Proof of Mercer's Theorem: Step 4 - Convergence

With Definitions of Convergence

Mercer's Theorem (Theorem 2 in the provided text) states that a continuous, symmetric, positive semi-definite kernel $k(x, y)$ on a compact domain $[a, b] \times [a, b]$ can be represented by an infinite series:

$$k(x, y) = \sum_{i=1}^{\infty} \lambda_i \psi_i(x) \psi_i(y)$$

The theorem explicitly mentions that "the convergence is **absolute and uniform**." Let's break down what these two terms mean in this context and why they are important.

Absolute Convergence
$\sum_{i=1}^{\infty} |\lambda_i \psi_i(x) \psi_i(y)|$ converges for all (x, y) in the domain $[a, b] \times [a, b]$.

Uniform Convergence With $f(x, y)$ on the domain, $\forall \epsilon > 0$, $\exists N_0$ (which depends *only* on ϵ) such that for all $N > N_0$ and for *all* $(x, y) \in D$, $|\sum_{i=1}^{N} \lambda_i \psi_i(x) \psi_i(y) - f(x, y)| < \epsilon$

Definitions of Convergence

Absolute Convergence

1. **Definition:** A series of functions $\sum_{i=1}^{\infty} f_i(x, y)$ is said to converge **absolutely** if the series of the absolute values of its terms, $\sum_{i=1}^{\infty} |f_i(x, y)|$, converges. In the context of Mercer's Theorem, this means that the series:
$$\sum_{i=1}^{\infty} |\lambda_i \psi_i(x) \psi_i(y)|$$
converges for all (x, y) in the domain $[a, b] \times [a, b]$.

2. **Significance:**
   - **Stronger Condition:** Absolute convergence is a stronger condition than ordinary convergence; if a series converges absolutely, it also converges in the usual sense.
   - **Term Rearrangement:** A key benefit is that the order of summation does not affect the sum, which is a desirable property for mathematical manipulations in the proof.

Uniform Convergence

1. **Definition:** A sequence of partial sums $S_N(x, y) = \sum_{i=1}^{N} f_i(x, y)$ converges **uniformly** to a function $f(x, y)$ on a set D if, for every $\epsilon > 0$, there exists an integer N_0 (which depends *only* on ϵ, not on x or y) such that for all $N > N_0$ and for *all* $(x, y) \in D$, we have:
$$|S_N(x, y) - f(x, y)| < \epsilon$$
Uniform convergence means that the partial sums $S_N(x, y)$ approach the limit function $k(x, y)$ at the same rate across the entire domain $[a, b] \times [a, b]$.

2. **Significance:**
   - **Preservation of Properties:** Uniform convergence ensures that many properties of the individual terms are preserved in the limit function. Since the individual terms are continuous functions, uniform convergence guarantees that the limit function $k(x, y)$ is also continuous, which is consistent with the theorem's assumption.

- **Interchange of Limits:** It allows for the interchange of limits and integrals/derivatives, essential for many operations in functional analysis related to the integral operator T_k.

In summary, the conditions of **absolute and uniform convergence** guarantee that the infinite sum is well-behaved, converging to a definite value consistently across the domain, and preserving essential properties like continuity.

Step 4: Establishing Absolute and Uniform Convergence

1. Define the Remainder Term

We introduce the truncated kernel, $r_n(x, y)$, which represents the remainder of the infinite series after summing the first n terms:

$$r_n(x, y) := k(x, y) - \sum_{i=1}^{n} \lambda_i \psi_i(x)\psi_i(y) = \sum_{i=n+1}^{\infty} \lambda_i \psi_i(x)\psi_i(y)$$

We will show that $r_n(x, y)$ converges uniformly to 0 as $n \to \infty$.

2. Positivity and Pointwise Convergence of the Diagonal

T_k is compact, symmetric, and positive, implying that the eigenvalues λ_i are all non-negative. The remainder operator, T_{r_n}, is also a positive operator with kernel $r_n(x, y)$. Consequently, $r_n(x, y)$ is a positive semi-definite kernel. The diagonal elements must be non-negative:

$$r_n(x, x) = k(x, x) - \sum_{i=1}^{n} \lambda_i \psi_i(x)\psi_i(x) \geq 0$$

This inequality implies that:

$$\sum_{i=1}^{n} \lambda_i \psi_i(x)^2 \leq k(x, x)$$

Since $k(x, y)$ is continuous on the compact domain, $M_k = \sup_{x\in[a,b]} k(x, x) < \infty$. Therefore, for all $x \in [a, b]$ and any n:

$$\sum_{i=1}^{n} \lambda_i \psi_i(x)^2 \leq M_k < \infty$$

Since the partial sums of the series $\sum_{i=1}^{\infty} \lambda_i \psi_i(x)^2$ are non-decreasing and bounded above, this series **converges pointwise** for every $x \in [a, b]$.

3. Proof of Absolute Convergence for $k(x, y)$

Since $\lambda_i \geq 0$, $|\lambda_i \psi_i(x)\psi_i(y)| = \lambda_i |\psi_i(x)||\psi_i(y)|$. For any finite $N \in \mathbb{N}$ and any $(x, y) \in [a, b] \times [a, b]$, we apply the Cauchy-Schwarz inequality for sums:

$$\left(\sum_{i=1}^{N} \lambda_i |\psi_i(x)||\psi_i(y)|\right)^2 \leq \left(\sum_{i=1}^{N} \lambda_i \psi_i(x)^2\right)\left(\sum_{i=1}^{N} \lambda_i \psi_i(y)^2\right)$$

Using the bound derived in point 2.:

$$\left(\sum_{i=1}^{N} \lambda_i |\psi_i(x)||\psi_i(y)|\right)^2 \leq M_k \cdot M_k = M_k^2$$

Taking the square root:

$$\sum_{i=1}^{N} |\lambda_i \psi_i(x)\psi_i(y)| \leq M_k$$

4. Proof of Uniform Convergence for $k(x, y)$

A. Uniform Convergence of the Diagonal Series via Dini's Theorem

Let $S'_n(x) = \sum_{i=1}^{n} \lambda_i \psi_i(x)^2$.

1. Each partial sum $S'_n(x)$ is a **continuous function** on $[a, b]$.
2. The sequence of partial sums $\{S'_n(x)\}_{n=1}^{\infty}$ is **monotonically increasing** for each x.
3. The sequence $S'_n(x)$ converges **pointwise** to $k(x, x)$.
4. The limit function $k(x, x)$ is **continuous** on the compact interval $[a, b]$.

By **Dini's Theorem**, these four conditions imply that the convergence must be **uniform**. Thus, $R_n(x) := \sum_{i=n+1}^{\infty} \lambda_i \psi_i(x)^2$ converges **uniformly to** 0 as $n \to \infty$.

B. Extending to Uniform Convergence of the Full Series

Uniform convergence of $R_n(x)$ means that for any $\epsilon' > 0$, there exists an integer N_0 such that for all $n > N_0$ and for all $x \in [a, b]$:

$$|R_n(x)| = \left| \sum_{i=n+1}^{\infty} \lambda_i \psi_i(x)^2 \right| < \epsilon'$$

Applying the Cauchy-Schwarz inequality to $r_n(x, y)$:

$$|r_n(x, y)|^2 = \left| \sum_{i=n+1}^{\infty} \lambda_i \psi_i(x) \psi_i(y) \right|^2 \leq \left(\sum_{i=n+1}^{\infty} \lambda_i \psi_i(x)^2 \right) \left(\sum_{i=n+1}^{\infty} \lambda_i \psi_i(y)^2 \right)$$

$$|r_n(x, y)|^2 \leq R_n(x) R_n(y)$$

For any chosen $\epsilon > 0$, let $\epsilon' = \sqrt{\epsilon}$. For $n > N_0$ (where N_0 corresponds to this ϵ'):

$$|r_n(x, y)|^2 < \epsilon' \cdot \epsilon' = (\sqrt{\epsilon})^2 = \epsilon$$

Taking the square root, we get:

$$|r_n(x, y)| < \sqrt{\epsilon}$$

Since this holds for all $n > N_0$ and for all $(x, y) \in [a, b] \times [a, b]$, the series $\sum_{i=1}^{\infty} \lambda_i \psi_i(x) \psi_i(y)$ converges **uniformly** to $k(x, y)$.

Therefore, the series representation of the kernel $k(x, y) = \sum_{i=1}^{\infty} \lambda_i \psi_i(x) \psi_i(y)$ converges absolutely and uniformly on $[a, b] \times [a, b]$. Q.E.D.

N pairs $(x_1, y_1), (x_2, y_2), ..., (x_N, y_N)$