Functional Analysis - Homework 7

Problem 1.

Let $(X, \|\cdot\|_X)$ be a Banach space and X^* its dual space. The sequence $(f_n)_{n\in\mathbb{N}} \subset X^*$ and $f \in X^*$. We denote weak* convergence by $f_n \xrightarrow{*} f$.

a)

Statement: $f_n \xrightarrow{*} f$ if and only if $f_n(x) \to f(x)$ for every $x \in X$.

Proof. $(\Rightarrow)$

Assume $f_n \xrightarrow{*} f$. By definition, $f_n \xrightarrow{*} f$ means f_n converges to f in the weak* topology $\sigma(X^*, X)$. The weak* topology is the coarsest topology on X^* that makes every evaluation functional $\hat{x} : X^* \to \mathbb{C}$, defined by $\hat{x}(g) = g(x)$ for a fixed $x \in X$, continuous. A sequence converges in a topology if and only if it converges with respect to every continuous linear functional. Therefore, for every $x \in X$, the continuous functional $\hat{x}$ satisfies:

$$\lim_{n\to\infty} \hat{x}(f_n) = \hat{x}(f) \implies \lim_{n\to\infty} f_n(x) = f(x). \tag{1}$$

$(\Leftarrow)$

Assume $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. Let V be an arbitrary basic open neighborhood of f in the $\sigma(X^*, X)$ topology.

$$V = \{g \in X^* : |g(x_i) - f(x_i)| < \epsilon_i \text{ for } i = 1, \dots, k\} \tag{2}$$

for some finite set of vectors $\{x_1, \dots, x_k\} \subset X$ and $\epsilon_1, \dots, \epsilon_k > 0$. Since $\lim_{n\to\infty} f_n(x_i) = f(x_i)$ for each i, for every $\epsilon_i > 0$, there exists N_i such that for all $n > N_i$, $|f_n(x_i) - f(x_i)| < \epsilon_i$. Let $N = \max\{N_1, \dots, N_k\}$. Then for all $n > N$, f_n satisfies all k conditions, meaning $f_n \in V$. Thus, f_n converges to f in the $\sigma(X^*, X)$ topology: $f_n \xrightarrow{*} f$. □

b)

Statement: If $f_n \xrightarrow{w} f$, then $f_n \xrightarrow{*} f$.

Proof. Let $J : X \to X^{**}$ be the canonical embedding defined by

$$(Jx)(\varphi) = \varphi(x), \quad \forall x \in X, \varphi \in X^*. \tag{3}$$

Weak convergence $f_n \xrightarrow{w} f$ in X^* means

$$f_n(\Phi) \to f(\Phi), \quad \forall \Phi \in X^{**}, \tag{4}$$

i.e., convergence with respect to the weak topology $\sigma(X^*, X^{**})$.

Since every $x \in X$ gives an element $Jx \in X^{**}$, we have, for each $x \in X$,

$$f_n(x) = f_n(Jx) \longrightarrow f(Jx) = f(x). \tag{5}$$

Thus $f_n(x) \to f(x)$ for every $x \in X$, which by part (a) meas

$$f_n \xrightarrow{*} f. \tag{6}$$

□

c)

Statement: If $f_n \xrightarrow{*} f$, then (f_n) is bounded and $||f||_{X^*} \leq \liminf_{n\to\infty} ||f_n||_{X^*}$.

Proof. Since $f_n \xrightarrow{*} f$, part (a) shows that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. Since every convergent sequence of complex numbers is bounded, for every fixed $x \in X$, the scalar sequence $(f_n(x))$ is bounded:

$$\sup_{n\in\mathbb{N}} |f_n(x)| < \infty \tag{7}$$

The sequence of functionals (f_n) is therefore pointwise bounded on X. Since X is a Banach space, with the Uniform Boundedness Principle applies, we have:

$$\sup_{n\in\mathbb{N}} \|f_n\|_{X^*} = M < \infty \tag{8}$$

Thus, (f_n) is bounded in X^*.

By the definition of the dual norm, for any $x \in X$ with $\|x\|_X \leq 1$:

$$|f_n(x)| \leq \|f_n\|_{X^*} \|x\|_X \leq \|f_n\|_{X^*} \tag{9}$$

Taking the limit inferior of both sides of the inequality:

$$\liminf_{n\to\infty} |f_n(x)| \leq \liminf_{n\to\infty} \|f_n\|_{X^*} \tag{10}$$

Since $f_n(x) \to f(x)$, $\liminf |f_n(x)| = \lim |f_n(x)| = |f(x)|$. Thus:

$$|f(x)| \leq \liminf_{n\to\infty} \|f_n\|_{X^*} \tag{11}$$

$$\therefore \sup_{\|x\|_X \leq 1} |f(x)| \leq \sup_{\|x\|_X \leq 1} \left(\liminf_{n\to\infty} \|f_n\|_{X^*}\right) \tag{12}$$

$$\therefore \|f\|_{X^*} \leq \liminf_{n\to\infty} \|f_n\|_{X^*} \tag{13}$$

□

d)

Statement: If $f_n \xrightarrow{*} f$ and $x_n \to x$, then $\langle f_n, x_n\rangle \to \langle f, x\rangle$.

Proof. We want to show $\lim_{n\to\infty} |f_n(x_n) - f(x)| = 0$. We use the triangle inequality trick:

$$|f_n(x_n) - f(x)| = |f_n(x_n) - f_n(x) + f_n(x) - f(x)| \tag{14}$$

$$\leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)| \tag{15}$$

For the second term of (15), since $f_n \xrightarrow{*} f$ and x is a fixed vector, by part (a):

$$\lim_{n\to\infty} |f_n(x) - f(x)| = 0 \tag{16}$$

For the first term of (15)

$$|f_n(x_n) - f_n(x)| = |f_n(x_n - x)| \tag{17}$$

$$\leq \|f_n\|_{X^*} \|x_n - x\|_X \tag{18}$$

From part (c), we know that if $f_n \xrightarrow{*} f$, the sequence of norms is uniformly bounded:

$$\sup_{n\in\mathbb{N}} \|f_n\|_{X^*} = M < \infty \tag{19}$$

Therefore,

$$\lim_{n\to\infty} |f_n(x_n) - f_n(x)| \leq \lim_{n\to\infty} (\|f_n\|_{X^*} \|x_n - x\|_X) \tag{20}$$

$$\leq M \cdot \lim_{n\to\infty} \|x_n - x\|_X \tag{21}$$

Since $x_n \to x$, $\lim_{n\to\infty} \|x_n - x\|_X = 0$.

$$0 \le \lim_{n\to\infty} |f_n(x_n) - f_n(x)| \le M \cdot 0 = 0 \tag{22}$$

Thus, both terms in (15) converge to 0. Then,

$$\lim_{n\to\infty} |f_n(x_n) - f(x)| = 0 \tag{23}$$

, which means $\langle f_n, x_n\rangle \to \langle f, x\rangle$. □

Problem 2. *Let $(X, ||\cdot||_X)$ be a Banach space and X^* its dual. Given sequences $(x_n)_n \subset X$ and $(f_n)_n \subset X^*$. If $f_n \xrightarrow{*} f$ and $x_n \xrightarrow{w} x$, then $\langle f_n, x_n\rangle \to \langle f, x\rangle$.*

The statement is false.

We give a counterexample in l^2.

Consider the Banach space $X = l^2(\mathbb{N})$, which is a Hilbert space and therefore reflexive, so $X^* = l^2$. The pairing is given by $\langle f, x\rangle = \sum_{k=1}^{\infty} f_k x_k$.

Let (e_n) be the standard basis vectors, where e_n has a 1 in the n-th position and 0 elsewhere.

Let $x_n = e_n$. With $x_n \xrightarrow{w} x$: For any $f = (f_k) \in l^2$, we have $\langle f, x_n\rangle = f_n$. Since $f \in l^2$, $f_n \to 0$ as $n \to \infty$. Thus, $x_n \to 0$, so $x = 0$.

Next, we analysis the function sequence. Let $f_n = e_n$. Note that $f_n \in X^*$. With $f_n \xrightarrow{*} f$, for any $x = (x_k) \in l^2 = X$, we have $\langle f_n, x\rangle = x_n$. Since $x \in l^2$, $x_n \to 0$ as $n \to \infty$. Thus, $f_n \xrightarrow{w^*} 0$, so $f = 0$.

We check the limit of the sequence of pairings $\langle f_n, x_n\rangle$:

$$\lim_{n\to\infty} \langle f_n, x_n\rangle = \lim_{n\to\infty} \langle e_n, e_n\rangle = \lim_{n\to\infty} (1) = 1. \tag{24}$$

The claimed limit is the pairing of the weak limits:

$$\langle f, x\rangle = \langle 0, 0\rangle = 0. \tag{25}$$

$$\therefore \langle f_n, x_n\rangle \not\to \langle f, x\rangle. \tag{26}$$

Problem 3.

Proof. The map $T : E \to F$ is a linear surjective isometry. Thus, we have: $\|T(x)\|_F = \|x\|_E$ for all $x \in E$.

Therefore, T is injective (since $\|T(x)\|_F = 0 \iff \|x\|_E = 0 \iff x = 0$), and then T is a bijection.

Its inverse $T^{-1} : F \to E$ also exists and is a linear isometry. Hense, both T and T^{-1} are bounded operators with operator norms $\|T\| = 1$ and $\|T^{-1}\| = 1$.

E is Reflexive $\implies$ F is Reflexive

Since T is an isometry, T maps the closed unit ball of E onto the closed unit ball of F: $T(B_E) = B_F$

Since E is reflexive, its closed unit ball B_E is $\sigma(E, E^*)$-compact.

Since T is norm-continuous, it is weakly continuous, as a linear map between normed spaces is norm-continuous if and only if it is weakly continuous.

Since B_E is $\sigma(E, E^*)$-compact and T is weakly continuous, the image $T(B_E) = B_F$ must be $\sigma(F, F^*)$-compact, as the continuous image of a compact set is compact.

Thus, B_F is weakly compact in F. Therefore, F is reflexive.

F is Reflexive $\implies$ E is Reflexive

As we have shown that T^{-1} is also a linear isometry. With the same process of the previous part, F is reflexive $\implies$ E is reflexive.

□
