Functional Analysis - Homework 10

Problem 1.

a)

Proof. **(⇒)** Assume A is compact.

Let (x_n) be a bounded sequence, so $\|x_n\| \leq M$ for some $M > 0$. Define $z_n = x_n/M$. Then $\|z_n\| \leq 1$, so $z_n \in B_1(0)$.

$(Az_n) \subset A(B_1(0))$ is relatively compact since A is compact. Thus, (Az_n) has a convergent subsequence (Az_{n_k}). Since $Ax_{n_k} = A(Mz_{n_k}) = M \cdot Az_{n_k}$, the sequence (Ax_{n_k}) also converges in Y.

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(⇐) Assume the sequential property holds. Let (y_n) be an arbitrary sequence in $K = \overline{A(B_1(0))}$. By the definition of closure, for each y_n, there exists $z_n \in A(B_1(0))$ such that $\|y_n - z_n\|_Y < 1/n$. Since $z_n \in A(B_1(0))$, there is $x_n \in B_1(0)$ (hence $\|x_n\| \leq 1$) such that $z_n = Ax_n$. The sequence (x_n) is bounded. By hypothesis, (x_n) has a subsequence (x_{n_k}) such that (Ax_{n_k}) converges to some $y_0 \in Y$. Let $z_{n_k} = Ax_{n_k}$. Consider the corresponding subsequence (y_{n_k}).

$$\|y_{n_k} - y_0\|_Y \leq \|y_{n_k} - z_{n_k}\|_Y + \|z_{n_k} - y_0\|_Y \tag{1}$$

As $k \to \infty$, $\|y_{n_k} - z_{n_k}\|_Y < 1/n_k \to 0$ and $\|z_{n_k} - y_0\|_Y \to 0$. Thus, (y_{n_k}) converges to y_0. Since every sequence in K has a convergent subsequence, K is compact, and A is a compact operator. □

b)

Proof. **Subspace Property:**

1. **Scalar Multiplication:** If $A \in \mathcal{K}(X,Y)$, then $\overline{A(B_1(0))}$ is compact. For $\alpha \in \mathbb{R}$, $(\alpha A)(B_1(0)) = \alpha \cdot A(B_1(0))$. Since multiplication by a scalar is continuous, the continuous image of a compact set is compact. Hence $\overline{(\alpha A)(B_1(0))} = \alpha \overline{A(B_1(0))}$ is compact, and $\alpha A \in \mathcal{K}(X,Y)$.

2. **Addition:** If $A, B \in \mathcal{K}(X,Y)$, then $\overline{A(B_1(0))}$ and $\overline{B(B_1(0))}$ are compact. The set $(A+B)(B_1(0)) \subset A(B_1(0)) + B(B_1(0))$. The set $K_A + K_B$, where $K_A = \overline{A(B_1(0))}$ and $K_B = \overline{B(B_1(0))}$, is the continuous image of the compact set $K_A \times K_B$ under vector addition, hence $K_A + K_B$ is compact. Since $\overline{(A+B)(B_1(0))}$ is a closed subset of a compact set $K_A + K_B$, it is compact. Thus $A + B \in \mathcal{K}(X,Y)$.

Closed Property: Assume Y is complete. Let (A_n) be a sequence in $\mathcal{K}(X,Y)$ such that $A_n \to A$ in the operator norm $\mathcal{L}(X,Y)$. We show A is compact using total boundedness.

Let $\epsilon > 0$. Since $A_n \to A$, there exists N such that $\|A_N - A\| < \epsilon$. Since A_N is compact, the set $K_N = A_N(B_1(0))$ is relatively compact, and thus totally bounded. K_N has a finite ϵ-net $\{y_1, \ldots, y_M\}$.

Now, consider $K = A(B_1(0))$. For any $y = Ax \in K$ (with $\|x\| \leq 1$), consider $y_N = A_N x \in K_N$. Since $\{y_1, \ldots, y_M\}$ is an ϵ-net for K_N, there is y_i such that $\|y_N - y_i\| < \epsilon$. By the triangle inequality:

$$\|y - y_i\| \leq \|Ax - A_N x\| + \|A_N x - y_i\| \tag{2}$$

$$\|Ax - A_N x\| \leq \|A - A_N\| \, \|x\| < \epsilon \cdot 1 = \epsilon \tag{3}$$

$$\|y - y_i\| < \epsilon + \epsilon = 2\epsilon \tag{4}$$

Thus, $\{y_1, \ldots, y_M\}$ is an ϵ-net for K. Since K is totally bounded, its closure $\overline{K}$ is compact (as Y is complete). Hence $A \in \mathcal{K}(X,Y)$, proving $\mathcal{K}(X,Y)$ is closed. □

c)

Proof. Since $A(X)$ is a finite-dimensional subspace of Y, it is a closed set in Y. The set $K = \overline{A(B_1(0))}$ is a subset of $A(X)$, and since $A(X)$ is closed, we have $K \subset A(X)$. Also, A is a bounded operator, so K is a bounded set. Since K is a closed and bounded set contained in the finite-dimensional space $A(X)$, K is compact. Thus, A is a compact operator. □

d)

Proof. **Case 1:** $A \in \mathcal{K}(X, Y)$**,** $B \in \mathcal{L}(Y, Z)$**.** Since A is compact, $K_A = \overline{A(B_1(0))}$ is a compact set in Y. Since B is bounded, it is continuous. The image of the compact set K_A under B, $B(K_A)$, is compact in Z. Since $(B \circ A)(B_1(0)) \subset B(A(B_1(0))) \subset B(K_A)$, the image of the unit ball under $B \circ A$ is relatively compact. Thus $B \circ A$ is compact.

Case 2: $A \in \mathcal{L}(X, Y)$**,** $B \in \mathcal{K}(Y, Z)$**.** Since A is bounded, $A(B_1(0))$ is a bounded set in Y. Let $M = \|A\|$. Then $A(B_1(0)) \subset M \cdot B_1(0)$ in Y. The image of the unit ball under $B \circ A$ satisfies:

$$(B \circ A)(B_1(0)) = B(A(B_1(0))) \subset B(M \cdot B_1(0)) = M \cdot B(B_1(0)) \tag{5}$$

The closure $\overline{(B \circ A)(B_1(0))} \subset M \cdot \overline{B(B_1(0))}$. Since B is compact, $\overline{B(B_1(0))}$ is compact. Since scaling by M is continuous, $M \cdot \overline{B(B_1(0))}$ is compact. Thus, $\overline{(B \circ A)(B_1(0))}$ is a closed subset of a compact set, hence it is compact, and $B \circ A$ is compact. $\square$

e)

Proof. Let (x_n) be an arbitrary bounded sequence in X. Since X is a reflexive space, the Eberlein-Smulian Theorem implies that the bounded set $\{x_n\}$ is weakly sequentially compact. Therefore, (x_n) has a weakly convergent subsequence (x_{n_k}), which converges to some $x_0 \in X$:

$$x_{n_k} \rightharpoonup x_0 \quad \text{in } X$$

By the hypothesis on the operator A, it maps this weakly convergent sequence to a norm-convergent sequence in Y:

$$\|Ax_{n_k} - Ax_0\|_Y \to 0 \quad \text{as } k \to \infty \tag{6}$$

Since the sequence of images (Ax_{n_k}) is norm-convergent, it satisfies the condition required by statement (a). Therefore, A is a compact operator. $\square$

Problem 2.

a)

Proof. For any $f \in L^2(\Omega)$:

$$\|Kf\|^2_{L^2(\Omega)} = \int_\Omega \left| \int_\Omega k(x, y) f(y) dy \right|^2 dx \tag{7}$$

Applying the Cauchy-Schwarz inequality to the inner integral (with respect to y):

$$\left| \int_\Omega k(x, y) f(y) dy \right|^2 \leq \left(\int_\Omega |k(x, y)|^2 dy \right) \left(\int_\Omega |f(y)|^2 dy \right) \tag{8}$$

$$= \left(\int_\Omega |k(x, y)|^2 dy \right) \|f\|^2_{L^2(\Omega)} \tag{9}$$

Substituting this back into Equation (7):

$$\|Kf\|^2_{L^2(\Omega)} \leq \int_\Omega \left[\left(\int_\Omega |k(x, y)|^2 dy \right) \|f\|^2_{L^2(\Omega)} \right] dx \tag{10}$$

$$= \|f\|^2_{L^2(\Omega)} \int_\Omega \int_\Omega |k(x, y)|^2 dy dx \tag{11}$$

$$= \|k\|^2_{L^2(\Omega \times \Omega)} \|f\|^2_{L^2(\Omega)} \tag{12}$$

Taking the square root gives the operator norm bound:

$$\|Kf\|_{L^2(\Omega)} \leq \|k\|_{L^2(\Omega \times \Omega)} \|f\|_{L^2(\Omega)} \tag{13}$$

Since $k \in L^2(\Omega \times \Omega)$, $\|k\|_{L^2(\Omega \times \Omega)} < \infty$. Thus, $\|Kf\|_{L^2(\Omega)} < \infty$, which proves that $Kf \in L^2(\Omega)$

$\square$

b)

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis for $L^2(\Omega)$. Then the tensor products $\{e_i(x)e_j(y)\}_{i,j=1}^{\infty}$ form an orthonormal basis for $L^2(\Omega \times \Omega)$. Since $k \in L^2(\Omega \times \Omega)$, we can approximate it by its partial Fourier sums. Define the approximating kernels k_n:

$$k_n(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} e_i(x) e_j(y) \tag{14}$$

This approximation converges in the L^2 norm: $\|k - k_n\|_{L^2(\Omega\times\Omega)} \to 0$ as $n \to \infty$.

Define the corresponding integral operators K_n:

$$\begin{aligned}
(K_n f)(x) &= \int_\Omega k_n(x, y) f(y) dy &\quad (15)\\
&= \int_\Omega \left(\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} e_i(x) e_j(y) \right) f(y) dy &\quad (16)\\
&= \sum_{i=1}^{n} e_i(x) \left(\sum_{j=1}^{n} c_{ij} \int_\Omega e_j(y) f(y) dy \right) &\quad (17)
\end{aligned}$$

The range of K_n is spanned by the finite set $\{e_1, \ldots, e_n\}$, hence K_n is a finite-rank operator and thus a compact operator for all n.

We bound the operator norm of the difference $K - K_n$ using the result from part (a) (Equation (13)):

$$\begin{aligned}
\|K - K_n\|_{\mathcal{L}} &= \sup_{\|f\|_{L^2(\Omega)} \le 1} \|(K - K_n) f\|_{L^2(\Omega)} &\quad (18)\\
&= \sup_{\|f\|_{L^2(\Omega)} \le 1} \left\| \int_\Omega (k(x, y) - k_n(x, y)) f(y) dy \right\|_{L^2(\Omega)} &\quad (19)\\
&\le \|k - k_n\|_{L^2(\Omega\times\Omega)} \cdot \sup_{\|f\|_{L^2(\Omega)} \le 1} \|f\|_{L^2(\Omega)} &\quad (20)\\
&= \|k - k_n\|_{L^2(\Omega\times\Omega)} &\quad (21)
\end{aligned}$$

Since $\|k - k_n\|_{L^2(\Omega\times\Omega)} \to 0$ as $n \to \infty$, we have $\|K - K_n\|_{\mathcal{L}} \to 0$.

The sequence of compact operators (K_n) converges to K in the operator norm. Since $L^2(\Omega)$ is a Hilbert space, the set of compact operators $\mathcal{K}(L^2(\Omega), L^2(\Omega))$ is a closed subspace of $\mathcal{L}(L^2(\Omega), L^2(\Omega))$. Therefore, the limit operator K must be compact. □

Problem 3.

a)

1. **Continuity Proof:**

$$\|Af\|^2_{L^2([a,b])} = \int_a^b \left| x^2 f(x) \right|^2 dx = \int_a^b x^4 \left| f(x) \right|^2 dx \tag{22}$$

Since $a \le 0 \le b$, the maximum value of x^2 on the interval $[a, b]$ is $M^2 = \max(a^2, b^2)$. For all $x \in [a, b]$, we have $x^2 \le M^2$, which implies $x^4 \le M^4$.

Substituting this bound into Equation (22):

$$\begin{aligned}
\|Af\|^2_{L^2([a,b])} &\le \int_a^b M^4 \left| f(x) \right|^2 dx &\quad (23)\\
&= M^4 \int_a^b \left| f(x) \right|^2 dx = M^4 \|f\|^2_{L^2([a,b])} &\quad (24)
\end{aligned}$$

Taking the square root, $\|Af\|_{L^2([a,b])} \le M^2 \|f\|_{L^2([a,b])}$. Since $M^2 = \max(a^2, b^2)$ is finite, A is continuous.

2. **Operator Norm $||A||$:** The operator norm is defined as $||A|| = \sup_{\|f\|_{L^2([a,b])}=1} \|Af\|_{L^2([a,b])}$. For a multiplication operator A by $g(x) = x^2$ on L^2, its norm is the essential supremum of the multiplier function g:
$$\|A\| = \left\|x^2\right\|_{L^\infty([a,b])} = \operatorname{ess\,sup}_{x\in[a,b]} \left|x^2\right| = \max_{x\in[a,b]} x^2 = \max(a^2, b^2). \tag{25}$$

b)

Proof. A complex number λ is an eigenvalue if there exists a non-zero function $f \in L^2([a,b];\mathbb{C})$ such that $Af = \lambda f$. The eigenvalue equation is:

$$x^2 f(x) = \lambda f(x) \quad \text{a.e. } x \in [a,b] \tag{26}$$

Thus,
$$(x^2 - \lambda)f(x) = 0 \quad \text{a.e. } x \in [a,b]$$
, which implies that $f(x)$ can be non-zero only on the set $E_\lambda = \left\{x \in [a,b] \mid x^2 = \lambda\right\}$.

The set E_λ contains at most two distinct roots, which means its Lebesgue measure is zero:

$$m(E_\lambda) = 0$$

If $f \in L^2([a,b];\mathbb{C})$ is an eigenfunction, it must be zero on the complement of E_λ. Since $m(E_\lambda) = 0$, f is zero almost everywhere on the entire interval $[a,b]$. In L^2 space, this means f is the zero function ($f = 0$). Since there is no non-zero eigenfunction, the operator A has no eigenvalues. □

c)

For a multiplication operator $A_g f = gf$ on L^2, the spectrum $\sigma(A_g)$ is the essential range of the multiplier function g. Here the multiplier is $g(x) = x^2$.

1. **Essential Range of x^2:** Since x^2 is continuous on the compact interval $[a,b]$, its essential range is the closed interval:
$$R = [\min_{x\in[a,b]} x^2, \max_{x\in[a,b]} x^2] = [0, \max(a^2, b^2)] = [0, \|A\|]. \tag{27}$$

2. **Proof of $\sigma(A) = [0, \|A\|]$:** The operator $A - \lambda I$ is invertible if and only if the function $g_\lambda(x) = \frac{1}{x^2-\lambda}$ is in $L^\infty([a,b])$.

   - **Case 1: $\lambda \in [0, \|A\|]$ (The range R)** Since $\lambda \in [0, \max(a^2, b^2)]$, there exists $x_0 \in [a,b]$ such that $x_0^2 = \lambda$. Thus, $x^2 - \lambda$ vanishes at x_0. This means that $g_\lambda(x)$ is unbounded (not in $L^\infty([a,b])$) in any neighborhood of x_0. Therefore, $A - \lambda I$ is not invertible, so $\lambda \in \sigma(A)$. This shows $R \subseteq \sigma(A)$.
   - **Case 2: $\lambda \notin [0, \|A\|]$ (The resolvent set)** The value λ is separated from the compact set $[0, \|A\|]$ by a positive distance δ. Let $\delta = \operatorname{dist}(\lambda, [0, \|A\|]) > 0$. The inverse operator $R_\lambda = (A - \lambda I)^{-1}$ is multiplication by $g_\lambda(x)$. Since
   $$|g_\lambda(x)| = \frac{1}{|x^2 - \lambda|} \le \frac{1}{\delta}$$
   for all $x \in [a,b]$, the function g_λ is bounded and thus $g_\lambda \in L^\infty([a,b])$. This means the inverse operator R_λ is bounded, so $A - \lambda I$ is invertible. Thus, $\lambda \notin \sigma(A)$.

Therefore, the spectrum of the multiplication operator A is:

$$\sigma(A) = [0, \|A\|] = [0, \max(a^2, b^2)]. \tag{28}$$