# Functional Analysis
# Homework 6

Deadline: October 21st

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1. **Problem 1** (6 points)

   Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. Let $T : D \subset X \mapsto Y$ be a linear operator with closed graph, show the equivalence of the following statements:

   a) $T$ is injective and its range $T(D)$ is closed in $(Y, \|\cdot\|_Y)$;

   b) There exists a $C > 0$ such that for every $x \in D : \|x\|_X \leq C \|Ax\|_Y$.

2. **Problem 2** (4 points)

   Let $(X, \|\cdot\|)$ be an infinite-dimensional normed vector space. Let $Y \subset X$ be a bounded subset with compact boundary. Prove that $Y$ has empty interior.

   *Hint:* Use the fact that unit sphere in an infinite-dimensional normed space is not compact.

3. **Problem 3** (4 points)

   Let $(X, \|\cdot\|_X)$ be a finite dimensional normed space with $\dim X = d$. Let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $X$. Prove that weak convergence $x_n \xrightarrow{w} x$ for $n \to \infty$ implies that $\|x_n - x\|_X \to 0$ for $n \to \infty$.

4. **Problem 4** (6 points)

   Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be Banach spaces. Let $T : X \mapsto Y$ be a linear operator. Prove the equivalence of the following statements:

   a) $T$ is continuous;

   b) Given a sequence $(x_n)_{n\in\mathbb{N}}$ in $X$, if $x_n \xrightarrow{w} x$ in $X$ then $Tx_n \xrightarrow{w} Tx$ in $Y$.