Functional Analysis
Homework 2

Deadline: September 23rd

1. **Problem 1** (2 points)

   Let X be a linear space and let $p : X \to [0, +\infty)$ such that:

   1. $p(x) = 0 \iff x = 0$;
   2. $p(\lambda x) = |\lambda| p(x)$ $\forall x \in X$ and $\lambda \in \mathbb{K}$.

   Show that p is a norm if and only if the set $B_X = \{x \in X : p(x) \leq 1\}$ is convex.

2. **Problem 2** (5 points)

   Given a vector space X over $\mathbb{R}$, we say that two norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if

   $$\exists C > 0 \; \forall x \in X : \; C^{-1} \|x\|' \leq \|x\| \leq C \|x\|' .$$

   a) Let X be a finite-dimensional vector space over $\mathbb{R}$. Show that all norms on X are equivalent;
   Hints: relate your norm to the ℓ_∞ norm.

3. **Problem 3** (5 points)

   Let V be a vector space over $\mathbb{R}$. A norm $\|\cdot\|$ is induced by a scalar product $\langle \cdot, \cdot \rangle$ (*i.e.*, there exists a scalar product $\langle \cdot, \cdot \rangle$ such that for every $x \in V$ $\|x\|^2 = \langle x, x \rangle$) if and only if the norm satisfies the parallelogram identity *i.e.*, for every $x, y \in V$

   $$\|x + y\|^2 + \|x - y\|^2 = 2 \|x\|^2 + 2 \|y\|^2 .$$

   Hint: If the norm satisfies the parallelogram identity, consider $\langle x, y \rangle = \frac{1}{4} \|x + y\|^2 - \frac{1}{4} \|x - y\|^2$. Prove that $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ first for $\lambda \in \mathbb{N}$ then in $\mathbb{Q}$ then in $\mathbb{R}$.

4. **Problem 4** (4 points)

   Prove the following statements.

   a) Let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on the vector space V. Show that the two norms are equivalent if and only if they induce the same topology;

   b) If V is a vector space on which we have two norms that generate the same topology, prove that either both of them are complete or none of them is complete;

c) Let $x, y \in \mathbb{R}$. Let $d_1(x,y) = |x-y|$ and $d_2(x,y) = |g(x)-g(y)|$ with $g(x) = \frac{x}{1+|x|}$ for $x \in \mathbb{R}$. Prove that d_1 and d_2 are metrics on $\mathbb{R}$ that generate the same topology but $(\mathbb{R}, d_1)$ is complete while $(\mathbb{R}, d_2)$ is not.

5. **Problem 5** (4 points)

a) Determine all the values $p \in [1, +\infty]$ such that the Banach space $\ell_p(\mathbb{N})$ is an Hilbert space (*i.e.*, the norm $\|\cdot\|_{\ell_p}$ is induced by some inner product $\langle \cdot, \cdot \rangle_p$);

b) Determine all the values $p \in [1, +\infty]$ such that the Banach space $L_p(\mathbb{R})$ is an Hilbert space(*i.e.*, the norm $\|\cdot\|_{L_p}$ is induced by some inner product $\langle \cdot, \cdot \rangle_p$).

Notice that $p = \infty$ is included in the argument.