

Functional Analysis - Homework 9

Problem 1. Let H be a Hilbert space and let $K \subset H$ be a non-empty closed convex set.

- a) Prove that for every $f \in H$ there exists an element $u \in K$ such that $\|f - u\| = \min_{v \in K} \|f - v\| = \text{dist}(f, K)$.
- b) Prove that (a) is equivalent to the property: $\text{Re} \langle f - u, v - u \rangle \leq 0$ for all $v \in K$.
- c) Use now (b) to prove the uniqueness of such u .
- d) Prove that P_K is contractive, i.e., $\|P_K f_1 - P_K f_2\| \leq \|f_1 - f_2\|$.

a)

Let $(v_n)_{n \in \mathbb{N}} \subset K$ be a minimizing sequence such that $\lim_{n \rightarrow \infty} \|f - v_n\| = \text{dist}(f, K)$.

With the parallelogram equality, for $m, k \in \mathbb{N}$ we have

$$\|v_m - v_k\|^2 = \|(f - v_m) - (f - v_k)\|^2 \quad (1)$$

$$= 2\|f - v_m\|^2 + 2\|f - v_k\|^2 - \|(f - v_m) + (f - v_k)\|^2 \quad (2)$$

$$= 2\|f - v_m\|^2 + 2\|f - v_k\|^2 - 4\left\|f - \frac{v_m + v_k}{2}\right\|^2 \quad (3)$$

Since K is convex, $\frac{v_m + v_k}{2} \in K$, and thus $\left\|f - \frac{v_m + v_k}{2}\right\| \geq \text{dist}(f, K)$.

$$\|v_m - v_k\|^2 \leq 2\|f - v_m\|^2 + 2\|f - v_k\|^2 - 4\text{dist}(f, K)^2 \quad (4)$$

$$= 2\left(\|f - v_m\|^2 - \text{dist}(f, K)^2\right) + 2\left(\|f - v_k\|^2 - \text{dist}(f, K)^2\right) \quad (5)$$

Since $\|f - v_n\| \rightarrow \text{dist}(f, K)$, the right-hand side tends to 0 as $m, k \rightarrow \infty$. This implies (v_n) is Cauchy. Because H is complete and K is closed, (v_n) converges to some $u \in K$ which satisfies $\|f - u\| = \text{dist}(f, K)$.

b)

Let **(1)** be $\|f - u\| = \min_{v \in K} \|f - v\|$ and **(2)** be $\text{Re} \langle f - u, v - u \rangle \leq 0$ for all $v \in K$.

(1) \implies (2) Assume u satisfies (1). For arbitrary $v \in K$, let $v_t = (1 - t)u + tv \in K$ for $t \in [0, 1]$. By the minimization property, $\|f - u\|^2 \leq \|f - v_t\|^2$.

$$\|f - u\|^2 \leq \|(f - u) - t(v - u)\|^2 \quad (6)$$

$$= \|f - u\|^2 - 2t \text{Re} \langle f - u, v - u \rangle + t^2 \|v - u\|^2 \quad (7)$$

$$\therefore 2t \text{Re} \langle f - u, v - u \rangle \leq t^2 \|v - u\|^2 \quad (8)$$

Dividing by $2t$ for $t > 0$ and taking the limit as $t \rightarrow 0^+$ yields $\text{Re} \langle f - u, v - u \rangle \leq 0$.

(2) \implies (1) Assume u satisfies (2). For any $v \in K$, we expand $\|f - v\|^2$:

$$\|f - v\|^2 = \|(f - u) - (v - u)\|^2 \quad (9)$$

$$= \|f - u\|^2 - 2 \text{Re} \langle f - u, v - u \rangle + \|v - u\|^2 \quad (10)$$

Since $\text{Re} \langle f - u, v - u \rangle \leq 0$ by (2), the term $-2 \text{Re} \langle f - u, v - u \rangle \geq 0$.

$$\|f - v\|^2 \geq \|f - u\|^2 + \|v - u\|^2 \quad (11)$$

Since $\|v - u\|^2 \geq 0$, we conclude that $\|f - v\| \geq \|f - u\|$, proving (1).

c)

Assume u_1 and u_2 are both minimizers in K . By part (b), they must satisfy the inequality (2).

- u_1 is the projection of f . Setting $v = u_2$: $\operatorname{Re} \langle f - u_1, u_2 - u_1 \rangle \leq 0$.
- u_2 is the projection of f . Setting $v = u_1$: $\operatorname{Re} \langle f - u_2, u_1 - u_2 \rangle \leq 0$.

Using $u_1 - u_2 = -(u_2 - u_1)$, the second inequality is equivalent to $\operatorname{Re} \langle f - u_2, u_2 - u_1 \rangle \geq 0$. Summing the two relations:

$$0 \geq \operatorname{Re} \langle f - u_1, u_2 - u_1 \rangle - \operatorname{Re} \langle f - u_2, u_2 - u_1 \rangle \quad (12)$$

$$0 \geq \operatorname{Re} \langle (f - u_1) - (f - u_2), u_2 - u_1 \rangle \quad (13)$$

$$0 \geq \operatorname{Re} \langle u_2 - u_1, u_2 - u_1 \rangle \quad (14)$$

$$\|u_2 - u_1\|^2 \leq 0 \quad (15)$$

Since the norm squared must be non-negative, $\|u_2 - u_1\|^2 = 0$, which implies $u_1 = u_2$.

d)

Let $u_1 = P_K f_1$ and $u_2 = P_K f_2$. With property (b):

- For f_1 and u_1 : set $v = u_2$. $\operatorname{Re} \langle f_1 - u_1, u_2 - u_1 \rangle \leq 0$.
- For f_2 and u_2 : set $v = u_1$. $\operatorname{Re} \langle f_2 - u_2, u_1 - u_2 \rangle \leq 0$, which implies $-\operatorname{Re} \langle f_2 - u_2, u_2 - u_1 \rangle \leq 0$.

Summing these two inequalities:

$$0 \geq \operatorname{Re} \langle f_1 - u_1, u_2 - u_1 \rangle - \operatorname{Re} \langle f_2 - u_2, u_2 - u_1 \rangle \quad (16)$$

$$0 \geq \operatorname{Re} \langle (f_1 - u_1) - (f_2 - u_2), u_2 - u_1 \rangle \quad (17)$$

$$0 \geq \operatorname{Re} \langle (f_1 - f_2) - (u_1 - u_2), u_2 - u_1 \rangle \quad (18)$$

$$0 \geq \operatorname{Re} \langle f_1 - f_2, u_2 - u_1 \rangle - \operatorname{Re} \langle u_1 - u_2, u_2 - u_1 \rangle \quad (19)$$

$$\|u_2 - u_1\|^2 \leq \operatorname{Re} \langle f_1 - f_2, u_2 - u_1 \rangle \leq |\langle f_1 - f_2, u_2 - u_1 \rangle| \quad (20)$$

Applying the Cauchy-Schwarz inequality, $|\langle x, y \rangle| \leq \|x\| \|y\|$:

$$\|u_1 - u_2\|^2 \leq \|f_1 - f_2\| \|u_2 - u_1\| \quad (21)$$

Dividing by $\|u_1 - u_2\|$ (assuming it is non-zero) yields the desired result: $\|u_1 - u_2\| \leq \|f_1 - f_2\|$.

Problem 2.

With $\overline{x - i} = x + i$ and $\overline{x + i} = x - i$,

$$\langle f_m, f_n \rangle = \int_{-\infty}^{\infty} f_m(x) \overline{f_n(x)} dx \quad (22)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x - i)^m}{(x + i)^{m+1}} \frac{(x + i)^n}{(x - i)^{n+1}} dx \quad (23)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x - i)^{m-n-1}}{(x + i)^{m-n+1}} dx \quad (24)$$

Let $g(z) = \frac{1}{\pi} \frac{(z - i)^{m-n-1}}{(z + i)^{m-n+1}}$. We use the Residue Theorem on a semi-circular contour in the upper half-plane. The integral is equal to $2\pi i \sum \operatorname{Res}(g, z_k)$, where z_k are poles in the upper half-plane, and the poles are at $z = \pm i$.

Case 1: $m = n$ The integrand simplifies to $g(z) = \frac{1}{\pi} \frac{(z - i)^{-1}}{(z + i)^1} = \frac{1}{\pi(z - i)(z + i)}$. The only pole in the upper half-plane is a simple pole at $z = i$.

$$\operatorname{Res}(g, i) = \lim_{z \rightarrow i} (z - i)g(z) = \lim_{z \rightarrow i} \frac{1}{\pi(z + i)} = \frac{1}{\pi(2i)} \quad (25)$$

$$\langle f_n, f_n \rangle = 2\pi i \cdot \operatorname{Res}(g, i) = 2\pi i \cdot \frac{1}{2\pi i} = 1 \quad (26)$$

Case 2: $m \neq n$ **1)** $m > n$ Let $k = m - n \geq 1$. The integrand is $g(z) = \frac{1}{\pi} \frac{(z-i)^{k-1}}{(z+i)^{k+1}}$. The only singularity is a pole at $z = -i$ of order $k+1$. Since this pole is in the lower half-plane, the upper half-plane contour encloses no singularities.

$$\langle f_m, f_n \rangle = 2\pi i \cdot (0) = 0 \quad (27)$$

2) $m < n$ Let $k = n - m \geq 1$. The integrand is $g(z) = \frac{1}{\pi} \frac{(z+i)^{k-1}}{(z-i)^{k+1}}$. The only singularity is a pole at $z = i$ of order $p = k+1$, which is in the upper half-plane. The residue is:

$$\text{Res}(g, i) = \frac{1}{k!} \lim_{z \rightarrow i} \frac{d^k}{dz^k} [(z-i)^{k+1} g(z)] \quad (28)$$

$$= \frac{1}{k!} \lim_{z \rightarrow i} \frac{d^k}{dz^k} \left[\frac{1}{\pi} (z+i)^{k-1} \right] \quad (29)$$

Since $k \geq 1$, the k -th derivative of the polynomial $(z+i)^{k-1}$ (which has degree $k-1$) is zero.

$$\frac{d^k}{dz^k} (z+i)^{k-1} = 0 \quad (30)$$

$$\therefore \text{Res}(g, i) = 0 \quad (31)$$

$$\langle f_m, f_n \rangle = 2\pi i \cdot \text{Res}(g, i) = 0 \quad (32)$$

Conclusion The functions satisfy $\langle f_m, f_n \rangle = 1$ if $m = n$ and 0 if $m \neq n$, thus they form an orthonormal set.