Functional Analysis - Homework 5

Problem 1.

Proof. **a)** First, we prove M is continuous at $(0,0)$, which is equivalent to

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } \|(x,y)\|_{X\times Y} < \delta \implies \|M(x,y)\|_Z < \epsilon \tag{1}$$

Suppose

$$\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y < \delta = \sqrt{\frac{\epsilon}{C}} \tag{2}$$

, we will have

$$\|M(x,y)\|_Z \leq C\,\|x\|_X\,\|y\|_Y \tag{3}$$
$$\leq C(\|x\|_X + \|y\|_Y)(\|x\|_X + \|y\|_Y) \tag{4}$$
$$= C\delta^2 = \epsilon \tag{5}$$

, which is exactly equation (1). Thus, M is continuous at $(0,0)$. Since M is bi-linear, the continuity holds for the whole space.

b)

Assume X is complete, which means it is Banach.

Define a linear opertor

$$L_y : X \to Z, L_y(x) := M(x,y) \tag{6}$$

with $x \in X, y \in Y$.

Since $y \mapsto M(x',y)$ is continuous and linear, $\|M(x',y)\|$ should be bounded with fixed $x' \in X$. Thus,

$$\sup_{\|y\|_Y=1} \|L_y(x')\|_Z = C_{x'} < \infty \tag{7}$$

, which implies pointwise boundedness of L_y for $\{y \in Y : \|y\|_Y = 1\}$.

With Uniform Boundedness Theorem, we have the operator boundedness:

$$\sup_{\|y\|_Y=1} \|L_y\| = C < \infty \tag{8}$$

$$\because L_y(x) = M(x,y) \tag{9}$$
$$= \|y\|_Y\, M(x, \frac{y}{\|y\|_Y}) \tag{10}$$
$$= \|y\|_Y\, M(x,u), \text{ where } u = \frac{y}{\|y\|_Y} \tag{11}$$
$$= \|y\|_Y\, L_u(x), \text{ where } \|u\| = 1 \tag{12}$$
$$\therefore \|L_y\| \leq \|y\|_Y\, C \tag{13}$$

Since $x \mapsto M(x,y')$ is continuous and linear, $\|M(x,y')\|$ is bounded with fixed $y' \in Y$.

$$\|M(x,y')\|_Z = \|L_{y'}(x)\|_Z \leq \|L_{y'}\|\,\|x\|_X \tag{14}$$
$$\therefore \|M(x,y)\|_Z \leq C\,\|x\|_X\,\|y\|_Y \tag{15}$$

□

Problem 2.

Proof. **(1) continuity $\implies$ closedness**

Suppose f is continuous. With a closed set V in R, $f^{-1}(V)$ should be closed in E. Therefore, as $\{\alpha\}$ is closed in R, $H_\alpha = f^{-1}(\{\alpha\})$ is closed in E.

(2) closedness $\implies$ continuity

Suppose there exists a quotient space E/H_0, with a norm

$$\|[x]\|_{E/H_0} = \inf_{y \in H_0} \|x - y\|_E \tag{16}$$

, with $x \in E$ and $[x]$ is the equivalence class of x.

Verify the norm as follows.

1. Homogeneity: $\forall \lambda \in \mathbb{R}, x \in E$,

$$\begin{aligned}
\|[\lambda x]\|_{E/H_0} &= \inf_{y \in H_0} \|\lambda x - y\|_E && (17)\\
&= \inf_{y' \in H_0} \|\lambda x - \lambda y'\|_E \text{, where } y' = \frac{y'}{\lambda} && (18)\\
&= \inf_{y' \in H_0} \lambda \|x - y'\|_E && (19)\\
&= \lambda \|[x]\|_{E/H_0} && (20)
\end{aligned}$$

2. triangle inequality: $\forall \lambda \in \mathbb{R}, x_1, x_2 \in E$,

$$\begin{aligned}
\|[x_1 + x_2]\|_{E/H_0} &= \inf_{y \in H_0} \|x_1 + x_2 - y\|_E && (21)\\
&= \inf_{y' \in H_0} \|x_1 - y'\|_E + \inf_{y' \in H_0} \|x_2 - y'\|_E \text{, where } y' = \frac{y}{2} && (22)\\
&= \|[x_1]\|_{E/H_0} + \|[x_2]\|_{E/H_0} && (23)
\end{aligned}$$

.

3. Positive definiteness:

$$\begin{aligned}
&\|[x]\|_{E/H_0} = 0 && (24)\\
&\inf_{y \in H_0} \|x - y\|_E = 0 && (25)\\
&\iff x \in H_0 \text{, since } H_0 \text{ is closed} && (26)\\
&\iff [x] = H_0 && (27)
\end{aligned}$$

Therefore, the norm exists.

Define a function

$$g : E/H_0 \to H, g([x]) := f(x) \tag{28}$$

, with $x \in E$.

Let $y \in H_0$ be the closest point to $x \in E$, $\|x - y\|_E = \inf_{y \in H_0} \|x - y\|_E$.

$$\begin{aligned}
\therefore |g([x])| &= |f(x)| && (29)\\
&= |f(x) - f(y)| \text{, since } f(y) = 0 && (30)\\
&= |f(x - y)| && (31)\\
&\le \|f\| \|x - y\|_E && (32)\\
&= \|f\| \inf_{y \in H_0} \|x - y\|_E && (33)\\
&= \|f\| \|[x]\|_{E/H_0} && (34)
\end{aligned}$$

, which implies g is bounded, and in turn it is continuous.

With the same variables, we can have

$$\begin{aligned}
\|[x]\|_{E/H_0} = \inf_{y \in H_0} \|x - y\|_E && (35)\\
\|x - 0\|_E \text{, since } f(0) = 0 && (36)\\
\|x\|_E && (37)
\end{aligned}$$

. Thus, the canonical projection map $h : E \to E/H_0, h(x) = [x]$ is also continuous.

Therefore, $f = g \circ h$ is continuous.

□

Problem 3.

Proof. Let non-zero $x \in Y^c$. $\forall y' \in X$ could be decomposed as $y' = ax + y$, with $a \in \mathbb{C}, y \in Y$.

For each y', a and y are unique, since if not so, there could be

$$y' = a_1 x + y_1 = a_2 x + y_2 \tag{38}$$

$$\implies (a_1 - a_2)x = y_2 - y_1 \in Y \tag{39}$$

$$\implies x \in Y, \text{ which is contradiction.} \tag{40}$$

Apparently, $a = 0$ for all $y' \in Y$.

Let norm $\|\cdot\|$ in X be

$$\|y'\| := \|ax\|_1 + \|y\|_2, \forall y' \in X \tag{41}$$

$$\text{where } y' = ax + y, x \in Y^c, y \in Y, a \in \mathbb{C} \tag{42}$$

Thus, when $y' \in Y$,

$$\|y'\| = \|0x\|_1 + \|y'\|_2 = \|y'\|_2 \tag{43}$$

$$\|\cdot\| |_Y = \|\cdot\|_2 \tag{44}$$

, meaning $\|\cdot\|$ is equivalent to $\|\cdot\|_1$ on Y as $\|\cdot\|_2$ does.

When $y' \in Y^c$,

$$\|y'\| = \|ax\|_1 + \|y\|_2 \tag{45}$$

Since $\|\cdot\|_2$ is equivalent to $\|\cdot\|_1$ on Y, $\exists C_1 > 0$

$$C_1^{-1} \|y\|_1 \leq \|y\|_2 \leq C_1 \|y\|_1 \tag{46}$$

$$\therefore \|ax\|_1 + C_1^{-1} \|y\|_1 \leq \|ax\|_1 + \|y\|_2 \tag{47}$$

$$\therefore (C_1 + 1)^{-1}(\|ax\|_1 + \|y\|_1) \leq \|ax\|_1 + \|y\|_2 \tag{48}$$

$$\because \|ax + y\|_1 \leq \|ax\|_1 + \|y\|_1 \tag{49}$$

$$\therefore (C_1 + 1)^{-1}(\|ax + y\|_1) \leq \|ax\|_1 + \|y\|_2 \tag{50}$$

$$\therefore (C_1 + 1)^{-1} \|y'\|_1 \leq \|y'\| \tag{51}$$

. Also, $\exists C_2 > 0$

$$C_2^{-1} \|y\|_2 \leq \|y\|_1 \leq C_2 \|y\|_2 \tag{52}$$

$$\text{Similarly, we have } (C_2 + 1)^{-1} \|y'\| \leq \|y'\|_1 \tag{53}$$

$$\therefore \|y'\| \leq (C_2 + 1) \|y'\|_1 \tag{54}$$

$$\therefore (C_1 + 1)^{-1} \|y'\|_1 \leq \|y'\| \leq (C_2 + 1) \|y'\|_1 \tag{55}$$

In a whole, $\|\cdot\|$ is equivalent to $\|\cdot\|_1$ on X and its restriction to Y is $\|\cdot\|_2$.

□