Functional Analysis
Homework 7

Deadline: October 28th

1. **Problem 1** (8 points)

   Let $(X, \|\cdot\|_X)$ be a Banach space. Denote with $X^\star$ its dual with the typical dual norm rendering it a Banach space. Given a sequence $(f_n)_{n\in\mathbb{N}}$, denote with $f_n \xrightarrow{\star} f$ convergence of f_n against f with respect to the weak topology $\sigma(X^\star, X)$. Prove the following statements:

   a) $f_n \xrightarrow{\star} f$ if and only if $f_n(x) \to f(x)$ for every $x \in X$;

   b) if $f_n \xrightarrow{w} f$ then $f_n \xrightarrow{\star} f$;

   c) If $f_n \xrightarrow{\star} f$ then f_n is bounded and $\|f\|_{X^\star} \le \liminf \|f_n\|$;

   d) if $f_n \xrightarrow{\star} f$ and $x_n \to x$ (strongly) then $\langle f_n, x_n\rangle \to \langle f, x\rangle$.

2. **Problem 2** (6 points)

   Let $(X, \|\cdot\|_X)$ be a Banach space and $X^\star$ its dual. Given a sequence $(x_n)_n$ with $x_n \in X$ and a sequence $(f_n)_n$ with $f_n \in X^\star$. Assume that $f_n \xrightarrow{\star} f$ and $x_n \xrightarrow{w} x$. Prove or find a counter-example for the following statement:

   $$\langle f_n, x_n\rangle \to \langle f, x\rangle.$$

3. **Problem 3** (6 points)

   Let $(E, \|\cdot\|_E)$ and $(E, \|\cdot\|_F)$ be Banach spaces. Let $T : E \to F$ be a linear surjective isometry. Prove that E is reflexive if and only if F is reflexive.