Problem 1 a)

We prove (a)⇔(c), (c)⇔(b), and (a)⇔(d).

(a) ⇔ (c)

- $(a) \Rightarrow (c)$. If f is continuous, then for any closed $B \subseteq Y$ its complement $Y \setminus B$ is open, so $f^{-1}(Y \setminus B)$ is open in X. But $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$, so $f^{-1}(B)$ is closed.
- $(c) \Rightarrow (a)$. If preimages of closed sets are closed, take any open $U \subseteq Y$. Then $Y \setminus U$ is closed, so $f^{-1}(Y \setminus U)$ is closed. But $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$, so $f^{-1}(U)$ is open. Hence f is continuous.

Thus (a) and (c) are equivalent.

(c) ⇔ (b)

- $(b) \Rightarrow (c)$. Let $C \subseteq Y$ be closed and set $A = f^{-1}(C) \subseteq X$. Then $f(A) \subseteq C$. By (b),

$$f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{C} = C,$$

so every $x \in \overline{A}$ satisfies $f(x) \in C$. Hence $\overline{A} \subseteq f^{-1}(C) = A$. Since $A \subseteq \overline{A}$ always, we obtain $A = \overline{A}$, i.e. A is closed. Therefore $f^{-1}(C)$ is closed for every closed C, so (c) holds.

- $(c) \Rightarrow (b)$. Fix $A \subseteq X$. Note $f(A) \subset \overline{f(A)}$, so

$$A \subseteq f^{-1}(\overline{f(A)}).$$

By (c) the set $f^{-1}(\overline{f(A)})$ is closed, and it contains A; hence it contains the closure of A:

$$\overline{A} \subseteq f^{-1}(\overline{f(A)}).$$

Apply f to both sides to get

$$f(\overline{A}) \subseteq f\big(f^{-1}(\overline{f(A)})\big) \subseteq \overline{f(A)},$$

the last inclusion because $f(f^{-1}(S)) \subseteq S$ for any $S \subseteq Y$. Thus (b) holds.

So (b) and (c) are equivalent.

(a) ⇔ (d)

- $(a) \Rightarrow (d)$. Suppose f is continuous. Fix $x \in X$ and let V be a neighbourhood of $f(x)$; by definition there exists an open set $W \subseteq Y$ with $f(x) \in W \subset V$. Then $U := f^{-1}(W)$ is open (by continuity), contains x, and $f(U) \subseteq W \subset V$. Thus U is the required neighbourhood of x.
- $(d) \Rightarrow (a)$. Suppose (d) holds. Let $B \subseteq Y$ be open and take any $x \in f^{-1}(B)$. Then $f(x) \in B$, so B is a neighbourhood of $f(x)$; by (d) there is a neighbourhood U of x with $f(U) \subset B$. Hence $U \subset f^{-1}(B)$, and so $f^{-1}(B)$ is a union of open neighbourhoods U of each of its points, i.e. $f^{-1}(B)$ is open. Therefore f is continuous.

Thus (a) and (d) are equivalent.

This completes Problem 1(a).

Problem 1 b)

Proof.

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies the ε-δ condition at every point:

$$\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } |t - x| < \delta \implies |f(t) - f(x)| < \varepsilon$$

Let $V \subseteq \mathbb{R}$ be open and pick any $x \in f^{-1}(V)$. Then $f(x) \in V$, so there exists $\varepsilon > 0$ with the open interval

$$I_\varepsilon(f(x)) = (f(x) - \varepsilon, f(x) + \varepsilon) \subset V$$

.

By (ε, δ)-continuity at x there is $\delta > 0$ with $|t - x| < \delta \implies |f(t) - f(x)| < \varepsilon$, i.e. $t \in (x - \delta, x + \delta)$ implies $f(t) \in I_\varepsilon(f(x)) \subset V$.

Thus, $(x - \delta, x + \delta)$ is an open neighbourhood of x contained in $f^{-1}(V)$. Since this is true for each $x \in f^{-1}(V)$, the preimage $f^{-1}(V)$ is open. Hence f is continuous in the topological sense.

Problem 2

Example. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^2$. Define

$$d(x, y) = \|x - y\|, \qquad d'(x, y) = \min\{1, \|x - y\|\}.$$

We claim d and d' are not equivalent.

Proof. Assume by contradiction there is $C > 0$ with $d(x, y) \le C\, d'(x, y)$ for all x, y. For any $R > C$ pick x, y with $\|x - y\| = R$. Then $d'(x, y) = \min\{1, R\} = 1$, so the inequality would give $R = d(x, y) \le C \cdot 1 = C$, contradicting $R > C$. Hence no such C exists, so the metrics are not equivalent.

Problem 3 a)

- **Positive definiteness.** Each term

  $$a_n(f, g) := 2^{-n} \frac{\|f - g\|_{C([0,n])}}{1 + \|f - g\|_{C([0,n])}}$$

  is nonnegative and equals 0 iff $\|f - g\|_{C([0,n])} = 0$ (i.e. $f = g$ on $[0, n]$). If $f \equiv g$ on all $[0, n]$ for every n then $f = g$ on $[0, \infty)$, so $d(f, g) = 0$ iff $f = g$. Also symmetry is obvious since $\|f - g\| = \|g - f\|$.

- **Triangle inequality.** We show for any nonnegative real numbers s, t

  $$\frac{s}{1 + s} + \frac{t}{1 + t} \ge \frac{s + t}{1 + s + t}.$$

  This is elementary algebra: compute the difference

  $$\frac{s}{1 + s} + \frac{t}{1 + t} - \frac{s + t}{1 + s + t} = \frac{st(s + t + 2)}{(1 + s)(1 + t)(1 + s + t)} \ge 0.$$

  Hence

  $$\frac{\|f - h\|_{[0,n]}}{1 + \|f - h\|_{[0,n]}} \le \frac{\|f - g\|_{[0,n]}}{1 + \|f - g\|_{[0,n]}} + \frac{\|g - h\|_{[0,n]}}{1 + \|g - h\|_{[0,n]}}.$$

  Multiply by 2^{-n} and sum over n to obtain

$$d(f,h) \le d(f,g) + d(g,h).$$

Thus d satisfies the triangle inequality.

Therefore d is a metric.

Problem 3 b)

Let V be a real vector space and d a metric on V.

- If d is induced by a norm $\|\cdot\|$ via $d(x,y) = \|x-y\|$, then for all $x,y,z \in V$ and $\lambda \in \mathbb{R}$

  $$d(x+z, y+z) = \|x+z-(y+z)\| = \|x-y\| = d(x,y)$$

  (translation-invariance), and

  $$d(\lambda x, \lambda y) = \|\lambda x - \lambda y\| = |\lambda|\,\|x-y\| = |\lambda|\,d(x,y)$$

  (homogeneity).
- Conversely, suppose d is translation-invariant and homogeneous. Define $\|v\| := d(v,0)$. Then $\|v\| \ge 0$, $\|v\| = 0 \iff v = 0$ (because d is a metric), and for scalar λ,

  $$\|\lambda v\| = d(\lambda v, 0) = d(\lambda v, \lambda 0) = |\lambda| d(v,0) = |\lambda|\|v\|.$$

  For the triangle inequality:

  $$\|u+v\| = d(u+v,0) = d(u+v,v) \le d(u,0) + d(v,0) = \|u\| + \|v\|,$$

  where we used translation invariance to reduce $d(u+v,v) = d(u,0)$. Thus $\|\cdot\|$ is a norm and $d(x,y) = \|x-y\|$.

So a metric is induced by a norm iff it is translation-invariant and homogeneous.

Problem 3 c)

No.

Reason. If d were induced by a norm $\|\cdot\|$, then it would be homogeneous: $d(\lambda f, \lambda g) = |\lambda| d(f,g)$ for every real λ. But for the metric in (a) the dependence on $\|f-g\|_{[0,n]}$ is via the function $\phi(t) = t/(1+t)$, which is not homogeneous: $\phi(|\lambda|t) \ne |\lambda|\phi(t)$ in general. Concretely, take any nonzero h and $\lambda > 0$. The n-th summand for $(\lambda f, \lambda g)$ equals

$$2^{-n} \frac{|\lambda|\|f-g\|_{[0,n]}}{1 + |\lambda|\|f-g\|_{[0,n]}}$$

which is not equal to $|\lambda|$ times

$$2^{-n} \frac{\|f-g\|_{[0,n]}}{1 + \|f-g\|_{[0,n]}}.$$

Therefore $d(\lambda f, \lambda g) \ne |\lambda| d(f,g)$ in general, so d is not homogeneous and hence cannot come from a norm.

Problem 4

Proof. For any indices n, m the triangle inequality gives

$$\big|d(x_n, \tilde{x}_n) - d(x_m, \tilde{x}_m)\big| \le d(x_n, x_m) + d(\tilde{x}_n, \tilde{x}_m).$$

Since (x_n) and $(\tilde{x}_n)$ are Cauchy, for each $\varepsilon > 0$ there exists N such that for all $n, m \ge N$ both $d(x_n, x_m) < \varepsilon/2$ and $d(\tilde{x}_n, \tilde{x}_m) < \varepsilon/2$. Hence the difference above is $< \varepsilon$ for $n, m \ge N$, so the real sequence $d(x_n, \tilde{x}_n)$ is Cauchy and therefore converges.

To deduce continuity of $d(\cdot, \cdot)$: let $(x_n, y_n) \to (x, y)$ in the product topology (i.e. $x_n \to x$ and $y_n \to y$). Then the same inequality

$$\big|d(x_n, y_n) - d(x, y)\big| \le d(x_n, x) + d(y_n, y)$$

shows the left-hand side tends to 0 as $n \to \infty$. Thus $d(x_n, y_n) \to d(x, y)$ and d is continuous.

Problem 5 a)

- Nonnegativity and symmetry: If $d(f, g) = 0$ then $d_Y(f(x), g(x)) = 0$ for all x, so $f(x) = g(x)$ for all x, hence $f = g$. Conversely equal functions give zero distance.
- Triangle inequality: for any $x \in X$,

$$d_Y(f(x), h(x)) \le d_Y(f(x), g(x)) + d_Y(g(x), h(x)) \le \sup_x d_Y(f, g) + \sup_x d_Y(g, h).$$

Taking sup over x yields $d(f, h) \le d(f, g) + d(g, h)$. Thus d is a metric.

(Compactness of X ensures the supremum is finite for continuous f, g.)

Problem 5 b)

Let (f_n) be a Cauchy sequence in $(C(X, Y), d)$. Then for each fixed $x \in X$ the sequence $\big(f_n(x)\big)$ is Cauchy in Y because

$$d_Y(f_n(x), f_m(x)) \le d(f_n, f_m) \xrightarrow[m,n\to\infty]{} 0.$$

Since Y is complete, for each x there exists a limit $f(x) := \lim_{n\to\infty} f_n(x) \in Y$. This defines a pointwise limit map $f : X \to Y$.

Next, the Cauchy property in the sup-metric implies uniform convergence: for given $\varepsilon > 0$ choose N with $d(f_n, f_m) < \varepsilon$ for all $n, m \ge N$. Fix $n \ge N$ and let $m \to \infty$; then $d(f_n, f) = \sup_x d_Y(f_n(x), f(x)) \le \varepsilon$. So $f_n \to f$ uniformly.

Uniform limit of continuous functions into a metric space is continuous (no compactness needed here): hence $f \in C(X, Y)$. Finally $d(f_n, f) \to 0$ by the uniform convergence, showing every Cauchy sequence converges in $(C(X, Y), d)$. Thus the space is complete.