Functional Analysis - Homework 9

Problem 1. *Let H be a Hilbert space and let $K \subset H$ be a non-empty closed convex set.*

a) Prove that for every $f \in H$ there exists an element $u \in K$ such that $\|f-u\| = \min_{v\in K}\|f-v\| =$ dist(f,K).

b) Prove that (a) is equivalent to the property: $\operatorname{Re}\langle f-u, v-u\rangle \le 0$ for all $v \in K$.

c) Use now (b) to prove the uniqueness of such u.

d) Prove that P_K is contractive, i.e., $\|P_K f_1 - P_K f_2\| \le \|f_1 - f_2\|$.

a)

Let $(v_n)_{n\in\mathbb{N}} \subset K$ be a minimizing sequence such that $\lim_{n\to\infty}\|f-v_n\| = \operatorname{dist}(f,K)$.
With the parallelogram equality, for $m,k \in \mathbb{N}$ we have

$$\begin{aligned}
\|v_m - v_k\|^2 &= \|(f-v_m)-(f-v_k)\|^2 &(1)\\
&= 2\|f-v_m\|^2 + 2\|f-v_k\|^2 - \|(f-v_m)+(f-v_k)\|^2 &(2)\\
&= 2\|f-v_m\|^2 + 2\|f-v_k\|^2 - 4\left\|f - \frac{v_m+v_k}{2}\right\|^2 &(3)
\end{aligned}$$

Since K is convex, $\frac{v_m+v_k}{2} \in K$, and thus $\left\|f-\frac{v_m+v_k}{2}\right\| \ge \operatorname{dist}(f,K)$.

$$\begin{aligned}
\|v_m - v_k\|^2 &\le 2\|f-v_m\|^2 + 2\|f-v_k\|^2 - 4\operatorname{dist}(f,K)^2 &(4)\\
&= 2\left(\|f-v_m\|^2 - \operatorname{dist}(f,K)^2\right) + 2\left(\|f-v_k\|^2 - \operatorname{dist}(f,K)^2\right) &(5)
\end{aligned}$$

Since $\|f-v_n\| \to \operatorname{dist}(f,K)$, the right-hand side tends to 0 as $m,k\to\infty$. This implies (v_n) is Cauchy. Because H is complete and K is closed, (v_n) converges to some $u \in K$ which satisfies $\|f-u\| = \operatorname{dist}(f,K)$.

b)

Let (**1**) be $\|f-u\| = \min_{v\in K}\|f-v\|$ and (**2**) be $\operatorname{Re}\langle f-u, v-u\rangle \le 0$ for all $v \in K$.

(**1**) $\implies$ (**2**) Assume u satisfies (1). For arbitrary $v \in K$, let $v_t = (1-t)u + tv \in K$ for $t \in [0,1]$. By the minimization property, $\|f-u\|^2 \le \|f-v_t\|^2$.

$$\begin{aligned}
\|f-u\|^2 &\le \|(f-u) - t(v-u)\|^2 &(6)\\
&= \|f-u\|^2 - 2t\operatorname{Re}\langle f-u, v-u\rangle + t^2\|v-u\|^2 &(7)\\
\therefore 2t\operatorname{Re}\langle f-u, v-u\rangle &\le t^2\|v-u\|^2 &(8)
\end{aligned}$$

Dividing by $2t$ for $t > 0$ and taking the limit as $t \to 0^+$ yields $\operatorname{Re}\langle f-u, v-u\rangle \le 0$.

(**2**) $\implies$ (**1**) Assume u satisfies (2). For any $v \in K$, we expand $\|f-v\|^2$:

$$\begin{aligned}
\|f-v\|^2 &= \|(f-u)-(v-u)\|^2 &(9)\\
&= \|f-u\|^2 - 2\operatorname{Re}\langle f-u, v-u\rangle + \|v-u\|^2 &(10)
\end{aligned}$$

Since $\operatorname{Re}\langle f-u, v-u\rangle \le 0$ by (2), the term $-2\operatorname{Re}\langle f-u, v-u\rangle \ge 0$.

$$\|f-v\|^2 \ge \|f-u\|^2 + \|v-u\|^2 \qquad (11)$$

Since $\|v-u\|^2 \ge 0$, we conclude that $\|f-v\| \ge \|f-u\|$, proving (1).

c)

Assume u_1 and u_2 are both minimizers in K. By part (b), they must satisfy the inequality (2).

- u_1 is the projection of f. Setting $v = u_2$: $\operatorname{Re}\langle f - u_1, u_2 - u_1\rangle \le 0$.
- u_2 is the projection of f. Setting $v = u_1$: $\operatorname{Re}\langle f - u_2, u_1 - u_2\rangle \le 0$.

Using $u_1 - u_2 = -(u_2 - u_1)$, the second inequality is equivalent to $\operatorname{Re}\langle f - u_2, u_2 - u_1\rangle \ge 0$. Summing the two relations:

$$0 \ge \operatorname{Re}\langle f - u_1, u_2 - u_1\rangle - \operatorname{Re}\langle f - u_2, u_2 - u_1\rangle \tag{12}$$
$$0 \ge \operatorname{Re}\langle (f - u_1) - (f - u_2), u_2 - u_1\rangle \tag{13}$$
$$0 \ge \operatorname{Re}\langle u_2 - u_1, u_2 - u_1\rangle \tag{14}$$
$$\|u_2 - u_1\|^2 \le 0 \tag{15}$$

Since the norm squared must be non-negative, $\|u_2 - u_1\|^2 = 0$, which implies $u_1 = u_2$.

d)

Let $u_1 = P_K f_1$ and $u_2 = P_K f_2$. With property (b):

- For f_1 and u_1: set $v = u_2$. $\operatorname{Re}\langle f_1 - u_1, u_2 - u_1\rangle \le 0$.
- For f_2 and u_2: set $v = u_1$. $\operatorname{Re}\langle f_2 - u_2, u_1 - u_2\rangle \le 0$, which implies $-\operatorname{Re}\langle f_2 - u_2, u_2 - u_1\rangle \le 0$.

Summing these two inequalities:

$$0 \ge \operatorname{Re}\langle f_1 - u_1, u_2 - u_1\rangle - \operatorname{Re}\langle f_2 - u_2, u_2 - u_1\rangle \tag{16}$$
$$0 \ge \operatorname{Re}\langle (f_1 - u_1) - (f_2 - u_2), u_2 - u_1\rangle \tag{17}$$
$$0 \ge \operatorname{Re}\langle (f_1 - f_2) - (u_1 - u_2), u_2 - u_1\rangle \tag{18}$$
$$0 \ge \operatorname{Re}\langle f_1 - f_2, u_2 - u_1\rangle - \operatorname{Re}\langle u_1 - u_2, u_2 - u_1\rangle \tag{19}$$
$$\|u_2 - u_1\|^2 \le \operatorname{Re}\langle f_1 - f_2, u_2 - u_1\rangle \le |\langle f_1 - f_2, u_2 - u_1\rangle| \tag{20}$$

Applying the Cauchy-Schwarz inequality, $|\langle x, y\rangle| \le \|x\|\,\|y\|$:

$$\|u_1 - u_2\|^2 \le \|f_1 - f_2\|\,\|u_2 - u_1\| \tag{21}$$

Dividing by $\|u_1 - u_2\|$ (assuming it is non-zero) yields the desired result: $\|u_1 - u_2\| \le \|f_1 - f_2\|$.

Problem 2.

With $\overline{x - i} = x + i$ and $\overline{x + i} = x - i$,

$$\langle f_m, f_n\rangle = \int_{-\infty}^{\infty} f_m(x)\overline{f_n(x)}dx \tag{22}$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x-i)^m}{(x+i)^{m+1}} \frac{(x+i)^n}{(x-i)^{n+1}} dx \tag{23}$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x-i)^{m-n-1}}{(x+i)^{m-n+1}} dx \tag{24}$$

Let $g(z) = \frac{1}{\pi}\frac{(z-i)^{m-n-1}}{(z+i)^{m-n+1}}$. We use the Residue Theorem on a semi-circular contour in the upper half-plane. The integral is equal to $2\pi i \sum \operatorname{Res}(g, z_k)$, where z_k are poles in the upper half-plane, and the poles are at $z = \pm i$.

Case 1: $m = n$ The integrand simplifies to $g(z) = \frac{1}{\pi}\frac{(z-i)^{-1}}{(z+i)^1} = \frac{1}{\pi(z-i)(z+i)}$. The only pole in the upper half-plane is a simple pole at $z = i$.

$$\operatorname{Res}(g, i) = \lim_{z\to i}(z - i)g(z) = \lim_{z\to i}\frac{1}{\pi(z+i)} = \frac{1}{\pi(2i)} \tag{25}$$
$$\langle f_n, f_n\rangle = 2\pi i \cdot \operatorname{Res}(g, i) = 2\pi i \cdot \frac{1}{2\pi i} = 1 \tag{26}$$

Case 2: $m \neq n$ **1)** $m > n$ Let $k = m - n \geq 1$. The integrand is $g(z) = \frac{1}{\pi}\frac{(z-i)^{k-1}}{(z+i)^{k+1}}$. The only singularity is a pole at $z = -i$ of order $k+1$. Since this pole is in the lower half-plane, the upper half-plane contour encloses no singularities.

$$\langle f_m, f_n \rangle = 2\pi i \cdot (0) = 0 \tag{27}$$

2) $m < n$ Let $k = n - m \geq 1$. The integrand is $g(z) = \frac{1}{\pi}\frac{(z+i)^{k-1}}{(z-i)^{k+1}}$. The only singularity is a pole at $z = i$ of order $p = k+1$, which is in the upper half-plane. The residue is:

$$\begin{aligned} \operatorname{Res}(g, i) &= \frac{1}{k!} \lim_{z \to i} \frac{d^k}{dz^k} \left[(z-i)^{k+1} g(z) \right] && (28) \\ &= \frac{1}{k!} \lim_{z \to i} \frac{d^k}{dz^k} \left[\frac{1}{\pi}(z+i)^{k-1} \right] && (29) \end{aligned}$$

Since $k \geq 1$, the k-th derivative of the polynomial $(z+i)^{k-1}$ (which has degree $k-1$) is zero.

$$\begin{aligned} \frac{d^k}{dz^k}(z+i)^{k-1} &= 0 && (30) \\ \therefore \operatorname{Res}(g, i) &= 0 && (31) \end{aligned}$$

$$\langle f_m, f_n \rangle = 2\pi i \cdot \operatorname{Res}(g, i) = 0 \tag{32}$$

Therefore, the functions satisfy $\langle f_m, f_n \rangle = 1$ if $m = n$ and 0 if $m \neq n$, thus they form an orthonormal set.