Functional Analysis - Homework 8

Problem 1.

To show the inequality, for $2 \leq p < \infty$,

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leq \frac{1}{2}\left(|a|^p + |b|^p\right). \tag{1}$$

, it is sufficient to prove the real-variable inequality for $t \in [0,1]$:

$$\left(\frac{1+t}{2}\right)^p + \left(\frac{1-t}{2}\right)^p \leq \frac{1+t^p}{2}. \tag{2}$$

Define the function $f(t)$ on $[0,1]$ by

$$f(t) = \frac{1}{2}\left(1+t^p\right) - \left[\left(\frac{1+t}{2}\right)^p + \left(\frac{1-t}{2}\right)^p\right]. \tag{3}$$

We aim to show that $f(t) \geq 0$.
At $t = 1$:

$$f(1) = \frac{1}{2}(1+1) - \left[1^p + 0^p\right]$$
$$= 1 - 1 = 0. \tag{4}$$

At $t = 0$:

$$f(0) = \frac{1}{2}(1) - \left[2^{-p} + 2^{-p}\right]$$
$$= \frac{1}{2} - 2^{-(p-1)}. \tag{5}$$

Since $p \geq 2$ implies $2^{-(p-1)} \leq \frac{1}{2}$, it follows that $f(0) \geq 0$.

$$f'(t) = \frac{p}{2}t^{p-1} - p\,2^{-p}\left[(1+t)^{p-1} - (1-t)^{p-1}\right]. \tag{6}$$

We verify that $f'(1) = 0$:

$$f'(1) = \frac{p}{2} - p\,2^{-p}\left[2^{p-1} - 0\right]$$
$$= \frac{p}{2} - \frac{p}{2} = 0. \tag{7}$$

$$f''(t) = \frac{p(p-1)}{2}\left[t^{p-2} - 2^{-(p-1)}\left((1+t)^{p-2} + (1-t)^{p-2}\right)\right]. \tag{8}$$

The function $\psi(s) = s^{p-2}$ is convex on $[0,\infty)$ when $p \geq 2$ (since $p - 2 \geq 0$). Set

$$x_1 = \frac{1+t}{2}, \qquad x_2 = \frac{1-t}{2},$$

so that $x_1, x_2 \in [0,1]$ and $x_1 + x_2 = 1$. Writing $t = x_1 - x_2$ and using the convexity of ψ together with the evenness of the power $(\cdot)^{p-2}$, one obtains the pointwise inequality (for $t \in [0,1]$)

$$t^{p-2} \leq 2^{-(p-1)}\left((1+t)^{p-2} + (1-t)^{p-2}\right), \tag{9}$$

which implies $f''(t) \leq 0$ for $t \in [0,1]$ when $p > 2$ (and the boundary case $p = 2$ may be checked directly).
Thus $f''(t) \leq 0$ on $[0,1]$, so f is concave on $[0,1]$. Since a concave function attains its minimum at the boundary, and we already have $f(0) \geq 0$ and $f(1) = 0$, it follows that $f(t) \geq 0$ for all $t \in [0,1]$.

Hence, the scalar inequality is proven. By integrating this result over the measure space $(\Omega, \mathcal{F}, \mu)$, we obtain

$$\int_{\Omega} \left[\left| \frac{f(x) + g(x)}{2} \right|^p + \left| \frac{f(x) - g(x)}{2} \right|^p \right] d\mu \leq \int_{\Omega} \frac{1}{2} \left(|f(x)|^p + |g(x)|^p \right) d\mu, \tag{10}$$

$$\left\| \frac{f + g}{2} \right\|_{L_p(\mu)}^p + \left\| \frac{f - g}{2} \right\|_{L_p(\mu)}^p \leq \frac{1}{2} \left(\|f\|_{L_p(\mu)}^p + \|g\|_{L_p(\mu)}^p \right). \tag{11}$$

This inequality implies that $L_p(\mu)$ is uniformly convex for $p \geq 2$.

Problem 2.

(1) **T is an isometry:** $\|Tf\|_{(L_q)^*} = \|f\|_{L_p}$.

By definition,

$$\|Tf\|_{(L_q)^*} = \sup_{\|g\|_{L_q} \leq 1} |\langle Tf, g \rangle| = \sup_{\|g\|_{L_q} \leq 1} \left| \int fg \, d\mu \right|. \tag{12}$$

Hölder's inequality gives

$$\left| \int fg \, d\mu \right| \leq \|f\|_{L_p} \|g\|_{L_q}, \tag{13}$$

so $\|Tf\|_{(L_q)^*} \leq \|f\|_{L_p}$.

For equality, assume $f \neq 0$ and set

$$g_0(x) = sgn(f(x)) \left(\frac{|f(x)|}{\|f\|_{L_p}} \right)^{p-1}. \tag{14}$$

Then

$$\|g_0\|_{L_q}^q = \int |g_0|^q \, d\mu = \frac{1}{\|f\|_{L_p}^p} \int |f|^p \, d\mu = 1, \tag{15}$$

so $\|g_0\|_{L_q} = 1$. Moreover,

$$\langle Tf, g_0 \rangle = \int fg_0 \, d\mu = \frac{1}{\|f\|_{L_p}^{p-1}} \int |f|^p \, d\mu = \|f\|_{L_p}. \tag{16}$$

Hence $\|Tf\|_{(L_q)^*} = \|f\|_{L_p}$; thus T is an isometry.

(2) $T(L_p)$ **is closed in** $(L_q)^*$.

Let $\{Tf_n\}$ be a Cauchy sequence in $T(L_p)$. Since T is an isometry,

$$\|Tf_n - Tf_m\|_{(L_q)^*} = \|f_n - f_m\|_{L_p}, \tag{17}$$

so $\{f_n\}$ is Cauchy in L_p. Because L_p is complete, $f_n \to f \in L_p$, and by continuity of T,

$$Tf_n \to Tf. \tag{18}$$

Thus the limit lies in $T(L_p)$, proving that $T(L_p)$ is closed in $(L_q)^*$.

(3) L_p **is reflexive for** $1 < p \leq 2$.

For $1 < p \leq 2$, we have $q \geq 2$. By Clarkson's inequality, $L_q(\mu)$ is uniformly convex for $q \geq 2$. Since every uniformly convex Banach space is reflexive, it is reflexive.

A dual space is reflexive if and only if its predual is reflexive, so $(L_q)^*$ is reflexive.

Closed subspaces of reflexive spaces are reflexive; thus $T(L_p)$ is reflexive. Since T is an isometric isomorphism between L_p and $T(L_p)$, it follows that L_p itself is reflexive.

Problem 3.

a) Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|x\| = \sqrt{\langle x, x \rangle}$. We need to show: for every $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in H$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$, such that

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \tag{19}$$

In Hilbert spaces,

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \tag{20}$$

$$\left\| \frac{x + y}{2} \right\|^2 = \frac{\|x\|^2 + \|y\|^2}{2} - \frac{\|x - y\|^2}{4}. \tag{21}$$

Since $\|x\|, \|y\| \leq 1$, we have $\frac{\|x\|^2 + \|y\|^2}{2} \leq 1$, and $\|x - y\| \geq \varepsilon$ gives $-\frac{\|x-y\|^2}{4} \leq -\frac{\varepsilon^2}{4}$.
Thus

$$\left\| \frac{x + y}{2} \right\|^2 \leq 1 - \frac{\varepsilon^2}{4}. \tag{22}$$

$$\left\| \frac{x + y}{2} \right\| \leq \sqrt{1 - \frac{\varepsilon^2}{4}}. \tag{23}$$

Set

$$\delta = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} > 0. \tag{24}$$

Then

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta, \tag{25}$$

which implies that every Hilbert space is uniformly convex.

b) L_∞ is not uniformly convex.

Let $X = L_\infty[0, 2]$ with the norm $\|f\|_\infty$ as the essential supremum, and consider the unit ball $B_1 = \{f \in X : \|f\|_\infty \leq 1\}$.

Define

$$x(t) = 1, \qquad y(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ -1, & 1 < t \leq 2. \end{cases} \tag{26}$$

Then $\|x\|_\infty = \|y\|_\infty = 1$, so $x, y \in B_1$. Compute:

$$(x - y)(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ 2, & 1 < t \leq 2, \end{cases} \qquad \|x - y\|_\infty = 2. \tag{27}$$

Choose $\varepsilon = 1.9 < 2$, so $\|x - y\|_\infty > \varepsilon$.

The midpoint satisfies

$$\frac{x(t) + y(t)}{2} = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & 1 < t \leq 2, \end{cases} \qquad \left\| \frac{x + y}{2} \right\|_\infty = 1. \tag{28}$$

Hence, for this pair x, y, $\|x - y\|_\infty > \varepsilon$ but $\left\| \frac{x+y}{2} \right\|_\infty = 1$, so for any $\delta > 0$, $\left\| \frac{x+y}{2} \right\|_\infty \not\leq 1 - \delta$.
Therefore, no $\delta > 0$ satisfies the definition of uniform convexity, and L_∞ is not uniformly convex.