

# CONVEX ANALYSIS WORKSHOP

RYUTO SAITO

## 1. CONVEX FUNCTIONS

### 1.1. Definitions.

**Definition 1.1.1** (Convex function): Let  $S \subseteq \mathbb{R}^n$ . A function  $f : S \rightarrow \mathbb{R} \cup \{\infty\}$  is convex if  $\text{dom } f$  is a convex set and

$$\forall \mathbf{x}, \mathbf{y} \in \text{dom } f, \forall t \in [0, 1], f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y}) \quad (1)$$

where  $\text{dom } f$  is the effective domain of  $f$ :

$$\text{dom } f := \{\mathbf{x} \in S \mid f(\mathbf{x}) < \infty\}. \quad (2)$$

**Definition 1.1.2** (Gradient): Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable function. The gradient of  $f$  at  $\mathbf{x} \in \mathbb{R}^n$ , denoted  $\nabla f(\mathbf{x})$ , is an  $n$ -dimensional vector whose entries are given by

$$(\nabla f(\mathbf{x}))_i := \frac{\partial f(\mathbf{x})}{\partial x_i}. \quad (3)$$

The gradient of  $f$  is the vector containing all the partial derivatives. Element  $i$  of the gradient is the partial derivative of  $f$  with respect to  $x_i$ .

### 1.2. Lemma.

**Lemma 1.2.1:** A differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is convex if and only if

$$f(y) \geq f(x) + f'(x)(y - x) \quad (4)$$

for all  $x$  and  $y$  in  $\text{dom } f$ .

### 1.3. Exercise.

**Proposition 1.3.1** (First-order convexity condition): Suppose  $f$  is differentiable. Then  $f$  is convex if and only if  $\text{dom } f$  is convex and

$$\forall \mathbf{x}, \mathbf{y} \in \text{dom } f, f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}). \quad (5)$$

*Proof:* Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,  $\mathbf{z} = t\mathbf{y} + (1-t)\mathbf{x}$  for  $t \in [0, 1]$ , and

$$g(t) = f(t\mathbf{y} + (1-t)\mathbf{x}). \quad (6)$$

Then, using chain rule,

$$\begin{aligned}
g'(t) &= \frac{d}{dt} f(t\mathbf{y} + (1-t)\mathbf{x}) \\
&= \frac{d}{dt} f(\mathbf{z}) \\
&= \sum_{i=1}^n \frac{d}{dt} z_i \frac{\partial}{\partial z_i} f(\mathbf{z}) \\
&= \left( \frac{\partial}{\partial z_1} f(\mathbf{z}), \dots, \frac{\partial}{\partial z_n} f(\mathbf{z}) \right) \begin{bmatrix} \frac{d}{dt} z_1 \\ \vdots \\ \frac{d}{dt} z_n \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial}{\partial z_1} f(\mathbf{z}) \\ \vdots \\ \frac{\partial}{\partial z_n} f(\mathbf{z}) \end{bmatrix}^T \begin{bmatrix} \frac{d}{dt} (ty_1 + (1-t)x_1) \\ \vdots \\ \frac{d}{dt} (ty_n + (1-t)x_n) \end{bmatrix} \\
&= \nabla f(\mathbf{z})^T \begin{bmatrix} y_1 - x_1 \\ \vdots \\ y_n - x_n \end{bmatrix} \\
&= \nabla f(t\mathbf{y} + (1-t)\mathbf{x})^T (\mathbf{y} - \mathbf{x}).
\end{aligned} \tag{7}$$

( $\Rightarrow$ ) Assume  $f$  is convex. Then,  $g$  is convex. By lemma, we have

$$g(1) \geq g(0) + g'(0), \tag{8}$$

which means that

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^T (\mathbf{y} - \mathbf{x}). \tag{9}$$

( $\Leftarrow$ ) Assume that (5) holds for any  $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$ . Let  $t_1, t_2 \in [0, 1]$ . Since  $\mathbf{x}, \mathbf{y} \in \mathbf{dom} f$ ,  $t_1\mathbf{y} + (1-t_1)\mathbf{x} \in \mathbf{dom} f$  and  $t_2\mathbf{y} + (1-t_2)\mathbf{x} \in \mathbf{dom} f$ . From (5),

$$\begin{aligned}
&f(t_1\mathbf{y} + (1-t_1)\mathbf{x}) \\
&\geq f(t_2\mathbf{y} + (1-t_2)\mathbf{x}) \\
&\quad + \nabla f(t_2\mathbf{y} + (1-t_2)\mathbf{x})(t_1\mathbf{y} + (1-t_1)\mathbf{x} - (t_2\mathbf{y} + (1-t_2)\mathbf{x})) \\
&= f(t_2\mathbf{y} + (1-t_2)\mathbf{x}) \\
&\quad + \nabla f(t_2\mathbf{y} + (1-t_2)\mathbf{x})((t_1 - t_2)\mathbf{y} + (1-t_1 - (1-t_2))\mathbf{x}) \\
&= f(t_2\mathbf{y} + (1-t_2)\mathbf{x}) \\
&\quad + \nabla f(t_2\mathbf{y} + (1-t_2)\mathbf{x})((t_1 - t_2)\mathbf{y} - (t_1 - t_2)\mathbf{x}) \\
&= f(t_2\mathbf{y} + (1-t_2)\mathbf{x}) \\
&\quad + \nabla f(t_2\mathbf{y} + (1-t_2)\mathbf{x})(t_1 - t_2)(\mathbf{y} - \mathbf{x}).
\end{aligned} \tag{10}$$

That is

$$g(t_1) \geq g(t_2) + g'(t_2)(t_1 - t_2). \tag{11}$$

From lemma, since  $g$  is convex, then  $f$  is convex.  $\square$

## REFERENCES

1. Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press (2004)

*Email address:* j2200071@gunma-u.ac.jp