

CONVEX ANALYSIS WORKSHOP

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1. CONVEX FUNCTIONS

I made this material referring to [1].

1.1. Definitions.

Definition 1.1.1 (Convex function): A function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ is *convex* if $\mathbf{dom} f$ is a convex set and

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{dom} f, \forall t \in [0, 1], f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y}) \quad (1)$$

where $\mathbf{dom} f$ is the effective domain of f :

$$\mathbf{dom} f := \{\mathbf{x} \mid f(\mathbf{x}) < \infty\}. \quad (2)$$

Definition 1.1.2 (Concave function): A function f is said to be *concave* if $-f$ is convex.

Definition 1.1.3 (Non-decreasing and non-increasing): A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called *non-decreasing* if

$$\forall a, b \in \mathbb{R}, a \leq b \implies f(a) \leq f(b). \quad (3)$$

Likewise, a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called *non-increasing* if

$$\forall a, b \in \mathbb{R}, a \leq b \implies f(a) \geq f(b). \quad (4)$$

1.2. Exercise.

Proposition 1.2.1 (Scalar composition): For $h : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$, $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$, define $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$f(\mathbf{x}) := h(g(\mathbf{x})) \quad (5)$$

with

$$\mathbf{dom} f = \{\mathbf{x} \in \mathbf{dom} g \mid g(\mathbf{x}) \in \mathbf{dom} h\}. \quad (6)$$

Then f is convex if

- h is convex and nondecreasing and g is convex, or
- h is convex and nonincreasing and g is concave.

Proof: First, we prove that f is convex if h is convex and nondecreasing and g is convex.

Let $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$, and $t \in [0, 1]$. Since $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$, we have $\mathbf{x}, \mathbf{y} \in \mathbf{dom} g$

and $g(\mathbf{x}), g(\mathbf{y}) \in \mathbf{dom} h$. From the convexity of $\mathbf{dom} g$, $t\mathbf{x} + (1-t)\mathbf{y} \in \mathbf{dom} g$. Then, since g is convex, we have

$$g(t\mathbf{x} + (1-t)\mathbf{y}) \leq tg(\mathbf{x}) + (1-t)g(\mathbf{y}). \quad (7)$$

Since $g(\mathbf{x}), g(\mathbf{y}) \in \mathbf{dom} h$ and $\mathbf{dom} h$ is convex, we have $tg(\mathbf{x}) + (1-t)g(\mathbf{y}) \in \mathbf{dom} h$. Then,

$$h(tg(\mathbf{x}) + (1-t)g(\mathbf{y})) < \infty. \quad (8)$$

Using the assumption that h is nondecreasing and (7), it follows that

$$h(g(t\mathbf{x} + (1-t)\mathbf{y})) \leq h(tg(\mathbf{x}) + (1-t)g(\mathbf{y})). \quad (9)$$

From (8) and (9), we get

$$h(g(t\mathbf{x} + (1-t)\mathbf{y})) < \infty, \quad (10)$$

which means $g(t\mathbf{x} + (1-t)\mathbf{y}) \in \mathbf{dom} h$. Since $t\mathbf{x} + (1-t)\mathbf{y} \in \mathbf{dom} g$ and $g(t\mathbf{x} + (1-t)\mathbf{y}) \in \mathbf{dom} h$, we get $t\mathbf{x} + (1-t)\mathbf{y} \in \mathbf{dom} f$. Therefore, $\mathbf{dom} f$ is convex set. From the convexity of h , we have

$$h(tg(\mathbf{x}) + (1-t)g(\mathbf{y})) \leq th(g(\mathbf{x})) + (1-t)h(g(\mathbf{y})). \quad (11)$$

From (9) and (11), we get

$$h(g(t\mathbf{x} + (1-t)\mathbf{y})) \leq th(g(\mathbf{x})) + (1-t)h(g(\mathbf{y})). \quad (12)$$

That is

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y}). \quad (13)$$

Then, we have shown that f is convex if h is convex and nondecreasing and g is convex.

Next, we prove that f is convex if h is convex and nonincreasing and g is concave. Let $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$, and $t \in [0, 1]$. Since $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$, we have $\mathbf{x}, \mathbf{y} \in \mathbf{dom} g$ and $g(\mathbf{x}), g(\mathbf{y}) \in \mathbf{dom} h$. From the convexity of $\mathbf{dom} -g$, $t\mathbf{x} + (1-t)\mathbf{y} \in \mathbf{dom} -g$. Then, since $-g$ is convex, we have

$$-g(t\mathbf{x} + (1-t)\mathbf{y}) \leq t(-g(\mathbf{x})) + (1-t)(-g(\mathbf{y})). \quad (14)$$

That is

$$g(t\mathbf{x} + (1-t)\mathbf{y}) \geq tg(\mathbf{x}) + (1-t)g(\mathbf{y}). \quad (15)$$

Since $g(\mathbf{x}), g(\mathbf{y}) \in \mathbf{dom} h$ and $\mathbf{dom} h$ is convex, we have $tg(\mathbf{x}) + (1-t)g(\mathbf{y}) \in \mathbf{dom} h$. Then,

$$h(tg(\mathbf{x}) + (1-t)g(\mathbf{y})) < \infty. \quad (16)$$

Using the assumption that h is nonincreasing and (15), it follows that

$$h(g(t\mathbf{x} + (1-t)\mathbf{y})) \leq h(tg(\mathbf{x}) + (1-t)g(\mathbf{y})). \quad (17)$$

From (16) and (17), we get

$$h(g(t\mathbf{x} + (1-t)\mathbf{y})) < \infty, \quad (18)$$

which means $g(t\mathbf{x} + (1-t)\mathbf{y}) \in \mathbf{dom} h$. Since $t\mathbf{x} + (1-t)\mathbf{y} \in \mathbf{dom} g$ and $g(t\mathbf{x} + (1-t)\mathbf{y}) \in \mathbf{dom} h$, we get $t\mathbf{x} + (1-t)\mathbf{y} \in \mathbf{dom} f$. Therefore, $\mathbf{dom} f$ is convex set. From the convexity of h , we have

$$h(tg(\mathbf{x}) + (1-t)g(\mathbf{y})) \leq th(g(\mathbf{x})) + (1-t)h(g(\mathbf{y})). \quad (19)$$

From (17) and (19), we get

$$h(g(t\mathbf{x} + (1-t)\mathbf{y})) \leq th(g(\mathbf{x})) + (1-t)h(g(\mathbf{y})). \quad (20)$$

That is

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y}). \quad (21)$$

Then, we have shown that f is convex if h is convex and nonincreasing and g is cocave.

□

REFERENCES

1. Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press (2004)

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