CONVEX ANALYSIS WORKSHOP

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1. Convex functions

1.1. Definitions.

Definition 1.1.1 (Convex function): Let $S \subseteq \mathbb{R}^n$. A function $f : S \to \mathbb{R} \cup \{\infty\}$ is convex if $\mathbf{dom}\, f$ is a convex set and

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, f, \forall t \in [0,1], f(t\boldsymbol{x} + (1-t)\boldsymbol{y} \leq tf(\boldsymbol{x}) + (1-t)f(\boldsymbol{y}))$$

where $\mathbf{dom}\, f$ is the effective domain of f:

$$\mathbf{dom}\, f \coloneqq \{\boldsymbol{x} \in S \mid f(\boldsymbol{x}) < \infty\}.$$

Definition 1.1.2 (Gradient): Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. The gradient of f at $\boldsymbol{x} \in \mathbb{R}^n$, denoted $\nabla f(\boldsymbol{x})$, is an n-dimensional vector whose entries are given by

$$(\nabla f(\boldsymbol{x}))_i \coloneqq \frac{\partial f(\boldsymbol{x})}{\partial x_i}.$$

The gradient of f is the vector containing all the partial derivatives. Element i of the gradient is the partial derivative of f with respect to x_i.

1.2. Lemma.

Lemma 1.2.1:

1.3. Exercise.

Proposition 1.3.1 (First-order convexity condition): Suppose f is differentiable. Then f is convex if and only if $\mathbf{dom}\, f$ is convex and

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{dom}\, f, f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}).$$

Proof:

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References

1. Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press (2004)

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