HW5

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1 Problem 1

1.1 Part 1

$$
\begin{array}{rrcl}
\min & 4y_1 + 7y_2 & & \\
\text{s.t.} & 2y_1 + y_2 & \geq & 5\ ; \\
 & 3y_1 + 2y_2 & \geq & 2\ ; \\
 & y_1 + 3y_2 & \geq & 5\ ; \\
 & \mathbf{y} & \geq & \mathbf{0}\ .
\end{array}
$$

1.2 Part 2

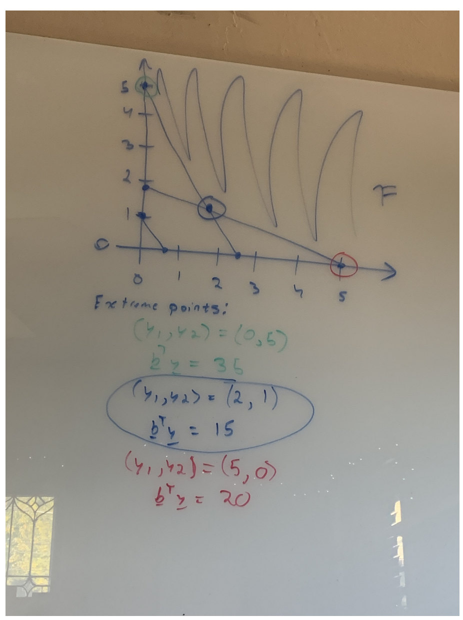

Figure 1: Graph of the dual problem.

The optimal solution of the dual problem is $(y_1, y_2) = (2, 1)$, which has an optimal value of 15.

1.3 Part 3

When $j = 1$,

$$x_1(2y_1 + y_2 - 5) = 0$$
$$x_1(2(2) + (1) - 5) = 0$$
$$0 = 0.$$

When $j = 2$,

$$x_2(3y_1 + 2y_2 - 2) = 0$$
$$x_2(3(2) + 2(1) - 2) = 0$$
$$6x_2 = 0$$
$$x_2 = 0.$$

When $j = 3$,

$$x_3(y_1 + 3y_2 - 5) = 0$$
$$x_3((2) + 3(1) - 5) = 0$$
$$0 = 0.$$

At this point, we know that $x_2 = 0$.

When $i = 1$,

$$y_1(2x_1 + 3x_2 + x_3 - 4) = 0$$
$$y_1(2x_1 + x_3 - 4) = 0$$
$$2(2x_1 + x_3 - 4) = 0$$
$$4x_1 + 2x_3 = 8.$$

When $i = 2$,

$$y_2(x_1 + 2x_2 + 3x_3 - 7) = 0$$
$$y_2(x_1 + 3x_3 - 7) = 0$$
$$x_1 + 3x_3 - 7 = 0$$
$$x_1 + 3x_3 = 7.$$

Solving $4x_1 + 2x_3 = 8$ and $x_1 + 3x_3 = 7$ gives us $x_1 = 1$ and $x_3 = 2$, with $x_2 = 0$. It can be verified that $\mathbf{c}^T\mathbf{x} = 15$ in the primal problem.

2 Problem 2

2.1 Part 1

Process 1 generates a profit of $4(38) + 3(33) - 51 = 200$. Process 2 generates a profit of $(38)+(33)-11 = 60$. Process 3 generates a profit of $3(38)+4(33)-40 = 206$. Thus we can formulate the LP as

$$\begin{array}{rrcl} -\min & \mathbf{c}^T\mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & = & \mathbf{b}\ ; \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

where

$$\mathbf{b} = \begin{pmatrix} 8000 \\ 5000 \end{pmatrix},\ \mathbf{c} = \begin{pmatrix} -200 \\ -60 \\ -206 \\ 0 \\ 0 \end{pmatrix},\ A = \begin{pmatrix} 3 & 1 & 5 & 1 & 0 \\ 5 & 1 & 3 & 0 & 1 \end{pmatrix}.$$

Step 1:

BFS: $\mathbf{x}_B = \begin{pmatrix} 1000 \\ 0 \\ 0 \\ 5000 \\ 0 \end{pmatrix}$

Objective value: $\mathbf{c}^T\mathbf{x} = -200000$

Reduced costs: $\bar{\mathbf{c}} = \begin{pmatrix} 0 \\ -20 \\ -86 \\ 0 \\ 40 \end{pmatrix}$

Index to enter: 2

Basic direction: $\mathbf{d} = \begin{pmatrix} -.2 \\ 1 \\ 0 \\ -.4 \\ 0 \end{pmatrix}$

Index to exit: 1

Step 2:

BFS: $\mathbf{x}_B = \begin{pmatrix} 0 \\ 5000 \\ 0 \\ 3000 \\ 0 \end{pmatrix}$

Objective value: $\mathbf{c}^T\mathbf{x} = -300000$

Reduced costs: $\bar{\mathbf{c}} = \begin{pmatrix} 100 \\ 0 \\ -26 \\ 0 \\ 60 \end{pmatrix}$

Index to enter: 3

Basic direction: $\mathbf{d} = \begin{pmatrix} 0 \\ -3 \\ 1 \\ -2 \\ 0 \end{pmatrix}$

Index to exit: 4

Step 3:

BFS: $\mathbf{x}_B = \begin{pmatrix} 0 \\ 500 \\ 1500 \\ 0 \\ 0 \end{pmatrix}$

Objective value: $\mathbf{c}^T\mathbf{x} = -339000$

Reduced costs: $\bar{\mathbf{c}} = \begin{pmatrix} 74 \\ 0 \\ 0 \\ 13 \\ 47 \end{pmatrix}$

We are now done. The optimal solution is $\mathbf{x}_B = \begin{pmatrix} 0 \\ 500 \\ 1500 \\ 0 \\ 0 \end{pmatrix}$ with an objective value of \$339000.

2.2 Part 2

Let $\mathbf{c}' = \mathbf{c} + \Delta \begin{pmatrix} -4 \\ -1 \\ -3 \\ 0 \\ 0 \end{pmatrix}$ denote the new cost vector associated with the changed price. In order for $B = \{2, 3\}$ to remain the optimal basis, we must insist that $\bar{\mathbf{c}}_N = \mathbf{c}'_N - (A_B^{-1} A_N)^T \mathbf{c}'_B \geq \mathbf{0}$.

$A_B = \begin{pmatrix} 1 & 5 \\ 1 & 3 \end{pmatrix}$ and $A_N = \begin{pmatrix} 3 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$.

$\bar{\mathbf{c}}_N = \mathbf{c}'_N - (A_B^{-1} A_N)^T \mathbf{c}'_B = \begin{pmatrix} -200 \\ -0 \\ -0 \end{pmatrix} + \Delta \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} - (\begin{pmatrix} 1 & 5 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix})^T (\begin{pmatrix} -60 \\ -206 \end{pmatrix} +$
$\Delta \begin{pmatrix} -1 \\ -3 \end{pmatrix}) = \begin{pmatrix} 74 + \Delta \\ 13 \\ 47 + \Delta \end{pmatrix} \geq \mathbf{0}.$
Thus the optimal solution will change only if $\Delta > 47$.

2.3 Part 3

The optimal solution will be the same as in Part 1. The optimal solution of $x_2 = 500$ and $x_3 = 1500$ only produces $3x_2 + 5x_3 = 9000 \leq 10000$, so all constraints are still satisfied.

3 Problem 3

3.1 Part 1

If x_1 is the number of units of special risk insurance, x_2 is the number of units of mortgage insurance, and x_3 is the number of units of long term care insurance, then the LP can be formulated as follows:

$$\begin{array}{lrcl} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0}. \end{array}$$

where $\mathbf{c} = \begin{pmatrix} 500 \\ 250 \\ 600 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 240 \\ 150 \\ 180 \end{pmatrix}$, and $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$.

3.2 Part 2

The dual problem can be written as

$$\begin{array}{lrcl} \min & \mathbf{b}^T\mathbf{y} & & \\ \text{s.t.} & A^T\mathbf{y} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq & \mathbf{0}. \end{array}$$

Solving the dual problem gives a solution of $\mathbf{y} = \begin{pmatrix} 0 \\ 140 \\ 80 \end{pmatrix}$. Then, we can list out all of the complementarity conditions in terms of $\mathbf{y}$.

$$x_1(2y_1 + 3y_2 + y_3 - 500) = x_1(3 \cdot 140 + 80 - 500) = 0 \cdot x_1 = 0$$

$$x_2(y_1 + y_2 + 2y_3 - 250) = x_2(140 + 2 \cdot 80 - 250) = 50x_2 = 0$$

$$x_3(y_1 + 2y_2 + 4y_3 - 600) = x_1(2 \cdot 140 + 4 \cdot 80 - 600) = 0 \cdot x_3 = 0$$

The first and third of the above equations are true for all values of x_1 and x_3, but the second equation is only true for $x_2 = 0$. Thus no mortgage insurance is sold.

3.3 Part 3

In order for B to be the optimal basis, $\mathbf{x}'_B = A_B^{-1}(\mathbf{b} + \Delta\mathbf{e}_3) = \mathbf{x}_B + \Delta A_B^{-1}\mathbf{e}_3 = \begin{pmatrix} 24 \\ 39 \\ 153 \end{pmatrix} + \Delta \begin{pmatrix} -.2 \\ .3 \\ .1 \end{pmatrix} \geq \mathbf{0}$. Thus $\Delta \leq \frac{-24}{-.2} = 120$, $\Delta \geq \frac{-39}{.3} = -130$, and $\Delta \geq \frac{-153}{.1} = -1530$. Since $\mathbf{b} \geq \mathbf{0}, -180 \leq \Delta \leq 120$.

3.4 Part 4

In order for B to be the optimal basis,
$\bar{\mathbf{c}}_N = \mathbf{c}_N - (A_B^{-1}A_N)^T(\mathbf{c}_B + \Delta\mathbf{e}_1) = \begin{pmatrix} -250 \\ 0 \\ 0 \end{pmatrix} - (\begin{pmatrix} 0 & .4 & -.2 \\ 0 & -.1 & .3 \\ 1 & -.7 & .1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix})^T \begin{pmatrix} -500 + \Delta \\ -600 \\ 0 \end{pmatrix} = \begin{pmatrix} 3290 \\ 140 - .4\Delta \\ 1700 + .2\Delta \end{pmatrix} \geq \mathbf{0}$.
Thus $\Delta \leq \frac{140}{.4} = 350$ and $\Delta \geq \frac{1700}{-.2} = -8500$.

4 Problem 4

Proof. Let P be defined as

$$\begin{array}{rrcl} \max & 0 & & \\ \text{s.t.} & A\mathbf{x} & = & \mathbf{b}\ . \end{array}$$

Clearly, P has a solution iff system 1 has a solution. We can represent $dual(P)$ as

$$\begin{array}{rrcl} \min & \mathbf{b}^T\mathbf{y} & & \\ \text{s.t.} & A^T\mathbf{y} & = & \mathbf{0} \\ & \mathbf{y} & \geq & \mathbf{0}\ . \end{array}$$

We start by showing that no more than one of the two systems can have a solution. We do this by showing that if system 2 has a solution, then system 1 cannot. Assume that $\hat{\mathbf{y}}$ is a solution to system 2. Then, for any $\lambda > 0$, $A^T(\lambda\hat{\mathbf{y}}) = ((\lambda\hat{\mathbf{y}})^T A)^T = \lambda(\hat{\mathbf{y}}^T A)^T = \lambda\mathbf{0} = \mathbf{0}$. Since $\hat{\mathbf{y}} \geq \mathbf{0}$ and $\lambda > 0$, $\lambda\hat{\mathbf{y}} \geq \mathbf{0}$. Thus $\lambda\hat{\mathbf{y}}$ is a feasible solution to $dual(P)$. Since $\mathbf{b}^T\hat{\mathbf{y}} < 0$ and $\lambda > 0$, $\mathbf{b}^T(\lambda\hat{\mathbf{y}}) = \lambda\mathbf{b}^T\hat{\mathbf{y}} < 0$. Thus, $\mathbf{b}^T(\lambda\hat{\mathbf{y}})$ has no minimum value, since λ can be arbitrarily large, so $dual(P)$ is unbounded and therefore P is infeasible. Thus system 1 does not have a solution.

Now we show that at least one of the two systems must have a solution. We do this by showing that if system 2 is infeasible, system 1 must be feasible.

Assume that system 2 is infeasible. Since $A^T\mathbf{0} = 0$ and $\mathbf{0} \geq \mathbf{0}$, $\mathbf{0} \in \mathcal{F}_{dual(P)} \neq \emptyset$. $dual(P)$ is not infeasible. Then, let $\hat{\mathbf{y}} \in \mathcal{F}_{dual(P)}$ be any feasible solution to $dual(P)$. $A^T\hat{\mathbf{y}} = \mathbf{0}$ and $\hat{\mathbf{y}} \geq \mathbf{0}$. Since system 2 is not feasible, it must be that $\mathbf{b}^T\hat{\mathbf{y}} \geq 0$. Then, $\min\{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} \in \mathcal{F}_{dual(P)}\} \geq 0$, therefore $dual(P)$ is not unbounded. $dual(P)$ must have an optimal solution, and therefore P has an optimal solution as well. So system 1 has a solution.

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