Functional Analysis
Homework 9

Deadline: November 11th

1. **Problem 1: Projections** (16 points)

   Let H be a Hilbert space and let $K \subset H$ be a non-empty closed convex set.

   a) Prove that for every $f \in H$ there exists an element $u \in K$ such that

   $$\|f - u\| = \min_{v \in K} \|f - v\| = \operatorname{dist}(f, K). \tag{1}$$

   To prove such a statement consider a minimising sequence (v_n) with $v_n \in K$ and prove that it is Cauchy.

   Such u is called the projection of f onto K and is usually denoted as follows

   $$u = P_K f = \pi_K f.$$

   b) Prove that (1) is equivalent to the following property:

   $$\langle f - u, v - u \rangle \quad v \in K. \tag{2}$$

   Hint: Assume that $u \in K$ and that it satisfies (1). Now take $w \in K$ and consider the family of objects $v = (1 - t)u + tw \in K$ with $t \in [0, 1]$. Relate now the norm of $f - u$ to the desired inner product using v.

   c) Use now (2) to prove the uniqueness of such u.

   d) Prove that P_K is contractive *i.e.*,

   $$\|P_K f_1 - P_K f_2\| \leq \|f_1 - f_2\|.$$

2. **Problem 2: An orthonormal set in $L_2(\mathbb{R})$** (4 points)

   Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{C}$ given by

   $$f_n(x) = \frac{1}{\sqrt{\pi}} \frac{(x - i)^n}{(x + i)^{n+1}}.$$

   Prove that it is orthonormal in $L_2(\mathbb{R})$ *i.e.*, prove that

   $$\int_{-\infty}^{+\infty} f_m(x)\overline{f_n(x)}dx = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{else.} \end{cases} \tag{3}$$