HW2

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October 4, 2019

1 Problem 1

Proof. Let $P = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$.
We want to show that $\forall \mathbf{x}, \mathbf{y} \in P, \forall \lambda \in (0,1), \lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in P$.
Let $\mathbf{x}, \mathbf{y} \in P$ and $\lambda \in (0,1)$ be given.
Since $\mathbf{x}, \mathbf{y} \in P$, $A\mathbf{x} = \mathbf{b} = A\mathbf{y}$.
$A(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = \lambda A\mathbf{x} + (1-\lambda)A\mathbf{y} = \lambda\mathbf{b} + (1-\lambda)\mathbf{b} = (\lambda + 1 - \lambda)\mathbf{b} = \mathbf{b}$.
Since $\mathbf{x}, \mathbf{y} \in P$, $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$.
Since $0 < \lambda < 1$, $\lambda\mathbf{x} \geq \mathbf{0}$.
Since $0 < (1-\lambda) < 1$, $(1-\lambda)\mathbf{y} \geq \mathbf{0}$.
Thus, $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \geq \mathbf{0}$.
$A(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = \mathbf{b}$ and $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \geq \mathbf{0}$, so $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in P$. $\square$

2 Problem 2

2.1 Part 1

False. Consider the LP defined by

$$\begin{array}{rlcl} \min & \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\mathbf{x} & & \\ \text{s.t.} & \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\mathbf{x} & = & 6\ ; \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

where $A \in \mathbb{R}^{1\times 3}$. The optimal BFS is $\mathbf{x} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$, which has only 1 positive variable. However, $\mathbf{x} = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix}$ is also an optimal solution with 2 positive variables.

2.2 Part 2

True.

Proof. Suppose $\mathbf{s}$ and $\mathbf{t}$ are unique optimal solutions. Let v be the optimal value, such that $\mathbf{c}^T\mathbf{s} = v = \mathbf{c}^T\mathbf{t}$. Let $V = \{\mathbf{x} \in P \mid \mathbf{c}^T\mathbf{x} = v\}$ be the set of all optimal solutions.

Claim: $\forall \lambda \in (0,1), \lambda\mathbf{s} + (1-\lambda)\mathbf{t} \in V$.
Proof of claim. Let $\lambda \in (0,1)$ be given. $\mathbf{s}, \mathbf{t} \in P$, therefore $\lambda\mathbf{s} + (1-\lambda)\mathbf{t} \in P$, since P is convex. $\mathbf{c}^T(\lambda\mathbf{s} + (1-\lambda)\mathbf{t}) = \lambda\mathbf{c}^T\mathbf{s} + (1-\lambda)\mathbf{c}^T\mathbf{t} = \lambda v + (1-\lambda)v = v$. Thus $\lambda\mathbf{s} + (1-\lambda)\mathbf{t} \in V$. ■

Define $f : (0,1) \to V$ by $f(\lambda) = \lambda\mathbf{s} + (1-\lambda)\mathbf{t}$. f is clearly injective, so $|V| \geq |(0,1)|$, and therefore V is uncountable. □

3 Problem 3

Here is a list of all extreme points of the form (x_1, x_2) and their objective values of the form $f(x_1, x_2)$.

$$f(4, 2.5) = 6.5$$
$$f(3, 3) = 6$$
$$f(4, 0) = 4$$
$$f(.5, 3) = 3.5$$
$$f(0, 2.5) = 2.5$$
$$f(0, 0) = 0$$

The optimal solution is at $(4, 2.5)$ with an optimal value of 6.5. The active constraints at this optimal solution are $x_1 \leq 4$ and $x_1 + 2x_2 \leq 9$.

4 Problem 4

4.1 Part 1

$$\begin{array}{lrcl} \min & \mathbf{c}^T\mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & = & \mathbf{b}\ ; \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

where

$$\mathbf{c} = \begin{pmatrix} 1 \\ 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \ A = \begin{pmatrix} 2 & 2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 \end{pmatrix},$$

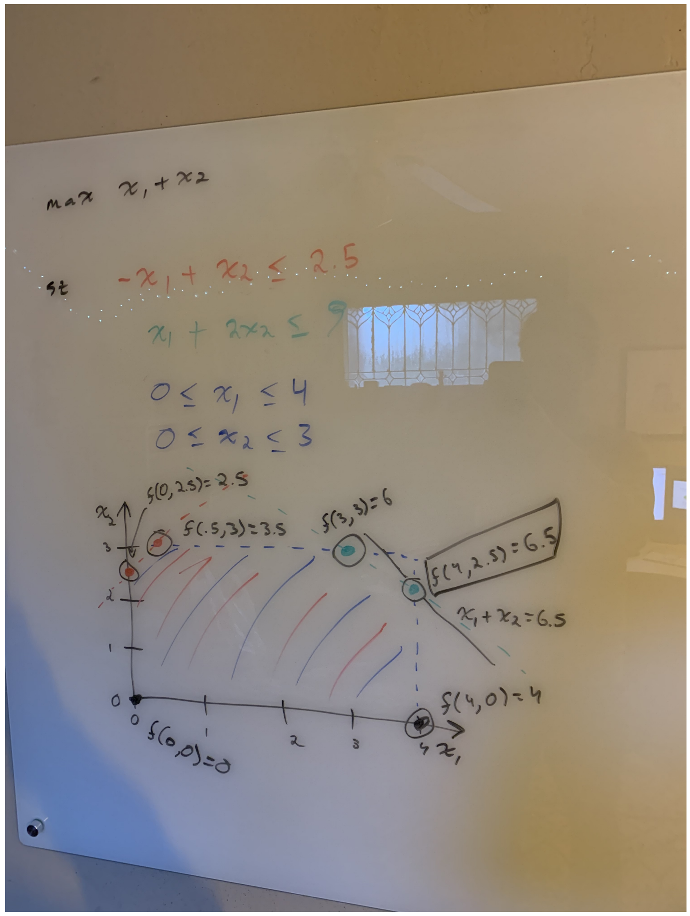

Figure 1: Graph of problem 3.

4.2 Part 2

Proof. Columns 4 and 5 of A are clearly linearly independent, therefore $rank(A) = 2$ and the image of $A\mathbf{x}$ is $\mathbb{R}^2$. Thus, there must exist some $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b} \in \mathbb{R}^2$. In other words, there exists a feasible solution to the LP. Since the rows of A are linearly independent and the full row rank of A is 2, the existence of a feasible solution implies the existence of a basic feasible solution, by the LP fundamental theorem.

Since $rank(A) = 2$, the basic partition of $A \in \mathbb{R}^{2\times 5}$ consists of B and N with $|B| = 2$ and $|N| = 3$. Then, the basic feasible solution $\mathbf{x}$ is obtained as

$$A\mathbf{x} = \begin{pmatrix} A_B & | & A_N \end{pmatrix} \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \mathbf{b}$$

with $|\mathbf{x}_B| = 2$, and $\mathbf{x}_N = \mathbf{0}_3$. Thus the only possible non-zero entries in $\mathbf{x}$

are the entries of $\mathbf{x}_B$, of which there are only 2. $\square$

4.3 Part 3

The basic solutions for each partition are as follows.

$$B = \{1,2\}: \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$B = \{1,3\}: \quad \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} \frac{5}{3} \\ 0 \\ \frac{2}{3} \\ 0 \\ 0 \end{pmatrix}$$

$$B = \{1,4\}: \quad \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

$$B = \{1,5\}: \quad \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$B = \{2,3\}: \quad \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ -1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} 0 \\ \frac{5}{2} \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$B = \{2,5\}: \quad \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$B = \{3,4\}: \qquad \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 5 \\ 0 \end{pmatrix}$$

$$B = \{3,5\}: \qquad \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 0 \\ -5 \end{pmatrix}$$

$$B = \{4,5\}: \qquad \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ -1 \end{pmatrix}$$

The basic feasible solutions are

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} \frac{5}{3} \\ 0 \\ \frac{2}{3} \\ 0 \\ 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

4.4 Part 4

Let $f : \mathbb{R}^5 \to \mathbb{R}$ be the objective function, defined as $f(\mathbf{x}) = \begin{pmatrix} 1 & 4 & 1 & 0 & 0 \end{pmatrix} \mathbf{x}$.

$$B = \{1,2\}: \qquad f(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}) = 5$$

$$B = \{1,3\}: \qquad f(\begin{pmatrix} \frac{5}{3} \\ 0 \\ \frac{2}{3} \\ 0 \\ 0 \end{pmatrix}) = \frac{7}{3}$$

$$B = \{1,4\}: \qquad f(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}) = 1$$

$$B = \{1,5\}: \qquad f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}\right) = 2$$

Thus the optimal solution is $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ with an optimal value of 5.

5 Problem 5

Let $A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ -1 & -1 \\ 1 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 6 \\ 18 \\ -1 \\ 7 \end{pmatrix}$.

Formulate the LP as

$$\begin{array}{lrcccc}
\max & & r & & & \\
\text{s.t.} & A\mathbf{y} & \leq & \mathbf{b}\ ; & & \\
 & \mathbf{0}_2 & \leq & \mathbf{y} & \leq & \begin{pmatrix} 11 \\ 10 \end{pmatrix}\ ; \\
\forall i \in \{1,..,4\}, & 0 & \leq & r & \leq & d_i\ ; \\
\forall i \in \{1,..,4\}, & d_i & \leq & \frac{1}{\sqrt{2}}(b_i - A(i,:)\mathbf{y})\ ; & &
\end{array}$$

where $\mathbf{y} \in \mathbb{R}^2, r \in \mathbb{R}$, and $\mathbf{d} \in \mathbb{R}^4$.

This gave me an optimal solution of $r = 4.5962$ (r is also the optimal value) as the maximum radius for a fountain centered at $\mathbf{y} = \begin{pmatrix} 5.0418 \\ 4.5418 \end{pmatrix}$ (with $\mathbf{d} = \begin{pmatrix} 4.5962 \\ 5.2737 \\ 5.3329 \\ 4.5962 \end{pmatrix}$).