Set: A well defined collection of objects

well defined meaning

Examples: $A = \{1, 2, 5, 6\}$, $B = \{a, b, c, d, \ldots, z\}$

$C = \{x \in \mathbb{R} : x \leq 2\}$, $\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{C}$

Empty Set: $\phi = \{\ \}$ eg. $A = \{x \in \mathbb{R} : x < 2 \text{ and } x > 2\}$

Subset: $A \subseteq B$ if $\forall\, x \in A \Rightarrow x \in B$.

Proper Subset: $A \subset B$ means $A \subset B$ and $A \neq B$

Improper Subset: $A \subseteq A$ means A is improper subset of A

Power Set: Set of all subsets of a set

ie. $P(A) = \{S : S \subseteq A\}$

Total no. of subsets of a set with n elements is 2^n.

$\therefore$ No. of elements in a power set is 2^n.

Operations on Sets:

(i) $A \cap B = \{x : x \in A \text{ and } x \in B\}$

(ii) $A \cup B = \{x : x \in A \text{ or } x \in B\}$

(iii) $A \setminus B = A - B = \{x : x \in A \text{ and } x \notin B\}$

$B \setminus A = B - A = \{x : x \in B \text{ and } x \notin A\}$

So, simply $A \setminus B \neq B \setminus A$.

Disjoint Sets: If $A \cap B = \phi$, then we say that A and B are disjoint.

Partition of a Set: A collection of sets $\{S_1, S_2, \ldots, S_n\}$ is said to be the partition of a set A if

$$S_1 \cup S_2 \cup \ldots \cup S_n = A$$

and $S_i \cap S_j = \phi \quad \forall\ i, j$

Cardinality of a Set: The no. of elements in a set is called its cardinality and is denoted by $|A|$.

Eg. When a set A has n-elements we say that $|A| = n$.

Then the set A is said to be n-set.

Cartesian Product of two Sets

$$A \times B = \{(x, y) : x \in A, y \in B\}$$

eg. $A = \{1, 2, 3\}$, $B = \{4, 6\}$

$$A \times B = \{(1,4), (1,6), (2,4), (2,6), (3,4), (3,6)\}$$

Here $|A \times B| = 3 \times 2 = 6$.

$$B \times A = \{(4,1), (4,2), (4,3), (6,1), (6,2), (6,3)\}$$

$|B \times A| = 6$

$\therefore$ we observe that $|A \times B| = |B \times A|$ but $A \times B$ may not be equal to $B \times A$.

Pigeonhole Principle: If p pigeons fly into H pigeonholes, then at least one pigeonhole contains at least $\lceil \frac{p}{h} \rceil$ pigeons, where $\lceil \cdot \rceil$ is greatest integer function.

Define first

Generalized Pigeonhole Principle: If 'n' pigeonholes are occupied by $kn+1$ or more pigeons then at least one pigeonhole is occupied by $k+1$ or more pigeons.

Q: Find the minimum no. of teachers in a college to be sure that four of them are born in the same month.

Soln: $n = 12$, $k+1 = 4 \Rightarrow k = 3$

$kn+1 = 12 \times 3 + 1 = 37$.

So, required number is 37.

Q Among 100 people at least how many will celebrate their birthday in the same month.

Soln: $p = 100$ (No. of people)

$h = 12$ (No. of months)

By using pigeonhole principle at least $\lceil \frac{p}{h} \rceil = \lceil \frac{100}{12} \rceil = 8+1 = 9$ people will celebrate their birthday in the same month.

Ex: How many cards to be selected to guarantee that at least 3 of them are of same suit.

Solⁿ: $p = ?$ (Total no. of cards)

$n = 4$ (no. of cards in same suit)

So, by PH- Principle

$\left\lceil \frac{p}{n} \right\rceil \geq 3$

$\Rightarrow \left\lceil \frac{p}{4} \right\rceil \geq 3 \Rightarrow p \geq 9.$

Ex What is the minimum number of students required in a class to be sure that at least 6 students will receive the same grade, if there are 5 grades (ABCDE)?

Solⁿ $p = ?$ (Total no. of students)

$n = 5$ (Total no. of grades)

So, by PH- Principle

$\left\lceil \frac{p}{n} \right\rceil \geq 6$

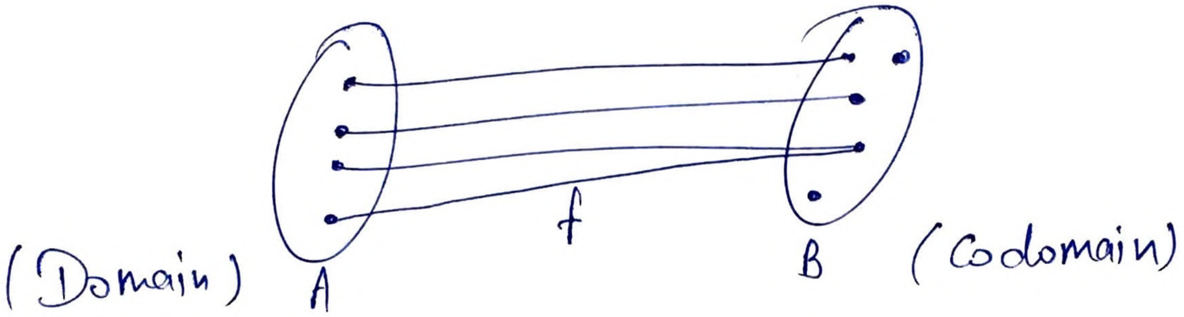
$\Rightarrow \left\lceil \frac{p}{5} \right\rceil \geq 6 \Rightarrow p \geq 26$

So, minimum no. of students is 26.

Function: A function on a non-empty set A is a rule that assigns each element of A ~~to~~ a unique element of the set B.

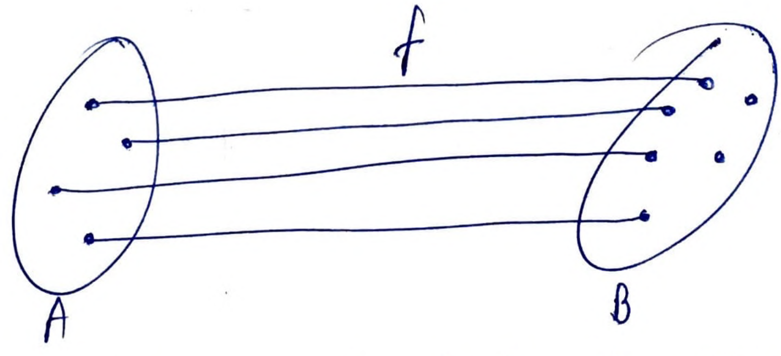
Notation $f : A \rightarrow B$

$\downarrow$ Domain $\quad \downarrow$ Codomain



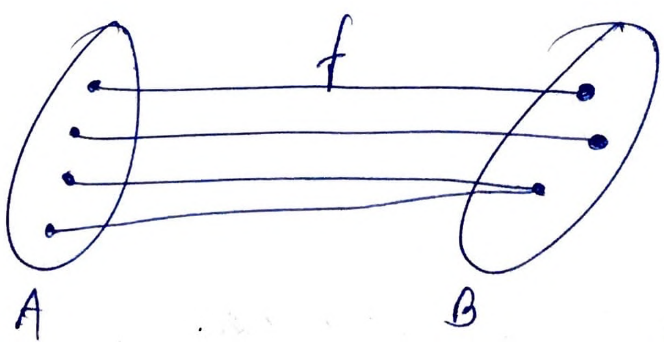
One-One function: if $x \neq y \Rightarrow f(x) \neq f(y) \;\forall\, x, y \in A$
(Injective function)

i.e. no two elements of A have the same image in B.



Otherwise, it is said to be a many-one function.

Onto function: if for every $y \in B$, $\exists\; x \in A$ s.t. $f(x) = y$.
(Surjective function)



Otherwise into.

Example 1. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x+3$

(i) f is a function since for every $x \in \mathbb{R}$, we have one $2x+3 \in \mathbb{R}$.

(ii) f is one-one, since if we take $x \neq y$ in $\mathbb{R}$, then $f(x) = 2x+3$ and $f(y) = 2y+3$ are never equal.

(iii) f is onto: since if we take any $y \in \mathbb{R}$ we have one $x = \frac{y-3}{2} \in \mathbb{R}$

such that $f(x) = 2\left(\frac{y-3}{2}\right) + 3$

$= y \in \mathbb{R}$.

Ex 2. $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$

(i) Here, f is a function since for every $x \in \mathbb{R}$, we are getting some number in $\mathbb{R}$ only.

(ii) f is not one-one: Since if we take two numbers 3 and -3 in $\mathbb{R}$ then by the function we get $f(3) = 3$ and $f(-3) = 3$

i.e. $f(-3) = f(3)$ but $-3 \neq 3$.

(iii) f is not onto: Since -ve numbers are not the Image of any $x \in \mathbb{R}$.

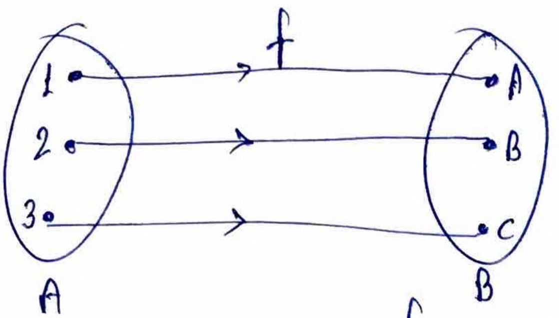
Bijective function: which is both one-one and onto

Sets with Same Cardinality :

Two sets A and B will have the same cardinality iff $\exists$ a 1-1, onto function b/w them.

i.e. $\exists\ f : A \xrightarrow[\text{onto}]{1-1} B$

eg. 1



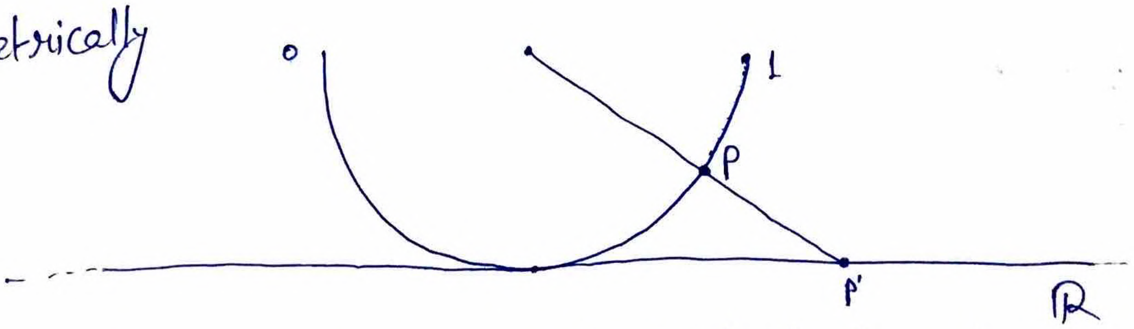
f is one-one and onto from A to B so $|A| = |B|$.

eg. 2 $(0,1)$ and $\mathbb{R}$ have the same cardinality.
In other words, we say that $(0,1)$ and $\mathbb{R}$ are equivalent.

$$f(x) = \begin{cases} \dfrac{2x-1}{x}, & 0 < x < \dfrac{1}{2} \\ \dfrac{2x-1}{1-x}, & \dfrac{1}{2} \le x < 1 \end{cases}$$

Clearly $f(x)$ defines a one-one onto function from $(0,1)$ to $\mathbb{R}$. Hence $(0,1) \sim \mathbb{R}$.

Geometrically



Bend $(0,1)$ into a semicircle and rest it tangentially on the real line $\mathbb{R}$. Then a line joining the centre of semicircle with point P on $(0,1)$ corresponds a point on $\mathbb{R}$. This establishes a one-one correspondence from $(0,1)$ to $\mathbb{R}$.

Sequence (Real Sequence) : A real sequence is a bijective function from $\mathbb{N}$ to $\mathbb{R}$.

Generally denoted by $x_n : \mathbb{N} \xrightarrow[\text{onto}]{1-1} \mathbb{R}$

$\{x_n\}_{n=1}^{\infty}$ or $(x_1, x_2, x_3, \ldots)$.

Constant Sequence : $x_n : \mathbb{N} \longrightarrow \mathbb{R}$ is s.t.b. a constant sequence if $x_n = c$ for all $n = 1, 2, 3, \ldots$

i.e $\{x_n\}_{n=1}^{\infty} = (c, c, c, \ldots)$

eg. $(1, 1, 1, \ldots)$

Finite Sequence : A finite sequence is a bijective function from a set $I_n = \{1, 2, \ldots, n\}$ to any set S.

eg. $\{x_n\}_{n=1}^{5} = (x_1, x_2, x_3, x_4, x_5)$

If a finite sequence contains n-elements, then it is called n-sequence.

Illustration : $(1, 2, 3, 3, 4, 5)$

and $(1, 2, 3, 4, 3, 5)$

both are not the same sequence.

So, in a sequence order of elements and repetition of elements both does matter.

Characteristic function: Let U be a universal set and A be its subset then the characteristic function is $\chi_A : U \to [0,1\}$ defined as

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Characteristic Sequence: Let U be an indexed n-set so that $U = \{x_1, x_2, ----, x_n\}$ and A be a subset of U then the characteristic sequence is defined as the function

$$\chi_A(i) = \begin{cases} 1 & \text{if } x_i \in A \\ 0 & \text{if } x_i \notin A. \end{cases}$$

Example 1.: $U = [1, 2, 3, 4, a, b, c\}$

$A = \{1, 2, 3, 4]$

$B = [a, b, c\}$

So, the characteristic sequences are defined as

$\chi_A = \{1, 1, 1, 1, 0, 0, 0\}$

$\chi_B = \{0, 0, 0, 0, 1, 1, 1]$

$\chi_U = [1, 1, 1, 1, 1, 1, 1\}.$

Example 2. $U = [1, 3, 5, 7, 9, 11, 13, 15, 17, 19]$

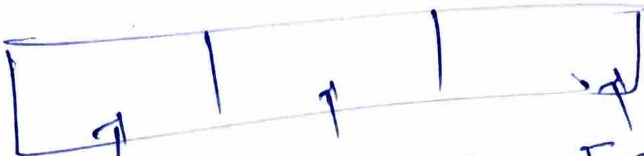
$A = [3, 5, 7, 11, 13, 17, 19\}$

$B = [3, 9, 15\}$

$\therefore$ $\chi_A = (0, 1, 1, 1, 0, 1, 1, 0, 1, 1)$

$\chi_B = (0, 1, 0, 0, 1, 0, 0, 1, 0, 0)$

Counting Principle : Suppose we need to construct a three character code in which first place should be filled by an alphabet, second place should be filled by an odd no. and third place by an even no. then the total no. of such possible codes is obtained as



Total 26 Alphabets   {1,3,5,7,9}   {0,2,4,6,8}

So, total no. of such codes will be

$26 \times 5 \times 5 = 650$.

Q How many three digit numbers are there in which the digits are not repeated.

Sol^n:

for this space we are left with only 8 digits

{1,2,....,9}
So total 9 possible digits

one digit has been used now the left digits are 9 in number

So, total no. of such numbers will be

$9 \times 9 \times 8 = 648$.

k - sequence on n-set : It is a function from the set $\{1,2,3,\dots,k\}$ into the set $X=\{1,2,3,\dots,n\}$.

Ex: $k\text{-set} = \{1,2,3,4,5\}$

$n\text{-set} = \{5,8,10,12,15\}$

$S = (5,5,8,8,10)$ is an example of k-sequence on n-sets.

Total number of k-sequences on n-set

$= n \times n \times \dots \times n$ (k times)

$= n^k$.

k - permutation on n-set : It is a k-sequence without repetition.

Ex: $k\text{-set} = \{1,2,3,4,5\}$

$n\text{-set} = \{5,8,10,12,15\}$

$S = \{10,12,8,5,15\}$ is an example of k-permut. on n-set.

Total no. of k-permutations on n-set

$= n \times (n-1) \times (n-2) \times \dots (n-(k-1))$

k- permutations on n-set : Arrangement of k objects from n-set with order is called a k-permutation on an n-set. (No repetition is allowed)

Total no. of k-permutations on n-set is

$$^nP_k = \frac{n!}{(n-k)!}$$

k- Combinations on n-set : Arrangement of k-objects from n-set without any order is called as a k-combination on n-set (No repetition is allowed)

Total no. of k-combinations on n-set is

$$^nC_k = \frac{n!}{(n-k)!}$$

n- factorial : no. of k-permutations on an n-set is s.t.b. n-factorial when $k=n$.

$$\therefore \quad n! = \begin{cases} n(n-1)(n-2)\ldots(3)(2)(1) & , \; n>0 \\ 1 & , \; n=0. \end{cases}$$

$n! = n(n-1)!$

$^nP_k = {}^nC_k \times k!$

Q There is a group of 6 boys and 4 girls

(a) In how many ways they can sit on a bench?

Sol^n $n=10, \quad k=10$

$$\therefore \; {}^nP_k = {}^{10}P_{10} = \frac{10!}{(10-10)!} = \frac{10!}{0!} = \frac{10!}{1}$$

(b) In how many ways they can sit on a bench if boys sit together and girls sit together?

Sol^n:

$$\underbrace{\times \; \times \; \times \; \times \; \times \; \times}_{\text{Boys}} \qquad \underbrace{\times \; \times \; \times \; \times}_{\text{Girls}}$$

6 boys can be arranged in 6! ways

4 girls can be arranged in 4! ways

Also boys and girls together can be interchanged

So, total no. of ways

$= 2 \times 6! \times 4!$

$= 2 \times 720 \times 24$

$= 34560.$

Q How many permutations of the letters A, B, C, D, E contain the string "ABC"?

Sol^n Fix ABC togethether then we are left with ABC, D, E i.e. 3 choices.

So, the required no. of ways will be 3!.

Q How many permutations of the letters A, B, C, D, E, F, G, H contain the following types:

(i) the string BA and FGH.

(ii) the string CAB and BED.

(iii) the string BCA and ABF.

Sol^n: (i) fixing BA and FGH we are left with the choices BA, FGH, C, D, E

i.e. total 5 choices.

∴ Reqd no. of ways will be $5! = 120$.

(ii) Similarly, here the total choices are CAB, BED, F, G, H. But here B is fixed priorly by A and posteriorly by E so we have to consider CABED as only one choice.

So reqd no. of ways will be $4! = 24$

(iii) There will be no choice since BCA and ABF can not happen at the same time. So reqd no. of ways $= 0$

Q How many bit strings of length 7 contains exactly 3 1's ?

Sol^n

x x x x x x x

We have 7 spaces to be filled by 3 ones

Now, we know that all 1's are the same so order of these 1's will not matter.

So, reqd no. of strings will be

$$= {}^7C_3 = \frac{7!}{3!(7-3)!} = \frac{7!}{3!\,4!}$$

$$= \frac{7\times 6\times 5\times 4!}{3\times 2\times 1\times 4!} = 35.$$

Q How many bit strings of length 10 contain

(i) exactly 3 0's.

(ii) exactly 4 1's.

(iii) AtLeast 2 1's

Sol^n (i) $${}^{10}C_3 = \frac{10!}{3!\,7!} = \frac{10\times 9\times 8\times 7!}{3\times 2\times 7!}$$

$$= 120$$

(ii) $${}^{10}C_4$$

(iii) $${}^{10}C_{10} - ({}^{10}C_0 + {}^{10}C_1)$$

Bad Banana Theorem : Suppose $0 < k \le n$

then $^{n-1}C_k + {}^{n-1}C_{k-1} = {}^nC_k$

Proof: $^{n-1}C_k + {}^{n-1}C_{k-1}$

$$= \frac{(n-1)!}{k!\,(n-1-k)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!}$$

$$= \left[\frac{(n-1)!}{k!\,(n-k-1)!} \times \frac{(n-k)}{(n-k)}\right] + \left[\frac{(n-1)!}{(n-k)!\,(k-1)!} \times \frac{k}{k}\right]$$

$$= \frac{(n-k)(n-1)!}{k!\,(n-k)!} + \frac{k\,(n-1)!}{(n-k)!\,k!}$$

$$= \frac{(n-k+k)\,(n-1)!}{k!\,(n-k)!}$$

$$= \frac{n\,(n-1)!}{k!\,(n-k)!}$$

$$= \frac{n!}{k!\,(n-k)!} = {}^nC_k$$

Bad Banana Theorem : Illustration

Suppose you want to by 6 bananas. There are 20 bananas at the store out of which 19 are good and 1 is bad.

Then any selection of 6 can either avoid the bad one or include the bad one. So this can be seen as the selection is done out of 19 as all 6 may be out of those good 19 or 5 may be out of those good 19.

So, we can write the selections in either way

$$^{20}C_6$$

or

$$^{19}C_6 + {}^{19}C_5$$

Also, we can confirm this by calculating their values.

$$^{20}C_6 = \frac{20!}{6!\,(20-6)!} = \frac{20!}{6!\,14!} = 38760$$

$$^{19}C_6 + {}^{19}C_5 = \frac{19!}{6!\,13!} + \frac{19!}{5!\,14!} = 38760.$$

This confirms the bad banana theorem