Functional Analysis - Homework 6

Problem 1. *Let X and Y be Banach spaces, and let $T : D \subset X \to Y$ be a linear operator with closed graph. Show that the following two statements are equivalent:*

(a) T is injective and $T(D)$ is closed in Y.

(b) $\exists C > 0$ such that $\|x\|_X \leq C \|Tx\|_Y \qquad \forall x \in D$.

Proof. **(b) $\implies$ (a)**

Let $x_1, x_2 \in X$ and set $Tx_1 = Tx_2$. We have

$$\|x_1 - x_2\|_X \leq C \|T(x_1 - x_2)\|_Y = C \|Tx_1 - Tx_1\|_Y = 0 \tag{1}$$
$$\therefore x_1 - x_2 = 0 \tag{2}$$
$$\therefore x_1 = x_2 \tag{3}$$

Thus T is injective.

Next, we prove that $T(D)$ is closed. Let $y_n = Tx_n \in T(D)$ be a sequence such that $y_n \to y$ in Y. Then, $\forall \epsilon > 0, \exists m, n > N$, such that $\|y_n - y_m\|_Y < \frac{\epsilon}{C}$, then we have

$$\|x_n - x_m\|_X \leq C \|Tx_n - Tx_m\|_Y = C \|y_n - y_m\|_Y < \epsilon \tag{4}$$

so (x_n) is Cauchy in X. Since X is Banach, there exists $x \in X$ with $x_n \to x$. Because the graph of T is closed and $Tx_n \to y$, we have (x, y) in the graph of T, so $x \in D$ and $Tx = y$. Thus $y \in T(D)$, so $T(D)$ is closed.

(a) $\Rightarrow$ (b)

With the assumption that T is injective we can define the inverse operator

$$S := T^{-1} : T(D) \to D \subset X. \tag{5}$$

Since $T(D)$ is a closed subspace of a Banach space Y, it is also a Banach space.

Let $y_n = Tx_n \to y$ in $T(D)$ and $Sy_n = x_n \to x$ in X. Since T has closed graph, (x, y) is in the graph of T, so $y = Tx$. Therefore (y, x) is in the graph of S, hence the graph of S is closed.

Let graph

$$Gr(S) = \{(y, x) \in Y \times X : x = Sy\} \tag{6}$$

with a norm

$$\|(y, x)\| = \|y\|_Y + \|x\|_X . \tag{7}$$

Since X and Y are Banach, $Gr(S)$ is Banach as well.

Define projections, $\forall x \in X, y \in Y$

$$\pi_1(y, x) = y \tag{8}$$
$$\pi_2(y, x) = x \tag{9}$$

.

π_1 is bounded since $\|y\|_Y \leq \|(y, x)\| = \|y\|_Y + \|x\|_X$. It is bijective as well. With the open mapping theorem, for any open set $O \subset Y \times X$, $\pi_1(O) \subset Y$, which implies π_1^{-1} is continuous.

Similar to π_1, π_2 is also bounded and continuous. Thus, $S = \pi_2 \circ \pi_1^{-1}$ is continuous and bounded.

That is, there exists $C > 0$ such that

$$\|Sy\|_X \leq C \|y\|_Y \qquad \forall y \in T(D) \tag{10}$$
$$\implies \|x\|_X \leq C \|Tx\|_Y \qquad \forall x \in D \tag{11}$$

which is (b).

□

Problem 2. *Let $(X, \|\cdot\|)$ be an infinite-dimensional normed space. Let $Y \subset X$ be bounded. Assume that the boundary ∂Y is compact. Prove that $int(Y) = \emptyset$.*

Proof. Assume for contradiction that $int(Y) \neq \emptyset$. Then there exists a point $x_0 \in X$ and $r > 0$ such that the open ball $B(x_0, r) \subset Y$.

For each $u \in S := \{v \in X : \|v\| = 1\}$ define

$$t(u) := \sup\{t \geq 0 : \ x_0 + tu \in Y\}. \tag{12}$$

Since $B(x_0, r) \subset Y$ we have $t(u) \geq r > 0$ for all u, and because Y is bounded each $t(u)$ is finite. Set

$$f(u) := x_0 + t(u)u, \tag{13}$$
$$V := \{f(u) : u \in S\} \subset \partial Y. \tag{14}$$

Injectivity. If $f(u) = f(v)$ then $t(u)u = t(v)v$. As $t(u), t(v) > 0$ this forces $u = v$, so f is injective. Thus $f : S \to V$ is a bijection.

Continuity of the inverse. For $y \in V$ we have $y - x_0 \neq 0$ and

$$f^{-1}(y) = \frac{y - x_0}{\|y - x_0\|} \in S, \tag{15}$$

which is a continuous map on V. Hence $f^{-1} : V \to S$ is continuous.

Continuity of f. Let $u_n \to u$ in S and $y_n := f(u_n) \in \partial Y$. Since ∂Y is compact, $\{y_n\}$ has a convergent subsequence $y_{n_k} \to y \in \partial Y$. Thus,

$$\frac{y_{n_k} - x_0}{\|y_{n_k} - x_0\|} = u_{n_k} \to u, \tag{16}$$

so the limit satisfies $\dfrac{y - x_0}{\|y - x_0\|} = u$.

By definition of $t(u)$ we have $y = x_0 + t(u)u = f(u)$. Thus every convergent subsequence of $\{y_n\}$ converges to $f(u)$, so the whole sequence $y_n \to f(u)$.

Therefore, f is sequentially continuous, and in turn it is continuous.

Combining bijectivity, continuity of f, and continuity of f^{-1}, we have that $f : S \to V$ is a homeomorphism.

Since S is not compact as a unit sphere in an infinite-dimensional normed space, V is not compact.

But $V \subset \partial Y$ and ∂Y was assumed compact, a contradiction. Hence $\operatorname{int} Y = \emptyset$.

$\square$

Problem 3. *Let $(X, \|\cdot\|_X)$ be a finite-dimensional normed space with $\dim(X) = d$. Let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Prove that weak convergence $x_n \xrightarrow{w} x$ for $n \to \infty$ implies that $\|x_n - x\|_X \to 0$ for $n \to \infty$.*

Proof. Since $\dim(X) = d < \infty$, X has a basis $\mathcal{B} = \{e_1, e_2, \ldots, e_d\}$.

Any vector $x \in X$ can be uniquely represented by its coordinates $x = \sum_{i=1}^{d} \alpha_i e_i$, where $\alpha_i \in \mathbb{K}$. Similarly, for the sequence, $x_n = \sum_{i=1}^{d} \alpha_i^{(n)} e_i$.

For each basis vector e_i, let the i-th element of the dual basis $\phi_i \in X^*$ defined by $\phi_i(e_j) = \delta_{ij}$. Thus,

$$\phi_i(x) = \phi_i\left(\sum_{j=1}^{d} \alpha_j e_j\right) = \alpha_i \tag{17}$$

In a finite-dimensional space, every linear functional is continuous, so $\phi_i \in X^*$.

$x_n \xrightarrow{w} x$ means

$$\lim_{n\to\infty} f(x_n) = f(x) \qquad \forall f \in X^* \tag{18}$$
$$\therefore \lim_{n\to\infty} \phi_i(x_n) = \phi_i(x) \tag{19}$$
$$\therefore \lim_{n\to\infty} \alpha_i^{(n)} = \alpha_i \tag{20}$$

Since X is finite-dimensional, all norms on X are equivalent. Specifically, $\|\cdot\|_X$ is equivalent to the L^1 norm:

$$\|x\|_1 = \left\|\sum_{i=1}^{d} \alpha_i e_i\right\|_1 := \sum_{i=1}^{d} |\alpha_i| \tag{21}$$

The equivalence of norms implies that there exists a constant $C > 0$ such that for any vector $v \in X$, $\|v\|_X \leq C \|v\|_1$.

Apply this equivalence, we can have

$$\|x_n - x\|_X = \left\|\sum_{i=1}^{d} (\alpha_i^{(n)} - \alpha_i) e_i\right\| \leq C \sum_{i=1}^{d} |\alpha_i^{(n)} - \alpha_i| \tag{22}$$

$$\therefore \lim_{n\to\infty} \|x_n - x\|_X \leq \lim_{n\to\infty} \left(C \sum_{i=1}^{d} |\alpha_i^{(n)} - \alpha_i|\right) \tag{23}$$

$$\therefore \lim_{n\to\infty} \|x_n - x\|_X \leq C \sum_{i=1}^{d} \left(\lim_{n\to\infty} |\alpha_i^{(n)} - \alpha_i|\right) \tag{24}$$

From (19), we know $\lim_{n\to\infty} |\alpha_i^{(n)} - \alpha_i| = 0$. Therefore,

$$\lim_{n\to\infty} \|x_n - x\|_X \leq C \sum_{i=1}^{d} 0 = 0 \tag{25}$$

$$\therefore \lim_{n\to\infty} \|x_n - x\|_X = 0 \tag{26}$$

, which is $\|x_n - x\|_X \to 0$ for $n \to \infty$. □

Problem 4. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces, and let $T : X \to Y$ be a linear operator. Prove the equivalence of the following statements:*

(a) T is continuous.

(b) Given a sequence $(x_n)_{n\in\mathbb{N}}$ in X, if $x_n \xrightarrow{w} x$ in X then $Tx_n \xrightarrow{w} Tx$ in Y.

Proof. **(a) $\implies$ (b)**

Assume T is continuous (i.e., bounded). Let (x_n) be a sequence such that $x_n \xrightarrow{w} x$ in X.

Let $g \in Y^*$, meaning it is a continuous linear functional on Y.

Let $f = g \circ T : X \to \mathbb{K}$, defined by $f(z) = g(Tz)$. f is linear and continuous, as T and g are linear and continuous. Thus, $f \in X^*$.

Since $x_n \xrightarrow{w} x$ in X, by the definition of weak convergence, we have

$$\lim_{n\to\infty} f(x_n) = f(x) \tag{27}$$

$$\therefore \lim_{n\to\infty} g(Tx_n) = g(Tx) \tag{28}$$

Since this holds for every $g \in Y^*$, we conclude that $Tx_n \xrightarrow{w} Tx$ in Y.

(b) $\implies$ (a)

Let $g \in Y^*$ with $\|g\|_{Y^*} = 1$, and $F = \{f_g = g \circ T : g \in Y^*, \|g\|_{Y^*} = 1\}$.

For fixed $x \in X, \forall f_g \in F$, $|f_g(x)|$ is bounded because

$$|f_g(x)| = |g(Tx)| \leq \|g\|_{Y^*} \|Tx\|_Y = \|Tx\|_Y < \infty. \tag{29}$$

By the statement (b), $\forall g \in Y^*, f_g = g \circ T$ is weakly sequentially continuous. Since f_g is linear, it is continuous and $f_g \in X^*$.

By the Uniform Boundedness Principle, F is uniformly bounded.

$$\sup_{f_g \in F} \|f_g\|_{X^*} < \infty \tag{30}$$

$$\therefore \sup_{\|g\|_{Y^*}=1} \|g \circ T\|_{X^*} < \infty. \tag{31}$$

Therefore,

$$\begin{aligned}
\|T\| &= \sup_{\|x\|_X=1} \|Tx\|_Y && (32)\\
&= \sup_{\|x\|_X=1} \left(\sup_{\|g\|_{Y^*}=1} |g(Tx)| \right) && (33)\\
&= \sup_{\|g\|_{Y^*}=1} \left(\sup_{\|x\|_X=1} |(g \circ T)(x)| \right) && (34)\\
&= \sup_{\|g\|_{Y^*}=1} \|g \circ T\|_{X^*} < \infty && (35)
\end{aligned}$$

, which means T is continuous.

$\square$ $\square$
