CSCI 2824 – Spring 2020 **Homework 8**

This assignment is due on Friday, Mar 13 to Gradescope by 11:59pm. You are expected to write up your solutions neatly and **use the coverpage**. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: On the **official CSCI 2824 cover page** of your assignment clearly write your full name, the lecture section you belong to (001 or 002), and your student ID number. You may **neatly** type your solutions for +2 extra credit on the assignment. You will lose *all* 5 style/neatness points if you fail to use the official cover page.

(1) The following are riffs on exam one questions that many students struggled with.

(a) Name two domains - with the same domain for both x and y - such that *both* the statements $\forall x \exists y \left(x^2 \geq y\right)$ and $\exists x \forall y \left(x^2 > y\right)$ are **true**? Next, name two domains - with the same domain for both x and y - such that *both* the statements $\forall x \exists y \left(x^2 \geq y\right)$ and $\exists x \forall y \left(x^2 > y\right)$ are **false**?

(b) Suppose sets A and B satisfy $A \cup B = A$. What can you conclude about sets A and B? Explain.

(c) Suppose sets A and B satisfy $A - B = A$. What can you conclude about sets A and B? Explain.

(2) Use divisibility and modular arithmetic to answer the following, showing all work:

(a) Find the greatest common divisor of $a = 8640$ and $b = 102816$.

(b) Determine whether $c = 733$ is prime or not by checking its divisibility by prime numbers up to $\sqrt{c}$.

(3) Consider the function $f(n) = 7n^4 + 22n^4 \log(n) - 5n^2 \log(n^2)$ which represents the complexity of some algorithm.

(a) Find a tight big-$\mathbf{O}$ bound of the form $g(n) = n^p$ for the given function f with some natural number p. What are the constants C and k from the big-$\mathbf{O}$ definition?

(b) Find a tight big-$\mathbf{\Omega}$ bound of the form $g(n) = n^p$ for the given function f with some natural number p. What are the constants C and k from the big-$\mathbf{\Omega}$ definition?

(c) Can we conclude that f is big-$\mathbf{\Theta}(n^p)$ for some natural number p?

(4) Consider the function $g(n) = 2^n + \dfrac{n(n+1)}{2} - \log\left(n^{n^n}\right)$ which represents the complexity of some algorithm.

(a) Between 2^n and $\frac{n(n+1)}{2}$, which function grows *asymptotically faster* as $n \to \infty$? Justify by computing an appropriate limit.

(b) Between 2^n and $\log\left(n^{n^n}\right)$, which function grows *asymptotically faster* as $n \to \infty$? Justify by computing an appropriate limit.

(c) Between $\frac{n(n+1)}{2}$ and $\log\left(n^{n^n}\right)$, which function grows *asymptotically faster* as $n \to \infty$? Justify by computing an appropriate limit.

(d) What is the order of g?

(5) Consider the following matrices.

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

For this problem, we will calculate product $P = ABC$. Note that matrix multiplication is associative, which means we can calculate the product P by first computing the matrix (AB), then multiplying this by C to obtain $P = (AB)C$. **Or** we could first compute the matrix (BC), then multiply it by A to obtain $P = A(BC)$. Now recall that to multiply an $m \times n$ matrix by an $n \times k$ matrix requires $m \times n \times k$ *multiplications.*

(a) Suppose A is $m \times n$, B is $n \times k$ and C is $k \times p$. How many multiplications are needed to calculate P in the order $(AB)C$? Do not just write down an expression; show your work/justification!

(b) For the same matrix dimensions specified in (a), how many multiplications are needed to calculate P in the order $A(BC)$? Again, do not just write down an expression.

(c) Based on the specific dimensions of A, B, and C in the problem description, which multiplication order would be the most efficient?

(d) Calculate $P = ABC$ using whichever order you specified in part (c).

(6) Use induction to show that $\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$. Be sure to state whether you're using weak or strong induction.

(7) Let $A_1, A_2, \ldots A_n$ be sets. Use induction to show that for $n \geq 2$, the cardinality of the union of n sets is always less than or equal to the sum of the cardinalities of those sets. In other words, show:

$$\left| \bigcup_{i=1}^{n} A_i \right| \leq \sum_{i=1}^{n} |A_i|$$

Be sure to state whether you're using weak or strong induction.
Hint: use the same rule that HW 4 #6 was based around.