Problem 1

a)

The following statements will be enough to show $\|f\|_{X,\infty}$ is a norm.

$$\|f\|_{X,\infty} \geq 0, = 0 \text{ iff } f = 0 \|\lambda f\|_{X,\infty} = |\lambda| \|f\|_{X,\infty} \|f + g\|_{X,\infty} \leq \|f\|_{X,\infty} + \|g\|_{X,\infty}$$

Since norm $\|f(d)\|_X \geq 0$, we have

$$\|f\|_{X,\infty} = \sup_{d \in D} \|f(d)\|_X \geq 0$$

When

$$\sup_{d \in D} \|f(d)\|_X = 0,$$

then $\|f(d)\|_X = 0$ uniformly, which means $f(d) = 0$ uniformly on D. Thus, the first condition is true.

$$\|\lambda f\|_{X,\infty} = \sup_{d \in D} \|\lambda f(d)\|_X = |\lambda| \sup_{d \in D} \|f(d)\|_X = |\lambda| \|f\|_{X,\infty},$$

which is equivalent to the second condition.

$$\|f+g\|_{X,\infty} = \sup_{d \in D} \|f(d)+g(d)\|_X \leq \sup_{d \in D} \big(\|f(d)\|_X + \|g(d)\|_X \big) \leq \sup_{d \in D} \|f(d)\|_X + \sup_{d \in D} \|g(d)\|_X = \|f\|_{X,\infty} + \|g\|_{X,\infty},$$

which is equivalent to the triangle inequality.

With the proof of the three conditions, $\|f\|_{X,\infty}$ is a norm.

b)

From a) we know the space is normed. Next, we show it is a vector space and complete to prove it's Banach.

To show it is a vector space, we will prove that, $\forall f, g \in B(D, X), \lambda \in \mathbb{K}$,

$$f + g \in B(D, X) \tag{1}$$
$$\lambda f \in B(D, X) \tag{2}$$

$$\because \|f + g\|_{X,\infty} \leq \|f\|_{X,\infty} + \|g\|_{X,\infty} < \infty \therefore f + g \in B(D, X)$$

$$\because \|\lambda f\|_{X,\infty} \leq |\lambda| \|f\|_{X,\infty} < \infty \therefore \lambda f \in B(D, X)$$

Thus, equations (4) and (5) hold, which means the target space is a vector space.

Next is the proof of completeness. Suppose there is a Cauchy sequence in the target space $(f_i)_{i \in \mathbb{N}}$. Then,

$$\forall \epsilon > 0, \exists m, n > N, \text{ s.t. } \|f_m - f_n\|_{X,\infty} < \epsilon.$$

Let $(f_i)_{i \in \mathbb{N}}$ be bounded by f. With the triangle inequality, we have

$$\|f\|_{X,\infty} \leq \|f - f_n\|_{X,\infty} + \|f_n\|_{X,\infty}.$$

$$\because \|f - f_n\|_{X,\infty} < \epsilon, \|f_n\|_{X,\infty} < \infty \therefore \|f\|_{X,\infty} < \infty \therefore f \in B(D, X).$$

Therefore, $B(D, X)$ is complete, and in turn it is Banach.

Problem 2

Proof of $\Rightarrow$:

Let $X = (x_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in a metric space (M, d) that converges at x. Thus, we have

$$\forall\epsilon > 0, \exists m > N : d(x_m, x) < \epsilon.$$

Create a subsequence X' of X by removing x_1. X' should converge to the same x as well, as removing x_1 has no effect on convergence.

Proof of $\Leftarrow$:

Let $X = (x_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in a metric space (M, d), and $Y = (y_j)_{j\in\mathbb{N}}$ be a convergent subsequence of X that converges to y.

$$\forall\epsilon > 0, \exists m > N : d(y_m, y) < \epsilon.$$

Also,

$$\forall\sigma > 0, \exists n, k > N : d(x_n, x_k) < \sigma.$$

Since Y is a subsequence of X, there exists an index mapping $y_j = x_{f(j)}$. Then,

$$d(x_{f(m)}, y) < \epsilon.$$

When m is large enough, $f(m) > N$, and we can use k to replace it. Then,

$$d(x_k, y) < \epsilon.$$

Therefore,

$$d(x_n, y) \leq d(x_k, y) + d(x_n, x_k) < \epsilon + \sigma,$$

which means X converges to y.

Problem 3

As $(X, \|\cdot\|_X)$ is a normed vector space, statement (a) is equivalent to completeness.

In X, let $(x_n)_{n\in\mathbb{N}}$ such that $\sum_{i=1}^{\infty}\|x_i\|_X < \infty$, $(y_n)_{n\in\mathbb{N}}$ with $y_n = \sum_{i=1}^{n} x_i$, and $(y'_n)_{n\in\mathbb{N}}$ with $y'_n = \sum_{i=1}^{n}\|x_i\|_X$.

(a) to (b)

Let $l, k \in \mathbb{N}$ be arbitrary large indices. Since $y'_n = \sum_{i=1}^{\infty}\|x_i\|_X$ is bounded, say by C, and each $\|x_i\|_X \geq 0$, we can always set k large enough to make y'_k close to $C - \epsilon$, with $\epsilon > 0$. Next, we can make another index $l > k$ to have $|y'_k - y'_l| < \epsilon$. Thus, (y'_n) is Cauchy.

$$\|y_k - y_l\|_X = \left\|\sum_{i=k}^{l} x_i\right\|_X \leq \sum_{i=k}^{l}\|x_i\|_X = |y'_k - y'_l| < \epsilon.$$

Thus, (y_n) is Cauchy as well. Since X is Banach, (y_n) converges and the limit exists.

(b) to (a)

Let (z_i) be an arbitrary Cauchy sequence in X. Collect indices i in the following style to gather a subsequence: - Begin with a real number d, find i_1, i_2 such that

$$\|z_{i_1} - z_{i_2}\|_X < \tfrac{d}{2}.$$

- Next, find i_3 such that

$$\|z_{i_2} - z_{i_3}\|_X < \tfrac{d}{2^2}.$$

- Iterate to find i_k with

$$\|z_{i_k} - z_{i_{k+1}}\|_X < \tfrac{d}{2^k}.$$

The resulting subsequence (z_{i_n}) should be one of the type in statement b). Therefore,

$$\sum_{k=1}^{\infty} \|z_{i_k} - z_{i_{k+1}}\|_X = \sum_{k=1}^{\infty} 2^{-k} d = d < \infty.$$

According to statement b),

$$\lim_{N \to \infty} \sum_{k=1}^{N} (z_{i_k} - z_{i_{k+1}}) \in X.$$

Therefore,

$$\lim_{N \to \infty} (z_{i_1} - z_{i_{N+1}}) \in X,$$

which means (z_{i_n}) converges in X. Using the result in Problem 2, we have (z_i) converges in X as well. Therefore, X is complete and hence Banach.

Problem 4

(a) to (b)

Let M be a meagre set in X. Then M^c should be a residual set. With the statement (a), we have

$$\overline{M^c} = X \therefore (\overline{M^c})^c = \ \therefore int((M^c)^c) = int(M) =$$

, which means (b) is true.

(b) to (c)

Let M be a meagre set in X.

If M is open and not empty, there should be a open set $V \subseteq M$. Thus, $V \subseteq int(M)$, which is contradictory to (b) saying $int(M) =$. Therefore, M cannot be both open and inempty.

If M is empty, it satisfies to be open and meagre. Thus, the empty set is the only subset that is open and meagre.

(c) to (d)

Let

$$A = \bigcup_{i \in \mathbb{N}} A_i$$

, with A_i are dense open sets in X. A is open since it is a union of open sets. Thus, the complement of A, A^c is closed.

$$A^c = (\bigcap_{i \in \mathbb{N}} A_i)^c = \bigcup_{i \in \mathbb{N}} A_i^c$$

Let $B = int(A^c)$. Since B is open, $B \cap A_i^c$ should be no where dense. Thus,

$$int(A^c) = A^c \cup int(A^c) = \bigcup_{i\in\mathbb{N}} (int(A^c) \cap A_i^c)$$

is a union of no where dense sets. Thus, $int(A^c)$ is meagre. $int(A^c)$ is meagre and open,

$$\therefore int(A^c) = \ \therefore \overline{A} = X$$

, which means A is dense.

(d) $\implies$ (a)

Let $M = \bigcup_{i\in\mathbb{N}} M_i$ be a meagre set in X, with M_i are nowhere dense and M^c a residual set.

$$M^c = (\bigcup_{i\in\mathbb{N}} M_i)^c = \bigcap_{i\in\mathbb{N}} M_i^c$$

The definition of M_i being nowhere dense is $\text{int}(\overline{M_i}) = \emptyset$.

Let $A_i = \overline{M_i}^c$. A_i is an open set (as the complement of a closed set).

Using the properties of complement, $\text{int}(\overline{M_i}) = \text{int}((A_i)^c) = (\overline{A_i})^c$. Since $\text{int}(\overline{M_i}) = \emptyset$:

$$(\overline{A_i})^c = \emptyset \implies \overline{A_i} = X$$

Thus, $A_i = \overline{M_i}^c$ is a **dense open set**.

Since $M_i \subseteq \overline{M_i}$, we have $M_i^c \supseteq \overline{M_i}^c = A_i$. The residual set M^c is the intersection of the M_i^c:

$$M^c = \bigcap_{i\in\mathbb{N}} M_i^c \supseteq \bigcap_{i\in\mathbb{N}} A_i$$

The intersection of the countable family of dense open sets $(A_i)_{i\in\mathbb{N}}$ is $G = \bigcap_{i\in\mathbb{N}} A_i$. By statement (c), $\overline{G} = X$, meaning G is dense. Since M^c contains a dense set G, M^c itself must be dense:

$$\overline{M^c} \supseteq \overline{G} = X \implies \overline{M^c} = X$$

Thus, M^c is dense, which means the statement (a) is true.