CSCI 2824 – Spring 2020 **Homework 8**

This assignment is due on Friday, Mar 13 to Gradescope by 11:59pm. You are expected to write up your solutions neatly and **use the coverpage**. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: On the **official CSCI 2824 cover page** of your assignment clearly write your full name, the lecture section you belong to (001 or 002), and your student ID number. You may **neatly** type your solutions for +2 extra credit on the assignment. You will lose *all* 5 style/neatness points if you fail to use the official cover page.

(1) The following are riffs on exam one questions that many students struggled with.

(a) Name two domains - with the same domain for both x and y - such that *both* the statements $\forall x \exists y \left(x^2 \geq y\right)$ and $\exists x \forall y \left(x^2 > y\right)$ are **true**? Next, name two domains - with the same domain for both x and y - such that *both* the statements $\forall x \exists y \left(x^2 \geq y\right)$ and $\exists x \forall y \left(x^2 > y\right)$ are **false**?

(b) Suppose sets A and B satisfy $A \cup B = A$. What can you conclude about sets A and B? Explain.

(c) Suppose sets A and B satisfy $A - B = A$. What can you conclude about sets A and B? Explain.

Solution:

(a) $\forall x \exists y \left(x^2 \geq y\right)$ and $\exists x \forall y \left(x^2 > y\right)$ are **TRUE**

(i) $x, y \,\exists\, \mathbb{Z} \geq 0$

(ii) $x, y \,\exists\, \mathbb{Z} \leq 0$

$\forall x \exists y \left(x^2 \geq y\right)$ and $\exists x \forall y \left(x^2 > y\right)$ are **FALSE**

(i) $0 < x, y < 1$

(ii) $-1 < x, y < 0$

(b) This tells us for sure that either $B = \emptyset$ or $B \subseteq A$

(c) This tells us that A and B must have no elements in common i.e. $A \cap B = \emptyset$. B may $= \emptyset$

(2) Use divisibility and modular arithmetic to answer the following, showing all work:

(a) Find the greatest common divisor of $a = 8640$ and $b = 102816$.

(b) Determine whether $c = 733$ is prime or not by checking its divisibility by prime numbers up to $\sqrt{c}$.

Solution:

(a) $a = 8640$ and $b = 102816$
$a = 2^6 \cdot 3^2 \cdot 5$ and $b = 2^5 \cdot 3^3 \cdot 7 \cdot 17$
$\gcd(a \cdot b) = d$ then $\gcd(ka, kb) = |k|\gcd(a, b)$
$\therefore \gcd(8640, 102816) = \gcd(2^6 \cdot 3^3 \cdot 5, 2^5 \cdot 3^3 \cdot 7 \cdot 17)$
$= 2^5 \cdot 3^3 \gcd(2 \cdot 5, 7 \cdot 17)$
$= 2^5 \cdot 3^3 \gcd(10, 119)$
$= 32 \cdot 27$
$= \mathbf{864}$

(b) $c = 733$, $\sqrt{c} = \sqrt{733} \approx 27$
Prime numbers $< 27 = 2, 3, 5, 7, 11, 13, 17, 19, 23$

And 733 is divisible by none of those evenly
$\therefore$ 733 **is a prime number**

(3) Consider the function $f(n) = 7n^4 + 22n^4 \log(n) - 5n^2 \log(n^2)$ which represents the complexity of some algorithm.

(a) Find a tight big-**O** bound of the form $g(n) = n^p$ for the given function f with some natural number p. What are the constants C and k from the big-**O** definition?

(b) Find a tight big-$\boldsymbol{\Omega}$ bound of the form $g(n) = n^p$ for the given function f with some natural number p. What are the constants C and k from the big-$\boldsymbol{\Omega}$ definition?

(c) Can we conclude that f is big-$\boldsymbol{\Theta}(n^p)$ for some natural number p?

Solution:

(a) $f(n) = 7n^4 + 22n^4 \log(n) - 5n^2 \log(n^2)$
$|f(n)| \leq |7n^4| + |22n^4 \log(n)| + |10n^2 \log(n)|$ $\forall n \exists \mathbb{N}$
$|f(n)| \leq 7n^5 + 22n^4 + 10n^2 n^3$ $\forall n \geq 1$
$|f(n)| \leq 39n^5$ $\forall n \geq 1$ Since $\log(n) \leq n$
$g(n) = n^5$ By definition of big-**O**
$\therefore P = 5$

(b) $f(n) = 7n^4 + 22n^4 \log(n) - 5n^2 \log(n^2)$
$f(n) = 7n^4 + 2n^2 \log(n)[11n^2 - 5]$
$n^2 \geq 0 \ \log(n) \geq 0 \ 11n^2 - 5 \geq 0$ $\forall n \exists \mathbb{N}$
$\therefore 2n^2 \log(n)[11n^2 - 5] \geq 0$ $\forall n \geq 1$
$f(n) \geq 7n^4$
$\therefore g(n) = n^4$ By the definition of big-$\boldsymbol{\Omega}$
$\therefore P = 4$

(c) **No**, we cannot conclude for some natural number p

(4) Consider the function $g(n) = 2^n + \dfrac{n(n+1)}{2} - \log\left(n^{n^n}\right)$ which represents the complexity of some algorithm.

(a) Between 2^n and $\frac{n(n+1)}{2}$, which function grows *asymptotically faster* as $n \to \infty$? Justify by computing an appropriate limit.

(b) Between 2^n and $\log\left(n^{n^n}\right)$, which function grows *asymptotically faster* as $n \to \infty$? Justify by computing an appropriate limit.

(c) Between $\frac{n(n+1)}{2}$ and $\log\left(n^{n^n}\right)$, which function grows *asymptotically faster* as $n \to \infty$? Justify by computing an appropriate limit.

(d) What is the order of g?

Solution:

(a) $2^n = n^2$ for $n = 4$
$2^n > n^2$ for $n \geq 5$
and $2^n > n^2 + n$ for $n \geq 5$
$\therefore 2^n \gg \frac{n(n+1)}{2}$ for $n \to \infty$

(b) $\log(n^{n^n}) = n \log(n^n) = n^2 \log(n)$
$\log(n) < n$ for $(n \to \infty)$
$n^2 \log(n) < n^3$ for $n \to \infty$
And because $2^n > n^3$ for $n \geq 10$
$2^n \gg n^3 \gg n \log(n)$ for $n \to \infty$

(c) $\log(n^{n^n}) = n\log(n^n) = n^2\log(n)$
$\frac{n^2+n}{2} = n^2\log(n)$ for $n \approx 4$
$n^3 \sim n^2\log(n) \gg \frac{n^2+n}{2}$ for $n >\sim 4$

(d) $g = 2^n + \frac{n(n+1)}{2} + n^2\log(n)$
$= 1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{3} + \ldots + 1$
If n is even: $= \frac{n!}{\frac{n}{2}!\frac{n}{2}!}$
If n is odd: $= \frac{n(n-1)(n-2)\ldots(n-(\frac{n}{2}+1))(\frac{n}{2})!}{\frac{n}{2}!\frac{n}{2}!}$
$\therefore$ minimum degree is $\frac{n}{2}$ for an even n and $\frac{n}{2}+1$ for n odd.

(5) Consider the following matrices.

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

For this problem, we will calculate product $P = ABC$. Note that matrix multiplication is associative, which means we can calculate the product P by first computing the matrix (AB), then multiplying this by C to obtain $P = (AB)C$. **Or** we could first compute the matrix (BC), then multiply it by A to obtain $P = A(BC)$. Now recall that to multiply an $m \times n$ matrix by an $n \times k$ matrix requires $m \times n \times k$ *multiplications.*

(a) Suppose A is $m \times n$, B is $n \times k$ and C is $k \times p$. How many multiplications are needed to calculate P in the order $(AB)C$? Do not just write down an expression; show your work/justification!

(b) For the same matrix dimensions specified in (a), how many multiplications are needed to calculate P in the order $A(BC)$? Again, do not just write down an expression.

(c) Based on the specific dimensions of A, B, and C in the problem description, which multiplication order would be the most efficient?

(d) Calculate $P = ABC$ using whichever order you specified in part (c).

Solution:

(a) $AB = (A \times B) = ((m \times n) \times (n \times k)) = (m \times n \times k)$ and will result in a matrix of size $m \times k$
$(AB \times C) = ((m \times k) \times (k \times p)) = (m \times k \times p)$ and will result in a matrix of size $m \times p$
$\therefore$ the amount of multiplications is $(\mathbf{m} \times \mathbf{n} \times \mathbf{k}) + (\mathbf{m} \times \mathbf{k} \times \mathbf{p})$

(b) $BC = (B \times C) = ((n \times k) \times (k \times p)) = (n \times k \times p)$ and will create a matrix of size $n \times p$
$(A \times (B \times C) = ((m \times n) \times (n \times p)) = (m \times n \times p)$ and creates a matrix of size $m \times p$
$\therefore$ the amount of multiplication is $(\mathbf{n} \times \mathbf{k} \times \mathbf{p}) + (\mathbf{m} \times \mathbf{n} \times \mathbf{p})$

(c) These specific dimensions: $A = C = 4 \times 2$, $B = 2 \times 4$
$m = 4,\ n = 2,\ k = 4,\ p = 2$
Case 1:
$(AB)C = (m \times n \times k) + (m \times k \times p)$
$= (4 \cdot 2 \cdot 4) + (4 \cdot 4 \cdot 2)$
$= (32) + (32) = 64$
Case 2:
$A(BC) = (n \times k \times p) + (m \times n \times p)$
$= (2 \cdot 4 \cdot 2) + (4 \cdot 2 \cdot 2)$
$= (16) + (16) = 32$
$\therefore A(BC)$ is more efficient

(d) We'll use $A(BC)$ like we established above, first calculate BC

$$B = \begin{bmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$BC = \begin{bmatrix} (2 + (2 \cdot 2) + 2 + 2) & (2 + (2 \cdot 2) + 2 + 0) \\ (1 + 2 + 1 + (2 \cdot 2)) & (1 + 2 + 1 + 2) \end{bmatrix}$$

$$BC = \begin{bmatrix} 10 & 8 \\ 8 & 4 \end{bmatrix}$$

$P = A(BC)$

$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}$ and $BC = \begin{bmatrix} 10 & 8 \\ 8 & 4 \end{bmatrix}$

$P = \begin{bmatrix} (10 + (4 \cdot 8)) & (8 + (4 \cdot 4)) \\ (10 + 8) & (8 + 4) \\ (10 + (4 \cdot 8)) & (8 + (4 \cdot 4) \\ (10 + 8) & (8 + 4) \end{bmatrix}$

$P = \begin{bmatrix} 42 & 24 \\ 18 & 12 \\ 42 & 24 \\ 18 & 12 \end{bmatrix}$

(6) Use induction to show that $\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$. Be sure to state whether you're using weak or strong induction.

Solution: Let $s(n) : \sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for $n = 1 :$
$\sum_{i=0}^{n} i^3 = 0^3 + 1^3 = 1$
$\frac{n^2(n+1)^2}{4} = \frac{1(1+1)^2}{4} = 1$
$\therefore s(1)$ is true
Assume $s(k) : \sum_{i=0}^{k} i^3 = \frac{k^2(k+1)^2}{4}$
Now, we will prove that $s(k+1)$ i.e. $\sum_{i=0}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$
$\sum_{i=0}^{k+1} i^3 = \sum_{i=0}^{k} i^3 + (k+1)^3$
$= \frac{k^2(k+1)^2}{4} + (k+1)^3$
$= (k+1)^2[\frac{k^2}{4} + (k+1)]$
$= (k+1)^2(\frac{k^2+4k+4}{4})$
$= \frac{(k+1)^2(k+2)^2}{4}$
$\therefore s(k+1)$ is true $\therefore s(n)$ is true by **weak** induction.

(7) Let $A_1, A_2, \ldots A_n$ be sets. Use induction to show that for $n \geq 2$, the cardinality of the union of n sets is always less than or equal to the sum of the cardinalities of those sets. In other words, show:

$$\left| \bigcup_{i=1}^{n} A_i \right| \leq \sum_{i=1}^{n} |A_i|$$

Be sure to state whether you're using weak or strong induction.
Hint: use the same rule that HW 4 #6 was based around.

Solution: For $n = 2 :$
$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
$\leq |A_1| + |A_2|$
So for $n = 2$ it is true.
We'll use **strong** induction on n
Let the result be true for $n \geq 2$:

$$|\bigcup_{i=1}^{n} A_i| = |(\bigcup_{i=1}^{n} A_i) \cup A_{n+1}| \leq |\bigcup_{i=1}^{n} A_i| + |A_{n+1}|$$

$$\leq \sum_{i=1}^{n} |A_i| + A_{n+1} = (\sum_{i=1}^{n+1} A_i)$$