The Arzelà-Ascoli Theorem and its Application to Mercer's Theorem

The **Arzelà-Ascoli Theorem** is a fundamental result in real analysis and functional analysis that provides conditions under which a sequence of continuous functions on a compact set will have a uniformly convergent subsequence. It is a powerful tool for proving the existence of solutions to differential equations, properties of integral operators, and other results involving function spaces.

High-Level Intuition

In finite-dimensional Euclidean space, a set is "compact" (meaning every sequence in the set has a convergent subsequence) if and only if it is **closed and bounded** (Heine-Borel theorem). However, in infinite-dimensional spaces, such as spaces of functions, being closed and bounded is not enough to guarantee compactness. Functions can be bounded but still "wiggle" infinitely fast, preventing convergence.

The Arzelà-Ascoli Theorem provides the **additional conditions** needed for a set of functions to be **relatively compact** (meaning its closure is compact) in a space of continuous functions. These conditions essentially limit how "wiggly" or "spread out" the functions in the set can be.

Formal Statement (Simplified for Context)

Let X be a compact metric space (e.g., a closed and bounded interval like $[a, b]$ in $\mathbb{R}$). Let $\mathcal{F}$ be a family (set) of continuous functions from X to $\mathbb{R}$ (or $\mathbb{C}$, or $\mathbb{R}^n$).

The Arzelà-Ascoli Theorem states that $\mathcal{F}$ is **relatively compact** in the space of continuous functions $C(X)$ (equipped with the uniform norm, $\|f\|_\infty = \sup_{x \in X} |f(x)|$) if and only if it satisfies two conditions:

1. **Pointwise Boundedness:** For each point $x \in X$, the set of values $\{f(x) : f \in \mathcal{F}\}$ is bounded. That is, for every $x \in X$, there exists a constant M_x such that $|f(x)| \leq M_x$ for all $f \in \mathcal{F}$.

   *Note: Often, in practical applications, a stronger condition called **uniform boundedness** is used: there exists a single constant M such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and all $x \in X$. For continuous functions on a compact domain, pointwise boundedness implies uniform boundedness.*

2. **Equicontinuity:** The family $\mathcal{F}$ is equicontinuous. This means that all functions in the family are "equally continuous" at every point. Formally, for every $\epsilon > 0$ and every $x \in X$, there exists a $\delta > 0$ such that for all $f \in \mathcal{F}$ and all $x' \in X$ with $d(x, x') < \delta$, we have $|f(x) - f(x')| < \epsilon$.

   Note: The key here is that δ depends only on ϵ and x, not on the individual function f. It ensures that no function in the family can vary too rapidly.

If these two conditions are met, then every sequence of functions in $\mathcal{F}$ has a subsequence that converges uniformly to a continuous function.

Relevance to Mercer's Theorem (as per text)

In **Step 1 of the proof of Mercer's Theorem**, the Arzelà-Ascoli Theorem is invoked to establish that the Hilbert-Schmidt integral operator T_k is a **compact operator**.

The statement in the text is: "Therefore, according to the Arzelà-Ascoli theorem (Arzelà, 1895), the image of the unit ball after applying the operator is compact. In other words, the operator T_k is compact."

Let's break down how this connection works:

1. **The Operator T_k:**
$$T_k f(x) := \int_a^b k(x,y) f(y)\, dy \quad \text{(Eq. 18)}$$
This operator maps functions $f \in L^2([a,b])$ to functions $T_k f \in C([a,b])$ (the space of continuous functions on $[a,b]$), provided $k(x,y)$ is continuous.

2. **Compact Operator Definition:** A linear operator $A : V \to W$ (between normed spaces) is called **compact** if it maps bounded sets in V to relatively compact sets in W. In the context of Mercer's Theorem, $V = L^2([a,b])$ and $W = C([a,b])$. We want to show that the image of the unit ball in $L^2([a,b])$ (i.e., the set $\mathcal{F} = \{T_k f \mid \|f\|_{L^2} \le 1\}$) is relatively compact in $C([a,b])$.

3. **Applying Arzelà-Ascoli:** To show that the set $\mathcal{F} = \{T_k f \mid \|f\|_{L^2} \le 1\}$ is relatively compact in $C([a,b])$, we need to verify the two conditions of Arzelà-Ascoli:

   - **Uniform Boundedness:** We need to show that there is a constant M such that for all f with $\|f\|_{L^2} \le 1$, and for all $x \in [a,b]$, $|T_k f(x)| \le M$.
$$\begin{aligned} |T_k f(x)| &= \left| \int_a^b k(x,y) f(y)\, dy \right| \\ &\le \left(\int_a^b |k(x,y)|^2\, dy \right)^{1/2} \left(\int_a^b |f(y)|^2\, dy \right)^{1/2} \quad \text{(Cauchy-Schwarz)} \\ &\le \left(\int_a^b |k(x,y)|^2\, dy \right)^{1/2} \cdot 1 \quad (\text{Since } \|f\|_{L^2} \le 1) \end{aligned}$$
     The kernel $k(x,y)$ is bounded: $\sup_{x,y} k(x,y) < \infty$. Let $K_{max} = \sup_{x,y} |k(x,y)|$. Then
$$\int_a^b |k(x,y)|^2\, dy \le \int_a^b K_{max}^2\, dy = K_{max}^2 (b-a)$$
$$|T_k f(x)| \le K_{max}\sqrt{b-a}.$$

   - **Equicontinuity:** We need to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all f with $\|f\|_{L^2} \le 1$, and for any $x_1, x_2 \in [a,b]$ with $|x_1 - x_2| < \delta$, we have $|T_k f(x_1) - T_k f(x_2)| < \epsilon$.
$$\begin{aligned} |T_k f(x_1) - T_k f(x_2)| &= \left| \int_a^b (k(x_1,y) - k(x_2,y)) f(y)\, dy \right| \\ &\le \left(\int_a^b |k(x_1,y) - k(x_2,y)|^2\, dy \right)^{1/2} \left(\int_a^b |f(y)|^2\, dy \right)^{1/2} \quad \text{(Cauchy-Schwarz)} \\ &\le \left(\int_a^b |k(x_1,y) - k(x_2,y)|^2\, dy \right)^{1/2} \quad (\text{Since } \|f\|_{L^2} \le 1) \end{aligned}$$
     Since $k(x,y)$ is **continuous** on the compact set $[a,b] \times [a,b]$, it is **uniformly continuous**.
$$\forall \eta > 0, \exists \delta > 0, \text{ s.t. } |x_1 - x_2| < \delta \implies \forall y \in [a,b], |k(x_1,y) - k(x_2,y)| < \eta,$$
     So, if $|x_1 - x_2| < \delta$, then
$$\int_a^b |k(x_1,y) - k(x_2,y)|^2\, dy < \int_a^b \eta^2\, dy = \eta^2 (b-a).$$
$$|T_k f(x_1) - T_k f(x_2)| < \eta\sqrt{b-a}$$

Since both uniform boundedness and equicontinuity are satisfied, the Arzelà-Ascoli Theorem ensures that the image of the unit ball under T_k is relatively compact in $C([a,b])$. This implies that T_k is a **compact operator**. The compactness of T_k is a critical prerequisite for applying the Spectral Theorem for compact self-adjoint operators in the next step of Mercer's proof.