Matrix Algebra Marathon

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1 Basic Matrix Identities

1.1 Definition

Definition 1.1.1. *The transpose of an $m \times n$ matrix $\boldsymbol{A}$, to be denoted by A^{T}, is the $n \times m$ matrix whose (j, i)-th entry is (i, j)-th entry of $\boldsymbol{A}$. Namely,*

$$\left(\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix}\right)^{\mathrm{T}} = \begin{bmatrix} A_{1,1} & \cdots & A_{m,1} \\ \vdots & \ddots & \vdots \\ A_{1,n} & \cdots & A_{m,n} \end{bmatrix} \tag{1}$$

Definition 1.1.2. *The inverse of a square $n \times n$ matrix $\boldsymbol{A}$, to be denoted by $\boldsymbol{A}^{-1}$, is the $n \times n$ matrix such that*

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}_n. \tag{2}$$

The matrix $\boldsymbol{A}$ is said to be non-singlar if $\boldsymbol{A}^{-1}$ exists.

1.2 Exercise

Exercise 2.5.2.

Let $\boldsymbol{A} \in \mathbb{R}^{m \times p}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{p \times n}$. Show that $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$.

Proof. Let $\boldsymbol{A} \in \mathbb{R}^{m \times p}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{p \times n}$. For all $i \in \mathbb{N}_m$, and $j \in \mathbb{N}_n$,

$$\begin{aligned} (\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}))_{i,j} &= \sum_{k=1}^{p} \boldsymbol{A}_{i,k}(\boldsymbol{B} + \boldsymbol{C})_{k,j} \\ &= \sum_{k=1}^{p} \boldsymbol{A}(\boldsymbol{B}_{k,j} + \boldsymbol{C}_{k,j}) \\ &= \sum_{k=1}^{p} (\boldsymbol{A}_{i,k}\boldsymbol{B}_{k,j} + \boldsymbol{A}_{i,k}\boldsymbol{C}_{k,j}) \\ &= \sum_{k=1}^{p} \boldsymbol{A}_{i,k}\boldsymbol{B}_{k,j} + \sum_{k=1}^{p} \boldsymbol{A}_{i,k}\boldsymbol{C}_{k,j} \\ &= \boldsymbol{AB}_{i,j} + \boldsymbol{AC}_{i,j} \\ &= (\boldsymbol{AB} + \boldsymbol{AC})_{i,j}. \end{aligned}$$

From this $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$ follows. □

Exercise 2.5.5.

Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$. Show that $(\boldsymbol{A} + \boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}$.

Proof. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$. For all $i \in \mathbb{N}_m$, and $j \in \mathbb{N}_n$,

$$\begin{aligned} \left((\boldsymbol{A} + \boldsymbol{B})^{\mathrm{T}}\right)_{i,j} &= (\boldsymbol{A} + \boldsymbol{B})_{j,i} \\ &= \boldsymbol{A}_{j,i} + \boldsymbol{B}_{j,i} \\ &= \boldsymbol{A}^{\mathrm{T}}_{i,j} + \boldsymbol{B}^{\mathrm{T}}_{i,j} \\ &= (\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}})_{i,j}. \end{aligned}$$

From this $(\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}$ follows. $\square$

Exercise 2.5.7.

Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times p}, \boldsymbol{C} \in \mathbb{R}^{p\times n}$. Show that $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$.

Proof. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times p}, \boldsymbol{C} \in \mathbb{R}^{p\times n}$. For all $i \in \mathbb{N}_m$, and $j \in \mathbb{N}_n$,

$$\begin{aligned}
((\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C})_{i,j} &= \sum_{k=1}^{p} (\boldsymbol{A}+\boldsymbol{B})_{i,k}\boldsymbol{C}_{k,j} \\
&= \sum_{k=1}^{p} (\boldsymbol{A}_{i,k}+\boldsymbol{B}_{i,k})\boldsymbol{C}_{k,j} \\
&= \sum_{k=1}^{p} (\boldsymbol{A}_{i,k}\boldsymbol{C}_{k,j}+\boldsymbol{B}_{i,k}\boldsymbol{C}_{k,j}) \\
&= \sum_{k=1}^{p} (\boldsymbol{A}_{i,k}\boldsymbol{C}_{k,j}) + \sum_{k=1}^{p} (\boldsymbol{B}_{i,k}\boldsymbol{C}_{k,j}) \\
&= (\boldsymbol{AC})_{i,j} + (\boldsymbol{BC})_{i,j} \\
&= (\boldsymbol{AC}+\boldsymbol{BC})_{i,j}.
\end{aligned}$$

From this $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$ follows. $\square$

Exercise 2.5.9.

Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{B} \in \mathbb{R}^{n\times p}$, and $\boldsymbol{C} \in \mathbb{R}^{n\times q}$. Show that

$$\boldsymbol{A}[\boldsymbol{B}, \boldsymbol{C}] = [\boldsymbol{AB}, \boldsymbol{AC}]. \tag{3}$$

Proof. Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{B} \in \mathbb{R}^{n\times p}$, and $\boldsymbol{C} \in \mathbb{R}^{n\times q}$.

$$\begin{aligned}
\boldsymbol{A}[\boldsymbol{B}, \boldsymbol{C}] &= \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} \begin{bmatrix} B_{1,1} & \cdots & B_{1,p} & C_{1,1} & \cdots & C_{1,q} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{n,1} & \cdots & B_{n,p} & C_{n,1} & \cdots & C_{n,q} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^{n} A_{1,i}B_{i,1} & \cdots & \sum_{i=1}^{n} A_{1,i}B_{i,p} & \sum_{i=1}^{n} A_{1,i}C_{i,1} & \cdots & \sum_{i=1}^{n} A_{1,i}C_{i,q} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n} A_{m,i}B_{i,1} & \cdots & \sum_{i=1}^{n} A_{m,i}B_{i,p} & \sum_{i=1}^{n} A_{m,i}C_{i,1} & \cdots & \sum_{i=1}^{n} A_{m,i}C_{i,q} \end{bmatrix} \\
&= [\boldsymbol{AB}, \boldsymbol{AC}]
\end{aligned}$$

$\square$

Exercise 2.5.11.

Let $\boldsymbol{A} \in \mathbb{R}^{m\times p}, \boldsymbol{B} \in \mathbb{R}^{m\times q}, \boldsymbol{C} \in \mathbb{R}^{p\times n}$, and $\boldsymbol{D} \in \mathbb{R}^{q\times n}$. Show that

$$[\boldsymbol{A}, \boldsymbol{B}] \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{D} \end{bmatrix} = \boldsymbol{AC} + \boldsymbol{BD}. \tag{4}$$

Proof. Let $\boldsymbol{A} \in \mathbb{R}^{m\times p}, \boldsymbol{B} \in \mathbb{R}^{m\times q}, \boldsymbol{C} \in \mathbb{R}^{p\times n}$, and $\boldsymbol{D} \in \mathbb{R}^{q\times n}$.

$$
\begin{aligned}
[\boldsymbol{A}, \boldsymbol{B}] \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{D} \end{bmatrix} &= \begin{bmatrix} A_{1,1} & \cdots & A_{1,p} & B_{1,1} & \cdots & B_{1,q} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,p} & B_{m,1} & \cdots & B_{m,q} \end{bmatrix} \begin{bmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{p.1} & \cdots & C_{p,n} \\ D_{1.1} & \cdots & D_{1,n} \\ \vdots & \ddots & \vdots \\ D_{q,1} & \cdots & D_{q,n} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^p A_{1,i}C_{i,1} + \sum_{i=1}^q B_{1,i}D_{i,1} & \cdots & \sum_{i=1}^p A_{1,i}C_{i,n} + \sum_{i=1}^q B_{1,i}D_{i,n} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^p A_{m,i}C_{i,1} + \sum_{i=1}^q B_{m,i}D_{i,1} & \cdots & \sum_{i=1}^p A_{m,i}C_{i,n} + \sum_{i=1}^q B_{m,i}D_{i,n} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^p A_{1,i}C_{i,1} & \cdots & \sum_{i=1}^p A_{1,i}C_{i,n} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^p A_{m,i}C_{i,1} & \cdots & \sum_{i=1}^p A_{m,i}C_{i,n} \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^q B_{1,i}D_{i,1} & \cdots & \sum_{i=1}^q B_{1,i}D_{i,n} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^q B_{m,i}D_{i,1} & \cdots & \sum_{i=1}^q B_{m,i}D_{i,n} \end{bmatrix} \\
&= \boldsymbol{AC} + \boldsymbol{BD}
\end{aligned}
$$

□

Exercise 2.5.13.

Let $\boldsymbol{A}^{(1)} \in \mathbb{R}^{m\times n_1}, \boldsymbol{A}^{(2)} \in \mathbb{R}^{m\times n_2}, \boldsymbol{B}^{(1,1)} \in \mathbb{R}^{n_1\times p_1}, \boldsymbol{B}^{(1,2)} \in \mathbb{R}^{n_1\times p_2}, \boldsymbol{B}^{(2,1)} \in \mathbb{R}^{n_2\times p_1}$, and $\boldsymbol{B}^{(2,2)} \in \mathbb{R}^{n_2\times p_2}$. Show that

$$
\begin{aligned}
&[\boldsymbol{A}^{(1)}, \boldsymbol{A}^{(2)}] \begin{bmatrix} \boldsymbol{B}^{(1,1)} & \boldsymbol{B}^{(1,2)} \\ \boldsymbol{B}^{(2,1)} & \boldsymbol{B}^{(2,2)} \end{bmatrix} \\
&= [\boldsymbol{A}^{(1)}\boldsymbol{B}^{(1,1)} + \boldsymbol{A}^{(2)}\boldsymbol{B}^{(2,1)}, \boldsymbol{A}^{(1)}\boldsymbol{B}^{(1,2)} + \boldsymbol{A}^{(2)}\boldsymbol{B}^{(2,2)}].
\end{aligned} \tag{5}
$$

Proof. Let $\boldsymbol{A}^{(1)} \in \mathbb{R}^{m\times n_1}, \boldsymbol{A}^{(2)} \in \mathbb{R}^{m\times n_2}, \boldsymbol{B}^{(1,1)} \in \mathbb{R}^{n_1\times p_1}, \boldsymbol{B}^{(1,2)} \in \mathbb{R}^{n_1\times p_2}, \boldsymbol{B}^{(2,1)} \in \mathbb{R}^{n_2\times p_1}$, and $\boldsymbol{B}^{(2,2)} \in \mathbb{R}^{n_2\times p_2}$.

$$
\begin{aligned}
&[\boldsymbol{A}^{(1)}, \boldsymbol{A}^{(2)}] \begin{bmatrix} \boldsymbol{B}^{(1,1)} & \boldsymbol{B}^{(1,2)} \\ \boldsymbol{B}^{(2,1)} & \boldsymbol{B}^{(2,2)} \end{bmatrix} \\
&= \begin{bmatrix} A^{(1)}_{1,1} & \cdots & A^{(1)}_{1,n_1} & A^{(2)}_{1,1} & \cdots & A^{(2)}_{1,n_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A^{(1)}_{m,1} & \cdots & A^{(1)}_{m,n_1} & A^{(2)}_{m,1} & \cdots & A^{(2)}_{m,n_2} \end{bmatrix} \begin{bmatrix} B^{(1,1)}_{1,1} & \cdots & B^{(1,1)}_{1,p_1} & B^{(1,2)}_{1,1} & \cdots & B^{(1,2)}_{1,p_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B^{(1,1)}_{n_1,1} & \cdots & B^{(1,1)}_{n_1,p_1} & B^{(1,2)}_{n_1,1} & \cdots & B^{(1,2)}_{n_1,p_2} \\ B^{(2,1)}_{1,1} & \cdots & B^{(2,1)}_{1,p_1} & B^{(2,2)}_{1,1} & \cdots & B^{(2,2)}_{1,p_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B^{(2,1)}_{n_2,1} & \cdots & B^{(2,1)}_{n_2,p_1} & B^{(2,2)}_{n_2,1} & \cdots & B^{(2,2)}_{n_2,p_2} \end{bmatrix} \\
&= \begin{bmatrix} \sum\limits_{i=1}^{n_1} A^{(1)}_{1,i}B^{(1,1)}_{i,1} + \sum\limits_{i=1}^{n_2} A^{(2)}_{1,i}B^{(2,1)}_{i,1} & \cdots & \sum\limits_{i=1}^{n_1} A^{(1)}_{1,i}B^{(1,1)}_{i,p_1} + \sum\limits_{i=1}^{n_2} A^{(2)}_{1,i}B^{(2,1)}_{i,p_1} & \sum\limits_{i=1}^{n_1} A^{(1)}_{1,i}B^{(1,2)}_{i,1} + \sum\limits_{i=1}^{n_2} A^{(2)}_{1,i}B^{(2,2)}_{i,1} & \cdots & \sum\limits_{i=1}^{n_1} A^{(1)}_{1,i}B^{(1,2)}_{i,p_2} + \sum\limits_{i=1}^{n_2} A^{(2)}_{1,i}B^{(2,2)}_{i,p_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{i=1}^{n_1} A^{(1)}_{m,i}B^{(1,1)}_{i,1} + \sum\limits_{i=1}^{n_2} A^{(2)}_{m,i}B^{(2,1)}_{i,1} & \cdots & \sum\limits_{i=1}^{n_1} A^{(1)}_{m,i}B^{(1,1)}_{i,p_1} + \sum\limits_{i=1}^{n_2} A^{(2)}_{m,i}B^{(2,1)}_{i,p_1} & \sum\limits_{i=1}^{n_1} A^{(1)}_{m,i}B^{(1,2)}_{i,1} + \sum\limits_{i=1}^{n_2} A^{(2)}_{m,i}B^{(2,2)}_{i,1} & \cdots & \sum\limits_{i=1}^{n_1} A^{(1)}_{m,i}B^{(1,2)}_{i,p_2} + \sum\limits_{i=1}^{n_2} A^{(2)}_{m,i}B^{(2,2)}_{i,p_2} \end{bmatrix} \\
&= \begin{bmatrix} \sum\limits_{i=1}^{n_1} A^{(1)}_{1,i}B^{(1,1)}_{i,1} & \cdots & \sum\limits_{i=1}^{n_1} A^{(1)}_{1,i}B^{(1,1)}_{i,p_1} & \sum\limits_{i=1}^{n_1} A^{(1)}_{1,i}B^{(1,2)}_{i,1} & \cdots & \sum\limits_{i=1}^{n_1} A^{(1)}_{1,i}B^{(1,2)}_{i,p_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{i=1}^{n_1} A^{(1)}_{m,i}B^{(1,1)}_{i,1} & \cdots & \sum\limits_{i=1}^{n_1} A^{(1)}_{m,i}B^{(1,1)}_{i,p_1} & \sum\limits_{i=1}^{n_1} A^{(1)}_{m,i}B^{(1,2)}_{i,1} & \cdots & \sum\limits_{i=1}^{n_1} A^{(1)}_{m,i}B^{(1,2)}_{i,p_2} \end{bmatrix} \\
&+ \begin{bmatrix} \sum\limits_{i=1}^{n_2} A^{(2)}_{1,i}B^{(2,1)}_{i,1} & \cdots & \sum\limits_{i=1}^{n_2} A^{(2)}_{1,i}B^{(2,1)}_{i,p_1} & \sum\limits_{i=1}^{n_2} A^{(2)}_{1,i}B^{(2,2)}_{i,1} & \cdots & \sum\limits_{i=1}^{n_2} A^{(2)}_{1,i}B^{(2,2)}_{i,p_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{i=1}^{n_2} A^{(2)}_{m,i}B^{(2,1)}_{i,1} & \cdots & \sum\limits_{i=1}^{n_2} A^{(2)}_{m,i}B^{(2,1)}_{i,p_1} & \sum\limits_{i=1}^{n_2} A^{(2)}_{m,i}B^{(2,2)}_{i,1} & \cdots & \sum\limits_{i=1}^{n_2} A^{(2)}_{m,i}B^{(2,2)}_{i,p_2} \end{bmatrix} \\
&= [\boldsymbol{A}^{(1)}\boldsymbol{B}^{(1,1)}, \boldsymbol{A}^{(1)}\boldsymbol{B}^{(1,2)}] + [\boldsymbol{A}^{(2)}\boldsymbol{B}^{(2,1)}, \boldsymbol{A}^{(2)}\boldsymbol{B}^{(2,2)}] \\
&= [\boldsymbol{A}^{(1)}\boldsymbol{B}^{(1,1)} + \boldsymbol{A}^{(2)}\boldsymbol{B}^{(2,1)}, \boldsymbol{A}^{(1)}\boldsymbol{B}^{(1,2)} + \boldsymbol{A}^{(2)}\boldsymbol{B}^{(2,2)}]
\end{aligned}
$$

□

Exercise 2.5.19.

Show that $(\boldsymbol{ABC})^{-1} = \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{n\times n}$ are non-singular.

Proof. Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{n\times n}$ be non-singular.

$$\begin{aligned}
(\boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1})(\boldsymbol{ABC}) &= \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}\boldsymbol{ABC} \\
&= \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{BC} \\
&= \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{I}_n\boldsymbol{BC} \\
&= \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{BC} \\
&= \boldsymbol{C}^{-1}(\boldsymbol{B}^{-1}\boldsymbol{B})\boldsymbol{C} \\
&= \boldsymbol{C}^{-1}\boldsymbol{I}_n\boldsymbol{C} \\
&= \boldsymbol{C}^{-1}\boldsymbol{C} \\
&= \boldsymbol{I}_n
\end{aligned}$$

Then, by definition 2.5.16. , we get $(\boldsymbol{ABC})^{-1} = \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$. □

Exercise 2.5.22.

Show the Woodbury formula

$$(\boldsymbol{A} + \boldsymbol{BD}^{-1}\boldsymbol{C})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} + \boldsymbol{CA}^{-1}\boldsymbol{B})^{-1}\boldsymbol{CA}^{-1}. \tag{6}$$

where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, and $\boldsymbol{D}$ are matrices with the correct size.

Proof. Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}, \boldsymbol{B}, \boldsymbol{C}$, and, $\boldsymbol{D}$ be matrices with the correct size.

$$\begin{aligned}
&(\boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} + \boldsymbol{CA}^{-1}\boldsymbol{B})^{-1}\boldsymbol{CA}^{-1})(\boldsymbol{A} + \boldsymbol{BD}^{-1}\boldsymbol{C}) \\
=&(\boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} + \boldsymbol{CA}^{-1}\boldsymbol{B})^{-1}\boldsymbol{CA}^{-1})\boldsymbol{A} \\
&+ (\boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} + \boldsymbol{CA}^{-1}\boldsymbol{B})^{-1}\boldsymbol{CA}^{-1})\boldsymbol{BD}^{-1}\boldsymbol{C} \\
=&\boldsymbol{I}_n - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} + \boldsymbol{CA}^{-1}\boldsymbol{B})^{-1}\boldsymbol{C} \\
&+ \boldsymbol{A}^{-1}\boldsymbol{BD}^{-1}\boldsymbol{C} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} + \boldsymbol{CA}^{-1}\boldsymbol{B})^{-1}\boldsymbol{CA}^{-1}\boldsymbol{BD}^{-1}\boldsymbol{C} \\
=&\boldsymbol{I}_n + \boldsymbol{A}^{-1}\boldsymbol{BD}^{-1}\boldsymbol{C} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} + \boldsymbol{CA}^{-1}\boldsymbol{B})^{-1}(\boldsymbol{DD}^{-1} + \boldsymbol{CA}^{-1}\boldsymbol{BD}^{-1})\boldsymbol{C} \\
=&\boldsymbol{I}_n + \boldsymbol{A}^{-1}\boldsymbol{BD}^{-1}\boldsymbol{C} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} + \boldsymbol{CA}^{-1}\boldsymbol{B})^{-1}(\boldsymbol{D} + \boldsymbol{CA}^{-1}\boldsymbol{B})\boldsymbol{D}^{-1}\boldsymbol{C} \\
=&\boldsymbol{I}_n + \boldsymbol{A}^{-1}\boldsymbol{BD}^{-1}\boldsymbol{C} - \boldsymbol{A}^{-1}\boldsymbol{BD}^{-1}\boldsymbol{C} \\
=&\boldsymbol{I}_n
\end{aligned}$$

Then, by definition 2.5.16. , we get (6). □

Exercise 2.5.25.

Let $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m]^{\mathrm{T}} \in \mathbb{R}^{m\times n}$. Note that i-th row of $\boldsymbol{A}$ is $\boldsymbol{a}_i^{\mathrm{T}}$. Show that, $\forall k \in \mathbb{N}_m$,

$$\boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{a}_k^{\mathrm{T}} \tag{7}$$

where $\boldsymbol{e}_k$ is a unit vector with k-th entry one and the other entries zero.

Proof. Let $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m]^{\mathrm{T}} \in \mathbb{R}^{m\times n}$ and $k \in \mathbb{N}_m$.

$$\begin{aligned}
\boldsymbol{e}_k^{\mathrm{T}}\boldsymbol{A} &= \boldsymbol{e}_k^{\mathrm{T}}\begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} \\
&= [A_{k,1}, \ldots, A_{k,n}]
\end{aligned}$$

Since i-th row of $\boldsymbol{A}$ is $\boldsymbol{a}^{\mathrm{T}}$,

$$[A_{k,1}, \ldots, A_{k,n}] = \boldsymbol{a}_k^{\mathrm{T}}.$$

Thus, (7) holds. □

Exercise 2.5.27.

Let $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m] \in \mathbb{R}^{n \times m}$. Note that i-th column of $\boldsymbol{A}$ is $\boldsymbol{a}_i$. Let $\boldsymbol{x} \in \mathbb{R}^n$ and $k \in \mathbb{N}_m$. Show that

$$\langle \boldsymbol{e}_k, \boldsymbol{A}^{\mathrm{T}} \boldsymbol{x} \rangle = \langle \boldsymbol{a}_k, \boldsymbol{x} \rangle \tag{8}$$

where $\boldsymbol{e}_k$ is a unit vector with k-th entry one and the other entries zero.

Proof. Let $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m] \in \mathbb{R}^{n \times m}$, $\boldsymbol{x} \in \mathbb{R}^n$ and $k \in \mathbb{N}_m$.

$$\begin{aligned}
\boldsymbol{A}^{\mathrm{T}} \boldsymbol{x} &= \begin{bmatrix} A_{1,1} & \cdots & A_{n,1} \\ \vdots & \ddots & \vdots \\ A_{1,m} & \cdots & A_{n,m} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{a}_1^{\mathrm{T}} \boldsymbol{x} \\ \vdots \\ \boldsymbol{a}_m^{\mathrm{T}} \boldsymbol{x} \end{bmatrix}
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle \boldsymbol{e}_k, \boldsymbol{A}^{\mathrm{T}} \boldsymbol{x} \rangle &= \boldsymbol{a}_k^{\mathrm{T}} \boldsymbol{x} \\
&= \langle \boldsymbol{a}_k, \boldsymbol{x} \rangle.
\end{aligned}$$

□