Statement (or Proposition) : A sentence which is either always true or always false is called a statement.

eg.

1.	Sky is blue	T	Stat.
2.	Blood is red	T	Stat.
3	It will rain tomorrow		Not stat.

Boolean Variable : It is a symbo (say p), which takes the boolean values.

eg. p : Sky is blue

Boolean Expression : The expression with Boolean variables and logical operations is called Boolean expression

Truth Table : It is table that gives the truth values corresponding to each possible combination of inputs for the logical operations.

Negation : Negation of a statement p gives T for the actual value F and F for actual value T.

eg. p : Sky is blue T

$\sim p$: Sky is not blue F

Truth Table

p	$\sim p$
T	F
F	T

Conjunction '$\wedge$'

[$p \wedge q = p$

$p \wedge q$: T if both p and q are T, otherwise false.

Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction '$\vee$' $p \vee q$ = p or q

$p \vee q$: T if at least one of p or q is true

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional Operator (Implication operator)

$p \Rightarrow q$: if p then q

p — Hypothesis, q — Implication

Truth Table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \Rightarrow q$ simply means that if hypothesis p is true and it is implying a true statement then $p \Rightarrow q$ is true.

Again if hypothesis p is true and it is implying a false statement then $p \Rightarrow q$ is false.

In case when the hypothesis is false then it is of no meaning whatever the implication is we cannot say $p \Rightarrow q$ to be false, so in each case we take $p \Rightarrow q$ is true.

Converse of $p \Rightarrow q$: is defined as $q \Rightarrow p$

eg. p: sky is blue
q: parrot is red

Then $p \Rightarrow q$: If sky is blue then parrot is red.

its converse is $q \Rightarrow p$: If parrot is red then sky is blue.

Inverse of $p \Rightarrow q$: is defined as $\sim p \Rightarrow \sim q$

eg: $p \Rightarrow q$: If sky is blue then parrot is red

its inverse is $\sim p \Rightarrow \sim q$: If sky is not blue then parrot is not red.

Contrapositive of $p \Rightarrow q$: is defined as $\sim q \Rightarrow \sim p$

eg: $p \Rightarrow q$: If sky is blue the parrot is red

its contrapositive is $\sim q \Rightarrow \sim p$: If parrot is not red then sky is not blue.

A Conditional expression is equivalent to its contrapositive.

i.e. $p \Rightarrow q \quad = \quad \sim q \Rightarrow \sim p$

This can be seen from their truth tables.

But $p \Rightarrow q$ and its inverse or converse are not equivalent.

Biconditional operator

$p \Leftrightarrow q$ is defined as p if and only if q.

$p \Leftrightarrow q$ means the both side implication

i.e. $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$

Truth Table:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

eg: p: Blood is red

q: sky is blue

$p \Leftrightarrow q$: Blood is red if and only if sky is blue

$p \Leftrightarrow q$ means that either both are true at the same time or both are false at the same time.

Tautology: An expression which is true for all possible inputs is called a tautology.

Ex: p: sky is blue

$\sim p$: sky is not blue

Then $\sim(p \wedge \sim p)$ is a tautology

p	$\sim p$	$p \wedge \sim p$	$\sim(p \wedge \sim p)$
T	F	F	T
F	T	F	T
~~F~~	~~T~~	~~F~~	~~T~~
~~F~~	~~T~~	~~F~~	~~T~~

Contradiction: An expression which is false for all possible inputs is called a contradiction.

Ex: p: sky is blue

$\sim p$: sky is not blue

Then $p \wedge \sim p$ is a contradiction

p	$\sim p$	$p \wedge \sim p$
T	F	F
~~T~~	~~F~~	~~F~~
~~F~~	~~T~~	~~F~~
F	T	F

De Morgan's Laws

1. $\sim(p \vee q) = \sim p \wedge \sim q$

2. $\sim(p \wedge q) = \sim p \vee \sim q$

1.

p	$\sim p$	q	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	F	T	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	T	F	T	F	T	T

2.

p	$\sim p$	q	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	F	T	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	T	F	T	F	T	T

Dual Statement (Expression): Two expressions are said to be dual of each other if either of them can be obtained from the other by replacing $\wedge$ by $\vee$, $\vee$ by $\wedge$, T by F, and F by T.

Example: 1. $(p \wedge q) \vee r$

its dual is $(p \vee q) \wedge r$

2. $\sim(p \vee q) \wedge (p \vee \sim(p \wedge r))$

its dual is $\sim(p \wedge q) \vee (p \wedge \sim(p \vee r))$

Principle of Duality: It states that the dual of any two equal expressions will also be equal.

Example: We have two equal expressions

$$\sim(p \vee q) = \sim p \wedge \sim q$$

Then by the principle of duality, their duals will also be equal i.e.

$$\sim(p \wedge q) = \sim p \vee \sim q.$$

Predicate : A predicate is an expression of one or more Boolean variable defined on ~~some~~ some specific domain.

This becomes a statement for some specific values of variables.

Example : Domain : $D =$ set of all fruits

Predicate : $P(x)$: x is mango.

Let $x =$ mango

So, $P(x)$ is now a statement which is true.

Example : $P(x)$: x is prime , $x \in \mathbb{N}$

$\therefore$ $P(2)$ is True , $P(4)$ is false

\# Simply the truth value of $P(x)$ depends on the value of x.

Example : $P(x)$: x is odd , $x \in \mathbb{N}$

$\therefore$ $P(1)$ is T , $P(2)$ is F.

Universal Quantifier: It is denoted by $\forall$ and read as 'for all' or 'for every' declares that a statement must be true for all variables within some domain.

Let $P(x)$ be a predicate and D the domain then the universal statement '$\forall x : P(x)$' means that '$P(x)$ is true for every x in D'.

Example: $\mathbb{N}$ = Set of all natural numbers

$P(x)$: $x+1 > 1$

'$\forall x : P(x)$' means that "$x+1>1$ is true for every x in $\mathbb{N}$".

Example 2.: $\mathbb{N}$ = Set of all natural numbers

$P(x)$: $3x$ is odd.

$\therefore$ '$\forall x : P(x)$' means that "for every $x \in \mathbb{N}$ $3x$ is odd."

or "$3x$ is odd is true for every $x \in \mathbb{N}$."

Existential Quantifier : It is denoted by '$\exists$' and read as 'there exists' or 'for at least one' declares that a statement must be true for at least one variable within the domain.

Let $P(x)$ be a predicate and D be the domain of x. Then the existential statement '$\exists x : P(x)$' means that "$P(x)$ is true for at least one x in D."

Example 1. : $\mathbb{N}$ = Set of natural numbers

$P(x)$, $x+1>1$

'$\exists x : P(x)$' means that "$x+1>1$ is true for at least one x in $\mathbb{N}$."

Example 2. : $\mathbb{N}$ = Set of natural numbers

$P(x) = 3x+1$ is odd

'$\exists x ; P(x)$' means that "$3x+1$ is odd is true for at least one x in $\mathbb{N}$."

Example 3. : Domain = Set of all birds

$P(x)$: x is a crow , $Q(x)$: x is black

Statement : All crows are not black.

Boolean Expression : $\exists x : P(x) \Rightarrow \sim Q(x)$

Nested Quantifier: If we use a quantifier that happens with the scope of another quantifier, it is called nested quantifier.

Example 1.

N = Set of all natural numbers,

$P(x)$: x is prime

'$\forall x \ \exists y : (y > x) \wedge P(y)$' means

" for every x there exists at least one y such that y is greater than x and y is prime.

Example 2. N = Set of natural numbers

$P(x,y)$: $y > x$

$\forall x \ \exists y : P(x,y)$ means that "for each $x \in \mathbb{N}$, there is at least one $y \in \mathbb{N}$ which is greater than x".

Negating Quantified Expressions : To negate quantified statement, we first negate all the quantifiers from left to right (keeping in the same order), then we negate the statements.

Example:

1. $\sim[\forall x : P(x)] = \exists x : \sim P(x)$

2. $\sim[\exists x : P(x)] = \forall x : \sim P(x)$

3. $\sim[\forall x, \exists y : P(x,y)] = \exists x, \forall y : \sim P(x,y)$

4. $\sim[\exists x, \forall y : P(x,y)] = \forall x, \exists y : \sim P(x,y)$.

\# "2x is an even number for every natural number x"

$\forall x : P(x)$ where $P(x)$: 2x is an even no. and $x \in \mathbb{N}$.

$\therefore \sim[\forall x : P(x)] = \exists x : \sim P(x)$

"there exists at least one x in N such that 2x is not an even number.