Department of Mathematics, School of advanced sciences

Winter Semester(2022-23) Instructor: Dr.Raghavendar K

Applications of Differential and Difference Equations (MAT2002)

Worksheet-

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1. For the following matrices, find

   (a) Eigenvalues, eigenvectors

   (b) Diagonalize or orthogonal diagonalize the matrix, what ever applicable(If exists).

      i. $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

      Eigenvalues: $\lambda_1 = 5, \lambda_2 = 1, \lambda_3 = 1,$

      Eigenvectors: $X_1 = [1, 1, 1]^T$, $X_2 = [-1, 0, 1]^T$, $X_3 = [-2, 1, 0]^T$.

      Conclusion: Three independent eigenvectors exists. A is diagonalizable.

      ii. $\begin{pmatrix} 1 & 1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

      Eigenvalues: $\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 1,$

      Eigenvectors: $X_1 = [1, -1, 1]^T$, $X_2 = [1, 2, 1]^T$.

      Conclusion: Three independent eigenvectors does not exists. Not diagonalizable.

      iii. $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

      Eigenvalues: $\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 1,$

      Eigenvectors: $X_1 = [1, 2, 0]^T$, $X_2 = [0, 0, 1]^T$, $X_3 = [-2, 1, 0]^T$.

      Conclusion: Three independent eigenvectors exists and the matrix is symmetric, hence the matrix is orthogonal diagonalizable.

      iv. $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

      Eigenvalues: $\lambda_1 = 4, \lambda_2 = 1, \lambda_3 = 1,$

      Eigenvectors: $X_1 = [1, 1, 1]^T$, $X_2 = [-1, 0, 1]^T$, $X_3 = [-1, 1, 0]^T$.

      Conclusion: Three independent eigenvectors exists and the matrix is symmetric, hence the matrix is orthogonal diagonalizable.

2. Verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}$. Find the inverse of A if exists.

3. If A is an $n \times n$ diagonalizable matrix and $A^2 = A$, then show that each eigenvalue of A is 0 or 1.

4. Let A be similar to B. Then show that (i) A^{-1} is similar toB^{-1}, (ii)A^m is similar to B^m for any positive integer m, (iii)$|A| = |B|$

5. Let A be a non-singular matrix, show that $A^T A^{-1}$ is symmetric if and only if $A^2 = (A^T)^2$

6. Verify whether the matrices $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$ are similar or not? If they are similar find a non singular matrix P such that $P^{-1}AP = B$.
   Hint: Let P be an 2×2 arbitrary matrix satisfies the relation $AP = PB$.
   $P = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

7. What kind of conic section (or pair of straight lines) is given by the quadratic form? Transform it to principal axes.Express $x^T = [x_1, x_2, \cdots, x_n]$ in terms of new coordinate vector $y^T = [y_1, y_2, \cdots, y_n]$.

   (a) $x_1^2 - 6x_2^2 + 24x_1x_2 = 5$
   Hint:Let $\vec{X}^T A \vec{X}$ be the matrix form of the quadratic form, where $A = \begin{pmatrix} 1 & 12 \\ 12 & -6 \end{pmatrix}$.
   Eigenvalues:$\lambda_1 = -15, \lambda_2 = 10$.
   Eigenvectors:$X_1 = [-3, 4]^T$,$X_2 = [4, 3]^T$.
   Conclusion: Two independent eigenvectors exists and the matrix is symmetric, hence the matrix is orthogonal diagonalizable. Canonical form is $-15y_1^2 + 10y_2^2 + y_3^2 = 5$.
   Conic section:Hyperbola.

   (b) $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3 = 1$
   Hint:Let $\vec{X}^T A \vec{X}$ be the matrix form of the quadratic form,
   where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$.
   Eigenvalues:$\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 1$,
   Eigenvectors:$X_1 = [0, -1, 1]^T$,$X_2 = [0, 1, 1]^T$,$X_3 = [1, 0, 0]^T$.
   Conclusion: Three independent eigen vectors exists and the matrix is symmertric, hence the matrix is orthogonal diagonalizable. Canonical form is $4y_1^2 + 2y_2^2 + y_3^2 = 1$.
   Conic section:Ellipsoid.