Functional Analysis
Homework 8

Deadline: November 4th

1. **Problem 1: Clarkson's inequality** (6 points)

   Let $2 \leq p < \infty$ prove that for every $f, g \in L_p(\mu)$

   $$\left\|\frac{f+g}{2}\right\|_{L_p(\mu)}^p + \left\|\frac{f-g}{2}\right\|_{L_p(\mu)}^p \leq \frac{1}{2}\left(\|f\|_{L_p(\mu)}^p + \|g\|_{L_p(\mu)}^p\right).$$

   Conclude that $L_p(\mu)$ spaces are reflexive with $p \geq 2$ via uniform convexity.

   Hint: prove that $|(a+b)/2|^p + |(a-b)/2|^p \leq 1/2(|a|^p + |b|^p)$.

2. **Problem 2: L_p is reflexive for $1 < p \leq 2$** (8 points)

   We will now give a direct proof of the reflexivity of $L_p(\mu)$ with $1 < p \leq 2$. Take $1 < p < \infty$ and consider the operator $T : L_p \to (L_q)^\star$ with $q = p/(p-1)$, (the Hölder conjugate of p) such that given $f \in L_p$ and $g \in L_q$

   $$\langle Tf, h\rangle = \int fg \, \mathrm{d}\mu.$$

   1. Prove that T is an isometry *i.e.*, $\|Tf\|_{(L_q)^\star} = \|f\|_{L_p}$ for every $f \in L_p$ (*Hint: use Hölder's inequality and find then a function f that achieves equality*);
   2. Use item 1. to prove that $T(L_p)$ is a closed subspace of $(L_q)^\star$;
   3. Conclude that L_p with $1 < p \leq 2$ is reflexive using the reflexivity of L_s with $s \geq 2$.

3. **Problem 3** (6 points)

   Recall the definition of uniform convexity: let $(X, \|\cdot\|_X)$ be a Banach space and let $B_1 = \{x \in X : \|x\|_X \leq 1\}$, then X is said to be uniformly convex if

   $$\forall \epsilon \ \exists \delta \ \forall x, y \in B_1 : \|x-y\|_X > \epsilon \implies \left\|\frac{x+y}{2}\right\|_X < 1 - \delta.$$

   a) Prove that Hilbert spaces are necessarily uniformly convex;

   b) Is L_∞ uniformly convex? Prove it or provide a counter-example.