Functional Analysis
Homework 10

Deadline: November 26th

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1. **Problem 1: Compact Operators** (10 points)

   Consider the following normed spaces with annexed norms X, Y, Z. Denote by

   $$\mathcal{K}(X,Y) = \{A \in \mathcal{L}(X,Y) \mid \overline{T(B_1(0))} \subset Y \text{ compact } \}$$

   the set of compact operators from X to Y. Prove the following statements:

   a) $A \in \mathcal{L}(X,Y)$ is compact iff every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Ax_{n_k})_{k\in\mathbb{N}}$ converges in Y;

   b) If Y is complete then $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$;

   c) Let $A \in \mathcal{L}(X,Y)$, if $A(X)$ is finite dimensional then $A \in \mathcal{K}(X,Y)$;

   d) Let $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,Z)$. If A or B is compact then $B \circ A$ is compact;

   e) If X is reflexive then any operator $A \in \mathcal{L}(X,Y)$ which maps weakly convergent sequences to norm-convergent sequences is a compact operator.

      Hint: Given a Banach space X and $A \subset X$, the Eberlein-Smulian Theorem says that A is weakly compact iff A is weakly sequentially compact (every sequence in A has a weakly convergent subsequence whose limit is also in A)

2. **Problem 2** (4 points)

   Let $d \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^d$ be a bounded subset. Given $k \in L^2(\Omega \times \Omega)$ consider the linear operator $K : L^2(\Omega) \to L^2(\Omega)$ defined as follows

   $$(Kf)(x) = \int k(x,y)f(y)dy.$$

   Prove that

   a) $Kf \in L^2$ if $f \in L^2$;

   b) K is a compact operator.

   Hint: use a result obtained in Problem 1.

3. **Problem 3** (6 points)

   Let $-\infty < a \leq 0 \leq b < \infty$ and let $A : L^2([a,b];\mathbb{C}) \to L^2([a,b];\mathbb{C})$ be the linear operator defined as follows:

   $$(Af)(x) = x^2 f(x).$$

a) Verify the continuity of A and compute its operator norm;

b) Prove that A has no eigenvalues;

c) Show that $\sigma(A) = [0, \|A\|]$.