Matrix Algebra Marathon I-D3

Takahiko Henmi

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シミュレーションにはnumpyを用いた．コードの先頭には次のコードが共通して含まれているものとする．

```
import numpy as np
import numpy.random as RX
import numpy.linalg as LA
```

2.1 Inner-Product of Vectors

Definition 2.1.1. The inner-product of vectors is defined as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{i=1}^{n} x_i y_i \tag{1}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

2.2 l_2-Norm

Definition 2.2.1. The l_2-norm of vectors is defined as

$$\|\boldsymbol{x}\| := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} \tag{2}$$

where $\boldsymbol{x} \in \mathbb{R}^n$.

Exercise 2.2.2. Show that, $\forall \boldsymbol{x} \in \mathbb{R}^n$

$$\|\boldsymbol{x}\|^2 = \sum_{i=1}^{n} x_i^2. \tag{3}$$

Proof.

$$\begin{aligned} \text{LHS} &\overset{(2.2.1)}{=} \langle \boldsymbol{x}, \boldsymbol{x} \rangle \\ &\overset{(2.1.1)}{=} \sum_{i=1}^{n} x_i \cdot x_i \\ &= \text{RHS} \end{aligned} \tag{4}$$

□

Exercise 2.2.4 (homogeneousness for l_2-norm)**.** For $\forall a \in \mathbb{R}, \forall \boldsymbol{x} \in \mathbb{R}^n$, derive the equality:

$$\|a\boldsymbol{x}\| = |a|\|\boldsymbol{x}\|. \tag{5}$$

Proof.

$$\begin{aligned} \text{LHS} &\overset{(2.2.2)}{=} \sqrt{\sum_{i=1}^{n} (ax_i)^2} \\ &= |a|\sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} \\ &\overset{(2.2.1)}{=} \text{RHS} \end{aligned} \tag{6}$$

□

Exercise 2.2.6 (definiteness for l_2-norm)**.** For $\forall \boldsymbol{x} \in \mathbb{R}^n$, show that

$$\|\boldsymbol{x}\| = 0 \Longleftrightarrow \boldsymbol{x} = \boldsymbol{0}. \tag{7}$$

Proof.

($\Longleftarrow$) $\boldsymbol{x} = \boldsymbol{0}$ ならば $\forall i, x_i = 0$ であるから，次が言える:

$$\|\boldsymbol{x}\| = \sum_{i=1}^{n} x_i = 0. \tag{8}$$

($\Longrightarrow$) 背理法で示す．いま，$\boldsymbol{x} \neq \boldsymbol{0}$ と仮定する．このとき，$\exists j, x_j \neq 0$ より，次が成り立つ:

$$\begin{aligned} \|\boldsymbol{x}\| &\overset{(2.2.2)}{=} \sqrt{\sum_{i=1}^{n} x_i^2} \\ &\geq \sqrt{x_j^2} \\ &= |x_j| \quad > \quad 0. \end{aligned} \tag{9}$$

これは $\|\boldsymbol{x}\| = 0$ とした前提に矛盾する．

以上より，命題は成り立つ． □

Exercise 2.2.8. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Derive the equality:

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\langle \boldsymbol{x}, \boldsymbol{y} \rangle. \tag{10}$$

Proof.

$$\begin{aligned} \text{LHS} &\overset{(2.2.2)}{=} \sum_{i=1}^{n} (x_i + y_i)^2 \\ &= \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + 2\sum_{i=1}^{n} x_i y_i \\ &\overset{(2.1.1, 2.2.2)}{=} \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\langle \boldsymbol{x}, \boldsymbol{y} \rangle \\ &= \text{RHS} \end{aligned} \tag{11}$$

□

Exercise 2.2.10 (Cauchy–Schwarz inequality)**.** Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Show that

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|. \tag{12}$$

Proof. $\|\boldsymbol{x}\|\|\boldsymbol{y}\| \geq 0$ であるから, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle < 0$ のとき, 式 (12) は成り立つ．以下，$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \geq 0$ とする．$L = (\|\boldsymbol{x}\|\|\boldsymbol{y}\|)^2 - \langle \boldsymbol{x}, \boldsymbol{y} \rangle^2$ とおく．このとき，次が成り立つ:

$$L \overset{\text{(Lagrange's identity)}}{=} \sum_{1 \leq i \leq j \leq n} (x_i y_j - x_j y_i)^2 \geq 0. \tag{13}$$

このことから，次のことが言える:

$$\begin{gathered} \langle \boldsymbol{x}, \boldsymbol{y} \rangle^2 \leq (\|\boldsymbol{x}\|\|\boldsymbol{y}\|)^2 \\ \Downarrow \\ |\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \big| \, \|\boldsymbol{x}\|\|\boldsymbol{y}\| \, \big| \\ \Downarrow \\ \langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|. \end{gathered} \tag{14}$$

以上より，いずれの場合も式 (12) は成り立つことが示せた． □

Simulation 2.2.12. Generate samples and verify the triangle inequality for l_2-norm:

$$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|. \tag{15}$$

Code:

```
n = 3
ntrys = 10000

for i in range(0, ntrys):
  x = RX.rand(n)
  y = RX.rand(n)
  assert LA.norm(x+y) <= LA.norm(x)+LA.norm(y)
```

Exercise 2.2.14. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ be unit vectors. Show that

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 1. \tag{16}$$

Proof. $\|\boldsymbol{x}\| = \|\boldsymbol{y}\| = 1$ であるので，次が成り立つ:

$$\begin{aligned} \text{LHS} &\overset{(2.2.10)}{\leq} \|\boldsymbol{x}\|\|\boldsymbol{y}\| \\ &= 1. \end{aligned} \tag{17}$$

□