CONVEX ANALYSIS WORKSHOP

RYUTO SAITO

1. Convex functions

I made this material referring to [1].

1.1. Definitions.

Definition 1.1.1 (Convex function): A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex if $\mathbf{dom}\, f$ is a convex set and

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, f, \forall t \in [0,1], f(t\boldsymbol{x} + (1-t)\boldsymbol{y} \leq tf(\boldsymbol{x}) + (1-t)f(\boldsymbol{y})) \quad (1)$$

where $\mathbf{dom}\, f$ is the effective domain of f:

$$\mathbf{dom}\, f := \{\boldsymbol{x} \mid f(\boldsymbol{x}) < \infty\}. \quad (2)$$

Definition 1.1.2 (Non-decreasing and non-increasing): A function $f : \mathbb{R} \to \mathbb{R}$ is called *non-decreasing* if

$$\forall a, b \in \mathbb{R}, a \leq b \Longrightarrow f(a) \leq f(b). \quad (3)$$

Likewise, a function $f : \mathbb{R} \to \mathbb{R}$ is called *non-increasing* if

$$\forall a, b \in \mathbb{R}, a \leq b \Longrightarrow f(a) \geq f(b). \quad (4)$$

1.2. Lemma.

Lemma 1.2.1: Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}, g : \mathbb{R} \to \mathbb{R} \cup \{\infty\}, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, t \in [0,1]$ and

$$g(t) = f(t\boldsymbol{y} + (1-t)\boldsymbol{x}). \quad (5)$$

Then, g is convex if and only if f is convex.

Proof: ($\Longrightarrow$) Let $\theta \in [0,1]$. For any $t_1, t_2 \in \mathbf{dom}\, g$,

$$\begin{aligned}
&g(\theta t_1 + (1-\theta)t_2) \\
&= f((\theta t_1 + (1-\theta)t_2)\boldsymbol{y} + (1 - (\theta t_1 + (1-\theta)t_2))\boldsymbol{x}) \\
&= f(\theta t_1 \boldsymbol{y} + (1-\theta)t_2 \boldsymbol{y} + \boldsymbol{x} - \theta t_1 \boldsymbol{x} - (1-\theta)t_2 \boldsymbol{x}) \\
&= f(\theta t_1 \boldsymbol{y} + \theta \boldsymbol{x} - \theta t_1 \boldsymbol{x} + (1-\theta)t_2 \boldsymbol{y} + (1-\theta)\boldsymbol{x} - (1-\theta)t_2 \boldsymbol{x}) \\
&= f(\theta(t_1 \boldsymbol{y} + (1-t_1)\boldsymbol{x}) + (1-\theta)(t_2 \boldsymbol{y} + (1-t_2)\boldsymbol{x})) \\
&\leq \theta f(t_1 \boldsymbol{y} + (1-t_1)\boldsymbol{x}) + (1-\theta) f(t_2 \boldsymbol{y} + (1-t_2)\boldsymbol{x}) \\
&= \theta g(t_1) + (1-\theta) g(t_2)
\end{aligned} \quad (6)$$

Thus, g is convex.

($\Longleftarrow$) Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, f$ and $t \in \mathbb{R}$. For any $\theta \in [0,1]$,

$$\begin{aligned} f(\theta \boldsymbol{y} + (1-\theta)\boldsymbol{x}) &= g(\theta) \\ &= g(\theta \cdot 1 + (1-\theta) \cdot 0) \\ &\leq \theta g(1) + (1-\theta) g(0) \\ &= \theta f(\boldsymbol{y}) + (1-\theta) f(\boldsymbol{x}) \end{aligned} \tag{7}$$

Thus, f is convex. □

1.3. **Exercise.**

Proposition 1.3.1 (Scalar composition): For $h : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, $g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, define $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ by

$$f(\boldsymbol{x}) := h(g(\boldsymbol{x})) \tag{8}$$

with

$$\mathbf{dom}\, f = \{\boldsymbol{x} \in \mathbf{dom}\, g \mid g(\boldsymbol{x}) \in \mathbf{dom}\, h\}. \tag{9}$$

Then f is convex if

- h is convex and nondecreasing and g is convex, or
- h is convex and nonincreasing and g is concave.

Proof: First, we prove that f is convex if h is convex and nondecreasing and g is convex. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, f$, and $t \in [0,1]$. Since $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, f$, we have $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, g$ and $g(\boldsymbol{x}), g(\boldsymbol{y}) \in \mathbf{dom}\, h$. From convexity of g, $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathbf{dom}\, g$. Then, since g is convex, we have

$$g(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \leq t g(\boldsymbol{x}) + (1-t) g(\boldsymbol{y}) \tag{10}$$

Next, we prove that f is convex if h is convex and nonincreasing and g is concave. □

References

1. Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press (2004)

Email address: j2200071@gunma-u.ac.jp