Homework 4

Due: Wednesday, November, 2020

Question 1. Using residue calculus, calculate

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{\sinh x} \mathrm{d}x$$

Since $x = 0$ is not a pole, I is a proper integral. Then

$$\begin{aligned} I &= P \int_{-\infty}^{\infty} \frac{\sin(x)}{\sinh x} \mathrm{d}x \\ &= \mathrm{Im}\ P \int_{-\infty}^{\infty} \frac{e^{ix}}{\sinh x} \mathrm{d}x \end{aligned}$$

Let $f(z) = \frac{1}{\sinh z}$. Since $|f(z)| \to 0$ as $R \to \infty$ on $C_{R+} : z = Re^{i\theta}$. By Jordan's lemma

$$| \int_{C_{R+}} \frac{e^{iz}}{\sinh z} \mathrm{d}z| \to 0 \text{ as } R \to \infty$$

Let us consider the poles. We have poles at $\sinh z = 0$. Let $z = iy$ for $y \in \mathbb{C}$. Then we want $\sinh(iy) = i \sin y = 0$. Thus the poles are at $z = in\pi$ for $n \in \mathbb{N}_0$. For $n = 0$, the pole is located on the contour.

$$\begin{aligned} I &= \mathrm{Im}\ P \oint \frac{e^{iz}}{\sinh z} \mathrm{d}z \\ &= \mathrm{Im}\ 2\pi i \sum_{n=1}^{\infty} \mathrm{Res} \left\{ \frac{e^{iz}}{\sinh z}; in\pi \right\} + \pi i \mathrm{Res} \left\{ \frac{e^{iz}}{\sinh z}; 0 \right\} \\ &= \mathrm{Im}\ 2\pi i \sum_{n=1}^{\infty} \left(\frac{e^{iz}}{\cosh z} \right)_{in\pi} + \pi i \left(\frac{e^{iz}}{\cosh z} \right)_0 \\ &= 2\pi \sum_{n=1}^{\infty} \frac{e^{-n\pi}}{\cos n\pi} + 2\pi - \pi \\ &= 2\pi \sum_{n=0}^{\infty} (-e)^{-n\pi} - \pi \\ &= \pi\left(\frac{2}{e^{-\pi} + 1} - 1\right) \text{ (geometric series formula)} \\ &= \pi \frac{1 - e^{-\pi}}{1 + e^{-\pi}} \\ &= \pi \frac{e^{\pi} - 1}{e^{\pi} + 1} \\ &= \pi \tanh\left(\frac{\pi}{2}\right) \end{aligned}$$

Question 2. Using residue calculus, calculate

$$I = \int_{-\infty}^{\infty} \frac{1+\cos(x)}{(x-\pi)^2}\mathrm{d}x$$

Since $x = \pi$ is not a pole, I is a proper integral. Then

$$\begin{aligned} I &= P\int_{-\infty}^{\infty} \frac{1+\cos(x)}{(x-\pi)^2}\mathrm{d}x \\ &= \text{Re } P\int_{-\infty}^{\infty} \frac{1+e^{ix}}{(x-\pi)^2}\mathrm{d}x \end{aligned}$$

Let $f(z) = \frac{1+e^{-iz}}{(z-\pi)^2}$. Since $|f(z)| \to 0$ as $R \to \infty$ on $C_{R+} : z = Re^{i\theta}$, by Jordan's lemma

$$|\int_{C_{R+}} e^{iz} f(z)\mathrm{d}z| = |\int_{C_{R+}} \frac{1+e^{iz}}{(z-\pi)^2}\mathrm{d}z| \to 0 \text{ as } R \to \infty$$

Let us consider the poles of $\frac{1+e^{iz}}{(z-\pi)^2}$. Taylor expand it at $z = \pi$

$$\begin{aligned} \frac{1+e^{iz}}{(z-\pi)^2} &= \frac{1+e^{i\pi}+ie^{i\pi}(z-\pi)+i^2e^{i\pi}\frac{(z-\pi)^2}{2!}+\cdots}{(z-\pi)^2} \\ &= \frac{-i}{z-\pi}+\frac{1}{2}+\cdots \end{aligned}$$

Thus $z = \pi$ is a simple pole and it is located on the contour, then

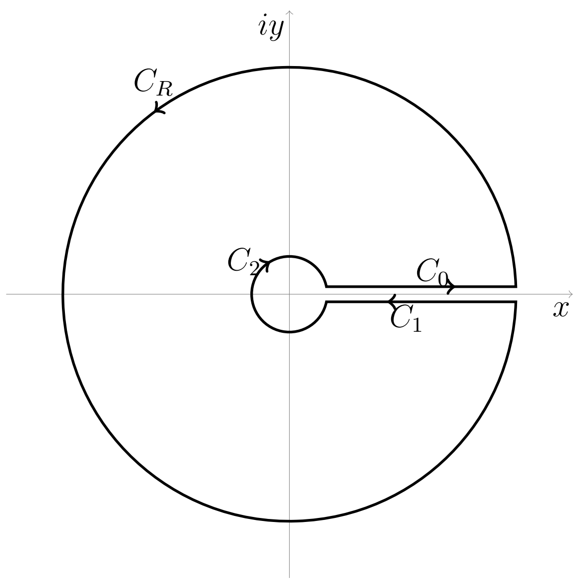
$$\begin{aligned} I &= \text{Re } P\oint \frac{1+e^{iz}}{(z-\pi)^2}\mathrm{d}z \\ &= \text{Re } \pi i \text{Res}\left\{\frac{1+e^{iz}}{(z-\pi)^2};\pi\right\} \\ &= \text{Re } \pi i \lim_{z-\pi}(z-\pi)(\frac{-i}{z-\pi}+\frac{1}{2}+\cdots) \\ &= \text{Re } \pi i(-i) \\ &= \pi \end{aligned}$$

Question 3. Evaluate the following integral using residue calculus:

$$I = \int_0^\infty \frac{x^a}{1 + 2x\cos(b) + x^2}\mathrm{d}x$$

where $-1 < a < 1, a \neq 0$, and $-\pi < b < \pi, b \neq 0$. Justify all key steps. Do not use the general formula for this integral.

The real integrand is single valued. If we convert it to a complex one, we will introduce multi-valueness. To make the complex integrand single-valued, we need to pick a branch, i.e. $0 \leq \arg z < 2\pi$. Then the integrand becomes discontinuous across the real positive axis.



Consider the closed contour $C = C_0 + C_1 + C_2 + C_R$. Let $f(z) = \frac{z^a}{1+2z\cos(b)+z^2}$. Since $|zf(z)| = O(\frac{1}{|z|^{1-a}}) \to 0$ as $|z| \to \infty$, then $|\int_{C_R} f(z)\mathrm{d}z| \to 0$ as $R \to \infty$.
Consider C_2: $z = \rho e^{i\theta}$, $\rho \to 0^+$. Then

$$\begin{aligned}\int_{C_2} f(z)\mathrm{d}z &= \int_{2\pi}^0 \frac{\rho^a e^{ia\theta}\rho e^{i\theta} i\mathrm{d}\theta}{1 + 2\rho e^{i\theta}\cos(b) + \rho^2 e^{2i\theta}} \\ &\to \int_{2\pi}^0 \rho^{a+1} e^{i(a+1)\theta} i\mathrm{d}\theta \to 0 \text{ as } \rho \to 0\end{aligned}$$

Consider C_1: $z = re^{2\pi i}$, then

$$\begin{aligned}\int_{C_1} f(z)\mathrm{d}z &= \int_R^0 \frac{(re^{2\pi i})^a e^{2\pi i}\mathrm{d}r}{1 + 2re^{2\pi i}\cos(b) + (re^{2\pi i})^2} \\ &= \int_R^0 \frac{r^a e^{2a\pi i}\mathrm{d}r}{1 + 2r\cos(b) + r^2} \\ &= -e^{2a\pi i}\int_0^\infty \frac{r^a\mathrm{d}r}{1 + 2r\cos(b) + r^2} \text{ as } R \to \infty \\ &= -e^{2a\pi i} I\end{aligned}$$

Consider C_0: $z = r$, then

$$\int_{C_0} f(z)\mathrm{d}z = \int_0^\infty \frac{r^a\mathrm{d}r}{1 + 2r\cos(b) + r^2} = I$$

Consider the poles of $f(z)$. We want to find the roots of $z^2 + 2z\cos(b) + 1 = 0$. We have

$$\begin{aligned}z^2 + 2z\cos(b) + 1 = (z + \cos(b))^2 + 1 - \cos^2(b) &= 0 \\ z &= \pm\sqrt{-(1 - \cos^2(b))} - \cos(b) \\ &= \pm i\sin(b) - \cos(b) \\ &= -e^{\pm bi}\end{aligned}$$

Then

$$\begin{aligned}\oint_C f(z)\mathrm{d}z &= 2\pi i \sum \operatorname{Res}\left\{\frac{z^a}{z^2 + 2z\cos(b) + 1}; -e^{\pm bi}\right\} \\ &= 2\pi i\left[\left(\frac{z^a}{2z + 2\cos(b)}\right)_{-e^{bi}} + \left(\frac{z^a}{2z + 2\cos(b)}\right)_{-e^{-bi}}\right] \\ &= 2\pi i\left[\frac{(-e)^{abi}}{-2e^{bi} + 2\cos(b)} + \frac{(-e)^{-abi}}{-2e^{-bi} + 2\cos(b)}\right] \\ &= 2\pi i\left[\frac{(e^{\pi i}e^{bi})^a}{-2i\sin(b)} + \frac{(e^{\pi i}e^{-bi})^a}{2i\sin(b)}\right] \\ &= 2\pi i\frac{e^{a\pi i}(-2i\sin(ab))}{2i\sin(b)} \\ &= -\frac{2\pi i e^{a\pi i}\sin(ab)}{\sin(b)}\end{aligned}$$

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\begin{aligned}
I = \frac{1}{1-e^{2a\pi i}} \oint_C f(z)\mathrm{d}z &= -\frac{2\pi i e^{a\pi i}\sin(ab)}{\sin(b)(1-e^{2a\pi i)}} \\
&= -\frac{2\pi i e^{a\pi i}\sin(ab)}{e^{a\pi i}\sin(b)(e^{-a\pi i}-e^{a\pi i})} \\
&= \frac{2\pi i\sin(ab)}{2i\sin(b)\sin(a\pi)} \\
&= \frac{\pi\sin(ab)}{\sin(b)\sin(a\pi)}
\end{aligned}
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