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Applications of Differential and Difference Equations.

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1)

i) ~~Given~~ Given:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

The Characteristic Eqn is:

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

On solving we get:

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

On solving ~~tis~~ this equation we get:

$\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = 1$

We know

$(A - \lambda I)\bar{X} = 0$

Where $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ represents the ~~eigen~~ eigenvector for the eigenvalue λ

For $\lambda = 5$:

$$\begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$\Rightarrow \cancel{3}-3x_1 + 2x_2 + 1x_3 = 0$

$\Rightarrow 1x_1 - 2x_2 + 1x_3 = 0$

$\Rightarrow 1x_1 + 2x_2 - 3x_3 = 0$

Using Cramers Rule we get

$$\frac{x_1}{\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & 2 \\ 1 & -2 \end{vmatrix}} \cancel{= k}$$

$$\Rightarrow \frac{x_1}{4} = \frac{-x_2}{-4} = \frac{x_3}{4}$$

$$\therefore \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\therefore \bar{x} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$= 4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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For $\lambda = 1$ we get :

$$\begin{bmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 + x_3 = 0 \; :$$

Putting $x_1 = 0$ we get :

$$\Rightarrow 2x_2 + x_3 = 0$$

$$\Rightarrow \frac{x_2}{x_3} = -\frac{1}{2}$$

$$\Rightarrow \frac{x_2}{-1} = \frac{x_3}{2}$$

$$\therefore \bar{X} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

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Putting $x_2 = 0$ we ~~gt~~ get :

$x_1 + x_3 = 0$

$\Rightarrow x_1 = -x_3$

$\Rightarrow \frac{x_1}{1} = \frac{x_3}{-1}$

$\therefore \bar{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\therefore$ The eigen vectors are :

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Since three independent eigenvectors exist, it is diagonalizable.

Let $\bar{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \bar{x}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \bar{x}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

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Then the modal matrix $P = [\bar{x}_1, \bar{x}_2\ \bar{x}_3]$ is :

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

Here, $|P| = 1(1-0) + 1(2+1)$

$= 4$

$\therefore$ It is non singular

To find $P^{-1}AP$:

$$P^{-1} = \frac{1}{|P|} \times \operatorname{adj}(P)$$

$$= \frac{1}{4} \times \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 3 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{-1}{2} & \frac{1}{4} \\ \frac{3}{4} & \frac{-1}{2} & \frac{-1}{4} \end{bmatrix}$$

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$$\therefore P^{-1}AP = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{-1}{2} & \frac{1}{4} \\ \frac{3}{4} & \frac{-1}{2} & \frac{-1}{4} \end{bmatrix} \times \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

On solving we get:

$$P^{-1}AP = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= D$

$\therefore$ A is a diagonalizable

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ii) Here :

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The characteristic eq^n is :

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & -2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

On solving we get :

$$-\lambda^3 - 3\lambda + 2 = 0$$

On solving we get :

$$\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 1$$

We know :

$$(A - \lambda I)\bar{X} = 0$$

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Where $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the eigenvector

for the eigenvalue λ

For $\lambda = -2$ we get :

$$\Rightarrow \begin{bmatrix} 1+2 & 1 & -2 \\ 1 & 0+2 & 1 \\ 0 & 1 & -1+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~On solving we get :~~

$\Rightarrow 3x_1 + 1x_2 - 2x_3 = 0$

$\Rightarrow 1x_1 + 2x_2 + 1x_3 = 0$

$\Rightarrow 0x_1 + 1x_2 + 1x_3 = 0$

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Using Cramer's Rule we get :

$$\frac{x_1}{\begin{vmatrix}1 & -2\\ 2 & 1\end{vmatrix}} = \frac{-x_2}{\begin{vmatrix}3 & -2\\ 1 & 1\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}3 & 1\\ 1 & 2\end{vmatrix}}$$

$$\rightarrow \frac{x_1}{5} = \frac{-x_2}{5} = \frac{x_3}{5}$$

$$\rightarrow \frac{x_1}{5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$$\therefore \bar{X} = \begin{bmatrix}5\\-5\\5\end{bmatrix} = 5\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

For $\lambda = 1$ we get :

$$\begin{bmatrix}1-1 & 1 & -2\\ 1 & 0-1 & 1\\ 0 & 1 & -1-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

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$$\Rightarrow \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 0x_1 + 1x_2 - 2x_3 = 0$

$\Rightarrow 1x_1 - x_2 + x_3 = 0$

Using Cramers Rule we get :

$$\frac{x_1}{\begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{-x_2}{2} = \frac{x_3}{-1}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{-2} = \frac{x_3}{-1}$$

$$\therefore \bar{X} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

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$\therefore$ The eigenvectors are :

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Since three independent eigenvectors do not exist, ~~they~~ the matrix A is not diagonalizable.

iii) Given :

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The Characteristic Eqn is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

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On solving we get:

$$\lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$$

On solving we get:

$$\lambda_1 = 6, \quad \lambda_2 = 3, \quad \lambda_3 = 1$$

We know $(A - \lambda I)\bar{X} = 0$

Where $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the eigenvector for the eigenvalue λ.

For $\lambda = 6$ we get:

$$\begin{bmatrix} 2-6 & 2 & 0 \\ 2 & 5-6 & 0 \\ 0 & 0 & 3-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -4x_1 + 2x_2 + 0x_3 = 0$$

$$\Rightarrow 2x_1 - x_2 + 0x_3 = 0$$

$$\Rightarrow 0x_1 + 0x_2 - 3x_3 = 0$$

Using Cramers Rule we get :

$$\cancel{\frac{x_1}{\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix}}} \quad \frac{x_1}{\begin{vmatrix} -1 & 0 \\ 0 & -3 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{3} = -\frac{x_2}{-6} = \frac{x_3}{0}$$

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{6} = \frac{x_3}{0}$$

$$\therefore x = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

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For $\lambda = 3$ we get,

$$\begin{bmatrix} 2-3 & 2 & 0 \\ 2 & 5-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x_1 + 2x_2 + 0x_3 = 0$ — (1)

$\Rightarrow 2x_1 + 2x_2 + 0x_3 = 0$ — (2)

Using Cramers Rule we get:

$$\frac{x_1}{\begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-6}$$

$$\therefore X = \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix} = -6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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For $\lambda = 1$ we get:

$$\begin{bmatrix} 2-1 & 2 & 0 \\ 2 & 5-1 & 0 \\ 0 & 0 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 + x_3 = 0$

$\Rightarrow 2x_1 + 4x_2 + 0x_3 = 0$

$\Rightarrow 0x_1 + 0x_2 + 2x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 4 \\ 0 & 0 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{8} = \frac{-x_2}{4} = \frac{x_3}{0}$$

$$\Rightarrow \frac{x_1}{8} = \frac{x_2}{-4} = \frac{x_3}{0}$$

$$\therefore \bar{x} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$\therefore$ The eigenvectors are :

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Since A is symmetric and it has 3 independent eigenvectors it is orthogonal diagonalizable

~~Here P~~ Let $\bar{X}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\bar{X}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

$$\bar{X}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

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$$\therefore \bar{P} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Since A is symmetric

$\bar{X}_1$ and $\bar{X}_2$ are orthogonal

$\Rightarrow \bar{X}_1^T \bar{X}_2 = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 + 0 + 0$$

$$= 0$$

$\bar{X}_{\cancel{3}1}$ and $\bar{X}_3$ are orthogonal

$\Rightarrow \bar{X}_1^T \cdot \bar{X}_3 = 0$ ~~are orthogonal~~

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 - 2 + 0$$

$$= 0$$

$\bar{X}_2$ and $\bar{X}_3$ are orthogonal

$\bar{X}_2 \cdot \bar{X}_3 = 0$

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$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 0 + 0 + 0$

$= 0$

$\Rightarrow N = \left[\frac{\bar{X}_1}{\|\bar{X}_1\|}, \frac{\bar{X}_2}{\|\bar{X}_2\|}, \frac{\bar{X}_3}{\|\bar{X}_3\|} \right]$

$\Rightarrow N = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{-1}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}$

To evaluate:

$N^T A N = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \end{bmatrix}$

$\times \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{-1}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}$

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On solving we get

$$N^T A N = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

$\therefore$ A is orthogonal diagonalizable.

iv) Given :

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

The ~~that~~ Characteristic Eq^n is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

On solving we get :

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

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On solving we get :

$\lambda_1 = 4$, $\lambda_2 = 1$, $\lambda_3 = 1$

We know $(A - \lambda I)\bar{X} = 0$

Where $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector for the eigenvalue λ.

For $\lambda = 4$ we get :

$$\begin{bmatrix} 2-4 & 1 & 1 \\ 1 & 2-4 & 1 \\ 1 & 1 & 2-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -2x_1 + x_2 + x_3 = 0$ — (1)

$\Rightarrow x_1 - 2x_2 + x_3 = 0$ — (2)

$\Rightarrow x_1 + x_2 - 2x_3 = 0$ — (3)

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Using Cramers Rule we get :

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{3} = \frac{-x_2}{-3} = \frac{x_2}{3}$$

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\therefore \bar{x} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 1$ we get :

$$\begin{bmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & 1 \\ 1 & 1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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If $x_1 + x_2 + x_3 = 0$

If $x_2 = 0$

$\Rightarrow x_1 + x_3 = 0$

$\Rightarrow x_1 = - x_3$

$\Rightarrow \frac{x_1}{1} = \frac{x_3}{-1}$

$\therefore \bar{X} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

If $x_3 = 0$

$x_1 + x_2 = 0$

$\Rightarrow x_1 = - x_2$

$\Rightarrow \frac{x_1}{1} = \frac{x_2}{-1}$

$\therefore \bar{X} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

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$\therefore$ The Eigenvectors are:

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}$$

Since A is symmetric and it has 3 independent ~~or~~ eigenvectors, it is orthogonal diagonalizable.

Let $X_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, X_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix}, X_3 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$

$$\therefore P = \begin{bmatrix}\bar{X_1} & X_2 & \bar{X_3}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1 & 1\\1 & 0 & -1\\1 & -1 & 0\end{bmatrix}$$

$$X_1^T X_2 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix} = 1+0-1$$
$$= 0$$

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$\therefore x_1$ and x_2 are orthogonal.

$$x_1^T x_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 - 1 + 0$$
$$= 0$$

$$x_2^T x_3 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 + 0 + 0$$
$$= 1 \neq 0$$

Let $x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

$x_1^T x_3 = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$\Rightarrow l + m + n = 0$ — (1)

$x_2^T x_3 = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

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$\Rightarrow l - n = 0$

$\Rightarrow l = n$ — ②

Putting ② in ① we get:

$n + m + n = 0$

$\Rightarrow m = -2n$

$\therefore \bar{X}_3 = \begin{bmatrix} n \\ -2n \\ n \end{bmatrix}$

$= \cancel{n \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}$ $n \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\cancel{\therefore \bar{X}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}$ $\therefore \bar{X}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\Rightarrow N = \begin{bmatrix} \frac{\bar{X}_1}{\|\bar{X}_1\|} & \frac{\bar{X}_2}{\|\bar{X}_2\|} & \frac{\bar{X}_3}{\|\bar{X}_3\|} \end{bmatrix}$

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$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

To evaluate:

$$N^T A N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \times$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

On solving we get:

$$N^T A N = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

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$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\therefore$ Matrix A is orthogonal diagonalizable.

2) To verify Cayley Hamilton's Theorem for the matrix $A = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}$

and to find A^{-1} if it exists

The characteristic Eq^n for the matrix is:

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & -4 & 4 \\ 12 & -11-\lambda & 12 \\ 4 & -4 & 5-\lambda \end{vmatrix} = 0$$

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On solving we get:

$$\lambda^3+\lambda^2-5\lambda+3=0$$

~~On so~~
According to Cayley Hamilton Theorem

$$A^3+A^2-5A+3I=0 \quad \text{—} \; 1$$

$$A^3 = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}$$

$$\times \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 28 \\ 84 & -83 & 84 \\ 28 & -28 & 29 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 8 & -8 \\ -24 & 25 & -24 \\ -8 & 8 & -7 \end{bmatrix}$$

$$A^3 + A^2 - 5A + 3I$$

$$= \begin{bmatrix} 29 & -28 & 28 \\ 84 & -83 & 84 \\ 28 & -28 & 29 \end{bmatrix} + \begin{bmatrix} -7 & 8 & -8 \\ -24 & 25 & -24 \\ -8 & 8 & -7 \end{bmatrix}$$

$$-5 \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On solving we get :

$$A^3 + A^2 - 5A + 3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence cayley hamilton theorem is verified for the matrix A.

From ① we have :

$$A^3 + A^2 - 5A + 3I = 0$$

Multiplying both sides with A^{-1} we get :

$$A^2 + A - 5I + 3A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{3}\left[5I - A - A^2\right]$$

$$= \frac{1}{3}\left[5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix} - \begin{bmatrix} -7 & 8 & -8 \\ -24 & 25 & -24 \\ -8 & 8 & -7 \end{bmatrix}\right]$$

$$= \frac{1}{3}\begin{bmatrix} 7 & -4 & 4 \\ 12 & -9 & 12 \\ 4 & -4 & 7 \end{bmatrix}$$

3) Given :

A is an $n \times n$ diagonalizable matrix, $A^2 = A$

To prove : the eigen value of A is 0 or 1

Let $\bar{X}$ be the eigenvector of A,

Then, $A\bar{X} = AA\lambda\bar{X}$ — (1) where λ is an eigenvalue.

$$\Rightarrow A^2\bar{X} = A\lambda\bar{X}$$

$$= \lambda A\bar{X}$$

From (1) $A\bar{X} = \lambda\bar{X}$

$$\Rightarrow A^2\bar{X} = \lambda^2\bar{X}$$

Since $A^2 = A$

$$\Rightarrow A\bar{X} = \lambda^2\bar{X}$$

From (1) $A\bar{X} = \lambda\bar{X}$

$$\Rightarrow \lambda\bar{X} = \lambda^2\bar{X}$$

$$\Rightarrow \lambda\bar{X} - \lambda^2\bar{X} = 0$$

$$\Rightarrow \lambda(1-\lambda)\bar{X} = 0$$

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From here

$\lambda = 0$ or $\lambda = 1$

4) ~~Given~~ Thus the eigenvalues of A are either 0 or 1

~~A is so~~

4) Given :

A is similar to B

To prove :

i) A^{-1} is similar to B^{-1}

ii) A^m is similar to B^m for any positive integer m

iii) $|A| = |B|$

i) Since A is similar to B there exists a non-singular matrix P such that

$$AP = PB$$

For similar matrices, the eigenvalues are the same :

$\therefore$ A and B have the eigenvalues λ_1, λ_2 - - -

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~~If~~ We know that if A has the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ - -

then A^{-1} has the eigenvalues $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$ —

In the same way since ~~A~~ B has the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ - -

B^{-1} has the eigenvalues $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$ —

Since A and B both have the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ - -

A^{-1} and B^{-1} will both have the eigenvalue $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ - —

Since the Eigenvalues of A^{-1} and B^{-1} are the same, they too are similar

ii) Since A and B are similar they have the same eigenvalues

Let the eigenvalues of A be λ_1, λ_2 - -

then the Eigenvalues of B will also be λ_1, λ_2 - -

We know that if the Eigen values of A is

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$\lambda_1, \lambda_2, \lambda_3$ -

then the Eigenvalues of A^m will be $\lambda_1^m, \lambda_2^m, \lambda_3^m$ ---

In a similar way ~~of the eigenvalues~~ since the eigenvalues of B are $\lambda_1, \lambda_2, \lambda_3$ --- the eigen values of B^m will be $\lambda_1^m, \lambda_2^m, \lambda_3^m$ --

since the Eigenvalues of A^m and B^m are the same they are similar.

Note : m is a positive integer -

iii) Since, $AP = PB$

$|AP| = |PB|$

$\Rightarrow |A| \times \cancel{|P|} = \cancel{|P|} \times |B|$

$\therefore |A| = |B|$

5) Given:

$|A| \neq 0$

To prove :

$A^T A^{-1} = (A^T A^{-1})^T$ only if $A^2 = (A^T)^2$

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$\Rightarrow A^T A^{-1} = (A^T A^{-1})^T$

$\Rightarrow A^T A^{-1} = (A^{-1})^T \cdot (A^T)^T$

$\Rightarrow A^T A^{-1} = (A^{-1})^T A$

$\Rightarrow (A^T A^{-1}) A = (A^{-1})^T A \cdot A$

$\Rightarrow A^T = (A^{-1})^T \cdot A^2$

$\Rightarrow A^T = (A^T)^{-1} \cdot A^2 \qquad \left((A^{-1})^T = (A^T)^{-1}\right)$

$\Rightarrow A^T (A^T) = A^T \left((A^T)^{-1} \cdot A^2\right)$

$\Rightarrow (A^T)^2 = A^2$

$\therefore A^T A^{-1}$ is symmetric only when

$(A^T)^2 = A^2$

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6) Given:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$$

To check:

Whether A and B are similar and to find a non-singular matrix P such that $P^{-1}AP = B$.

Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Since $P^{-1}AP = B$

$\Rightarrow AP = PB$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+2c & b+2d \\ 2a+c & 2b+d \end{bmatrix} = \begin{bmatrix} b & 3a+2b \\ d & 3c+2d \end{bmatrix}$$

$\Rightarrow a+2c = b$ — (1)

$\Rightarrow b+2d = 3a+2b$ — (2)

$\Rightarrow 2a+c = d$ — (3)

$\Rightarrow 2b+d = 3c+2d$ — (4)

Putting $a=1$ and ~~B=1~~ $b=1$ in ① we get:

$1+2c=1$
$\Rightarrow 2c=0$
$\therefore c=0$

Putting the values of a and b in ② we get:

$1+2d=3+2$
$\Rightarrow 2d=4$
$\therefore d=2$

Since the values of a, b, c, d satisfy ③, ④ it is a valid solution for the system of equations.

$\therefore$ The matrix P is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

$|P| = 2-0 = 2$

Since $|P| \neq 0$ the matrices A and B are similar.

7

a) Given:

$$x_1^2 - 6x_2^2 + 24x_1x_2 = 5$$

On converting it to matrix form we get:

$$A = \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix}$$

The characteristic Eq^n is:

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 12 \\ 12 & -6-\lambda \end{vmatrix} = 0$$

On solving we get: $\lambda^2 + 5\lambda - 150 = 0$

$$\Rightarrow \cancel{\lambda^2 + \lambda +} (\lambda+15)(\lambda-10) = 0$$

$$\therefore \lambda = -15, 10$$

We know $(A - \lambda I)\bar{X} = 0$

Where $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the eigenvector for the eigenvalue λ.

For $\lambda = -15$ we get:

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$$\begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From here,

$16x_1 + 12x_2 = 0$ — (1)
$12x_1 + 9x_2 = 0$ — (2)

From (1):

$16x_1 = -12x_2$

$\Rightarrow \frac{x_1}{-12} = \frac{x_2}{16}$

$\therefore \bar{x} = \begin{bmatrix} -12 \\ 16 \end{bmatrix}$

$= 4 \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

For $\lambda = 10$ we get:

$$\begin{bmatrix} -9 & 12 \\ 12 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From here:

$-9x_1 + 12x_2 = 0$ — (1)
$12x_1 - 16x_2 = 0$ — (2)

From (1):

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$9x_1 = 12x_2$

$\Rightarrow \frac{x_1}{12} = \frac{x_2}{9}$

$\therefore \bar{x} = \begin{bmatrix} 12 \\ 9 \end{bmatrix}$

$= 3 \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$\therefore$ The eigenvectors are $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $\cancel{\frac{4}{3}} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Let $\bar{X}_1 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $\bar{X}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Since the matrix is symmetric and two independent eigenvectors exist, it is orthogonal diagonalizable

$\Rightarrow \bar{N} = \begin{bmatrix} \frac{\bar{X}_1}{||\bar{X}_1||} & \frac{\bar{X}_2}{||\bar{X}_2||} \end{bmatrix}$

$= \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

To evaluate :

$N^T A N$

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$\Rightarrow N^TAN = \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \times \begin{bmatrix} 1 & 12 \\ 12 & -6 \end{bmatrix} \times$

$$\begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

On solving we get:

$$N^TAN = \begin{bmatrix} -15 & 0 \\ 0 & 10 \end{bmatrix} = D$$

On converting this to canonical form we get:

$$-15y_1^2 + 10y_2^2 = 5$$

$$\Rightarrow -3y_1^2 + 2y_2^2 = 1$$

$\therefore$ The conic section is a hyperbola.

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b) Given :

$x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3 = 1$

let $\bar{X}^T A \bar{X}$ be the matrix form of the quadratic form where:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \quad \bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic Eq^n of A is :

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

On solving we get :

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

On solving we get.

$$\lambda = 1, 2, 4$$

We know $(A - \lambda I)\bar{X} = 0$ Where $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is

the eigenvector for the eigen value λ

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For $\lambda = 1$ we get:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From here:

$$0x_1 + 2x_2 - x_3 = 0$$

$$0x_1 - x_2 + 2x_3 = 0$$

Using Cramers Rule we get:

$$\frac{x_1}{\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}} = \frac{-x_1}{\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{3} = \frac{-x_1}{0} = \frac{x_3}{0}$$

$$\therefore \bar{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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For $\lambda = 2$ we get :

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From here we get :

$$-x_1 + 0x_2 + 0x_3 = 0 \quad - (1)$$

$$0x_1 + 1x_2 - x_3 = 0 \quad - (2)$$

$$0x_1 - 1x_2 + x_3 = 0 \quad - (3)$$

Using cramers rule we get :

$$\frac{x_1}{\begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{0} = \frac{-x_2}{1} = \frac{x_3}{-1}$$

$$\Rightarrow \bar{x} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$= -1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

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For $\lambda = 4$ we get :

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

On solving we get :

$$-3X_1 + 0X_2 + 0X_3 = 0 \quad \text{— (1)}$$

$$0X_1 - X_2 - X_3 = 0 \quad \text{— (2)}$$

$$0X_1 - X_2 + X_3 = 0 \quad \text{— (3)}$$

Using Cramers Rule we get :

$$\frac{X_1}{\begin{vmatrix} 0 & 0 \\ -1 & -1 \end{vmatrix}} = \frac{-X_2}{\begin{vmatrix} -3 & 0 \\ 0 & -1 \end{vmatrix}} = \frac{X_3}{\begin{vmatrix} -3 & 0 \\ 0 & -1 \end{vmatrix}}$$

$$\Rightarrow \frac{X_1}{0} = \frac{-X_2}{3} = \frac{X_3}{3}$$

$$\therefore X = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

∴ The Eigen vectors are :

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Since the matrix ~~is a~~ is symmetric and 3 independent eigenvectors exist, it is orthogonal diagonalizable.

Let $\bar{X}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \bar{X}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \bar{X}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} \bar{X}_1, & \bar{X}_2, & \bar{X}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} \frac{\bar{X}_1}{\|\bar{X}_1\|}, & \frac{\bar{X}_2}{\|\bar{X}_2\|}, & \frac{\bar{X}_3}{\|\bar{X}_3\|} \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

To evaluate:

$N^T A N$

$$\Rightarrow N^T A N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

On solving we get:

$$N^T A N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D$$

Let $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

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then $Y^T D Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}_{1\times 3} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\cancel{= \begin{bmatrix} y_1 & 0 & 0 \\ 0 & 2y_2 & 0 \\ 0 & 0 & 4y_3 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}$$

$$= \begin{bmatrix} y_1 & 2y_2 & 4y_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \begin{bmatrix} y_1^2 + 2y_2^2 + 4y_3^2 \end{bmatrix}$$

On converting to canonical form we get.

$$y_1^2 + 2y_2^2 + 4y_3^2 = 1$$

The conic section is an ellipsoid.