Lecture 15: The Lasso

Reading: Sections 6.2.2-6.2.3

STATS 202: Data mining and analysis

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Shrinkage methods

The idea is to perform a linear regression, while *regularizing* or *shrinking* the coefficients $\hat{\beta}$ toward 0.

Why would shrunk coefficients be better?

- This introduces *bias*, but may significantly decrease the *variance* of the estimates. If the latter effect is larger, this would decrease the test error.
- There are Bayesian motivations to do this: the prior tends to shrink the parameters.

Ridge regression

Ridge regression solves the following optimization:

$$\min_{\beta} \quad \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{i,j} \right)^2 + \lambda \sum_{j=1}^{p} \beta_j^2$$

In blue, we have the RSS of the model.

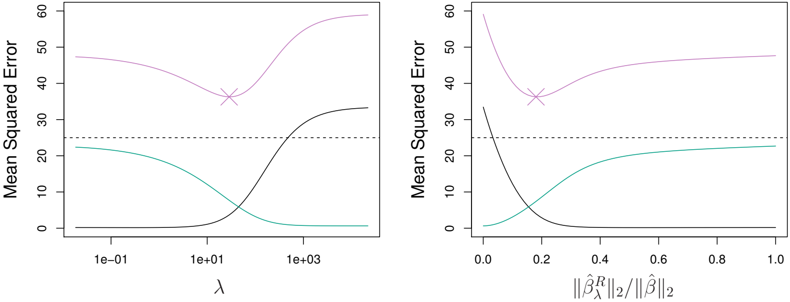
In red, we have the squared ℓ_2 norm of β, or $\|\beta\|_2^2$.

The parameter λ is a tuning parameter. It modulates the importance of fit vs. shrinkage.

We find an estimate $\hat{\beta}^R_\lambda$ for many values of λ and then choose it by cross-validation.

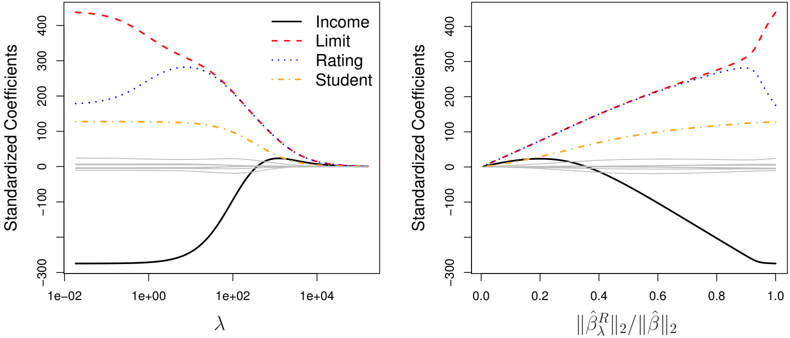
Bias-variance tradeoff

In a simulation study, we compute bias, variance, and test error as a function of λ.

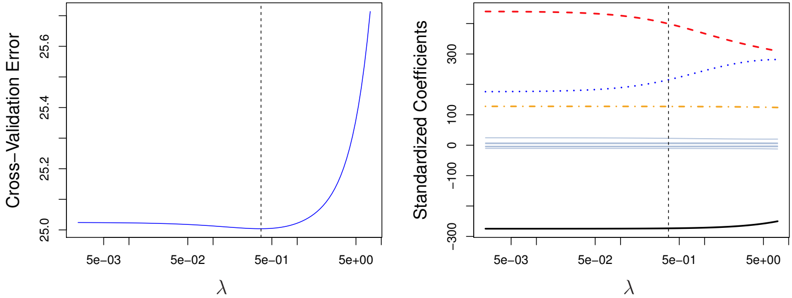


Example. Ridge regression

Ridge regression of `default` in the Credit dataset.



Selecting λ by cross-validation



The Lasso

Lasso regression solves the following optimization:

$$\min_{\beta} \quad \sum_{i=1}^{n}\left(y_i - \beta_0 - \sum_{j=1}^{p}\beta_j x_{i,j}\right)^2 + \lambda \sum_{j=1}^{p}|\beta_j|$$

In blue, we have the RSS of the model.

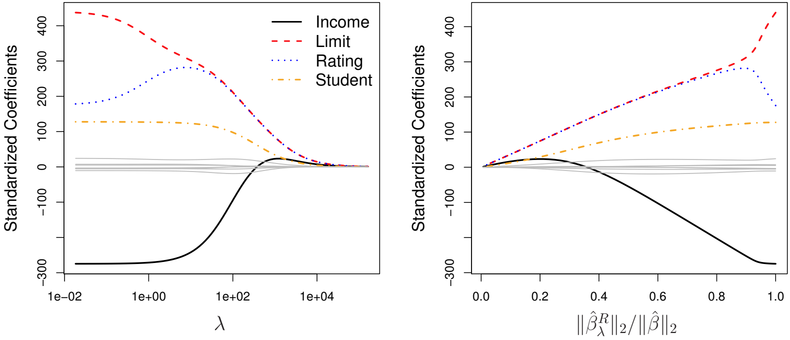
In red, we have the ℓ_1 norm of β, or $\|\beta\|_1$.

Why would we use the Lasso instead of Ridge regression?

- Ridge regression shrinks all the coefficients to a non-zero value.
- The Lasso shrinks some of the coefficients all the way to zero. Alternative to best subset selection or stepwise selection!

Example. Ridge regression

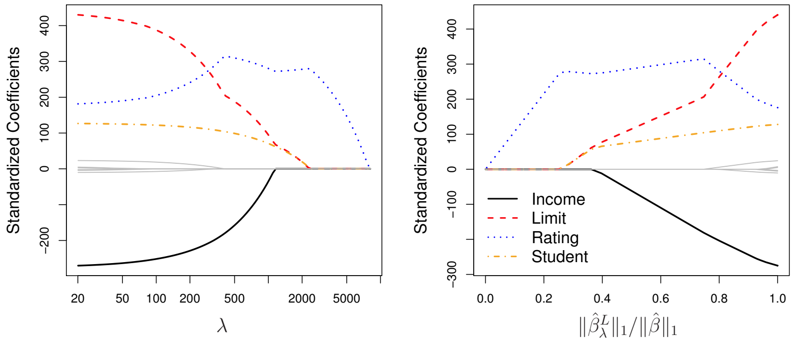
Ridge regression of `default` in the Credit dataset.



A lot of pesky small coefficients throughout the regularization path.

Example. The Lasso

Lasso regression of `default` in the Credit dataset.



Those coefficients are shrunk to zero.

An alternative formulation for regularization

- **Ridge:** for every λ, there is an s such that $\hat{\beta}_\lambda^R$ solves:

$$\underset{\beta}{\text{minimize}} \left\{ \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{i,j} \right)^2 \right\} \text{ subject to } \sum_{j=1}^{p} \beta_j^2 < s.$$

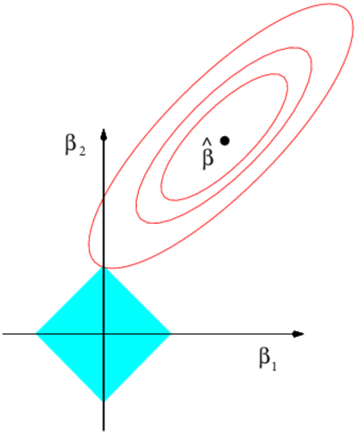
- **Lasso:** for every λ, there is an s such that $\hat{\beta}_\lambda^L$ solves:

$$\underset{\beta}{\text{minimize}} \left\{ \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{i,j} \right)^2 \right\} \text{ subject to } \sum_{j=1}^{p} |\beta_j| < s.$$

- **Best subset:**

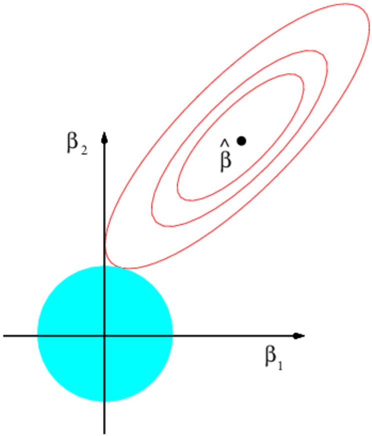
$$\underset{\beta}{\text{minimize}} \left\{ \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{i,j} \right)^2 \right\} \text{ s.t. } \sum_{j=1}^{p} \mathbf{1}(\beta_j \neq 0) < s.$$

Visualizing Ridge and the Lasso with 2 predictors



The Lasso

◆ : $\sum_{j=1}^{p} |\beta_j| < s$

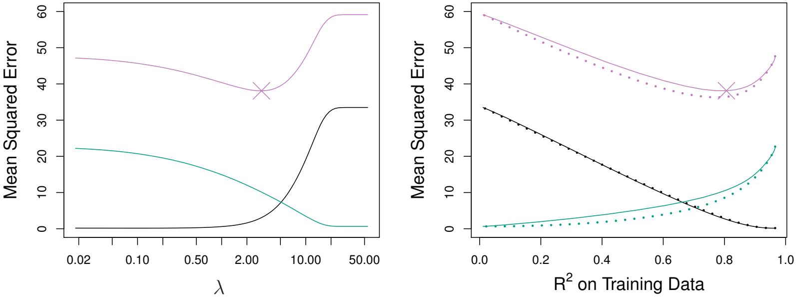


Ridge Regression

● : $\sum_{j=1}^{p} \beta_j^2 < s$

When is the Lasso better than Ridge?

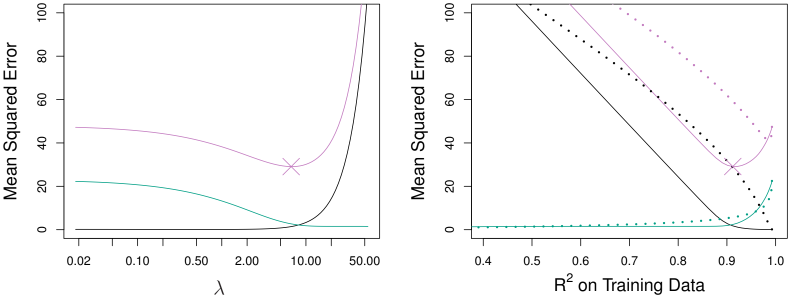
Example 1. Most of the coefficients are non-zero.



- Bias, Variance, MSE. The Lasso (—), Ridge ($\cdots$).
- The bias is about the same for both methods.
- The variance of Ridge regression is smaller, so is the MSE.

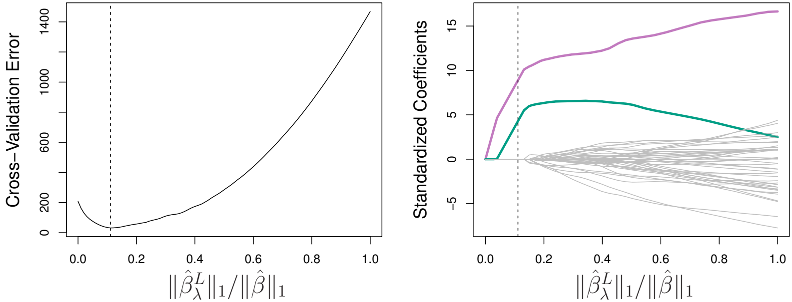
When is the Lasso better than Ridge?

Example 2. Only 2 coefficients are non-zero.



- Bias, Variance, MSE. The Lasso (—), Ridge ($\cdots$).
- The bias, variance, and MSE are lower for the Lasso.

Choosing λ by cross-validation



A very special case

Suppose $n = p$ and our matrix of predictors is $\mathbf{X} = I$.

Then, the objective function in Ridge regression can be simplified:

$$\sum_{j=1}^{p}(y_j - \beta_j)^2 + \lambda \sum_{j=1}^{p} \beta_j^2$$

and we can minimize the terms that involve each β_j separately:

$$(y_j - \beta_j)^2 + \lambda\beta_j^2.$$

It is easy to show that

$$\hat{\beta}_j^R = \frac{y_j}{1+\lambda}.$$

A very special case

Similar story for the Lasso; the objective function is:

$$\sum_{j=1}^{p}(y_j - \beta_j)^2 + \lambda \sum_{j=1}^{p} |\beta_j|$$

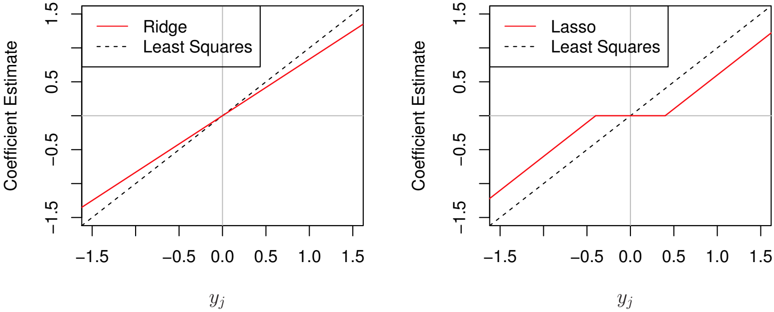
and we can minimize the terms that involve each β_j separately:

$$(y_j - \beta_j)^2 + \lambda|\beta_j|.$$

It is easy to show that

$$\hat{\beta}_j^L = \begin{cases} y_j - \lambda/2 & \text{if } y_j > \lambda/2; \\ y_j + \lambda/2 & \text{if } y_j < -\lambda/2; \\ 0 & \text{if } |y_j| < \lambda/2. \end{cases}$$

Lasso and Ridge coefficients as a function of λ



Bayesian interpretations

Ridge: $\hat{\beta}^R$ is the posterior mean, with a Normal prior on β.

Lasso: $\hat{\beta}^L$ is the posterior mode, with a Laplace prior on β.