathbf{dom}\, f$ is a convex set and

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, f, \forall t \in [0,1], f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \le t f(\boldsymbol{x}) + (1-t) f(\boldsymbol{y}) \tag{1}$$

where $\mathbf{dom}\, f$ is the effective domain of f:

$$\mathbf{dom}\, f \coloneqq \{\boldsymbol{x} \mid f(\boldsymbol{x}) < \infty\}. \tag{2}$$

Definition 1.1.2 (Concave function): A function f is said to be *concave* if $-f$ is convex.

Definition 1.1.3 (Non-decreasing and non-increasing): A function $f : \mathbb{R} \to \mathbb{R}$ is called *non-decreasing* if

$$\forall a, b \in \mathbb{R}, a \le b \Longrightarrow f(a) \le f(b). \tag{3}$$

Likewise, a function $f : \mathbb{R} \to \mathbb{R}$ is called *non-increasing* if

$$\forall a, b \in \mathbb{R}, a \le b \Longrightarrow f(a) \ge f(b). \tag{4}$$

1.2. Exercise.

Proposition 1.2.1 (Scalar composition): For $h : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, $g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, define $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ by

$$f(\boldsymbol{x}) \coloneqq h(g(\boldsymbol{x})) \tag{5}$$

with

$$\mathbf{dom}\, f = \{\boldsymbol{x} \in \mathbf{dom}\, g \mid g(\boldsymbol{x}) \in \mathbf{dom}\, h\}. \tag{6}$$

Then f is convex if

- h is convex and nondecreasing and g is convex, or
- h is convex and nonincreasing and g is concave.

Proof: First, we prove that f is convex if h is convex and nondecreasing and g is convex.
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, f$, and $t \in [0,1]$. Since $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, f$, we have $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, g$

and $g(\boldsymbol{x}), g(\boldsymbol{y}) \in \mathbf{dom}\, h$. From the convexity of $\mathbf{dom}\, g$, $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathbf{dom}\, g$. Then, since g is convex, we have

$$g(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \leq tg(\boldsymbol{x}) + (1-t)g(\boldsymbol{y}). \tag{7}$$

Since $g(\boldsymbol{x}), g(\boldsymbol{y}) \in \mathbf{dom}\, h$ and $\mathbf{dom}\, h$ is convex, we have $tg(\boldsymbol{x}) + (1-t)g(\boldsymbol{y}) \in \mathbf{dom}\, h$. Then,

$$h(tg(\boldsymbol{x}) + (1-t)g(\boldsymbol{y})) < \infty. \tag{8}$$

Using the assumption that h is nondecreasing and (7), it follows that

$$h(g(t\boldsymbol{x} + (1-t)\boldsymbol{y})) \leq h(tg(\boldsymbol{x}) + (1-t)g(\boldsymbol{y})). \tag{9}$$

From (8) and (9), we get

$$h(g(t\boldsymbol{x} + (1-t)\boldsymbol{y})) < \infty, \tag{10}$$

which means $g(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \in \mathbf{dom}\, h$. Since $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathbf{dom}\, g$ and $g(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \in \mathbf{dom}\, h$, we get $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathbf{dom}\, f$. Therefore, $\mathbf{dom}\, f$ is convex set. From the convexity of h, we have

$$h(tg(\boldsymbol{x}) + (1-t)g(\boldsymbol{y})) \leq th(g(\boldsymbol{x})) + (1-t)h(g(\boldsymbol{y})). \tag{11}$$

From (9) and (11), we get

$$h(g(t\boldsymbol{x} + (1-t)\boldsymbol{y})) \leq th(g(\boldsymbol{x})) + (1-t)h(g(\boldsymbol{y})). \tag{12}$$

That is

$$f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \leq tf(\boldsymbol{x}) + (1-t)f(\boldsymbol{y}). \tag{13}$$

Then, we have shown that f is convex if h is convex and nondecreasing and g is convex.

Next, we prove that f is convex if h is convex and nonincreasing and g is concave. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, f$, $t \in [0,1]$ and $g : \mathbb{R}^n \to \mathbb{R}$. Since $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, f$, we have $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, g$ and $g(\boldsymbol{x}), g(\boldsymbol{y}) \in \mathbf{dom}\, h$. Recall that $g(\boldsymbol{x}), g(\boldsymbol{y}), g(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \in \mathbb{R}$, we have $-g(\boldsymbol{x}) < \infty$, $-g(\boldsymbol{y}) < \infty$ and $g(t\boldsymbol{x} + (1-t)\boldsymbol{y}) < \infty$. That is $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, {-g}$ and $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathbf{dom}\, g$. Since $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathbf{dom}\, g$, $\mathbf{dom}\, g$ is convex. From the convexity of $\mathbf{dom}\, {-g}$, $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathbf{dom}\, {-g}$. Then, since $-g$ is convex, we have

$$-g(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \leq t(-g(\boldsymbol{x})) + (1-t)(-g(\boldsymbol{y})). \tag{14}$$

Multiplying this inequality by -1 yields

$$g(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \geq tg(\boldsymbol{x}) + (1-t)g(\boldsymbol{y}). \tag{15}$$

Since $g(\boldsymbol{x}), g(\boldsymbol{y}) \in \mathbf{dom}\, h$ and $\mathbf{dom}\, h$ is convex, we have $tg(\boldsymbol{x}) + (1-t)g(\boldsymbol{y}) \in \mathbf{dom}\, h$. Then,

$$h(tg(\boldsymbol{x}) + (1-t)g(\boldsymbol{y})) < \infty. \tag{16}$$

Using the assumption that h is nonincreasing and (15), it follows that

$$h(g(t\boldsymbol{x} + (1-t)\boldsymbol{y})) \leq h(tg(\boldsymbol{x}) + (1-t)g(\boldsymbol{y})). \tag{17}$$

From (16) and (17), we get

$$h(g(t\boldsymbol{x} + (1-t)\boldsymbol{y})) < \infty, \tag{18}$$

which means $g(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \in \textbf{dom}\, h$. Since $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \textbf{dom}\, g$ and $g(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \in \textbf{dom}\, h$, we get $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \textbf{dom}\, f$. Therefore, $\textbf{dom}\, f$ is convex set. From the convexity of h, we have

$$h(tg(\boldsymbol{x}) + (1-t)g(\boldsymbol{y})) \le th(g(\boldsymbol{x})) + (1-t)h(g(\boldsymbol{y})). \tag{19}$$

From (17) and (19), we get

$$h(g(t\boldsymbol{x} + (1-t)\boldsymbol{y})) \le th(g(\boldsymbol{x})) + (1-t)h(g(\boldsymbol{y})). \tag{20}$$

That is

$$f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \le tf(\boldsymbol{x}) + (1-t)f(\boldsymbol{y}). \tag{21}$$

Then, we have shown that f is convex if h is convex and nonincreasing and g is concave.

□

References

1. Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press (2004)

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