Functional Analysis
Homework 5

Deadline: October 14th

1. **Problem 1** (10 points)

   Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We consider the space $(X \times Y, \|\cdot\|_{X\times Y}$ where $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_X$. Consider a bi-linear map $M : X \times Y \to Z$.

   a) Show that if $\exists C > 0 \ \forall (x,y) \in X \times Y : \|M(x,y)\|_Z \le C \|x\|_X \|y\|_Y$ then B is continuous;

   b) Assume completeness of X, and that the maps $x \mapsto M(x, y')$ and $y \mapsto M(x', y)$ are continuous for every $x' \in X$ and $y' \in Y$. Show that the norm of $M(x,y)$ is bounded as in a) for all x, y.

   Hint: Use the Uniform Boundedness Principle (you do not need the collection of operators to be countable in its most general formulation).

2. **Problem 2** (4 points)

   Let f be a linear functional not identically equal to 0. Prove that, given $\alpha \in \mathbb{R}$, the affine hyperplane $H_\alpha = [f = \alpha] = \{x \in E : f(x) = \alpha\}$ is closed if and only if f is continuous.

3. **Problem 3** (6 points)

   Let $(X, \|\cdot\|_1)$ be a normed space and consider a linear subspace $Y \subseteq X$. Consider now a norm $\|\cdot\|_2$ that is equivalent to $\|\cdot\|_1$ on Y. Prove the existence of a norm $\|\cdot\|$ on X which is equivalent to $\|\cdot\|_1$ on X and whose restriction to Y is $\|\cdot\|_2$.