Functional Analysis - Homework 4

Problem 1.

Proof. Let $X = [1, 2] \subset \mathbb{R}$, a complete metric space.

Let $\epsilon > 0$, $m \in \mathbb{N}$, and

$$A_m = \{x \in X : \forall n \geq m, |f(nx)| \leq \epsilon\}$$

Since f is continuous, A is closed.

$\forall x \in X$, since $\lim_{n\to\infty}(nx) = 0$

$$\exists n > N, |f(nx)| \leq \epsilon$$

, meaning x should be contained in one of A_ms.

Thus,

$$X = \bigcup_m A_m$$

With Baire's category Theorem, at least one A_ms contains an open ball $B(b_0, r)$, where $b_0 \in X, r > 0$.

Therefore,

$$\forall n > m, b < r, |f(n(b_0 + b))| \leq \epsilon \tag{1}$$

$$\because n\frac{b_0 + r}{b_0 + b} > m \therefore |f(n(b_0 + r))| \leq \epsilon \tag{2}$$

For $x \in (0, \infty)$, with $n > \frac{m}{x}(b_0 + r)$, we have

$$|f(nx)| = \left|f(\frac{m}{x}(b_0 + r)x)\right| = \left|f(\frac{m}{x}(b_0 + r)x)\right| \leq \epsilon.$$

, which means $\lim_{t\to\infty} f(t) = 0$.

For $x = 0$, from $\lim_{n\to\infty} f(n0) = 0$, we know $f(0) = 0$.

In a whole, For $x \in [0, \infty)$, $\lim_{t\to\infty} f(t) = 0$.

□

Problem 2.

Proof. **1. Linearity of $c_0(X)$**

Let $x = (x_n), y = (y_n) \in c_0(X)$ and $\lambda \in \mathbb{K}$.

- Addition: $x + y = (x_n + y_n)$. By the triangle inequality on X:

$$0 \leq \lim_{n\to\infty} \|x_n + y_n\| \leq \lim_{n\to\infty} (\|x_n\| + \|y_n\|) = \lim_{n\to\infty} \|x_n\| + \lim_{n\to\infty} \|y_n\| = 0 \tag{3}$$

$$\therefore \lim_{n\to\infty} \|x_n + y_n\| = 0 \tag{4}$$

  Thus, $x + y \in c_0(X)$.

- Scalar Multiplication: $\lambda x = (\lambda x_n)$. By the properties of a norm:

$$\lim_{n\to\infty} \|\lambda x_n\| = \lim_{n\to\infty} |\lambda| \|x_n\| = |\lambda| \lim_{n\to\infty} \|x_n\| = |\lambda| \cdot 0 = 0$$

  Thus $\lambda x \in c_0(X)$.

Hence, $c_0(X)$ is a linear space.

2. X is Banach $\implies$ completeness of $c_0(X)$

Assume X is a Banach space. We show that $c_0(X)$ is complete with respect to the norm $\|x\| = \sup_{n\in\mathbb{N}} \|x_n\|_X$.

Let $(x^{(k)})_{k\in\mathbb{N}}$ be a Cauchy sequence in $c_0(X)$, where $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}, \ldots)$.

Since $(x^{(k)})_{k\in\mathbb{N}}$ is Cauchy, for every $\epsilon > 0$, there exists an integer K such that for all $k, j \geq K$:

$$\left\|x^{(k)} - x^{(j)}\right\| = \sup_{n\in\mathbb{N}} \left\|x_n^{(k)} - x_n^{(j)}\right\|_X < \epsilon \tag{5}$$

- **Step 1: Determine the limit element** x. For a fixed $n \in \mathbb{N}$, the sup-norm condition (5) implies that the component sequence $(x_n^{(k)})_{k\in\mathbb{N}}$ is Cauchy in X, since:

$$\left\|x_n^{(k)} - x_n^{(j)}\right\|_X \leq \sup_{m\in\mathbb{N}} \left\|x_m^{(k)} - x_m^{(j)}\right\|_X < \epsilon \quad \text{for all } k, j \geq K \tag{6}$$

  Since X is a Banach space, it is complete. Therefore, for each n, the sequence $(x_n^{(k)})_{k\in\mathbb{N}}$ converges to some limit $x_n \in X$:

$$x_n = \lim_{k\to\infty} x_n^{(k)} \tag{7}$$

  We define the candidate limit element as $x = (x_n)_{n\in\mathbb{N}}$.

- **Step 2: Show** $x \in c_0(X)$. We must show $\lim_{n\to\infty} \|x_n\|_X = 0$. Fix $k \geq K$. Taking the limit as $j \to \infty$ in the inequality $\left\|x_n^{(k)} - x_n^{(j)}\right\|_X \leq \epsilon$, which is a consequence of the Cauchy condition (5), we get:

$$\left\|x_n^{(k)} - x_n\right\|_X \leq \epsilon \quad \text{for all } n \in \mathbb{N} \tag{8}$$

  Now, use the triangle inequality on X:

$$\|x_n\|_X \leq \left\|x_n - x_n^{(k)}\right\|_X + \left\|x_n^{(k)}\right\|_X \leq \epsilon + \left\|x_n^{(k)}\right\|_X \tag{9}$$

  Since $x^{(k)} \in c_0(X)$, we know $\lim_{n\to\infty} \left\|x_n^{(k)}\right\|_X = 0$. Thus, for this fixed k, there exists an integer N such that for all $n \geq N$:

$$\left\|x_n^{(k)}\right\|_X < \epsilon \tag{10}$$

  Combining the two preceding inequalities (for $\|x_n\|_X$ and $\left\|x_n^{(k)}\right\|_X$) for $n \geq N$:

$$\|x_n\|_X < \epsilon + \epsilon = 2\epsilon \tag{11}$$

  Since $\epsilon > 0$ was arbitrary, this proves $\lim_{n\to\infty} \|x_n\|_X = 0$, so $x \in c_0(X)$.

- **Step 3: Show** $x^{(k)} \to x$ **in** $c_0(X)$. The inequality $\left\|x_n^{(k)} - x_n\right\|_X \leq \epsilon$ (derived in Step 2 for $k \geq K$) holds for all $n \in \mathbb{N}$. Taking the supremum over n:

$$\left\|x^{(k)} - x\right\| = \sup_{n\in\mathbb{N}} \left\|x_n^{(k)} - x_n\right\|_X \leq \epsilon \quad \text{for all } k \geq K \tag{12}$$

  Therefore, $x^{(k)}$ converges to x in $c_0(X)$.

Since every Cauchy sequence in $c_0(X)$ converges to an element in $c_0(X)$, the space $c_0(X)$ is a Banach space.

□

Problem 3.

Proof. **(a) $\implies$ (b)**

We define a new space

$$Z = \{(x_n)_{n\in\mathbb{N}} : \|x_n\| \to 0\}$$

and norm $\|(x_n)_{n\in\mathbb{N}}\| = \sup_n \|x_n\|$.

According to the proof of Problem 2, Z is Banach.

We define an operator $S_k : Z \to Y$

$$S_k(z) = T_k(x_k - x_{k+1}).$$

Therefore,

$$\begin{aligned}
\|S_k(z)\| &= \|T_k(x_k - x_{k+1})\| && (13)\\
&\le \|T_k\| \, \|x_k - x_{k+1}\| && (14)\\
&\le \|T_k\| \, (\|x_k\| + \|x_{k+1}\|) && (15)\\
&\le \|T_k\| \cdot 2 \sup_{n\in\mathbb{N}} \|x_n\| && (16)
\end{aligned}$$

Therefore,

$$\|S_k\| \le 2\,\|T_k\| < \infty$$

as T_k is continuous.

Since $T_k(x_k) \to 0$ in norm, S_k should be the same.

$$\lim_{k\to\infty} \|S_k(z)\| = 0 \tag{17}$$

$$\therefore \sup_{k\in\mathbb{N}} \|S_k(z)\| < \infty \tag{18}$$

With the Uniform Boundedness Principle, we have

$$\sup_{k\in\mathbb{N}} \|S_k\| < \infty \tag{19}$$

For any $t \in X$ with $\|t\| \le 1$, define the sequence $z' = (x_n)_{n\in\mathbb{N}}$, where $x_n = t$ if $n = k$, and $x_n = 0$ else. Then, $\sup_n x_n = \|t\| \le 1$. Applying the operator S_k,

$$\|S_k(z')\| = \|T_k(x_k - x_{k+1})\| = \|T_k(x_k)\| \tag{20}$$

$$\therefore \|S_k\| \ge \sup_{\|t\|\le 1} \|T_k(t)\| = \|T_k\| \tag{21}$$

Since $\sup_{k\in\mathbb{N}} \|S_k\| < \infty$,

$$\|T_k\| < \infty$$

Proof: (b) $\implies$ (a) Assume (b) holds, i.e., $M = \sup_{n\in\mathbb{N}} \|T_n\| < \infty$. Assume $\sum_{n=1}^{\infty} x_n$ is a norm convergent series. Let s be the sum. The sequence of partial sums $s_N = \sum_{n=1}^{N} x_n$ converges to s. This implies that the terms of the series must converge to zero: $\lim_{n\to\infty} x_n = 0$ in the norm of X.

$$\|T_n(x_n)\|_Y \le \|T_n\| \cdot \|x_n\|_X$$

Since $\sup_{n\in\mathbb{N}} \|T_n\| = M < \infty$, we have:

$$0 \le \|T_n(x_n)\|_Y \le M \cdot \|x_n\|_X$$

Since $\lim_{n\to\infty} \|x_n\|_X = 0$ and M is a finite constant, we have $\lim_{n\to\infty} M \cdot \|x_n\|_X = 0$. By the Squeeze Theorem, $\lim_{n\to\infty} \|T_n(x_n)\|_Y = 0$. Thus, $T_n(x_n) \to 0$ in norm.

□

Problem 4.

Proof. **(a) $\implies$ (b):** The set of p-absolutely norm convergent sequences $E = \ell_p(X)$ is a Banach space with the norm $\|(x_n)\|_p = (\sum_{n=1}^{\infty} \|x_n\|^p)^{1/p}$.

(a) states that for every sequence $(x_n) \in E$, the sum $\sum_{n=1}^{\infty} x_n^*(x_n)$ converges. This allows us to define a linear functional $T : E \to \mathbb{K}$ by:

$$T((x_n)) = \sum_{n=1}^{\infty} x_n^*(x_n)$$

Since T is a well-defined linear map on the entire Banach space E, the Uniform Boundedness Principle implies that T is bounded and in turn continuous. We have $T \in E^*$.

With Dual Space Isomorphism, the dual space of $\ell_p(X)$ is isometrically isomorphic to $\ell_q(X^*)$:

$$\ell_p(X)^* \cong \ell_q(X^*)$$

(x_n^*) corresponds to the functional T, and the norm of T in the dual space is precisely the ℓ_q norm of the sequence (x_n^*) in the dual sequence space:

$$\|T\|_{E^*} = \left(\sum_{n=1}^{\infty} \|x_n^*\|^q \right)^{1/q}$$

Since T is continuous, $\|T\|_{E^*} < \infty$, which means $\sum_{n=1}^{\infty} \|x_n^*\|^q < \infty$. Thus, (b) holds.

(b) $\implies$ (a): Assume (b) holds: $(x_n^*) \in \ell_q(X^*)$, so $A = (\sum_{n=1}^{\infty} \|x_n^*\|^q)^{1/q} < \infty$.

Assume the series $\sum_{n=1}^{\infty} x_n$ is p-absolutely norm convergent: $(x_n) \in \ell_p(X)$, so

$$B = \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} < \infty$$

With the Holder's Inequality, we have:

$$\sum_{n=1}^{\infty} |x_n^*(x_n)| \leq \sum_{n=1}^{\infty} \|x_n^*\| \cdot \|x_n\| \tag{22}$$

$$\leq \left(\sum_{n=1}^{\infty} \|x_n^*\|^q \right)^{1/q} \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} = A \cdot B < \infty \tag{23}$$

(c) $\iff$ (d)

Let $X = \mathbb{K}$. Then the dual space X^* is also $\mathbb{K}$.

A continuous linear functional $x_n^* : \mathbb{K} \to \mathbb{K}$ is simply multiplication by a scalar $x_n \in \mathbb{K}$. The norm of this functional is $||x_n^*|| = |x_n|$. The input vector x_n from statement (a) is now a scalar y_n.

Thus, statement (a) becomes: Given a series $\sum_{n=1}^{\infty} y_n$ such that $\sum_{n=1}^{\infty} |y_n|^p < \infty$ (i.e., $(y_n) \in \ell_p$) one has that the series $\sum_{n=1}^{\infty} x_n y_n$ converges. This is precisely statement (d). So we have (d) $\implies$ (a).

And statement (b) becomes: The series $\sum_{n=1}^{\infty} x_n^*$ is q-absolutely norm convergent i.e., $\sum_{n=1}^{\infty} \|x_n^*\|^q < \infty$. Since $\|x_n^*\| = |x_n|$, this simplifies to $\sum_{n=1}^{\infty} |x_n|^q < \infty$, which means $(x_n) \in \ell_q$. This is precisely statement (c). So we have (d) $\implies$ (a).

Since (a) is equivalent to (b), (d) is equivalent to (c). □