

## Functional Analysis - Homework 4

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### Problem 1.

*Proof.* Let  $X = [1, 2] \subset \mathbb{R}$ , a complete metric space.

Let  $\epsilon > 0$ ,  $m \in \mathbb{N}$ , and

$$A_m = \{x \in X : \forall n \geq m, |f(nx)| \leq \epsilon\}$$

Since  $f$  is continuous,  $A$  is closed.

$\forall x \in X$ , since  $\lim_{n \rightarrow \infty} f(nx) = 0$

$$\exists n > N, |f(nx)| \leq \epsilon$$

, meaning  $x$  should be contained in one of  $A_m$ s.

Thus,

$$X = \bigcup_m A_m$$

With Baire's category Theorem, at least one  $A_m$ s contains an open ball  $B(b_0, r)$ , where  $b_0 \in X$ ,  $r > 0$ . Therefore,

$$\forall n > m, b < r, |f(n(b_0 + b))| \leq \epsilon \quad (1)$$

$$\because n \frac{b_0 + r}{b_0 + b} > m \therefore |f(n(b_0 + r))| \leq \epsilon \quad (2)$$

For  $x \in (0, \infty)$ , with  $n > \frac{m}{x}(b_0 + r)$ , we have

$$|f(nx)| = \left| f\left(\frac{m}{x}(b_0 + r)x\right) \right| = \left| f\left(\frac{m}{x}(b_0 + r)x\right) \right| \leq \epsilon.$$

, which means  $\lim_{t \rightarrow \infty} f(t) = 0$ .

For  $x = 0$ , from  $\lim_{n \rightarrow \infty} f(n0) = 0$ , we know  $f(0) = 0$ .

In a whole, For  $x \in [0, \infty)$ ,  $\lim_{t \rightarrow \infty} f(t) = 0$ .

□

### Problem 2.

*Proof.* 1. Linearity of  $c_0(X)$ : Let  $x = (x_n), y = (y_n) \in c_0(X)$  and  $\lambda \in \mathbb{K}$ .

- Addition:  $x + y = (x_n + y_n)$ . By the triangle inequality on  $X$ :

$$0 \leq \lim_{n \rightarrow \infty} \|x_n + y_n\| \leq \lim_{n \rightarrow \infty} (\|x_n\| + \|y_n\|) = \lim_{n \rightarrow \infty} \|x_n\| + \lim_{n \rightarrow \infty} \|y_n\| = 0 \quad (3)$$

$$\therefore \lim_{n \rightarrow \infty} \|x_n + y_n\| = 0 \quad (4)$$

Thus,  $x + y \in c_0(X)$ .

- Scalar Multiplication:  $\lambda x = (\lambda x_n)$ . By the properties of a norm:

$$\lim_{n \rightarrow \infty} \|\lambda x_n\| = \lim_{n \rightarrow \infty} |\lambda| \|x_n\| = |\lambda| \lim_{n \rightarrow \infty} \|x_n\| = |\lambda| \cdot 0 = 0$$

Thus  $\lambda x \in c_0(X)$ .

Hence,  $c_0(X)$  is a linear space.

2.  $X$  is Banach  $\implies$  completeness of  $c_0(X)$  :

Assume  $X$  is a Banach space. Let  $(x^{(k)})_{k \in \mathbb{N}}$  be a Cauchy sequence converging to  $x$  in  $c_0(X)$  with respect to the sup-norm, where  $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots)$ .

$$\forall \epsilon > 0, \exists k > N \text{ s.t. } \|x^{(k)} - x\| < \epsilon,$$

$$\therefore \epsilon > \|x^{(k)} - x\| = \sup_{n \in \mathbb{N}} \|x_n^{(k)} - x_n\| \quad (5)$$

$$\therefore \|x_n^{(k)} - x_n\| < \epsilon \quad (6)$$

$$\therefore \|x_n\|_X \leq \|x_n - x_n^{(K)}\|_X + \|x_n^{(K)}\|_X < \epsilon + \|x_n^{(K)}\|_X \quad (7)$$

Since  $x^{(K)} \in c_0(X)$ ,  $\lim_{n \rightarrow \infty} \|x_n^{(K)}\|_X = 0$ . Therefore, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $\|x_n^{(K)}\|_X < \epsilon$ . Hence, for  $n \geq N$ :

$$\|x_n\|_X \leq \epsilon + \epsilon = 2\epsilon$$

Since  $\epsilon > 0$  was arbitrary, this proves  $\lim_{n \rightarrow \infty} \|x_n\|_X = 0$ , so  $x \in c_0(X)$ . Since every Cauchy sequence in  $c_0(X)$  converges to an element in  $c_0(X)$ , the space  $c_0(X)$  is Banach.  $\square$

### Problem 3.

*Proof.* (a)  $\implies$  (b)

We define a new space

$$Z = \{(x_n)_{n \in \mathbb{N}} : \|x_n\| \rightarrow 0\}$$

and norm  $\|(x_n)_{n \in \mathbb{N}}\| = \sup_n \|x_n\|$ .

According to the proof of Problem 2,  $Z$  is Banach.

We define an operator  $S_k : Z \rightarrow Y$

$$S_k(z) = T_k(x_k - x_{k+1}).$$

Therefore,

$$\|S_k(z)\| = \|T_k(x_k - x_{k+1})\| \quad (8)$$

$$\leq \|T_k\| \|x_k - x_{k+1}\| \quad (9)$$

$$\leq \|T_k\| (\|x_k\| + \|x_{k+1}\|) \quad (10)$$

$$\leq \|T_k\| \cdot 2 \sup_{n \in \mathbb{N}} \|x_n\| \quad (11)$$

Therefore,

$$\|S_k\| \leq 2 \|T_k\| < \infty$$

as  $T_k$  is continuous.

Since  $T_k(x_k) \rightarrow 0$  in norm,  $S_k$  should be the same.

$$\lim_{k \rightarrow \infty} \|S_k(z)\| = 0 \quad (12)$$

$$\therefore \sup_{k \in \mathbb{N}} \|S_k(z)\| < \infty \quad (13)$$

With the Uniform Boundedness Principle, we have

$$\sup_{k \in \mathbb{N}} \|S_k\| < \infty \quad (14)$$

For any  $t \in X$  with  $\|t\| \leq 1$ , define the sequence  $z' = (x_n)_{n \in \mathbb{N}}$ , where  $x_n = t$  if  $n = k$ , and  $x_n = 0$  else. Then,  $\sup_n \|x_n\| = \|t\| \leq 1$ . Applying the operator  $S_k$ ,

$$\|S_k(z')\| = \|T_k(x_k - x_{k+1})\| = \|T_k(x_k)\| \quad (15)$$

$$\therefore \|S_k\| \geq \sup_{\|t\| \leq 1} \|T_k(t)\| = \|T_k\| \quad (16)$$

Since  $\sup_{k \in \mathbb{N}} \|S_k\| < \infty$ ,

$$\|T_k\| < \infty$$

**Proof: (b)  $\implies$  (a)** Assume (b) holds, i.e.,  $M = \sup_{n \in \mathbb{N}} \|T_n\| < \infty$ . Assume  $\sum_{n=1}^{\infty} x_n$  is a norm convergent series. Let  $s$  be the sum. The sequence of partial sums  $s_N = \sum_{n=1}^N x_n$  converges to  $s$ . This implies that the terms of the series must converge to zero:  $\lim_{n \rightarrow \infty} x_n = 0$  in the norm of  $X$ .

$$\|T_n(x_n)\|_Y \leq \|T_n\| \cdot \|x_n\|_X$$

Since  $\sup_{n \in \mathbb{N}} \|T_n\| = M < \infty$ , we have:

$$0 \leq \|T_n(x_n)\|_Y \leq M \cdot \|x_n\|_X$$

Since  $\lim_{n \rightarrow \infty} \|x_n\|_X = 0$  and  $M$  is a finite constant, we have  $\lim_{n \rightarrow \infty} M \cdot \|x_n\|_X = 0$ . By the Squeeze Theorem,  $\lim_{n \rightarrow \infty} \|T_n(x_n)\|_Y = 0$ . Thus,  $T_n(x_n) \rightarrow 0$  in norm. □

**Problem 4.**

*Proof. (a)  $\implies$  (b):* The set of  $p$ -absolutely norm convergent sequences  $E = \ell_p(X)$  is a Banach space with the norm  $\|(x_n)\|_p = (\sum_{n=1}^{\infty} \|x_n\|^p)^{1/p}$ .

(a) states that for every sequence  $(x_n) \in E$ , the sum  $\sum_{n=1}^{\infty} x_n^*(x_n)$  converges. This allows us to define a linear functional  $T : E \rightarrow \mathbb{K}$  by:

$$T((x_n)) = \sum_{n=1}^{\infty} x_n^*(x_n)$$

Since  $T$  is a well-defined linear map on the entire Banach space  $E$ , the Uniform Boundedness Principle implies that  $T$  is bounded and in turn continuous. We have  $T \in E^*$ .

With Dual Space Isomorphism, the dual space of  $\ell_p(X)$  is isometrically isomorphic to  $\ell_q(X^*)$ :

$$\ell_p(X)^* \cong \ell_q(X^*)$$

$(x_n^*)$  corresponds to the functional  $T$ , and the norm of  $T$  in the dual space is precisely the  $\ell_q$  norm of the sequence  $(x_n^*)$  in the dual sequence space:

$$\|T\|_{E^*} = \left( \sum_{n=1}^{\infty} \|x_n^*\|^q \right)^{1/q}$$

Since  $T$  is continuous,  $\|T\|_{E^*} < \infty$ , which means  $\sum_{n=1}^{\infty} \|x_n^*\|^q < \infty$ . Thus, (b) holds.

**(b)  $\implies$  (a):** Assume (b) holds:  $(x_n^*) \in \ell_q(X^*)$ , so  $A = (\sum_{n=1}^{\infty} \|x_n^*\|^q)^{1/q} < \infty$ .

Assume the series  $\sum_{n=1}^{\infty} x_n$  is  $p$ -absolutely norm convergent:  $(x_n) \in \ell_p(X)$ , so

$$B = \left( \sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} < \infty$$

With the Holder's Inequality, we have:

$$\sum_{n=1}^{\infty} |x_n^*(x_n)| \leq \sum_{n=1}^{\infty} \|x_n^*\| \cdot \|x_n\| \tag{17}$$

$$\leq \left( \sum_{n=1}^{\infty} \|x_n^*\|^q \right)^{1/q} \left( \sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} = A \cdot B < \infty \tag{18}$$

**(c)  $\iff$  (d)**

Let  $X = \mathbb{K}$ . Then the dual space  $X^*$  is also  $\mathbb{K}$ .

A continuous linear functional  $x_n^* : \mathbb{K} \rightarrow \mathbb{K}$  is simply multiplication by a scalar  $x_n \in \mathbb{K}$ . The norm of this functional is  $\|x_n^*\| = |x_n|$ . The input vector  $x_n$  from statement (a) is now a scalar  $y_n$ .

Thus, statement (a) becomes: Given a series  $\sum_{n=1}^{\infty} y_n$  such that  $\sum_{n=1}^{\infty} |y_n|^p < \infty$  (i.e.,  $(y_n) \in \ell_p$ ) one has that the series  $\sum_{n=1}^{\infty} x_n y_n$  converges. This is precisely statement (d). So we have (d)  $\implies$  (a).

And statement (b) becomes: The series  $\sum_{n=1}^{\infty} x_n^*$  is  $q$ -absolutely norm convergent i.e.,  $\sum_{n=1}^{\infty} \|x_n^*\|^q < \infty$ . Since  $\|x_n^*\| = |x_n|$ , this simplifies to  $\sum_{n=1}^{\infty} |x_n|^q < \infty$ , which means  $(x_n) \in \ell_q$ . This is precisely statement (c). So we have (d)  $\implies$  (a).

Since (a) is equivalent to (b), (d) is equivalent to (c). □