Functional Analysis
Homework 3

Deadline: September 30th

1. **Problem 1** (6 points)

   Let D be a set and let $(X, \|\cdot\|_X)$ be a Banach space. Given a function $f : D \to X$ consider the following functional

   $$\|f\|_{X,\infty} = \sup_{d\in D} \|f(d)\|_X .$$

   Show that

   a) $\|f\|_{X,\infty}$ is a norm;

   b) $(B(D,X), \|\cdot\|_{X,\infty})$ is Banach,

   where

   $$B(D,X) = \left\{ f : D \to X : \|f\|_{X,\infty} < \infty \right\} .$$

2. **Problem 2** (4 points)

   Show that a Cauchy sequence converges if and only if it has a convergent subsequence.

3. **Problem 3** (6 points)

   let $(X, \|\cdot\|_X)$ be a normed vector space. Prove that the following statements are equivalent:

   a) $(X, \|\cdot\|_X)$ is Banach.

   b) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\sum_{i=1}^{\infty} \|x_n\|_X < \infty$, the limit $\lim_{N\to\infty} \sum_{i=1}^{N} x_n$ exists.

   Hint: Use the result in Problem 2.

4. **Problem 4** (4 points)

   Let (X, d) be a metric space and consider a subset $A \subset X$. Denote with $\overline{A}$ its closure, $\mathrm{int}A$ its interior, and with A^c its complement. We say that A is

   - dense, if $\overline{A} = X$;
   - nowhere dense, if $\mathrm{int}(\overline{A}) = \emptyset$;
   - meagre, if $A = \cup_{n\in\mathbb{N}} A_n$ is a countable union of nowhere dense sets A_n;
   - residual, if A^c is meagre.

Show that the following statements are equivalent

a) Every residual set is dense in X.

b) The interior of every meagre set is empty.

c) The empty set is the only subset of X that is open *and* meagre.

d) Countable intersections of dense open sets are dense.

Hint: Show that a) $\implies$ b) $\implies$ c) $\implies$ d) $\implies$ a).

Notice that subsets of meagre sets are meagre and that A is dense in $X \iff \overline{A} = X \iff \operatorname{int}(X \setminus A) = \emptyset$.