Problem 1

Part 1: p is a norm $\implies$ B_X is convex.

Assume p is a norm. This implies that the triangle inequality holds: $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$. To prove that B_X is convex, we must show that for any $x, y \in B_X$ and any $\alpha \in [0,1]$, the vector $\alpha x + (1-\alpha)y$ is also in B_X.

By the definition of B_X, if $x, y \in B_X$, then $p(x) \leq 1$ and $p(y) \leq 1$. Using the properties of the function p:

$$\begin{aligned} p(\alpha x + (1-\alpha)y) &\leq p(\alpha x) + p((1-\alpha)y) \\ &= \|\alpha\| p(x) + \|1-\alpha\| p(y) \\ &= \alpha p(x) + (1-\alpha) p(y) \end{aligned}$$

Since $p(x) \leq 1$ and $p(y) \leq 1$, we have:

$$\alpha p(x) + (1-\alpha)p(y) \leq \alpha \cdot 1 + (1-\alpha) \cdot 1 = \alpha + 1 - \alpha = 1$$

Therefore, $p(\alpha x + (1-\alpha)y) \leq 1$, which means $\alpha x + (1-\alpha)y \in B_X$. This proves that B_X is convex.

Part 2: B_X is convex $\implies$ p is a norm.

To prove that p is a norm, we only need to show that the triangle inequality holds, as the other properties are given. That is, we must prove $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

Let $x, y \in X$ be non-zero vectors. If $p(x) = 0$ or $p(y) = 0$, the inequality holds trivially. Consider the vectors $u = \frac{x}{p(x)}$ and $v = \frac{y}{p(y)}$. Using property 2 of p: $p(u) = p(\frac{1}{p(x)}x) = \|\frac{1}{p(x)}\| p(x) = \frac{p(x)}{p(x)} = 1$. Similarly, $p(v) = p(\frac{1}{p(y)}y) = \frac{p(y)}{p(y)} = 1$. Thus, $u \in B_X$ and $v \in B_X$.

Since B_X is convex, any convex combination of u and v is in B_X. Let $\alpha = \frac{p(x)}{p(x)+p(y)}$. Clearly, $\alpha \in [0,1]$. Consider the convex combination $\alpha u + (1-\alpha)v$:

$$\alpha u + (1-\alpha)v = \frac{p(x)}{p(x)+p(y)} \frac{x}{p(x)} + \frac{p(y)}{p(x)+p(y)} \frac{y}{p(y)} = \frac{x+y}{p(x)+p(y)}$$

Since this vector is a convex combination of u and v, it must be in B_X. By the definition of B_X, this implies:

$$p\left(\frac{x+y}{p(x)+p(y)}\right) \leq 1$$

Using property 2 of p again:

$$\|\frac{1}{p(x)+p(y)}\| p(x+y) \leq 1$$

Since $p(x) + p(y) > 0$, we can simplify:

$$\frac{p(x+y)}{p(x)+p(y)} \leq 1 \implies p(x+y) \leq p(x) + p(y)$$

This is the triangle inequality, which proves that p is a norm.

Problem 2

Given a finite-dimensional vector space X over $\mathbb{R}$, we must show that all norms on X are equivalent.

Let X be an n-dimensional vector space over $\mathbb{R}$. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for X. Any vector $x \in X$ can be uniquely written as $x = \sum_{i=1}^n \xi_i e_i$ for some scalars $\xi_i \in \mathbb{R}$. We will relate any given norm on X to the l_∞ norm defined as $\|x\|_\infty = \max_{1\le i\le n}\|\xi_i\|$.

First, let's show that any norm $\|\cdot\|$ is equivalent to the l_∞ norm. We need to find constants $C_1, C_2 > 0$ such that for any $x \in X$, $C_1\|x\|_\infty \le \|x\| \le C_2\|x\|_\infty$.

Part 1: Proving $\|x\| \le C_2\|x\|_\infty$.

Let $x = \sum_{i=1}^n \xi_i e_i$. By the triangle inequality and the homogeneity of the norm:

$$\begin{aligned}\|x\| = \|\sum_{i=1}^n \xi_i e_i\| &\le \sum_{i=1}^n \|\xi_i e_i\| \\ &= \sum_{i=1}^n \|\xi_i\| \cdot \|e_i\|\end{aligned}$$

Let $M = \max_{1\le i\le n}\|e_i\|$. Then we have:

$$\|x\| \le \sum_{i=1}^n \|\xi_i\| M \le \sum_{i=1}^n \left(\max_{1\le j\le n}\|\xi_j\|\right) M = \left(\max_{1\le j\le n}\|\xi_j\|\right)\sum_{i=1}^n M = \|x\|_\infty \cdot (nM)$$

Let $C_2 = nM$. This gives us $\|x\| \le C_2\|x\|_\infty$.

Part 2: Proving $C_1\|x\|_\infty \le \|x\|$.

Let $S = \{x \in X : \|x\|_\infty = 1\}$. This set is a unit sphere with respect to the l_∞ norm. The set S is closed and bounded in $\mathbb{R}^n$, which makes it compact. Consider the function $f(x) = \|x\|$. We proved in Part 1 that f is a continuous function from $(X, \|\cdot\|_\infty)$ to $\mathbb{R}$. Since S is compact and f is continuous, f must attain a minimum value on S. Let $C_1 = \min_{x\in S}\|x\|$. Since $x \in S$, $x \ne 0$, so $\|x\| > 0$. This means $C_1 > 0$. Now, consider any non-zero vector $x \in X$. Let $y = \frac{x}{\|x\|_\infty}$. Then $\|y\|_\infty = 1$, so $y \in S$. By the definition of C_1, we have $\|y\| \ge C_1$.

$$\|\frac{x}{\|x\|_\infty}\| \ge C_1 \implies \frac{\|x\|}{\|x\|_\infty} \ge C_1 \implies \|x\| \ge C_1\|x\|_\infty$$

This gives us $C_1\|x\|_\infty \le \|x\|$.

Combining the two parts, we have $C_1\|x\|_\infty \le \|x\| \le C_2\|x\|_\infty$. This shows that any norm $\|\cdot\|$ is equivalent to the l_∞ norm. The equivalence of norms is an equivalence relation. Since every norm on X is equivalent to the l_∞ norm, they are all equivalent to each other.

Problem 3

Part 1: A scalar product exists $\implies$ the parallelogram identity holds.

Assume a norm is induced by a scalar product, so there exists a scalar product $\langle\cdot,\cdot\rangle$ such that $\|x\|^2 = \langle x, x\rangle$ for all $x \in V$. Using the properties of the scalar product (linearity and conjugate symmetry):

$$\begin{aligned}&\|x+y\|^2 + \|x-y\|^2 \\ &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle\end{aligned}$$

$$
\begin{aligned}
&= (\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle) + (\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle) \\
&= (\langle x,x\rangle + \langle y,y\rangle + 2\Re\langle x,y\rangle) + (\langle x,x\rangle + \langle y,y\rangle - 2\Re\langle x,y\rangle) \\
&= 2\langle x,x\rangle + 2\langle y,y\rangle \\
&= 2\|x\|^2 + 2\|y\|^2
\end{aligned}
$$

The provided solution's calculation is a bit off in the third line, but the result is correct. This proves the parallelogram identity.

Part 2: The parallelogram identity holds $\implies$ a scalar product exists.

Assume the parallelogram identity holds. We must construct a scalar product $\langle\cdot,\cdot\rangle$ that induces the norm. We define the inner product using the polarization identity:

$$\langle x,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

We need to verify that this definition satisfies the properties of a scalar product:

1. **Symmetry:** $\langle x,y\rangle = \langle y,x\rangle$

$$
\begin{aligned}
\langle y,x\rangle &= \frac{1}{4}(\|y+x\|^2 - \|y-x\|^2) \\
&= \frac{1}{4}(\|x+y\|^2 - \|-(x-y)\|^2) \\
&= \frac{1}{4}(\|x+y\|^2 - \|-1\|^2\|x-y\|^2) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \langle x,y\rangle
\end{aligned}
$$

2. **Linearity in the first argument:** $\langle \lambda x,y\rangle = \lambda\langle x,y\rangle$ and $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$.

   - **Additivity:** $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$.

     First, we derive the identity $\langle a+b,c\rangle + \langle a-b,c\rangle = 2\langle a,c\rangle$ from the parallelogram identity. By the definition of the inner product and the parallelogram identity:

$$
\begin{aligned}
&4(\langle a+b,c\rangle + \langle a-b,c\rangle) \\
&= (\|a+b+c\|^2 - \|a+b-c\|^2) + (\|a-b+c\|^2 - \|a-b-c\|^2) \\
&= (\|(a+c)+b\|^2 + \|(a+c)-b\|^2) - (\|(a-c)+b\|^2 + \|(a-c)-b\|^2)
\end{aligned}
$$

     (by the parallelogram identity)

$$
\begin{aligned}
&= (2\|a+c\|^2 + 2\|b\|^2) - (2\|a-c\|^2 + 2\|b\|^2) \\
&= 2\|a+c\|^2 - 2\|a-c\|^2 \\
&= 2(4\langle a,c\rangle) = 8\langle a,c\rangle
\end{aligned}
$$

     Dividing by 4, we get the identity: $\langle a+b,c\rangle + \langle a-b,c\rangle = 2\langle a,c\rangle$.

     Now, for any $x,y,z \in V$, let $a = \frac{x+y}{2}$ and $b = \frac{x-y}{2}$. Then $a+b = x$ and $a-b = y$. Using the identity with $a = \frac{x+y}{2}$, $b = \frac{x-y}{2}$ and $c = z$:

$$\langle x,z\rangle + \langle y,z\rangle = \langle a+b,c\rangle + \langle a-b,c\rangle = 2\left\langle \frac{x+y}{2}, z\right\rangle$$

     From the identity, with $b = a$, we get $\langle 2a,c\rangle + \langle 0,c\rangle = 2\langle a,c\rangle$, so $\langle 2a,c\rangle = 2\langle a,c\rangle$. Setting $a = \frac{x+y}{2}$ we have $\langle x+y,z\rangle = 2\langle \frac{x+y}{2}, z\rangle$. Thus, we have proven $\langle x,z\rangle + \langle y,z\rangle = \langle x+y,z\rangle$.

   - **Homogeneity:** $\langle \lambda x,y\rangle = \lambda\langle x,y\rangle$.

- **Case 1: $\lambda \in \mathbb{N}$ (integers).** We prove $\langle nx, y\rangle = n\langle x, y\rangle$ by induction. Base case $n = 1$ is trivial. Assume the identity holds for $n-1$. Let $a = (n-1)x, b = x, c = y$. Using the identity $\langle a + b, c\rangle + \langle a - b, c\rangle = 2\langle a, c\rangle$:

$$\langle (n-1)x + x, y\rangle + \langle (n-1)x - x, y\rangle = 2\langle (n-1)x, y\rangle$$

$$\langle nx, y\rangle + \langle (n-2)x, y\rangle = 2\langle (n-1)x, y\rangle$$

By the induction hypothesis, $\langle (n-2)x, y\rangle = (n-2)\langle x, y\rangle$ and $\langle (n-1)x, y\rangle = (n-1)\langle x, y\rangle$.

$$\langle nx, y\rangle + (n-2)\langle x, y\rangle = 2(n-1)\langle x, y\rangle$$

$$\langle nx, y\rangle = (2n - 2 - n + 2)\langle x, y\rangle = n\langle x, y\rangle$$

- **Case 2: $\lambda \in \mathbb{Q}$ (rational numbers).** Let $\lambda = p/q$ where $p \in \mathbb{Z}, q \in \mathbb{N}, q \neq 0$. From Case 1, we know $\langle q \cdot \frac{p}{q}x, y\rangle = \langle px, y\rangle = p\langle x, y\rangle$. Also, $\langle q \cdot \frac{p}{q}x, y\rangle = q\langle \frac{p}{q}x, y\rangle$.

$$\therefore q\left\langle \frac{p}{q}x, y\right\rangle = p\langle x, y\rangle \implies \left\langle \frac{p}{q}x, y\right\rangle = \frac{p}{q}\langle x, y\rangle$$

- **Case 3: $\lambda \in \mathbb{R}$ (real numbers).** The function $f(\lambda) = \langle \lambda x, y\rangle = \frac{1}{4}(\|\lambda x + y\|^2 - \|\lambda x - y\|^2)$ is continuous because the norm $\|\cdot\|$ is continuous. Since the identity holds for all rational λ and the function is continuous, it must hold for all real λ as well.

3. **Positive-definiteness:** $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$.

$$\langle x, x\rangle = \frac{1}{4}(\|x + x\|^2 - \|x - x\|^2) = \frac{1}{4}(\|2x\|^2 - \|0\|^2)$$

Since $\|\cdot\|$ is a norm, $\|2x\|^2 = (2\|x\|)^2 = 4\|x\|^2$ and $\|0\| = 0$.

$$\langle x, x\rangle = \frac{1}{4}(4\|x\|^2) = \|x\|^2$$

Since $\|x\|^2 \geq 0$ and $\|x\|^2 = 0 \iff x = 0$, the positive-definiteness property holds.

Problem 4

Part a) Two norms are equivalent if and only if they induce the same topology.

Part 1: Equivalent norms $\implies$ same topology.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on a vector space V. This means there exist positive constants C_1, C_2 such that

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1, \forall x \in V$$

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the topologies induced by $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. A set $U \subseteq V$ is open in $\mathcal{T}_1$ if for every $x \in U$, there exists an open ball $B_1(x, r) = \{y \in V : \|y - x\|_1 < r\}$ such that $B_1(x, r) \subseteq U$.

We need to show that U is also open in $\mathcal{T}_2$. Given $B_1(x, r)$, we need to find an open ball $B_2(x, s)$ such that $B_2(x, s) \subseteq B_1(x, r)$. We have $\|y - x\|_2 \leq C_2\|y - x\|_1$. From the equivalence relation, we have $\|y - x\|_1 \leq \frac{1}{C_1}\|y - x\|_2$. If we choose a radius s such that $s < C_1 r$, then for any $y \in B_2(x, s)$, we have $\|y - x\|_2 < s$.

Therefore, $\|y - x\|_1 < \frac{s}{C_1} < \frac{C_1 r}{C_1} = r$.

This means $B_2(x, s) \subseteq B_1(x, r)$. The same logic applies in the reverse direction by using the other inequality. Hence, the two topologies are the same.

Part 2: Same topology $\implies$ equivalent norms.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology. This means that the identity map from $(V, \|\cdot\|_1)$ to $(V, \|\cdot\|_2)$ is a homeomorphism. A homeomorphism is a continuous bijection with a continuous inverse.

Continuity of the identity map implies that for any $x \in V$, the inverse image of any open ball in $\mathcal{T}_2$ is open in $\mathcal{T}_1$. This is equivalent to showing that for every $r > 0$, the open ball $B_2(0, r)$ contains an open ball $B_1(0, r')$ for some $r' > 0$. The set $B_2(0, 1)$ is open in $\mathcal{T}_2$, so it is open in $\mathcal{T}_1$. Since $0 \in B_2(0, 1)$, there must be an $r > 0$ such that $B_1(0, r) \subseteq B_2(0, 1)$. This implies that for any x with $\|x\|_1 < r$, we have $\|x\|_2 < 1$.

Now, for any non-zero $x \in V$, the vector $x' = \frac{x}{\|x\|_1} \cdot \frac{r}{2}$ has $\|x'\|_1 = \frac{r}{2} < r$. Thus, $\|x'\|_2 < 1$.

$$\|\frac{r}{2\|x\|_1} x\|_2 < 1 \implies \frac{r}{2\|x\|_1}\|x\|_2 < 1 \implies \|x\|_2 < \frac{2}{r}\|x\|_1$$

This gives one side of the equivalence. The other side is proven symmetrically.

Part b) If two norms on a vector space generate the same topology, prove that either both are complete or neither is complete.

Let $(V, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ be two normed spaces with equivalent norms. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(V, \|\cdot\|_1)$. This means for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n > N$, $\|x_m - x_n\|_1 < \epsilon$.

Since the norms are equivalent, there exists a constant $C > 0$ such that $\|x\|_2 \leq C\|x\|_1$. Then for $m, n > N$, we have $\|x_m - x_n\|_2 \leq C\|x_m - x_n\|_1 < C\epsilon$.

Since $C\epsilon$ can be made arbitrarily small, this shows that (x_n) is also a Cauchy sequence in $(V, \|\cdot\|_2)$. A sequence is Cauchy in one norm if and only if it is Cauchy in the other. Therefore, a sequence converges in one norm if and only if it converges in the other. This means that if every Cauchy sequence in $(V, \|\cdot\|_1)$ converges to a limit in V, then every Cauchy sequence in $(V, \|\cdot\|_2)$ also converges to a limit in V. Hence, if one space is complete, the other must be as well.

Part c) A complete space with an equivalent norm that is not complete.

Let $d_1(x, y) = \|x - y\|$ and $d_2(x, y) = \|g(x) - g(y)\|$ with $g(x) = \frac{x}{1+\|x\|}$.

1. d_1 **and** d_2 **are metrics:** d_1 is the standard metric on $\mathbb{R}$. To show d_2 is a metric, we verify the three properties:
    - $d_2(x, y) \geq 0$: Trivial, since $\|g(x) - g(y)\| \geq 0$.
    - $d_2(x, y) = 0 \iff x = y$: $d_2(x, y) = 0 \iff g(x) = g(y)$. $g(x)$ is a strictly increasing function, so it is injective. Thus $g(x) = g(y) \iff x = y$.
    - Triangle inequality: $d_2(x, z) \leq d_2(x, y) + d_2(y, z)$. For $x, y, z \in \mathbb{R}$,

      $$\|g(x) - g(z)\| \leq \|g(x) - g(y)\| + \|g(y) - g(z)\|$$

      because the standard metric on $(-1, 1)$ is induced by the standard metric on $\mathbb{R}$. The function g preserves distances in a way.
2. d_1 **and** d_2 **generate the same topology:** We can show that the identity mapping from $(\mathbb{R}, d_1)$ to $(\mathbb{R}, d_2)$ is a homeomorphism.
    - Continuity: The inverse image of a d_2-open ball is d_1-open. An open ball $B_2(x, \epsilon)$ is an open interval in $(-1, 1)$. Its inverse image under g is an open interval in $\mathbb{R}$, which is a d_1-open set.
    - Inverse continuity: The inverse mapping is also continuous for the same reason.
3. $(\mathbb{R}, d_1)$ **is complete:** Every Cauchy sequence of real numbers converges to a real number. So $(\mathbb{R}, d_1)$ is a complete metric space.

4. $(\mathbb{R}, d_2)$ **is not complete:** Consider the sequence $x_n = n$ in $(\mathbb{R}, d_2)$. $d_2(x_n, x_m) = \|g(n) - g(m)\| = \|\frac{n}{1+n} - \frac{m}{1+m}\|$. As $n, m \to \infty$, both $\frac{n}{1+n} \to 1$ and $\frac{m}{1+m} \to 1$. Thus, $d_2(x_n, x_m) \to 0$. This means (x_n) is a Cauchy sequence in $(\mathbb{R}, d_2)$. However, the sequence $x_n = n$ does not converge to any limit in $\mathbb{R}$. Therefore, $(\mathbb{R}, d_2)$ is not complete.

Problem 5

Part a) For which $p \in [1, +\infty]$ is $l_p(\mathbb{N})$ a Hilbert space?

A Banach space is a Hilbert space if and only if its norm is induced by an inner product. By Problem 3, this is true if and only if the norm satisfies the parallelogram identity. The norm on $l_p(\mathbb{N})$ for $x = (\xi_n)_{n\in\mathbb{N}} \in l_p$ is given by:

$$\|x\|_p = \left(\sum_{n=1}^{\infty} \|\xi_n\|^p\right)^{1/p} \qquad (1 \le p < \infty)$$

And for $p = \infty$:

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} \|\xi_n\|$$

We need to test the parallelogram identity:

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 2\|x\|_p^2 + 2\|y\|_p^2$$

Let's consider a simple case. Let $x = (1, 0, 0, \ldots)$ and $y = (0, 1, 0, \ldots)$. Then: $x + y = (1, 1, 0, \ldots)$ $x - y = (1, -1, 0, \ldots)$

The left-hand side (LHS) of the identity:

$$\begin{gathered}
\|x+y\|_p^2 + \|x-y\|_p^2 \\
= \left((\|1\|^p + \|1\|^p)^{1/p}\right)^2 + \left((\|1\|^p + \|-1\|^p)^{1/p}\right)^2 \\
= ((1+1)^{1/p})^2 + ((1+1)^{1/p})^2 = (2^{1/p})^2 + (2^{1/p})^2 \\
= 2^{2/p} + 2^{2/p} = 2 \cdot 2^{2/p} = 2^{1+2/p}
\end{gathered}$$

The right-hand side (RHS) of the identity:

$$2\|x\|_p^2 + 2\|y\|_p^2 = 2((1^p)^{1/p})^2 + 2((1^p)^{1/p})^2$$

$$= 2(1)^2 + 2(1)^2 = 4$$

For the identity to hold, we need $2^{1+2/p} = 4$. $2^{1+2/p} = 2^2 \implies 1 + \frac{2}{p} = 2 \implies \frac{2}{p} = 1 \implies p = 2$.

If $p \neq 2$, the identity does not hold, so the norm is not induced by an inner product. The only value of p for which $l_p(\mathbb{N})$ is a Hilbert space is $p = 2$. This space is $l_2(\mathbb{N})$.

The case $p = \infty$ can also be tested with the same vectors: $\|x\|_\infty = 1$, $\|y\|_\infty = 1$, $\|x + y\|_\infty = \sup(1, 1) = 1$, $\|x - y\|_\infty = \sup(1, 1) = 1$.

LHS: $\|x + y\|_\infty^2 + \|x - y\|_\infty^2 = 1^2 + 1^2 = 2$. RHS: $2\|x\|_\infty^2 + 2\|y\|_\infty^2 = 2(1)^2 + 2(1)^2 = 4$. $2 \neq 4$, so $l_\infty(\mathbb{N})$ is not a Hilbert space.

Part b) For which $p \in [1, +\infty]$ is $L_p(\mathbb{R})$ a Hilbert space?

This is similar to part a). The space $L_p(\mathbb{R})$ consists of measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\|f\|_p = (\int_{-\infty}^{\infty} \|f(x)\|^p dx)^{1/p} < \infty$.

For two functions $f, g \in L_p(\mathbb{R})$, let $f(x) = 1$ for $x \in [0, 1]$ and $f(x) = 0$ otherwise, and $g(x) = 1$ for $x \in [2, 3]$ and $g(x) = 0$ otherwise. Then:

$$\|f\|_p^p = \int_0^1 1^p dx = 1 \implies \|f\|_p = 1$$

$$\|g\|_p^p = \int_2^3 1^p dx = 1 \implies \|g\|_p = 1$$

$f(x) + g(x) = 1$ on $[0, 1] \cup [2, 3]$ and 0 otherwise.

$$\|f + g\|_p^p = \int_0^1 1^p dx + \int_2^3 1^p dx = 1 + 1 = 2 \implies \|f + g\|_p = 2^{1/p}$$

$f(x) - g(x) = 1$ on $[0, 1]$ and -1 on $[2, 3]$.

$$\|f - g\|_p^p = \int_0^1 \|1\|^p dx + \int_2^3 \|-1\|^p dx = 1 + 1 = 2 \implies \|f - g\|_p = 2^{1/p}$$

Now check the parallelogram identity:

LHS:

$$|f + g|_p^2 + |f - g|_p^2 = (2^{1/p})^2 + (2^{1/p})^2 = 2^{2/p} + 2^{2/p} = 2 \cdot 2^{2/p} = 2^{1+2/p}$$

RHS:

$$2|f|_p^2 + 2|g|_p^2 = 2(1)^2 + 2(1)^2 = 4$$

Again, the identity holds only if $2^{1+2/p} = 4$, which implies $p = 2$. For $p = \infty$, let $f(x) = 1$ for $x \in [0, 1]$ and $g(x) = 1$ for $x \in [2, 3]$.

$|f|_\infty = 1$, $|g|_\infty = 1$, $|f + g|_\infty = 1$, $|f - g|_\infty = 1$.

LHS: $1^2 + 1^2 = 2$.

RHS: $2(1)^2 + 2(1)^2 = 4$.

The identity does not hold. Therefore, the only value of p for which $L_p(\mathbb{R})$ is a Hilbert space is $p = 2$. This space is $L_2(\mathbb{R})$.