# Functional Analysis
# Homework 4

Deadline: October 7th

1. **Problem 1** (4 points)

   Let $f \in C_0([0,\infty))$ be a continuous function such that
   $$\forall x \in [0,\infty) : \lim_{n\to\infty} f(nx) = 0.$$
   Prove that $\lim_{t\to\infty} f(t) = 0$.

   *Hint:* Use Baire's category Theorem similarly to the proof of the Uniform Boundedness Principle.

2. **Problem 2** (4 points)

   Let $X$ be a normed space. Denote with $c_0(X) = \{(x_n)_{n\in\mathbb{N}} : \|x_n\| \to 0\}$. Show that $c_0$ is linear. Moreover, given the norm $\|x\| = \sup_{n\in\mathbb{N}} \|x_n\|$ for $x = (x_n)_{n\in\mathbb{N}} \in c_0(X)$, prove that if $X$ is Banach then $c_0(X)$ is Banach.

3. **Problem 3** (6 points)

   Let $X$ be Banach and $Y$ be a normed space. Let $(T_n : X \to Y)_{n\in\mathbb{N}}$ be a sequence of continuous and linear operators. Prove the equivalence between the following statements:

   a) For each norm convergent series $\sum_{n=1}^{\infty} x_n$ one has that $T_n(x_n) \to 0$ in norm;

   b) $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$.

   *Hint:* Look at differences of elements $x_i$ and define a new operator with domain $c_0(X)$.

   *Hint 2:* Use the Uniform Boundedness Principle.

4. **Problem 4** (6 points)

   Let $1 < p < \infty$ and let $q$ be the Hölder conjugate of $p$ *i.e.*, s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Let $X$ be a Banach space and let $(x_n^\star)_{n\in\mathbb{N}}$ a sequence of linear and continuous functionals. Prove the equivalence of the following two statements

   a) Given a series $\sum_{n=1}^{\infty} x_n$ such that $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$ (this is known as $p$-absolutely norm convergence) one has that the series $\sum_{n=1}^{\infty} x^\star(x_n)$ converges;

   b) The series $\sum_{n=1}^{\infty} x_n^\star$ is $q$-absolutely norm convergent *i.e.*, $\sum_{n=1}^{\infty} \|x_n^\star\|^q < \infty$.

Using the equivalence you just proved, prove that if $(x_n)_{n\in\mathbb{N}}$ is a sequence of scalars then the following assertions are equivalent

c) $(x_n)_{n\in\mathbb{N}} \in \ell_q$;

c) for each $(y_n)_{n\in\mathbb{N}} \in \ell_p$ the series $\sum_{n=1}^{\infty} x_n y_n$ converges.