HW4

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1 Problem 1

1.1 Part 1

$$
\begin{array}{lrcl}
\min & 2x_1 + 3.5x_2 + 8x_3 + 1.5x_4 & & (\mathit{cost}) \\
\text{s.t.} & 4x_1 + 6x_2 + 20x_3 + x_4 & \geq & 25 \ ; \ (\mathit{protein}) \\
 & 8x_1 + 12x_2 + 30x_4 & \geq & 40 \ ; \ (\mathit{carbohydrates}) \\
 & 130x_1 + 120x_2 + 150x_3 + 70x_4 & \geq & 400 \ ; \ (\mathit{Calories}) \\
 & \mathbf{x} & \geq & \mathbf{0} \ .
\end{array}
$$

where x_1 is the number of servings of bread, x_2 is the number of servings of milk, x_3 is the number of servings of fish, and x_4 is the number of servings of bread. Each constraint describes the minimum amount of protein/carbs/Calories needed for an adult.

1.2 Part 2

The optimal solution is $x_1 = 1.55$, $x_2 = 0$, $x_3 = .894$, and $x_4 = .92$ with an optimal value of 11.632.

1.3 Part 3

$$
\begin{array}{lrcl}
\max & 25y_1 + 40y_2 + 400y_3 & & \\
\text{s.t.} & 4y_1 + 8y_2 + 130y_3 & \leq & 2 \ ; \\
 & 6y_1 + 12y_2 + 120y_3 & \leq & 3.5 \ ; \\
 & 20y_1 + 150y_3 & \leq & 8 \ ; \\
 & y_1 + 30y_2 + 70y_3 & \leq & 1.5 \ .
\end{array}
$$

This problem could represent a pharmaceutical company that sells each nutrient in pill form. y_1 is the price per gram of protein, y_2 is the price per gram of carbohydrates, and y_3 is the price for a Calorie. The company wants to maximize the profit it can get from an adult with the nutritional requirements outlined in the problem, but they must also ensure that the equivalent pill form of any of the four foods (bread, milk, fish, potato) does not cost more than the original food item itself. Otherwise nobody would choose to buy the pills instead. For example, if 4 grams of protein, 8 grams of carbohydrates, and 130 Calories cost more than \$2 in pill form, then A person would just buy some bread instead. Similar reasoning describes the remaining constraints.

1.4 Part 4

The optimal solution to the dual is $y_1 = .3904$, $y_2 = .034$, and $y_3 = .0013$ with an optimal value of 11.632.

2 Problem 2

$$
\begin{array}{lrcl}
\max & d_1y_1 + d_2y_2 + d_3y_3 + d_4y_4 + d_5y_5 + d_6y_6 + d_7y_7 & & \\
\text{s.t.} & y_1 + y_2 + y_3 + y_4 + y_5 & \leq & 1\ ; \\
 & y_2 + y_3 + y_4 + y_5 + y_6 & \leq & 1\ ; \\
 & y_3 + y_4 + y_5 + y_6 + y_7 & \leq & 1\ ; \\
 & y_1 + y_4 + y_5 + y_6 + y_7 & \leq & 1\ ; \\
 & y_1 + y_2 + y_5 + y_6 + y_7 & \leq & 1\ ; \\
 & y_1 + y_2 + y_3 + y_6 + y_7 & \leq & 1\ ; \\
 & y_1 + y_2 + y_3 + y_4 + y_7 & \leq & 1\ ; \\
 & y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6,\ y_7 & \geq & 0\ .
\end{array}
$$

Assume the hospital pays each employee the same, and each employee gets a one paycheck for their 5 day shift. Then, it is easy to see that the "cost" in our original formulation is in units of paychecks. If x_i people start working on day i, then that is an additional $c_ix_i = x_i$ paychecks that the hospital will have to make that week.

Suppose there is a firm offering nursing service for hospitals. y_i is the price that the firm charges on day i, again in units of paychecks. Then, the firm will be paying y_i paychecks to each of the d_i employees that are needed to work on day i, so the revenue for day i is d_iy_i. Thus, the firm wants to maximize revenue over all seven days, but subject to the constraints that they are not more expensive than if the hospital just hired the staff directly. Each row j in the dual represents the total cost the hospital will have to pay over the five day period, per employee, starting on day j. If any of these costs come out to be greater than 1 paycheck, then the hospital could just hire d_j employees on that day and pay them each 1 paycheck, which would be cheaper. So the cost must be less than or equal to the amount the hospital would normally pay the employees to work that week. Obviously, these payments would have to be non-negative.

3 Problem 3

3.1 Part 1

We set up weight matrix $w = \begin{pmatrix} 0 & 8 & 7 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Then, we formulate the LP as

$$\begin{array}{rrcl} \max & & v & \\ \text{s.t.} & x_{1,2} - x_{2,3} - x_{2,4} & = & 0 \ ; \\ & x_{1,3} + x_{2,3} - x_{3,4} & = & 0 \ ; \\ & x_{1,2} + x_{1,3} - v & = & 0 \ ; \\ & v - x_{2,4} - x_{3,4} & = & 0 \ ; \\ & \mathbf{x} & \leq & \mathbf{w} \ ; \\ & \mathbf{x} & \geq & \mathbf{0_{4\times 4}} \ . \end{array}$$

Where v is the total flow and $x_{i,j}$ is the flow on edge (i, j). Row 1 specifies that the flow entering vertex 2 must equal the flow leaving vertex 2, row 2 specifies the same thing but about vertex 3, row 3 specifies that v is the total flow leaving the source, and row 4 specifies that v is the total flow entering the sink. Clearly, the flow on each edge cannot exceed the max capacity of that edge, and flows cannot be negative.

I got an optimal value of 13 with $v = 13$ and $x = \begin{pmatrix} 0 & 6 & 7 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

3.2 Part 2

$$\begin{array}{lrcl} \min & w_{1,2}z_{1,2} + w_{1,3}z_{1,3} + w_{2,3}z_{2,3} + w_{2,4}z_{2,4} + w_{3,4}z_{3,4} & & \\ \text{s.t.} & z_{1,2} & \geq & y_1 - y_2 \ ; \\ & z_{1,3} & \geq & y_1 - y_3 \ ; \\ & z_{2,3} & \geq & y_2 - y_3 \ ; \\ & z_{2,4} & \geq & y_2 - y_4 \ ; \\ & z_{3,4} & \geq & y_3 - y_4 \ ; \\ & y_1 - y_4 & = & 1 \ ; \\ & \mathbf{z} & \geq & \mathbf{0_{4\times 4}} \ . \end{array}$$

I got an optimal value of 13 with $z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

3.3 Part 3

The maximum flow of 13 is achieved with weights of $x_{1,2} = 8$, $x_{1,3} = 7$, $x_{2,3} = 2$, $x_{2,4} = 4$, and $x_{3,5} = 12$. The minimum cut partitions the nodes in V into $S = \{1, 2\}$ and $V \backslash S = \{3, 4\}$ having a cut weight of 13.