Linearized scalar equation in spherical GR background

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Notation: $G = c = 1, \epsilon_{0123} = 1$**.**

Existence of scalarization can be analyzed using the linearized scalar equation

$$\left(1 - \tfrac{2m}{r}\right) \Phi_{rr} + \left(1 - \tfrac{2m}{r}\right) \left(\nu_r + \tfrac{1}{r-2m} \left(2 - \tfrac{3m}{r} - m_r\right)\right) \Phi_r + \left(f_2 R_{\rm GB} - U_2\right) \Phi = 0, \tag{1}$$

under the spherical GR background spacetime

$$ds^2 = -e^{2\nu} dt^2 + \left(1 - \tfrac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2}$$

where ν and m are known solutions to the GR TOV equations

$$\begin{aligned} \nu_r &= \tfrac{4\pi r^2 p}{r-2m} + \tfrac{m}{r(r-2m)}, \\ m_r &= 4\pi r^2 \epsilon, \\ p_r &= -(\epsilon + p) \left(\tfrac{4\pi r^2 p}{r-2m} + \tfrac{m}{r(r-2m)}\right). \end{aligned} \tag{3}$$

Using Eq. (3), Eq. (1) can be written as

$$\left(1 - \tfrac{2m}{r}\right) \Phi_{rr} + \left(1 - \tfrac{2m}{r}\right) \left(\tfrac{2}{r-2m} \left(1 - \tfrac{m}{r}\right) - \tfrac{4\pi r^2(\epsilon - p)}{r-2m}\right) \Phi_r + \left(f_2 R_{\rm GB} - U_2\right) \Phi = 0, \tag{4}$$

and the Gauss-Bonnet term can be written as

$$R_{\rm GB} = \tfrac{48 m^2}{r^6} - \tfrac{128\pi\epsilon\left(m + 2\pi r^3 p\right)}{r^3}. \tag{5}$$

For numerical calculation, use a length unit r_0 to define the dimensionless quantities

$$x \equiv \tfrac{r}{r_0}, \quad y \equiv \tfrac{m}{r_0}, \quad w \equiv 4\pi r_0^2 \epsilon, \quad v \equiv 4\pi r_0^2 p, \quad f_2 \equiv \tfrac{f_2}{r_0^2}, \quad U_2 \equiv r_0^2 U_2, \tag{6}$$

then the dimensionless equation (4) is

$$\left(1 - \tfrac{2y}{x}\right) \Phi_{xx} + \left(1 - \tfrac{2y}{x}\right) \left(\tfrac{2}{x-2y} \left(1 - \tfrac{y}{x}\right) - \tfrac{x^2(w-v)}{x-2y}\right) \Phi_x + \left(f_2\, r_0^4 R_{\rm GB} - U_2\right) \Phi = 0, \tag{7}$$

with

$$r_0^4 R_{\rm GB} = \tfrac{48 y^2}{x^6} - \tfrac{32 w \left(y + \frac{1}{2} v x^3\right)}{x^3}. \tag{8}$$

I. BOUNDARY CONDITIONS

A. At the center

We are looking for solutions for Eq. (4) regular at the center of the star, namely that Φ has the series expansion

$$\Phi = \sum_{n=0}^{\infty} \Phi_n x^n. \tag{9}$$

We know that physical solutions to the TOV equations (3) have series expansion

$$m = \tfrac{4}{3}\pi\epsilon_c r^3 + \tfrac{2}{5}\pi\epsilon_c'' r^5 + ..., \quad y = \tfrac{1}{3}w_c x^3 + \tfrac{1}{10}w_c'' x^5 + ... \equiv \sum_{n=3,\,\mathrm{odd}}^{\infty} y_n x^n,$$
$$p = p_c - \tfrac{2}{3}\pi\left(\epsilon_c^2 + 4\epsilon_c p_c + 3p_c^2\right) r^2 + ..., \quad v = v_c - \tfrac{1}{6}\left(w_c^2 + 4w_c v_c + 3v_c^2\right) x^2 + ... \equiv \sum_{n=0,\,\mathrm{even}}^{\infty} v_n x^n,$$
$$\epsilon = \epsilon_c + \tfrac{1}{2}\epsilon_c'' r^2 + ..., \quad w = w_c + \tfrac{1}{2}w_c'' x^2 + ... \equiv \sum_{n=0,\,\mathrm{even}}^{\infty} w_n x^n, \tag{10}$$

at the center, where $\epsilon'' \equiv \frac{d^2\epsilon}{dr^2}$, $w'' \equiv \frac{d^2w}{dx^2}$, and the expansions of p and ϵ are related by the EOS

$$\epsilon = \epsilon(p), \quad \frac{d\epsilon}{dr} = \frac{d\epsilon}{dp}\frac{dp}{dr}, \quad \frac{d^2\epsilon}{dr^2} = \frac{d^2\epsilon}{dp^2}\left(\frac{dp}{dr}\right)^2 + \frac{d\epsilon}{dp}\frac{d^2p}{dr^2}, \quad .... \tag{11}$$

Define

$$a(r) := r - 2m, \quad b(r) := 2\left(1 - \tfrac{m}{r}\right) - 4\pi r^2(\epsilon - p), \quad c(r) := r(f_2 R_{\mathrm{GB}} - U_2). \tag{12}$$

They have the expansions

$$\begin{aligned}
a(r) &= r - \tfrac{8}{3}\pi\epsilon_c r^3 - \tfrac{4}{5}\pi\epsilon_c'' r^5 + \dots\,,\\
b(r) &= 2 - 4\pi\left(\tfrac{5}{3}\epsilon_c - p_c\right) r^2 - 2\pi\left(\tfrac{7}{5}\epsilon_c'' - p_c''\right) r^4 + \dots\,,\\
c(r) &= -\left(\tfrac{256}{3}\pi^2 f_2 \epsilon_c(\epsilon_c + 3p_c) + U_2\right) r - \tfrac{128}{3}\pi^2 f_2\left(2\epsilon_c''\epsilon_c + 3\epsilon_c'' p_c + 3\epsilon_c p_c''\right) r^3 + \dots\,,
\end{aligned} \tag{13}$$

or the dimensionless version

$$a(x) = \sum_{n=1,\,\mathrm{odd}}^{\infty} a_n x^n, \quad b(x) = \sum_{n=0,\,\mathrm{even}}^{\infty} b_n x^n, \quad c(x) = \sum_{n=1,\,\mathrm{odd}}^{\infty} c_n x^n. \tag{14}$$

Therefore, the differential equation (7) implies

$$\sum_{n=1,\,\mathrm{odd}}^{\infty} a_n x^n \sum_{n=2}^{\infty} n(n-1)\Phi_n x^{n-2} + \sum_{n=0,\,\mathrm{even}}^{\infty} b_n x^n \sum_{n=1}^{\infty} n\Phi_n x^{n-1} + \sum_{n=1,\,\mathrm{odd}}^{\infty} c_n x^n \sum_{n=0}^{\infty} \Phi_n x^n = 0. \tag{15}$$

The recurrance relations for Φ_n are

$$\begin{aligned}
x^0 :&\quad b_0\Phi_1 = 0,\\
x :&\quad 2a_1\Phi_2 + 2b_0\Phi_2 + c_1\Phi_0 = 0,\\
x^2 :&\quad 6a_1\Phi_3 + 3b_0\Phi_3 + b_2\Phi_1 + c_1\Phi_1 = 0,\\
x^3 :&\quad 12a_1\Phi_4 + 2a_3\Phi_2 + 4b_0\Phi_4 + 2b_2\Phi_2 + c_1\Phi_2 + c_3\Phi_0 = 0,\\
&\dots\\
x^n :&\quad \sum_{i=0,\,\mathrm{even}}^{n-1} (n-i+1)(n-i)a_{i+1}\Phi_{n-i+1} + \sum_{i=0,\,\mathrm{even}}^{n} (n-i+1)b_i\Phi_{n-i+1} + \sum_{i=0,\,\mathrm{even}}^{n-1} c_{i+1}\Phi_{n-i-1} = 0,
\end{aligned} \tag{16}$$

which restricts the expansion of Φ to even terms.

B. At infinity

Outside the star, the GR metric takes the form of the Schwarzschild spactime. The Gauss-Bonnet term is proportional to $\frac{1}{r^6}$ so can be neglected when considering the asymptotic behavior of the scalar. Then, Eq. (4) has the asymptotic solution

$$\Phi \to \frac{C_+}{r} e^{\sqrt{U_2}r} + \frac{C_-}{r} e^{-\sqrt{U_2}\,r}. \tag{17}$$

For $U_2 > 0$, the physical solution has the nontrivial requirement $C_+ = 0$.

II. NUMERICAL RESULTS

A. Exterior solutions

Outside the star, taking the radius of the star as the length unit, Eq.(7) becomes

$$\left(1-\tfrac{2\mathcal{C}}{x}\right)\Phi_{xx}+\tfrac{2}{x}\left(1-\tfrac{\mathcal{C}}{x}\right)\Phi_x+\left(f_2\, r_0^4 R_{\rm GB}-U_2\right)\Phi=0, \tag{18}$$

with $r_0^4 R_{\rm GB}=\frac{48\mathcal{C}^2}{x^6}$, where $\mathcal{C}$ is the compactness of the star. The equation is solved from $x=1$ using shooting method.

B. Interior solutions

1. *Uniform star*

Consider the toy model where ϵ is constant. Set $r_0=\sqrt{\frac{3}{8\pi\epsilon}}$ for the moment, then the dimensionless GR quantities are

$$y=\tfrac{1}{2}x^3,\quad w=\tfrac{3}{2},\quad v=\tfrac{3\eta\sqrt{1-x^2}-1}{2(1-\eta\sqrt{1-x^2})},\quad \nu'=\tfrac{2x}{1-x^2}(v+\tfrac{1}{2}), \tag{19}$$

where the integral constant η is related to v_c by

$$\eta=\frac{v_c+\frac{1}{2}}{v_c+\frac{3}{2}}. \tag{20}$$

Inside the star, Eq.(7) becomes

$$\left(1-x^2\right)\Phi_{xx}+\left(\tfrac{2}{x}-\tfrac{5}{2}x+xv\right)\Phi_x+\left(f_2\, r_0^4 R_{\rm GB}-U_2\right)\Phi=0, \tag{21}$$

with

$$r_0^4 R_{\rm GB}=-12\left(1+2v\right)=-\tfrac{24\eta\sqrt{1-x^2}}{1-\eta\sqrt{1-x^2}}<0. \tag{22}$$

Outside the star, Eq.(7) becomes

$$\left(1-\tfrac{2y_M}{x}\right)\Phi_{xx}+\tfrac{2}{x}\left(1-\tfrac{y_M}{x}\right)\Phi_x+\left(f_2\, r_0^4 R_{\rm GB}-U_2\right)\Phi=0, \tag{23}$$

with

$$y_M=\tfrac{1}{2}x_M^3=\tfrac{1}{2}\left(1-\tfrac{1}{9\eta^2}\right)^{\frac{3}{2}}, \tag{24}$$

and

$$r_0^4 R_{\rm GB}=\tfrac{48y_M^2}{x^6}>0. \tag{25}$$

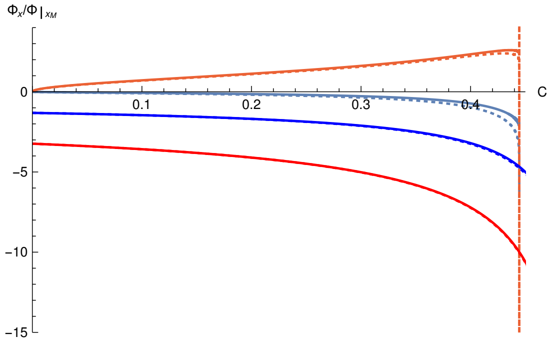
We numerically solve the interior equation (21) from $x_{\rm min}$ close to the center. The boundary condition can be obtained from the recurrance relation in (16)

$$\Phi_2=-\tfrac{c_1}{6}\Phi_0=\left(\tfrac{8}{9}f_2 w_c(w_c+3v_c)+\tfrac{1}{6}U_2\right)\Phi_0. \tag{26}$$

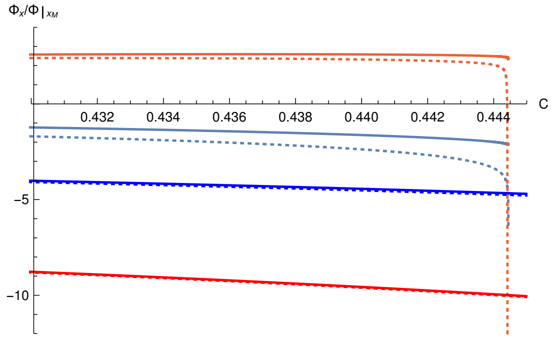
Take $\Phi(x_{\rm min})=1$, then

$$\Phi'(x_{\rm min})=2\Phi_2 x_{\rm min}=\left(8f_2(\tfrac{1}{2}+v_c)+\tfrac{1}{3}U_2\right)x_{\rm min}. \tag{27}$$

To connect the exterior solution and the interior solution, Φ'/Φ has to be matched at the surface of the star. Figures 1 and 3 shows examples of the change of Φ'/Φ at the surface of the star with respect to star compactness $\mathcal{C}$.



(a) $\Phi'(x_M)/\Phi(x_M)$ vs. $\mathcal{C}$.



(b) Left plot near $\mathcal{C} = \frac{4}{9}$. The branch that goes to infinity for $U_2 = 5$, $f_2 = -0.06$ is dropped.

FIG. 1: Change of Φ'/Φ at the surface of the star with respect to star compactness $\mathcal{C}$ for $U_2 = 0.1$ (blue curves) and 5 (red curves). For $U_2 = 0.1$, the curves shown have $f_2 = -0.05$ (dotted curves) and -0.04 (solid curves). For $U_2 = 5$, the curves shown have $f_2 = -0.06$ (dotted curves) and -0.05 (solid curves). The curves in deep colors show the exterior values of Φ'/Φ, while the curves in light colors show the interior values of Φ'/Φ.

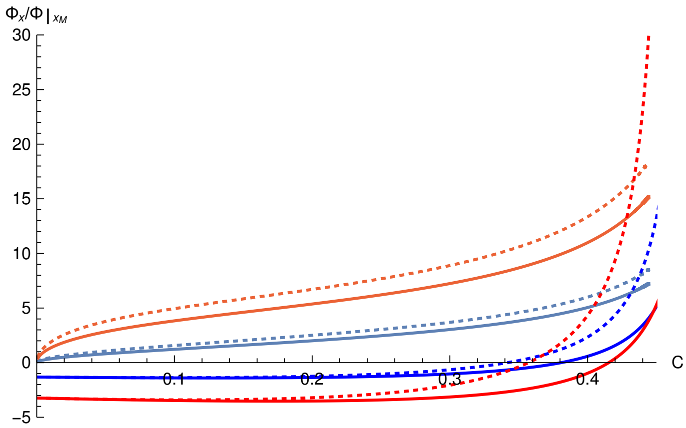


FIG. 2: $\Phi'(x_M)/\Phi(x_M)$ vs. $\mathcal{C}$.

FIG. 3: Change of Φ'/Φ at the surface of the star with respect to star compactness $\mathcal{C}$ for $U_2 = 0.1$ (blue curves) and 5 (red curves). For $U_2 = 0.1$, the curves shown have $f_2 = 0.8$ (dotted curves) and 0.6 (solid curves). For $U_2 = 5$, the curves shown have $f_2 = 3$ (dotted curves) and 2 (solid curves). The curves in deep colors show the exterior values of Φ'/Φ, while the curves in light colors show the interior values of Φ'/Φ.