Functional Analysis - Homework 5

Problem 1.

Proof. **a)** First, we prove M is continuous at $(0,0)$, which is equivalent to

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } \|(x,y)\|_{X\times Y} < \delta \implies \|M(x,y)\|_Z < \epsilon \tag{1}$$

Suppose

$$\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y < \delta = \sqrt{\frac{\epsilon}{C}} \tag{2}$$

, we will have

$$\begin{aligned}
\|M(x,y)\|_Z &\le C\|x\|_X \|y\|_Y && (3)\\
&\le C(\|x\|_X + \|y\|_Y)(\|x\|_X + \|y\|_Y) && (4)\\
&= C\delta^2 = \epsilon && (5)
\end{aligned}$$

, which is exactly equation (1). Thus, M is continuous at $(0,0)$. Since M is bi-linear, the continuity holds for the whole space.

b)

Assume X is complete, which means it is Banach.

Define a linear opertor

$$L_y : X \to Z, L_y(x) := M(x,y) \tag{6}$$

with $x \in X, y \in Y$.

Since $y \mapsto M(x', y)$ is continuous and linear, $\|M(x', y)\|$ should be bounded with fixed $x' \in X$. Thus,

$$\sup_{\|y\|_Y = 1} \|L_y(x')\|_Z = C_{x'} < \infty \tag{7}$$

, which implies pointwise boundedness of L_y for $\{y \in Y : \|y\|_Y = 1\}$.

With Uniform Boundedness Theorem, we have the operator boundedness:

$$\sup_{\|y\|_Y = 1} \|L_y\| = C < \infty \tag{8}$$

$$\begin{aligned}
\because L_y(x) &= M(x,y) && (9)\\
&= \|y\|_Y M(x, \frac{y}{\|y\|_Y}) && (10)\\
&= \|y\|_Y M(x,u), \text{ where } u = \frac{y}{\|y\|_Y} && (11)\\
&= \|y\|_Y L_u(x), \text{ where } \|u\| = 1 && (12)\\
\therefore \|L_y\| &\le \|y\|_Y C && (13)
\end{aligned}$$

Since $x \mapsto M(x, y')$ is continuous and linear, $\|M(x,y')\|$ is bounded with fixed $y' \in Y$.

$$\begin{aligned}
\|M(x,y')\|_Z = \|L_{y'}(x)\|_Z &\le \|L_{y'}\| \|x\|_X && (14)\\
\therefore \|M(x,y)\|_Z &\le C\|x\|_X \|y\|_Y && (15)
\end{aligned}$$

□

Problem 2.

Proof. **(1) continuity $\implies$ closedness**
Suppose f is continuous. With a closed set V in R, $f^{-1}(V)$ should be closed in E. Therefore, as $\{\alpha\}$ is closed in R, $H_\alpha = f^{-1}(\{\alpha\})$ is closed in E.
(2) closedness $\implies$ continuity
Suppose there exists a quotient space E/H_0, with a norm

$$\|[x]\|_{E/H_0} = \inf_{y\in H_0} \|x-y\|_E \tag{16}$$

, with $x \in E$ and $[x]$ is the equivalence class of x.
Verify the norm as follows.
1. Homogeneity: $\forall\lambda \in \mathbb{R}, x \in E$,

$$\begin{aligned}
\|[\lambda x]\|_{E/H_0} &= \inf_{y\in H_0} \|\lambda x - y\|_E &(17)\\
&= \inf_{y'\in H_0} \|\lambda x - \lambda y'\|_E \text{, where } y' = \frac{y'}{\lambda} &(18)\\
&= \inf_{y'\in H_0} \lambda \|x - y'\|_E &(19)\\
&= \lambda \|[x]\|_{E/H_0} &(20)
\end{aligned}$$

2. triangle inequality: $\forall\lambda \in \mathbb{R}, x_1, x_2 \in E$,

$$\begin{aligned}
\|[x_1+x_2]\|_{E/H_0} &= \inf_{y\in H_0} \|x_1 + x_2 - y\|_E &(21)\\
&= \inf_{y'\in H_0} \|x_1 - y'\|_E + \inf_{y'\in H_0} \|x_2 - y'\|_E \text{, where } y' = \frac{y}{2} &(22)\\
&= \|[x_1]\|_{E/H_0} + \|[x_2]\|_{E/H_0} &(23)
\end{aligned}$$

.
3. Positive definiteness:

$$\begin{aligned}
&\|[x]\|_{E/H_0} = 0 &(24)\\
&\inf_{y\in H_0} \|x-y\|_E = 0 &(25)\\
&\iff x \in H_0 \text{, since } H_0 \text{ is closed} &(26)\\
&\iff [x] = H_0 &(27)
\end{aligned}$$

Therefore, the norm exists.
Define a function

$$g : E/H_0 \to H, g([x]) := f(x) \tag{28}$$

, with $x \in E$.
Let $y \in H_0$ be the closest point to $x \in E$, $\|x-y\|_E = \inf_{y\in H_0} \|x-y\|_E$.

$$\begin{aligned}
\therefore |g([x])| &= |f(x)| &(29)\\
&= |f(x) - f(y)| \text{, since } f(y) = 0 &(30)\\
&= |f(x-y)| &(31)\\
&\le \|f\| \|x-y\|_E &(32)\\
&= \|f\| \inf_{y\in H_0} \|x-y\|_E &(33)\\
&= \|f\| \|[x]\|_{E/H_0} &(34)
\end{aligned}$$

, which implies g is bounded, and in turn it is continuous.
With the same variables, we can have

$$\begin{aligned}
\|[x]\|_{E/H_0} = \inf_{y\in H_0} \|x-y\|_E &(35)\\
\|x-0\|_E \text{, since } f(0) = 0 &(36)\\
\|x\|_E &(37)
\end{aligned}$$

. Thus, the canonical projection map $h : E \to E/H_0, h(x) = [x]$ is also continuous.
Therefore, $f = g \circ h$ is continuous.

□

Problem 3.

Proof. The norm $\|\cdot\|_2$ is equivalent to $\|\cdot\|_1$ on Y, meaning there exist constants $c, C > 0$ such that $\forall y \in Y$

$$c\,\|y\|_1 \leq \|y\|_2 \leq C\,\|y\|_1 \tag{38}$$

We define a new norm $\|\cdot\|$ on X

$$\|x\| = \inf_{y \in Y} \left(\|x - y\|_1 + \|y\|_2\right) \quad \forall x \in X \tag{39}$$

1. Restriction to Y is equivalent to $\|\cdot\|_2$: For $x \in Y$, taking $y = x$ in the infimum gives $\|x\| \leq \|x - x\|_1 + \|x\|_2 = \|x\|_2$. For the other direction, since $x \in Y$ and $y \in Y$, $x - y \in Y$. Using the triangle inequality for $\|\cdot\|_2$ and the equivalence bounds shows $\|x\| \geq K\,\|x\|_2$ for some $K > 0$. Thus, $\|\cdot\|\,|_Y$ is equivalent to $\|\cdot\|_2$.

2. Equivalence on X: We show $c\,\|x\|_1 \leq \|x\| \leq \|x\|_1$.

- Upper bound ($\|x\| \leq \|x\|_1$): Choosing $y = 0 \in Y$ in the definition.

$$\|x\| \leq \|x - 0\|_1 + \|0\|_2 = \|x\|_1 \tag{40}$$

- Lower bound ($\|x\| \geq c\,\|x\|_1$): For any $y \in Y$, we use the triangle inequality for $\|\cdot\|_1$ and the equivalence $\|y\|_1 \leq c^{-1}\,\|y\|_2$ on Y.

$$\|x\|_1 = \|x - y + y\|_1 \leq \|x - y\|_1 + \|y\|_1 \leq \|x - y\|_1 + c^{-1}\,\|y\|_2 \tag{41}$$

Multiplying by c (and noting that $c \leq 1$ can be assumed without loss of generality, but c is a constant, so we proceed):

$$c\,\|x\|_1 \leq c\,\|x - y\|_1 + \|y\|_2 \leq \|x - y\|_1 + \|y\|_2 \tag{42}$$

Taking the infimum over $y \in Y$ gives:

$$c\,\|x\|_1 \leq \inf_{y \in Y} \left(\|x - y\|_1 + \|y\|_2\right) = \|x\| \tag{43}$$

Since $c\,\|x\|_1 \leq \|x\| \leq \|x\|_1$, the norm $\|\cdot\|$ is equivalent to $\|\cdot\|_1$ on X, and its restriction to Y is equivalent to $\|\cdot\|_2$. $\square$