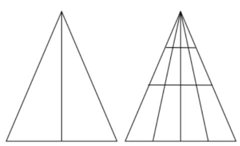
CSCI 2824 – Spring 2020 **Homework 10**

This assignment is due on Wednesday, Apr 22 to Gradescope by 11:59pm. You are expected to write up your solutions neatly and **use the coverpage**. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: On the **official CSCI 2824 cover page** of your assignment clearly write your full name, the lecture section you belong to (001 or 002), and your student ID number. You may **neatly** type your solutions for +2 extra credit on the assignment. You will lose *all* 5 style/neatness points if you fail to use the official cover page.

(1) A common "brainteaser" puzzle asks the following: suppose we draw an isosceles triangle, and then draw k lines from the central vertex to various unique points on the opposite edge. Suppose we also draw n unique lines through the triangle, each parallel to the opposing edge. We are going to count how many distinct triangles have been created.

For example, the $n = 0$ and $k = 1$ triangle is shown on the left. It contains 3 distinct triangles: the outer lines form a triangle, and the left and right sides also form triangles. The $n = 2$, $k = 3$ triangle is shown at right.



What is the total of number of triangles as a function of n and k? Use a sentence to explain why your result is the case.

Solution:
As you draw more and more vertical lines, the number of triangles increases linearly, in other words:
triangles due to only vertical $= \sum_{n=1}^{k+1} n$
triangles due to only horizontal $= (n+1)\sum_{p=1}^{k+1} p$

$\therefore$ total triangles $= (n+1)\frac{(k+1)(k+2)}{2}$

(2) Ask yourself: when you draw cards from a deck, does the order matter? Answer the following:

(a) You have a standard 52 card deck. If you draw two cards at random, what is the probability that they are both hearts?

(b) You reshuffle your cards into the deck, but unfortunately now a mischievous dog (a portly Beagle named Lola), decides to eat one of the cards! Assuming that the missing card is a diamond, now when you draw two cards at random what is the probability that they are both hearts?

(c) Suppose you have your now-51 card deck, and you randomly draw two cards from it. Both are hearts. Given this, what is the probability that the missing card is a diamond?

(d) Are the events (two cards drawn at random are both spades) and (the missing card is a diamond) independent? Justify your answer.

Solution:

(a) $\frac{\binom{13}{2}}{\binom{52}{2}} = \mathbf{0.0588}$

(b) P(both are spades / missing card is diamond)

$$= \frac{0.0588 \cdot \frac{\binom{13}{1}}{\binom{52}{1}}}{52 \cdot \frac{\binom{13}{1}}{\binom{52}{1}}} = \mathbf{0.0588}$$

(c) $= \frac{\binom{13}{52} \cdot 0.0588}{0.0588} = \mathbf{0.25}$

(d) $P(A/B) = P(A)$ and $P(B/A) = P(B)$
A and B are both **independent**
Where $A =$ both are spades and $B =$ missing card is a diamond

(3) In Monopoly, your token is allowed to leave the "jail" cell if you roll doubles: you roll two 6-sided dice and each shows the same face. Zach hates being in jail, because it reminds him of watching "The Wire" on TV. So he invents a couple of weighted dice that are *not independent*. In particular, if you roll either die on its own it's a fair die: each outcome has probability $1/6$. But if you roll one die and then the other, the red die will take the same outcome as the blue die exactly half the time: all other outcomes are equally likely.

(a) Suppose you roll a 3 on the blue die. What is the probability distribution of the red die *given* this outcome on the blue die?

(b) Find the full probability distribution for the value of the *sum* of the two faces of the dice.

(c) What is the probability you roll doubles?

(d) What is the probability that you roll a 7 as the sum of the two dice?

Solution:

(a)

x	1	2	3	4	5	6
$P(x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

(b)

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{5}{60}$	$\frac{2}{60}$	$\frac{7}{60}$	$\frac{4}{60}$	$\frac{9}{60}$	$\frac{6}{60}$	$\frac{9}{60}$	$\frac{4}{60}$	$\frac{7}{60}$	$\frac{2}{60}$	$\frac{5}{60}$

(c) Each pair has a prob of $\frac{1}{12}$
$\therefore P = 6 \times \frac{1}{12} = \mathbf{\frac{1}{2}}$

(d) From table : $P = \frac{6}{60} = \mathbf{\frac{1}{12}}$

(4) You've learned something in CSCI2824, and it's this: "Always bring Mudkip." So you collected some Mudkips for your Pokémon collection. Your Mudkips can be either blue (B) or purple (P) (exclusive) in coloration, and can use either water (W) attacks or ground (G) attacks (exclusive). 80% of your Mudkips are blue: the rest are purple. Of your blue Mudkips, 80% use water attacks. Of your purple Mudkips, 55% use water attacks.

(a) Suppose you pick a Mudkip at random from your Pokébox. What is $P(G)$, the probability that your random selection uses ground attacks?

(b) Suppose you pick a Mudkip at random and it happens to use a ground attack. Given this information, what's the probability that the Mudkip is purple?

Solution:

(a) $P(B) = 0.80$
$P(P) = 0.20$
$P(W|B) = 0.80$
$P(W|P) = 0.20$
$P(G) + P(W) = 1$
Total Probability Law states:
$P(W) = P(W|B) \times P(B) + P(B) + P(W|P) \times P(P)$
$P(W) = 0.75 \therefore P(G) = 1 - 0.75 = \mathbf{0.25}$

(b) Total probability:
$P(P) = P(P|G) \times P(G) + P(P|W) \times P(W)$
$P(P|G) = \frac{P(P) - P(P|W) \times P(W)}{P(G)}$
Bayes theorem:
$P(P|W) = \frac{P(W|P) \times P(P)}{P(W)} = 0.1467$
$\therefore P(P|G) = \mathbf{0.36}$

(5) It's day 2576 of post-apocalyptic quarantine, and you're bored. Really bored. For entertainment, you've taken to exchanging text messages with your best friend - or at least they were your best friend 2577 days ago - that are just contiguous strings of the four emojis: 😉 😇 🐱 ☕. For example, one such text message might read "🐱☕😉☕😇🐱".

(a) Find a recurrence relation for the number of possible length-n emoji strings that do not contain two consecutive cat emojis, 🐱🐱.

(b) What are the initial conditions for the recurrence relation?

(c) Find a closed-form solution to the recurrence relation you found in part (a) by solving for the roots of the characteristic polynomial and then using initial conditions to determine the constants.

(d) Use your closed form expression to determine the number of length-7 emoji strings that do not contain 2 consecutive cat emojis.

Solution:

(a) Case 1: the first emoji is not a cat, in which case the remaining $n-1$ emojis can be any valid string of length $n-1$.
$\therefore$ there would be $3T_{n-1}$ valid strings.
Case 2: the first emoji is a cat, in which case the second emoji would also not be. Then the remaining $n-2$ characters could be any string of length $n-2$.
$\therefore \mathbf{T_n = 3T_{n-1} + 3T_{n-2}}$

(b) With no emojis we get a single option $\mathbb{T}(\nvdash) = \nVdash$
With one emoji we have $\mathbf{T(1) = 4}$

(c) To find closed : make $T(n) = x^n$
$x^n = 3x^{n-1} + 3x^{n-2}$
$x^2 - 3x - 3 = 0$
$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-3)}}{2(1)}$
$= \frac{3 \pm \sqrt{21}}{2}$
$\therefore T(n) = A(\frac{3 \pm \sqrt{21}}{2})^n + B(\frac{3 \pm \sqrt{21}}{2})^n$ for $T(0) = 1$ and $T(1) = 4$
$1 = A + B$
$4 = A(\frac{3 \pm \sqrt{21}}{2})^1 + B(\frac{3 \pm \sqrt{21}}{2})^1$
$(3 + \sqrt{21})A + (3 - \sqrt{21})B = 8$
$2\sqrt{21}B = -5 + \sqrt{21}$

$A = 1 - B$
$A = 1 - (\frac{\sqrt{21}-5}{2\sqrt{2}}$
$= \frac{\sqrt{21}+5}{2\sqrt{21}}$
$\therefore \mathbf{T(n) = (\frac{\sqrt{21}+5}{2\sqrt{21}})(\frac{3+\sqrt{21}}{2})^n + (\frac{\sqrt{21}-5}{2\sqrt{21}})(\frac{3-\sqrt{21}}{2})^n}$

(d) Using above equation $T(7) = (\frac{\sqrt{21}+5}{2\sqrt{21}})(\frac{3+\sqrt{21}}{2})^7 + (\frac{\sqrt{21}-5}{2\sqrt{21}})(\frac{3-\sqrt{21}}{2})^7$
$= \frac{1}{256}(2624672)$
$\approx \mathbf{10253}$

(6) Consider the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2} + n^2$ with initial conditions $a_0 = 0$ and $a_1 = 7$.

Find a closed form solution for the given recurrence relation. In your solution, put a box around each of the following, and clearly label them:

(a) the characteristic polynomial

(b) the solution to the associated homogeneous recurrence relation ($a_n^{(h)}$)

(c) the full particular solution **guess** that you are plugging into the full nonhomogeneous recurrence relation ($a_n^{(p)}$)

(d) the full general solution (with unknown coefficients still) (a_n)

(e) the full solution to the initial value problem (having now solved for any unknown coefficients)

Solution:

(a) Characteristic polynomial : $\mathbf{x^2 - 2x - 3}$

(b) $\therefore$ characteristic equation : $x = -1, 3$
$\mathbf{a_n^{(h)} = c_1(-1)^n + c_2(3)^n}$

(c) Because RHS is n^2, we assume the particular solution as:
$\mathbf{a_n^{(p)} = An^2 + Bn + C}$

(d) Plugging into each other :
$n^2 = (An^2 + Bn + C) - 2(A(n-1)^2 + B(n-1) + C) - 3(A(n-2)^2 + B(n-2) + C)$
-or-
$n^2 = -4An^2 + (-4B + 16A)n + (-4C + 8B - 14A)$
Comparing both sides :
$-4A = 1 \rightarrow A = -\frac{1}{4}$
$-4B + 16A = 0 \rightarrow B = 4A = 1$
$-4C + 8B - 14A = 0 \rightarrow C = \frac{1}{4}(8B - 14A) = \frac{-9}{8}$
$\therefore a_n^{(p)} = -\frac{1}{4}n^2 - n - \frac{9}{8}$
$\therefore$ general solution:
$\mathbf{a_n = a_n^{(h)} + a_n^{(p)} = c_1(1)^n + c_2(3)^n - \frac{1}{4}n^2 - n - \frac{9}{8}}$

(e) Using initial conditions :
$a_0 = c_1 + c_2 - \frac{9}{8} = 0 \rightarrow c_1 + c_2 = \frac{9}{8}$
$a_1 = -c_1 + 3c_2 - \frac{1}{4} - 1 - \frac{9}{8} = 7 \rightarrow -c_1 + 3c_2 = \frac{75}{8}$
$\therefore c_1 = \frac{-3}{2}, c_2 = \frac{21}{8}$
$\therefore \mathbf{a_n = \frac{-3}{2}(-1)^n + \frac{21}{8}(3)^n - \frac{1}{4}n^2 - n - \frac{9}{8}}$