Funnctional Analysis
Homework 1

Deadline: September 16th

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1. **Problem 1** (5 points)

   a) Let X and Y be topological spaces; let $f : X \to Y$, prove that the following statements are equivalent:

      (a) f is continuous;

      (b) Given $A \subset X$, one has that $f(\bar{A}) \subset \overline{f(A)}$;

      (c) Given a closed set $B \subset Y$, one has that $f^{-1}(B)$ is closed in X;

      (d) For every $x \in X$ and every neighborhood V of $f(x)$, there is a neighboorhood U of x such that $f(U) \subset V$.

   b) Show that, if $X = Y = \mathbb{R}$ then the usual (ϵ, δ) definition of continuity over the real line implies the topological definition of continuity via open sets with respect to the standard topology;

2. **Problem 2** (3 points)

   Let X be a space, we say that two metrics d and d' on X are equivalent if

   $$\exists C > 0 \; \forall x_1, x_2 \in X : \; C^{-1} d'(x_1, x_2) \le d(x_1, x_2) \le C d'(x_1, x_2).$$

   Provide two metrics on $\mathbb{R}^2$ that are *not* equivalent.

3. **Problem 3** (4 points)

   Let V be a vector space over $\mathbb{R}$. Prove the following equivalences.

   a) Consider the vector space of continuous functions $f : [0, \infty) \to \mathbb{R}$. Prove that

   $$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - g\|_{C^0([0,n])}}{1 + \|f - g\|_{C^0([0,n])}},$$

   where $\|f\|_{C^0([0,n])} = \sup_{x \in [0,n]} |f(x)|$ defines a metric.

   b) A metric $d(\cdot, \cdot)$ is induced by a norm $\|\cdot\|$ (*i.e.*, there exists a norm $\|\cdot\|$ such that for every $x, y \in V : d(x, y) = \|x - y\|$) if and only if the metric is translation invariant and homogeneous *i.e.*, for every $x, y, z \in V$ and every $\lambda \in \mathbb{R}$:

   $$d(x + z, y + z) = d(x, y)$$
   $$d(\lambda x, \lambda y) = |\lambda| d(x, y).$$

c) Is the metric d defined in point a) induced by a norm?

4. **Problem 4** (2 points)

   Let $(x_n)_{n\in\mathbb{N}}$ and $(\tilde{x}_n)_{n\in\mathbb{N}}$ be two Cauchy sequences in a metric space (X, d). Prove that the sequence $(d(x_n, \tilde{x}_n))_{n\in\mathbb{N}}$ converges. Deduce that the map $d : X \times X \to \mathbb{R}$ is continuous with respect to the product topology.

5. **Problem 5** (6 points)

   Let (X, d_X) be a compact metric space and (Y, d_Y) be a complete metric space. Consider the space of continuous functions from X to Y, denoted by $C(X, Y)$. Define the following:

$$d(f, g) = \sup_{x\in X} d_Y(f(x), g(x)), \quad f, g \in C(X, Y).$$

   a) Show that d is a metric on $C(X, Y)$;

   b) Show that $(C(X, Y), d)$ is a complete metric space.