Argument: An argument is a process by which a conclusion is formed from a given set of statements called premises.

Example:

P_1 → Premises
P_2 → Premises
P_3 → Premises

Q → Conclusion

This argument can be seen as a statement:

$$[P_1 \wedge P_2 \wedge P_3] \Rightarrow Q$$

Valid Argument: An argument is said to be valid if the conjunction of all the premises implies conclusion. i.e.

If the expression $[P_1 \wedge P_2 \wedge P_3] \Rightarrow Q$ is a tautology.

Example 1. (Modus Ponens)

$P \Rightarrow Q$	If it is raining then the streets are wet
P	It is raining
Q	Streets are wet

Here $(P \Rightarrow Q) \wedge P \Rightarrow Q$ is always true and this can be seen from the truth table.

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \wedge P$	$(P \Rightarrow Q) \wedge P \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Example 2. (Modus Tollens)

$$\begin{array}{c} P \Rightarrow Q \\ \sim Q \\ \hline \sim P \end{array}$$

If it is raining then the streets are wet
The streets are not wet

It is not raining

Here $(P \Rightarrow Q) \wedge \sim Q \Rightarrow \sim P$ is always true and this can be seen from the truth table

P	Q	$P \Rightarrow Q$	$\sim Q$	$(P \Rightarrow Q) \wedge \sim Q$	$[(P \Rightarrow Q) \wedge \sim Q] \Rightarrow \sim P$
T	T	T	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	T

Example 3 : (Conditional Syllogism)

$P \to Q$	If it is raining then the streets are wet
$Q \to R$	If the streets are wet then she'll wear galoshes
$P \to R$	If it is raining the she'll wear galoshes

$(P \Rightarrow Q) \wedge (Q \Rightarrow R) \Rightarrow P \Rightarrow R$

Fallacy : If an argument is not valid it is called as fallacy.

Example:

$P \Rightarrow Q$	It it is raining then the streets are wet
Q	The streets are wet
P	It is raining (Not necessarily true)

This argument is not valid.

We can see this from the truth table:

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \wedge Q$	$(P \Rightarrow Q) \wedge Q \Rightarrow P$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Proof: A proof is a valid argument where all the premises are True.

Techniques for Proof:

1. Direct Proof (Hypothesis $\rightarrow$ Conclusion)
2. Indirect Proof (Proof by Contradiction)
3. Mathematical Induction

1. Direct Proof Examples

Statement: If n is an odd integer then n^2 is also an odd integer.

Proof: Given hypothesis: n is an odd integer

Conclusion to proove: n^2 is an odd integer.

Suppose $n = 2k+1$, $k \in \mathbb{Z}$

then
$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \\ &= 2(m) + 1, \quad \text{where } m = 2k^2 + 2k \in \mathbb{Z} \end{aligned}$$

an odd number

Thus, n^2 is an odd number.

Statement: Product of two integers is also an Integer.

Proof: Hypothesis: Two integers are given

To Proove: Their product is an integer

Let m and n be two integers, then these are the whole numbers (+ve or -ve) which are not fractions.

Then if we multiply such two numbers, we again get a number without fraction. So the product m·n is again an integer.

Statement: Product of two perfect squares is a perfect square.

Proof: Hypothesis: Given two perfect squares.

To proove: Their product is a perfect square.

Let m and n be two perfect squares, then

$m = r^2 \quad , \quad r \in \mathbb{Z}$

$n = s^2 \quad , \quad s \in \mathbb{Z}$

Then $m \cdot n = r^2 s^2$

$= (rs)^2$

$= p^2 \quad , \text{ where } p = r \cdot s \in \mathbb{Z}$

Thus product of two perfect squares is again a perfect square.

2. <u>Indirect method of Proof examples</u>

(Proof by contradiction)

Hypothesis $\Longrightarrow$ Conclusion

or $\sim$ Conclusion $\Longrightarrow$ $\sim$ Hypothesis

<u>Statement</u>: If $3n+2$ is odd, then n is odd.

<u>Proof</u>: Hypothesis : $3n+2$ is odd

To prove : n is odd

To prove by contradiction let us suppose that n is not an odd number

$\Rightarrow$ n is an even number

$\Rightarrow$ $3n$ is an even number

$\Rightarrow$ $3n+2$ is an even number

Thus, we get a contradiction to the hypothesis, therefore our consideration is wrong

Hence, n is an odd number.

Statement: $\sqrt{2}$ is an irrational number

Proof: Suppose that $\sqrt{2}$ is not an irrational no.

$\Rightarrow$ $\sqrt{2}$ is a rational number

$\Rightarrow$ $\sqrt{2} = p/q$, $p, q \in \mathbb{Z}$ & $q \neq 0$

so that $g.c.d.(p, q) = 1$

Now $\sqrt{2} = \frac{p}{q}$

$\Rightarrow p = \sqrt{2} q$

$\Rightarrow p^2 = 2q^2$

$\Rightarrow$ p^2 is an even number

$\Rightarrow$ p is an even number

$\Rightarrow$ $p = 2m$ for some $m \in \mathbb{Z}$

Now, again $\sqrt{2} = \frac{p}{q} = \frac{2m}{q}$

$\Rightarrow q = \sqrt{2} m$

$\Rightarrow q^2 = 2m^2$

$\Rightarrow$ q^2 is an even number

$\Rightarrow$ q is an even number

$\Rightarrow$ $q = 2n$ for some $n \in \mathbb{Z}$

Thus $g.c.d(p, q) = g.c.d(2m, 2n) = 2 \neq 1$

Thus, there is a contradiction to the hypothesis.

Hence our assumption was wrong. i.e $\sqrt{2}$ is an irrational number.

Principle of Mathematical Induction

Let $P(n)$ be a given statement for $n \in \{0, 1, 2, \ldots\} = S$

Then the statement will be true for every value of the set if

(i) $P(n)$ is true for some initial value i.e. for $n = 0, 1$ or 2.

(ii) Suppose, $P(n)$ is true for some $n = k \in S$ then $P(n)$ is true for $n = k+1$.

Example: The sum of first n natural numbers is $\frac{n(n+1)}{2}$.

Proof: Let $P(n): 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

To prove by Mathematical Induction:

(i) $P(1): 1 = \frac{1(1+1)}{2}$, so the statement is true for $n = 1$.

(ii) $P(k): 1 + 2 + \ldots + k = \frac{k(k+1)}{2} \longrightarrow (1)$

Suppose that it is true

Now, $P(k+1) = 1 + 2 + \ldots + k + k+1$

$= \frac{k(k+1)}{2} + k+1$ [using (1)]

$= \frac{k(k+1) + 2k + 2}{2} = \frac{(k+1)(k+2)}{2}$

Thus $P(k+1)$ is true whenever $P(k)$ is true. Hence by P.M.I $P(n)$ is true for all $n \in S$.

Example 2: Show that for all integer $n \geq 5$ we have $n^2 < 2^n$.

Proof: (i) for $n=5$, we have

$$5^2 < 2^5 \text{ i.e. } 25 < 32$$

$\therefore$ $P(n) = n^2 < 2^n$ is true for $n=5$

(ii) Assume that $P(n)$ is true for $n=k$.

i.e. $k^2 < 2^k \rightarrow (1)$

(iii) Then, we know that

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&< k^2 + 2k + k && [\because k \geq 5] \\
&= k^2 + 3k \\
&< k^2 + k.k && [\because k \geq 5] \\
&= 2k^2 \\
&< 2.2^k && [\text{by } (1)] \\
&= 2^{k+1}
\end{aligned}$$

Thus $P(n)$ is true for $n=k+1$.

Hence by P.M.I. $P(n)$ is true for all $n \geq 5$.