Matrix Algebra Marathon

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1 Inner-Product of Vectors

1.1 Definition

Definition 1.1.1. *The inner-product of vectors is defined as*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \coloneqq \sum_{i=1}^{n} x_i y_i \tag{1}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

1.2 Exercise

Exercise 2.1.3.

For $\forall a \in \mathbb{R}$ and $\forall \boldsymbol{x}, \forall \boldsymbol{y} \in \mathbb{R}^n$, show that

$$\langle a\boldsymbol{x}, \boldsymbol{y} \rangle = a\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, a\boldsymbol{y} \rangle. \tag{2}$$

Proof. For all $a \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{y} \in \mathbb{R}^n$,

$$\langle a\boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} (ax_i) y_i = a \sum_{i=1}^{n} x_i y_i = a\langle \boldsymbol{x}, \boldsymbol{y} \rangle$$

and

$$\langle \boldsymbol{x}, a\boldsymbol{y} \rangle = \sum_{i=1}^{n} x_i (ay_i) = a \sum_{i=1}^{n} x_i y_i = a\langle \boldsymbol{x}, \boldsymbol{y} \rangle.$$

Thus,

$$\langle a\boldsymbol{x}, \boldsymbol{y} \rangle = a\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, a\boldsymbol{y} \rangle.$$

□

Exercise 2.1.6.

For $\forall \boldsymbol{x}, \forall \boldsymbol{y}, \forall \boldsymbol{z} \in \mathbb{R}^n$, show that

$$\langle \boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, \boldsymbol{z} \rangle. \tag{3}$$

Proof. For all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$,

$$\begin{aligned}\langle \boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z} \rangle &= \sum_{i=1}^{n} x_i(y_i + z_i) = \sum_{i=1}^{n} (x_i y_i + x_i z_i) = \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} x_i z_i \\ &= \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, \boldsymbol{z} \rangle.\end{aligned}$$

□

2 ℓ_2-Norm

2.1 Definition

Definition 2.1.1. *The ℓ_2-norm of vectors is defined as*

$$\|\boldsymbol{x}\| \coloneqq \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} \tag{4}$$

where $\boldsymbol{x} \in \mathbb{R}^n$.

2.2 Exercise

Exercise 2.2.4.

For $\forall a \in \mathbb{R}, \forall \boldsymbol{x} \in \mathbb{R}^n$, derive the equality:

$$\|a\boldsymbol{x}\| = |a|\|\boldsymbol{x}\| \tag{5}$$

where $|a|$ denotes the absolute value of a.

Proof. For all $a \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n$,

$$\begin{aligned}\|a\boldsymbol{x}\| &= \sqrt{\langle a\boldsymbol{x}, a\boldsymbol{x} \rangle} = \sqrt{\sum_{i=1}^{n} (ax_i)(ax_i)} = \sqrt{a^2 \sum_{i=1}^{n} x_i^2} = \sqrt{a^2}\sqrt{\sum_{i=1}^{n} x_i^2} \\ &= |a|\sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = |a|\|\boldsymbol{x}\|.\end{aligned}$$

□

Exercise 2.2.8.

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Derive the equality:

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\langle \boldsymbol{x}, \boldsymbol{y} \rangle. \tag{6}$$

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

$$\begin{aligned}
\|\boldsymbol{x}+\boldsymbol{y}\|^2 &= \left(\sqrt{\langle \boldsymbol{x}+\boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y}\rangle}\right)^2 \\
&= \langle \boldsymbol{x}+\boldsymbol{y}, \boldsymbol{x}+\boldsymbol{y}\rangle \\
&= \sum_{i=1}^{n}(x_i+y_i)^2 \\
&= \sum_{i=1}^{n}(x_i^2+2x_iy_i+y_i^2) \\
&= \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} 2x_iy_i + \sum_{i=1}^{n} y_i^2 \\
&= \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_iy_i + \sum_{i=1}^{n} y_i^2 \\
&= \langle \boldsymbol{x}, \boldsymbol{x}\rangle + \langle \boldsymbol{y}, \boldsymbol{y}\rangle + 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle \\
&= \left(\sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}\right)^2 + \left(\sqrt{\langle \boldsymbol{y}, \boldsymbol{y}\rangle}\right)^2 + 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle \\
&= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle
\end{aligned}$$

□

Exercise 2.2.10.

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Show that

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle \le \|\boldsymbol{x}\|\|\boldsymbol{y}\|. \tag{7}$$

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, t \in \mathbb{R}$. Consider the function

$$\begin{aligned}
f(t) &= \|\boldsymbol{x}+t\boldsymbol{y}\|^2 \\
&= \langle \boldsymbol{x}+t\boldsymbol{y}, \boldsymbol{x}+t\boldsymbol{y}\rangle \\
&= \sum_{i=1}^{n}(x_i+ty_i)^2 \\
&= \sum_{i=1}^{n}\left(x_i^2+2tx_iy_i+(ty_i)^2\right) \\
&= \sum_{i=1}^{n} x_i^2 + 2t\sum_{i=1}^{n} x_iy_i + t^2\sum_{i=1}^{n} y_i^2 \\
&= \langle \boldsymbol{y}, \boldsymbol{y}\rangle t^2 + 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle t + \langle \boldsymbol{x}, \boldsymbol{x}\rangle \ge 0.
\end{aligned}$$

Therefore, the quadratic equation

$$\langle \boldsymbol{y}, \boldsymbol{y}\rangle t^2 + 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle t + \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0$$

has at most one solution. This implies that its discriminant must be less or equal to zero, that is

$$(2\langle \boldsymbol{x}, \boldsymbol{y}\rangle)^2 - 4\langle \boldsymbol{y}, \boldsymbol{y}\rangle\langle \boldsymbol{x}, \boldsymbol{x}\rangle \leq 0.$$

Hence

$$\begin{aligned}
4\langle \boldsymbol{x}, \boldsymbol{y}\rangle^2 &\leq 4\langle \boldsymbol{x}, \boldsymbol{x}\rangle\langle \boldsymbol{y}, \boldsymbol{y}\rangle \\
\langle \boldsymbol{x}, \boldsymbol{y}\rangle^2 &\leq \langle \boldsymbol{x}, \boldsymbol{x}\rangle\langle \boldsymbol{y}, \boldsymbol{y}\rangle \\
\langle \boldsymbol{x}, \boldsymbol{y}\rangle^2 &\leq \left(\sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}\right)^2 \left(\sqrt{\langle \boldsymbol{y}, \boldsymbol{y}\rangle}\right)^2 \\
\langle \boldsymbol{x}, \boldsymbol{y}\rangle^2 &\leq \|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2,
\end{aligned}$$

so

$$-\|\boldsymbol{x}\|\|\boldsymbol{y}\| \leq \langle \boldsymbol{x}, \boldsymbol{y}\rangle \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|.$$

This also implies

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|.$$

□

Exercise 2.2.14.

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ be unit vectors (i.e. $\|\boldsymbol{x}\| = \|\boldsymbol{y}\| = 1$) . Show that

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle \leq 1. \tag{8}$$

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ be unit vectors. It follows from (7) that

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\| = 1 \cdot 1 = 1.$$

□

3 ℓ_1-Norm of Vectors

3.1 Definition

Definition 3.1.1. *The ℓ_1-norm of vectors is defined as*

$$\|\boldsymbol{x}\|_1 \coloneqq \sum_{i=1}^{n} |x_i| \tag{9}$$

where $\boldsymbol{x} \in \mathbb{R}^n$.

3.2 Exercise

Exercise 2.3.3.

For $\forall \boldsymbol{x} \in \mathbb{R}^n$, show that

$$\|\boldsymbol{x}\|_1 \geq 0. \tag{10}$$

Proof. Let $\boldsymbol{x} \in \mathbb{R}^n$. Since the absolute value is always either a positive number or zero, for all $i \in \mathbb{N}_n$,

$$|x_i| \geq 0.$$

Then

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i| \geq 0.$$

□

4 ℓ_∞-Norm of Vectors

4.1 Definition

Definition 4.1.1. *The ℓ_∞-norm of vectors is defined as*

$$\|\boldsymbol{x}\|_\infty \coloneqq \max_{i \in \mathbb{N}_n} |x_i| \tag{11}$$

where $\boldsymbol{x} \in \mathbb{R}^n$.

4.2 Exercise

Exercise 2.4.3.

For $\forall \boldsymbol{x} \in \mathbb{R}^n$, show that

$$\|\boldsymbol{x}\|_\infty \geq 0. \tag{12}$$

Proof. Let $\boldsymbol{x} \in \mathbb{R}^n$ and

$$M = \max_{i \in \mathbb{N}_n} |x_i|.$$

Then, for all $j \in \mathbb{N}_n$,

$$M \geq |x_j|$$

and

$$|x_j| \geq 0.$$

Thus,

$$\|\boldsymbol{x}\|_\infty = \max_{i \in \mathbb{N}_n} |x_i| = M \geq |x_j| \geq 0.$$

□