CSCI 2824
Spring 2020

Write **clearly**:

Name:	Ryan Taylor
Student ID:	109290202
Section number:	002
Assignment:	Homework #8

Read the following:

- This cover sheet must be included as the first page for all written homework submissions to CSCI 2824.
- Fill out all of the fields above.
- Submit your written homework assignment to the electronic dropbox. You will receive graded feedback through the same mechanism.
- If you type up your homework assignment using MS Word or LaTeX, then you can earn two extra credit points per homework assignment. You **must** use properly formatted equations and nice-looking text in order to be eligible for this extra credit point. If you type it up and do not format equations properly or do not use the cover sheet (for example), you might still lose the style/neatness points.
- By submitting this assignment, you are agreeing that you have abided by the **CU Honor Code**, and that this submission constitutes your own original work.

CSCI 2824 – Spring 2020 **Homework 8**

This assignment is due on Friday, Mar 13 to Gradescope by 11:59pm. You are expected to write up your solutions neatly and **use the coverpage**. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: On the **official CSCI 2824 cover page** of your assignment clearly write your full name, the lecture section you belong to (001 or 002), and your student ID number. You may **neatly** type your solutions for +2 extra credit on the assignment. You will lose *all* 5 style/neatness points if you fail to use the official cover page.

(1) The following are riffs on exam one questions that many students struggled with.

(a) Name two domains - with the same domain for both x and y - such that *both* the statements $\forall x \exists y \left(x^2 \geq y\right)$ and $\exists x \forall y \left(x^2 > y\right)$ are **true**? Next, name two domains - with the same domain for both x and y - such that *both* the statements $\forall x \exists y \left(x^2 \geq y\right)$ and $\exists x \forall y \left(x^2 > y\right)$ are **false**?

(b) Suppose sets A and B satisfy $A \cup B = A$. What can you conclude about sets A and B? Explain.

(c) Suppose sets A and B satisfy $A - B = A$. What can you conclude about sets A and B? Explain.

Solution:

(a) $\forall x \exists y \left(x^2 \geq y\right)$ and $\exists x \forall y \left(x^2 > y\right)$ are **TRUE**

(i) $x, y \, \exists \, \mathbb{Z} \geq 0$

(ii) $x, y \, \exists \, \mathbb{Z} \leq 0$

$\forall x \exists y \left(x^2 \geq y\right)$ and $\exists x \forall y \left(x^2 > y\right)$ are **FALSE**

(i) $0 < x, y < 1$

(ii) $-1 < x, y < 0$

(b) This tells us for sure that either $B = \emptyset$ or $B \subseteq A$

(c) This tells us that A and B must have no elements in common i.e. $A \cap B = \emptyset$. B may $= \emptyset$

(2) Use divisibility and modular arithmetic to answer the following, showing all work:

(a) Find the greatest common divisor of $a = 8640$ and $b = 102816$.

(b) Determine whether $c = 733$ is prime or not by checking its divisibility by prime numbers up to $\sqrt{c}$.

Solution:

(a) $a = 8640$ and $b = 102816$
$a = 2^6 \cdot 3^2 \cdot 5$ and $b = 2^5 \cdot 3^3 \cdot 7 \cdot 17$
$\gcd(a \cdot b) = d$ then $\gcd(ka, kb) = |k|\gcd(a, b)$
$\therefore \gcd(8640, 102816) = \gcd(2^6 \cdot 3^3 \cdot 5, 2^5 \cdot 3^3 \cdot 7 \cdot 17)$
$= 2^5 \cdot 3^3 \gcd(2 \cdot 5, 7 \cdot 17)$
$= 2^5 \cdot 3^3 \gcd(10, 119)$
$= 32 \cdot 27$
$= \mathbf{864}$

(b) $c = 733$, $\sqrt{c} = \sqrt{733} \approx 27$
Prime numbers $< 27 = 2, 3, 5, 7, 11, 13, 17, 19, 23$

And 733 is divisible by none of those evenly
$\therefore$ 733 **is a prime number**

(3) Consider the function $f(n) = 7n^4 + 22n^4 \log(n) - 5n^2 \log(n^2)$ which represents the complexity of some algorithm.

(a) Find a tight big-**O** bound of the form $g(n) = n^p$ for the given function f with some natural number p. What are the constants C and k from the big-**O** definition?

(b) Find a tight big-$\boldsymbol{\Omega}$ bound of the form $g(n) = n^p$ for the given function f with some natural number p. What are the constants C and k from the big-$\boldsymbol{\Omega}$ definition?

(c) Can we conclude that f is big-$\boldsymbol{\Theta}(n^p)$ for some natural number p?

Solution:

(a) $f(n) = 7n^4 + 22n^4 \log(n) - 5n^2 \log(n^2)$
$|f(n)| \leq |7n^4| + |22n^4 \log(n)| + |10n^2 \log(n)|$ $\quad \forall n \exists \mathbb{N}$
$|f(n)| \leq 7n^5 + 22n^4 + 10n^2 n^3$ $\quad \forall n \geq 1$
$|f(n)| \leq 39n^5$ $\quad \forall n \geq 1$ Since $\log(n) \leq n$
$g(n) = n^5$ $\quad$ By definition of big-**O**
$\therefore P = 5$

(b) $f(n) = 7n^4 + 22n^4 \log(n) - 5n^2 \log(n^2)$
$f(n) = 7n^4 + 2n^2 \log(n)[11n^2 - 5]$
$n^2 \geq 0$ $\log(n) \geq 0$ $11n^2 - 5 \geq 0$ $\quad \forall n \exists \mathbb{N}$
$\therefore 2n^2 \log(n)[11n^2 - 5] \geq 0$ $\quad \forall n \geq 1$
$f(n) \geq 7n^4$
$\therefore g(n) = n^4$ $\quad$ By the definition of big-$\boldsymbol{\Omega}$
$\therefore P = 4$

(c) **No**, we cannot conclude for some natural number p

(4) Consider the function $g(n) = 2^n + \dfrac{n(n+1)}{2} - \log\left(n^{n^n}\right)$ which represents the complexity of some algorithm.

(a) Between 2^n and $\frac{n(n+1)}{2}$, which function grows *asymptotically faster* as $n \to \infty$? Justify by computing an appropriate limit.

(b) Between 2^n and $\log\left(n^{n^n}\right)$, which function grows *asymptotically faster* as $n \to \infty$? Justify by computing an appropriate limit.

(c) Between $\frac{n(n+1)}{2}$ and $\log\left(n^{n^n}\right)$, which function grows *asymptotically faster* as $n \to \infty$? Justify by computing an appropriate limit.

(d) What is the order of g?

Solution:

(a) $2^n = n^2$ for $n = 4$
$2^n > n^2$ for $n \geq 5$
and $2^n > n^2 + n$ for $n \geq 5$
$\therefore 2^n \gg \frac{n(n+1)}{2}$ for $n \to \infty$

(b) $\log(n^{n^n}) = n \log(n^n) = n^2 \log(n)$
$\log(n) < n$ for $(n \to \infty)$
$n^2 \log(n) < n^3$ for $n \to \infty$
And because $2^n > n^3$ for $n \geq 10$
$2^n \gg n^3 \gg n \log(n)$ for $n \to \infty$

(c) $\log(n^{n^n}) = n\log(n^n) = n^2\log(n)$
$\frac{n^2+n}{2} = n^2\log(n)$ for $n \approx 4$
$n^3 \sim n^2\log(n) \gg \frac{n^2+n}{2}$ for $n >\sim 4$

(d) $g = 2^n + \frac{n(n+1)}{2} + n^2\log(n)$
$= 1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{3} + \ldots + 1$
If n is even: $= \frac{n!}{\frac{n}{2}!\frac{n}{2}!}$
If n is odd: $= \frac{n(n-1)(n-2)\ldots(n-(\frac{n}{2}+1))(\frac{n}{2})!}{\frac{n}{2}!\frac{n}{2}!}$
$\therefore$ minimum degree is $\frac{n}{2}$ for an even n and $\frac{n}{2}+1$ for n odd.

(5) Consider the following matrices.

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

For this problem, we will calculate product $P = ABC$. Note that matrix multiplication is associative, which means we can calculate the product P by first computing the matrix (AB), then multiplying this by C to obtain $P = (AB)C$. **Or** we could first compute the matrix (BC), then multiply it by A to obtain $P = A(BC)$. Now recall that to multiply an $m \times n$ matrix by an $n \times k$ matrix requires $m \times n \times k$ *multiplications.*

(a) Suppose A is $m \times n$, B is $n \times k$ and C is $k \times p$. How many multiplications are needed to calculate P in the order $(AB)C$? Do not just write down an expression; show your work/justification!

(b) For the same matrix dimensions specified in (a), how many multiplications are needed to calculate P in the order $A(BC)$? Again, do not just write down an expression.

(c) Based on the specific dimensions of A, B, and C in the problem description, which multiplication order would be the most efficient?

(d) Calculate $P = ABC$ using whichever order you specified in part (c).

Solution:

(a) $AB = (A \times B) = ((m \times n) \times (n \times k)) = (m \times n \times k)$ and will result in a matrix of size $m \times k$
$(AB \times C) = ((m \times k) \times (k \times p)) = (m \times k \times p)$ and will result in a matrix of size $m \times p$
$\therefore$ the amount of multiplications is $(\mathbf{m} \times \mathbf{n} \times \mathbf{k}) + (\mathbf{m} \times \mathbf{k} \times \mathbf{p})$

(b) $BC = (B \times C) = ((n \times k) \times (k \times p)) = (n \times k \times p)$ and will create a matrix of size $n \times p$
$(A \times (B \times C) = ((m \times n) \times (n \times p)) = (m \times n \times p)$ and creates a matrix of size $m \times p$
$\therefore$ the amount of multiplication is $(\mathbf{n} \times \mathbf{k} \times \mathbf{p}) + (\mathbf{m} \times \mathbf{n} \times \mathbf{p})$

(c) These specific dimensions: $A = C = 4 \times 2$, $B = 2 \times 4$
$m = 4$, $n = 2$, $k = 4$, $p = 2$
Case 1:
$(AB)C = (m \times n \times k) + (m \times k \times p)$
$= (4 \cdot 2 \cdot 4) + (4 \cdot 4 \cdot 2)$
$= (32) + (32) = 64$
Case 2:
$A(BC) = (n \times k \times p) + (m \times n \times p)$
$= (2 \cdot 4 \cdot 2) + (4 \cdot 2 \cdot 2)$
$= (16) + (16) = 32$
$\therefore A(BC)$ is more efficient

(d) We'll use $A(BC)$ like we established above, first calculate BC

$$B = \begin{bmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$BC = \begin{bmatrix} (2 + (2 \cdot 2) + 2 + 2) & (2 + (2 \cdot 2) + 2 + 0) \\ (1 + 2 + 1 + (2 \cdot 2)) & (1 + 2 + 1 + 2) \end{bmatrix}$$

$$BC = \begin{bmatrix} 10 & 8 \\ 8 & 4 \end{bmatrix}$$

$P = A(BC)$

$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}$ and $BC = \begin{bmatrix} 10 & 8 \\ 8 & 4 \end{bmatrix}$

$P = \begin{bmatrix} (10 + (4 \cdot 8)) & (8 + (4 \cdot 4)) \\ (10 + 8) & (8 + 4) \\ (10 + (4 \cdot 8)) & (8 + (4 \cdot 4) \\ (10 + 8) & (8 + 4) \end{bmatrix}$

$P = \begin{bmatrix} 42 & 24 \\ 18 & 12 \\ 42 & 24 \\ 18 & 12 \end{bmatrix}$

(6) Use induction to show that $\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$. Be sure to state whether you're using weak or strong induction.

Solution: Let $s(n) : \sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for $n = 1 :$
$\sum_{i=0}^{n} i^3 = 0^3 + 1^3 = 1$
$\frac{n^2(n+1)^2}{4} = \frac{1(1+1)^2}{4} = 1$
$\therefore s(1)$ is true
Assume $s(k) : \sum_{i=0}^{k} i^3 = \frac{k^2(k+1)^2}{4}$
Now, we will prove that $s(k+1)$ i.e. $\sum_{i=0}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$
$\sum_{i=0}^{k+1} i^3 = \sum_{i=0}^{k} i^3 + (k+1)^3$
$= \frac{k^2(k+1)^2}{4} + (k+1)^3$
$= (k+1)^2[\frac{k^2}{4} + (k+1)]$
$= (k+1)^2(\frac{k^2+4k+4}{4})$
$= \frac{(k+1)^2(k+2)^2}{4}$
$\therefore s(k+1)$ is true $\therefore s(n)$ is true by **weak** induction.

(7) Let $A_1, A_2, \ldots A_n$ be sets. Use induction to show that for $n \geq 2$, the cardinality of the union of n sets is always less than or equal to the sum of the cardinalities of those sets. In other words, show:

$$\left| \bigcup_{i=1}^{n} A_i \right| \leq \sum_{i=1}^{n} |A_i|$$

Be sure to state whether you're using weak or strong induction.
Hint: use the same rule that HW 4 #6 was based around.

Solution: For $n = 2 :$
$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
$\leq |A_1| + |A_2|$
So for $n = 2$ it is true.
We'll use **strong** induction on n
Let the result be true for $n \geq 2$:

$$|\bigcup_{i=1}^{n} A_i| = |(\bigcup_{i=1}^{n} A_i) \cup A_{n+1}| \leq |\bigcup_{i=1}^{n} A_i| + |A_{n+1}|$$

$$\leq \sum_{i=1}^{n} |A_i| + A_{n+1} = (\sum_{i=1}^{n+1} A_i)$$