# HW7

Ry Wiese
wiese176@umn.edu

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## 1 Problem 1

### 1.1 Part 1

False. Let $f(x) = -ln(x)$ and $g(x) = e^{x^2}$. Both $f$ and $g$ are convex. However, $(f \circ g)(x) = -ln(e^{x^2}) = -x^2$ is concave.

### 1.2 Part 2

True.

*Proof.* Assume $f$ is convex and nondecreasing. Assume $g$ is convex.
Let $\mu_1, \mu_2 \in dom[f \circ g]$ be given, and let $\lambda$ be given with $0 < \lambda < 1$.
Since $g$ is convex, $g(\lambda\mu_1 + (1-\lambda)\mu_2) \leq \lambda g(\mu_1) + (1-\lambda)g(\mu_2)$.
Therefore, since $f$ is non-decreasing,
$f(g(\lambda\mu_1 + (1-\lambda)\mu_2)) \leq f(\lambda g(\mu_1) + (1-\lambda)g(\mu_2))$.
Since $f$ is convex, $f(\lambda g(\mu_1) + (1-\lambda)g(\mu_2)) \leq \lambda f(g(\mu_1)) + (1-\lambda)f(g(\mu_2))$.
Putting this together, we have
$(f \circ g)(\lambda\mu_1 + (1-\lambda)\mu_2) = f(g(\lambda\mu_1 + (1-\lambda)\mu_2)) \leq f(\lambda g(\mu_1) + (1-\lambda)g(\mu_2)) \leq \lambda f(g(\mu_1)) + (1-\lambda)f(g(\mu_2)) = \lambda(f \circ g)(\mu_1) + (1-\lambda)(f \circ g)(\mu_2)$.
Thus $f \circ g$ is convex.

$\square$

### 1.3 Part 3

False. Let $f(x) = -e^x$ and let $g(x) = x^2$. $f$ is concave and non-increasing, $g$ is convex, but $(f \circ g)(x) = -e^{x^2}$ is concave.

### 1.4 Part 4

True. I'm struggling to come up with a formal proof. However, it seems to me that $x \cdot f(x)$ is asymptotically bound below by the function $x$, which is convex, and that since $f(x)$ is non-negative and non-decreasing, $x \cdot f(x)$ would have to curve upward at a rate faster than $x$ which would make it also convex.

### 1.5 Part 5

True. $\frac{d}{dx}(-log(-f(x))) = -\frac{1}{|-f(x)|} \cdot -f'(x) = -f(x)^{-1}f'(x)$ since $f(x) < 0$. $\frac{d^2}{dx^2}(-log(-f(x))) = \frac{d}{dx}(-f(x)^{-1}f'(x)) = -[\frac{d}{dx}(f(x)^{-1}) \cdot f'(x) + f(x)^{-1} \cdot \frac{d}{dx}(f'(x))] = \frac{f'(x)^2}{f(x)^2} - \frac{f''(x)}{f(x)} = \frac{f'(x)^2 - f(x) \cdot f''(x)}{f(x)^2} \geq 0$, since $f'(x)^2 \geq 0$, $f(x)^2 \geq 0$, and $-f(x) \cdot f''(x) \geq 0$ (since $f(x) < 0$ and since $f$ is convex $\therefore f''(x) \geq 0$).

## 2 Problem 2

### 2.1 Part 1

Denote the Hessian of $f$ as $H$ where $H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} \frac{e^{x_1+x_2}}{(e^{x_1}+e^{x_2})^2} & \frac{-e^{x_1+x_2}}{(e^{x_1}+e^{x_2})^2} \\ \frac{-e^{x_1+x_2}}{(e^{x_1}+e^{x_2})^2} & \frac{e^{x_1+x_2}}{(e^{x_1}+e^{x_2})^2} \end{pmatrix}$. The determinants of the principal submatrices of $H$ are

$$det(H_1) = \frac{e^{x_1+x_2}}{(e^{x_1}+e^{x_2})^2} \geq 0$$

and

$$det(H_2) = (\frac{e^{x_1+x_2}}{(e^{x_1}+e^{x_2})^2})^2 - (\frac{-e^{x_1+x_2}}{(e^{x_1}+e^{x_2})^2})^2 = 0 \geq 0$$

The determinants of all principal submatrices are non-negative, so $H$ is a PSD matrix. Thus $f$ is convex.

### 2.2 Part 2

$$\begin{array}{llcll} \min & e^{a-b} & & & \\ \text{s.t.} & -10 & \leq & a & \leq & 3\ ; \\ & e^{2a-\frac{b}{2}} & + & e^{\frac{b}{2}-c} & \leq & 1; \\ & & & a-b & = & 2c \end{array}$$

### 2.3 Part 3

I got an optimal solution of $x = e^{-10}$, $y = e^{-5}$, and $z = e^{-2.5}$ and an optimal value of 0.00673795. This can be reproduced by running the Matlab file P3.m.

# 3 Problem 3

## 3.1 Part 1

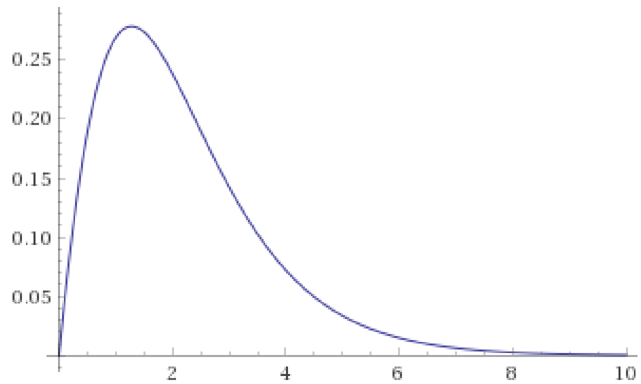


Figure 1: Graph of $r(p)$

Let $u_1 = 4$ and let $u_2 = 8$. Let $l(p)$ be the line segment connecting $r(u_1)$ and $r(u_2)$. We can clearly see that, for all points $p_0$ along the line segment between $u_1$ and $u_2$, $r(p_0) \leq l(p_0)$. Thus $r$ is not concave.

## 3.2 Part 2

$$\lambda = \frac{e^{-p}}{1+e^{-p}}$$
$$\lambda(1+e^{-p}) = e^{-p}$$
$$\lambda + \lambda e^{-p} = e^{-p}$$
$$\lambda e^{p} + \lambda = 1$$
$$\lambda e^{p} = 1 - \lambda$$
$$e^{p} = \frac{1-\lambda}{\lambda}$$
$$p = ln(\frac{1-\lambda}{\lambda})$$

$p\lambda(p) = ln(\frac{1-\lambda}{\lambda}) \cdot \lambda$

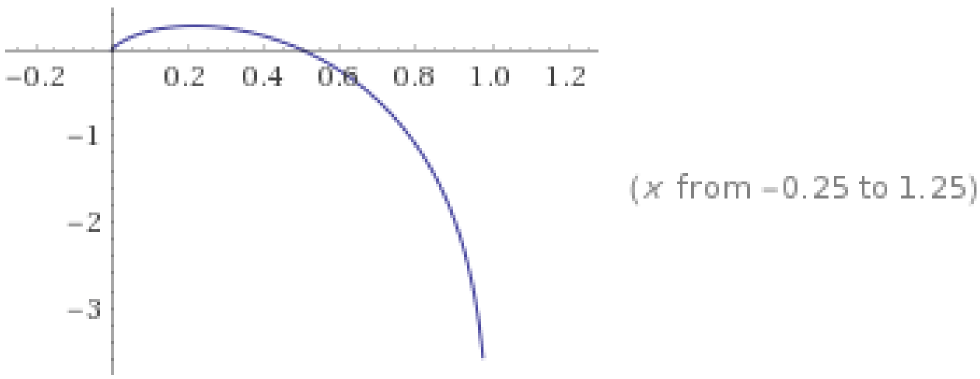


Figure 2: Graph of $r(\lambda)$

Clearly, this function is concave.

## 3.3 Part 3

### 3.3.1 Main Condition

$$\frac{1}{\lambda - 1} + ln(\frac{1}{\lambda} - 1) + \eta_1 - \eta_2 = 0$$

### 3.3.2 Dual Feasibility

$$\eta_1, \eta_2 \geq 0$$

### 3.3.3 Complementarity

$$\eta_1(\lambda - 1) = 0$$

$$\eta(-\lambda) = 0$$

### 3.3.4 Primal Feasibility

$$0 \leq \lambda \leq 1$$

### 3.3.5 Transformed Main Condition

$$\frac{1}{\frac{e^{-p}}{1+e^{-p}} - 1} + ln(\frac{1}{\frac{e^{-p}}{1+e^{-p}}} - 1) + \eta_1 - \eta_2 = 0$$

### 3.3.6 Transformed Dual Feasibility

$$\eta_1, \eta_2 \geq 0$$

### 3.3.7 Transformed Complementarity

$$\eta_1\left(\frac{e^{-p}}{1+e^{-p}} - 1\right) = 0$$

$$\eta\left(-\frac{e^{-p}}{1+e^{-p}}\right) = 0$$

### 3.3.8 Transformed Primal Feasibility

$$0 \leq \frac{e^{-p}}{1+e^{-p}} \leq 1$$

# 4 Problem 4

The two roots I found were 1.7000 and 5.4069. This can be reproduced by running the Matlab file P4.m.