HW3

Ry Wiese
wiese176@umn.edu

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1 Problem 1

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 240 \\ 150 \\ 180 \end{pmatrix}, \; c = \begin{pmatrix} 500 \\ 250 \\ 600 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Step 1:
Basis: $B = \{4, 5, 6\}$
BFS: $\mathbf{x}_B = A_B^{-1}\mathbf{b} = \begin{pmatrix} 240 \\ 150 \\ 180 \end{pmatrix}$
Objective value: $\mathbf{c}^T\mathbf{x} = 0$
Reduced costs: $\bar{\mathbf{c}}_N = \mathbf{c}_N - (A_B^{-1}A_N)^T\mathbf{c}_B = \begin{pmatrix} -500 \\ -250 \\ -600 \end{pmatrix}$
Variable to enter: x_1, since $\bar{\mathbf{c}}_{N_1} = -500$ is the first negative value in $\bar{\mathbf{c}}_N$
Variable to exit: x_5, since $\mathbf{d}_B = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$, $\theta_B = \begin{pmatrix} 120 \\ 50 \\ 180 \end{pmatrix}$ with $\theta_{B_5} = 50$
We change to basis $\{1, 4, 6\}$ since $\bar{\mathbf{c}}$ is not semipositive.

Step 2:
Basis: $B = \{1, 4, 6\}$
BFS: $\mathbf{x}_B = A_B^{-1}\mathbf{b} = \begin{pmatrix} 50 \\ 140 \\ 130 \end{pmatrix}$
Objective value: $\mathbf{c}^T\mathbf{x} = -25,000$
Reduced costs: $\bar{\mathbf{c}}_N = \mathbf{c}_N - (A_B^{-1}A_N)^T\mathbf{c}_B = \begin{pmatrix} -83 \\ -267 \\ -167 \end{pmatrix}$
Variable to enter: x_2, since $\bar{\mathbf{c}}_{N_2} = -83$ is the first negative value in $\bar{\mathbf{c}}_N$

Variable to exit: x_6, since $\mathbf{d}_B = \begin{pmatrix} -.3 \\ -.3 \\ -1.7 \end{pmatrix}$, $\theta_B = \begin{pmatrix} 150 \\ 420 \\ 78 \end{pmatrix}$ with $\theta_{B_6} = 78$
We change to basis $\{1, 2, 4\}$ since $\bar{\mathbf{c}}$ is not semipositive.

Step 3:
Basis: $B = \{1, 2, 4\}$
BFS: $\mathbf{x}_B = A_B^{-1}\mathbf{b} = \begin{pmatrix} 24 \\ 78 \\ 114 \end{pmatrix}$
Objective value: $\mathbf{c}^T\mathbf{x} = -31,500$
Reduced costs: $\bar{\mathbf{c}}_N = \mathbf{c}_N - (A_B^{-1}A_N)^T\mathbf{c}_B = \begin{pmatrix} -100 \\ -150 \\ -50 \end{pmatrix}$
Variable to enter: x_3, since $\bar{\mathbf{c}}_{N_3} = -100$ is the first negative value in $\bar{\mathbf{c}}_N$
Variable to exit: x_2, since $\mathbf{d}_B = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$, $\theta_B = \begin{pmatrix} - \\ 39 \\ - \end{pmatrix}$ with $\theta_{B_2} = 39$
We change to basis $\{1, 3, 4\}$ since $\bar{\mathbf{c}}$ is not semipositive.

Step 4:
Basis: $B = \{1, 3, 4\}$
BFS: $\mathbf{x}_B = A_B^{-1}\mathbf{b} = \begin{pmatrix} 24 \\ 39 \\ 153 \end{pmatrix}$
Objective value: $\mathbf{c}^T\mathbf{x} = -35,400$
Reduced costs: $\bar{\mathbf{c}}_N = \mathbf{c}_N - (A_B^{-1}A_N)^T\mathbf{c}_B = \begin{pmatrix} 50 \\ 140 \\ 80 \end{pmatrix}$
The current basis, $B = \{1, 3, 4\}$ is optimal because $\bar{\mathbf{c}} \geq \mathbf{0}$.

Thus the optimal solution is $x_1 = 24, x_2 = 0$, and $x_3 = 39$ for an optimal value of 35,400 (since the original problem was maximization).

2 Problem 2

Step 1:
Basis: $B = \{5, 6\}$
BFS: $\mathbf{x}_B = A_B^{-1}\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Objective value: $\mathbf{c}^T\mathbf{x} = 0$

Reduced costs: $\bar{\mathbf{c}}_N = \mathbf{c}_N - (A_B^{-1} A_N)^T \mathbf{c}_B = \begin{pmatrix} -2 \\ -3 \\ 1 \\ 12 \end{pmatrix}$
Variable to enter: x_1, since $\bar{\mathbf{c}}_{N_1} = -2$ is the first negative value in $\bar{\mathbf{c}}_N$
Variable to exit: x_6, since $\mathbf{d}_B = \begin{pmatrix} 2 \\ -.3 \end{pmatrix}$, $\theta_B = \begin{pmatrix} - \\ 0 \end{pmatrix}$ with $\theta_{B_6} = 0$
We change to basis $\{1, 5\}$ since $\bar{\mathbf{c}}$ is not semipositive.

Step 2:
Basis: $B = \{1, 5\}$
BFS: $\mathbf{x}_B = A_B^{-1}\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Objective value: $\mathbf{c}^T\mathbf{x} = 0$
Reduced costs: $\bar{\mathbf{c}}_N = \mathbf{c}_N - (A_B^{-1} A_N)^T \mathbf{c}_B = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 6 \end{pmatrix}$
Variable to enter: x_3, since $\bar{\mathbf{c}}_{N_3} = -1$ is the first negative value in $\bar{\mathbf{c}}_N$
$\mathbf{d}_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \geq \mathbf{0}$, therefore the problem is unbounded.

3 Problem 3

Let $B \subseteq \{1, .., n+m\}$ be the basis returned by Φ. Suppose B can be partitioned into $\{P, Q\}$ where $P \subseteq \{1, .., n\}$ (indices of non-artificial variables) and $Q \subseteq \{n+1, .., n+m\}$ (indices of artificial variables). Let $A' = (A|I_m)$ be the matrix used in Φ. Since the optimal value $\sum_{j=1}^m x_{n+j} = 0$ and $\mathbf{x} = 0$, $\begin{pmatrix} x_n + 1 \\ ... \\ x_n + m \end{pmatrix} = \mathbf{0}$.
Then, if $\mathbf{x}$ is a BFS of Φ, $A'\mathbf{x} = A \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix} + I_m \begin{pmatrix} x_n + 1 \\ ... \\ x_n + m \end{pmatrix} = A \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$.
Thus the basis vectors defined in Q do not contribute anything to the BFS, as they have scalar coefficients of 0. This means we can replace the indices in Q with *any* indices in B that do not create a linear dependence.

4 Problem 4

My algorithm is contained in the Simplex function, which uses Simplex_Helper as a helper function.You should be able to use Simplex without needing to touch Simplex_Helper at all. I have also created a function Test(m, n) which randomly generates A, $\mathbf{b}$, and $\mathbf{c}$, prints the output of the actual solution using cvx, and then prints my solution using my algorithm.