Homework 5

Due: Wednesday, November 11, 2020

Question 1. (AF 4.1.2) Evaluate the integrals

$$\frac{1}{2\pi i}\oint_C f(z)dz$$

where C is the unit circle centered at the origin with f(z) given below. Do these problems by both

(i) enclosing the singular points inside C

(ii) enclosing the singular points outside C (by including the point at infinity)

Show that you obtain the same result in both cases.

(a) $\frac{z^2+1}{z^2-a^2}, a^2 < 1$

(b) $\frac{z^2+1}{z^3}$.

(c) $z^2 e^{-1/z}$

Hint: the point at infinity is defined as $t = 1/z \to 0$

(a)

$$\begin{aligned}
\frac{1}{2\pi i}\oint_C \frac{z^2+1}{z^2-a^2}dz &= \frac{1}{2\pi i}2\pi i\sum \text{Res}(\frac{z^2+1}{z^2-a^2};\pm a) \\
&= \left(\frac{z^2+1}{2z}\right)_a + \left(\frac{z^2+1}{2z}\right)_{-a} \\
&= 0 \\
\frac{1}{2\pi i}\oint_C \frac{z^2+1}{z^2-a^2}dz &= \text{Res}(\frac{z^2+1}{z^2-a^2};\infty) \\
&= \text{Res}(\frac{1}{t^2}\frac{\frac{1}{t^2}+1}{\frac{1}{t^2}-a^2};0) \\
&= \text{Res}(\frac{1+t^2}{t^2(1-t^2a^2)};0) \\
&= \lim_{t\to 0}\frac{d}{dt}(\frac{1+t^2}{1-t^2a^2}) \\
&= \lim_{t\to 0}\frac{(1-t^2a^2)2t+2a^2t(1+t^2)}{(1-t^2a^2)^2} \\
&= 0
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{1}{2\pi i}\oint_C \frac{z^2+1}{z^3}dz &= \frac{1}{2\pi i}2\pi i\mathrm{Res}(\frac{z^2+1}{z^3};0) \\
&= \lim_{z\to 0}\frac{1}{(3-1)!}\frac{d^2}{dz}(z^2+1) \\
&= 1 \\
\frac{1}{2\pi i}\oint_C \frac{z^2+1}{z^3}dz &= \mathrm{Res}(\frac{z^2+1}{z^3};\infty) \\
&= \mathrm{Res}(\frac{1}{t}+t;0) \\
&= \lim_{t\to 0} 1+t^2 \\
&= 1
\end{aligned}$$

(c) We have $z^2e^{-1/z} = z^2(1-\frac{1}{z}+\frac{1}{2z^2}-\frac{1}{6z^3}+\cdots) = z^2-z+\frac{1}{2}-\frac{1}{6z}+\cdots$, where $z=0$ is an essential pole.

$$\begin{aligned}
\frac{1}{2\pi i}\oint_C z^2e^{-1/z}dz &= \frac{1}{2\pi i}2\pi i\mathrm{Res}(z^2e^{-1/z};0) \\
&= -\frac{1}{6} \\
\frac{1}{2\pi i}\oint_C z^2e^{-1/z}dz &= \mathrm{Res}(z^2e^{-1/z};\infty) \\
&= \mathrm{Res}(\frac{e^{-t}}{t^4};0) \\
&= \lim_{t\to 0}\frac{1}{(4-1)!}\frac{d^3}{dt}(e^{-t}) \\
&= -\frac{1}{6}
\end{aligned}$$

Question 2. Find the Fourier transform of $f(t) = \begin{cases} 1 \text{ for } -a < t < a \\ 0 \text{ otherwise} \end{cases}$ Then, do the inverse transform using techniques of contour integration, e.g. Jordan's lemma, principal values, etc.

$$F(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} f(t) dt = \int_{-a}^{a} e^{i\lambda t} \cdot 1 dt = \frac{1}{i\lambda}[e^{i\lambda t}]_{-a}^{a} = \frac{e^{i\lambda a} - e^{-i\lambda a}}{i\lambda} = \frac{2\sin(a\lambda)}{\lambda}$$

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\lambda t} 2\sin(a\lambda)}{\lambda} d\lambda \\ &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-i\lambda t}(e^{ia\lambda} - e^{-ia\lambda})}{\lambda} d\lambda \\ &= \frac{1}{2\pi i} \left(P \int_{-\infty}^{\infty} \frac{e^{-i\lambda(t-a)}}{\lambda} d\lambda - P \int_{-\infty}^{\infty} \frac{e^{-i\lambda(t+a)}}{\lambda} d\lambda \right) \end{aligned}$$

For $-a < t < a$, $-(t-a) > 0$ and $-(t+a) < 0$. Let $g(z) = \frac{1}{z}$. Since $|g(z)| \to 0$ as $|z| \to \infty$, by Jordan's lemma we can complete the contours in the UHP for the first integral and LHP for the second, where the simple pole $z = 0$ is on both contours.

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \left(P \oint \frac{e^{-iz(t-a)}}{z} dz + P \oint \frac{e^{-iz(t+a)}}{z} dz \right) \\ &= \frac{1}{2\pi i} \pi i \left(\text{Res}\left(\frac{e^{-iz(t-a)}}{z}; 0 \right) + \text{Res}\left(\frac{e^{-iz(t+a)}}{z}; 0 \right) \right) \\ &= \frac{1}{2}(1+1) \\ &= 1 \end{aligned}$$

For $t > a$, $-(t-a) < 0$ and $-(t+a) < 0$, we complete both contours in the LHP and they cancel out each other. For $t < -a$, $-(t-a) > 0$ and $-(t+a) > 0$, we complete both contours in the UHP and they again cancel out. Thus $f(t) = 0$ for $t > a$ and $t < -a$.

For $t = a$, $f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-ia\lambda} \sin(a\lambda)}{\lambda} d\lambda$. Let $g(z) = \frac{\sin(az)}{z}$. Since $|g(z)| \to 0$ as $|z| \to \infty$, by Jordan's lemma we can complete the contour in the LHP because $-a < 0$.

$$f(t) = \frac{1}{\pi} \oint \frac{e^{-iaz} \sin(az)}{z} dz$$

Since $z = 0$ is not a singularity, $f(t) = 0$.
For $t = -a$, $f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{ia\lambda} \sin(a\lambda)}{\lambda} d\lambda$. Similarly by Jordan's lemma we can complete the contour in the UHP because $a > 0$.

$$f(t) = \frac{1}{\pi} \oint \frac{e^{iaz} \sin(az)}{z} dz$$

Since $z = 0$ is not a singularity, $f(t) = 0$. Therefore $f(t) = 1$ for $-a < t < a$ and 0 otherwise.