Homework 1
Due: Wednesday, October 14, 2020

Question 1. (AF 1.1.1: b-d) Express each of the following in polar exponential form:
(b) $-i = e^{-i\pi/2}$
$r = \sqrt{(-1)^2} = 1$, $\theta = \tan^{-1}(\frac{-1}{0}) = -\frac{\pi}{2}$
(c) $1 + i = \sqrt{2}e^{i\pi/4}$
$r = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\theta = \tan^{-1}(\frac{1}{1}) = \frac{\pi}{4}$
(d) $\frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{i\pi/3}$
$r = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = 1$, $\theta = \tan^{-1}(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}) = \frac{\pi}{3}$

Question 2. (AF 1.1.2) Express each of the following in the form of a + bi, where a and b are real.
(a) $e^{2+i\pi/2} = e^2e^{i\pi/2} = e^2(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})) = e^2(0 + i) = e^2i$
(b) $\frac{1}{1+i} = \frac{1-i}{(1-i)(1+i)} = \frac{1-i}{2}$
(c) $(1 + i)^3 = (1 + i)^2(1 + i) = (1 + 2i - 1)(1 + i) = -2 + 2i$
(d) $|3 + 4i| = \sqrt{3^2 + 4^2} = 5$
(e) $\cos(i\pi/4 + c)$, where c is real.

$$\begin{aligned}\cos(i\pi/4 + c) &= (e^{i(i\pi/4+c)} + e^{-i(i\pi/4+c)})/2 \\ &= \frac{1}{2}e^{-4/\pi}(\cos(c) + i\sin(c)) + \frac{1}{2}e^{4/\pi}(\cos(-c) + i\sin(-c)) \\ &= \frac{1}{2}\cos(c)(e^{-4/\pi} + e^{4/\pi}) + \frac{i}{2}\sin(c)(e^{-4/\pi} - e^{4/\pi})\end{aligned}$$

Question 3. (AF 1.1.3: a, b) Solve for the roots of the following equation:
(a) $z^3 = 4$
$z^3 = 4 = 4e^{0i} \Rightarrow z = 4^{1/3}e^{i(0+2n\pi)/3}$, where $n = 0, 1, -1$
$z_1 = 4^{1/3},\ z_2 = 4^{1/3}e^{2i\pi/3},\ z_3 = 4^{1/3}e^{-2i\pi/3}$
(b) $z^4 = -1$
$z^4 = -1 = e^{\pi i} \Rightarrow z = e^{i(\pi+2n\pi)/4}$, where $n = 0, -1, 1, -2$
$z_1 = e^{i\pi/4},\ z_2 = e^{-i\pi/4},\ z_3 = e^{3i\pi/4},\ z_4 = e^{-3i\pi/4}$

Question 4. (AF 1.1.4: a, d, e, f) Establish the following result:
(a) $(z + w)^* = z^* + w^*$
Let $z = a + bi$ and $w = c + di$ where a, b, c, d are real. $(z + w)^* = (a + c) - (b + d)i$ and $z^* + w^* = a - bi + c - di = (a + c) - (b + d)i = (z + w)^*$
(d) $\text{Re}(z) \le |z|$
Let $z = a + bi$ where a, b are real. $|z|^2 = a^2 + b^2 \ge a^2 = (Re(z))^2 \Rightarrow Re(z) \le |z|$
(e) $|wz^* + w^*z| \le 2|wz|$

Let $z = a + bi$ and $w = c + di$ where a, b, c, d are real.

$$\begin{aligned}|wz^* + w^*z| &= |(c+di)(a-bi) + (c-di)(a+bi)| \\ &= |ac - bci + adi + bd + ac + bci - adi + bd| \\ &= 2|ac + bd| \\ &= 2\sqrt{(ac)^2 + (bd)^2 + 2abcd}\end{aligned}$$

$$\begin{aligned}2|wz| &= 2|(c+di)(a+bi)| \\ &= 2|ac + adi + bci - bd| \\ &= 2|(ac - bd) + (ad + bc)i| \\ &= 2\sqrt{(ac-bd)^2 + (ad+bc)^2} \\ &= 2\sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2}\end{aligned}$$

Since $(ad - bc)^2 = (ad)^2 + (bc)^2 - 2abcd \geq 0$, $(ad)^2 + (bc)^2 \geq 2abcd$. Therefore $|wz^* + w^*z| \leq 2|wz|$
(f) $|z_1||z_2| = |z_1 z_2|$
Let $z_1 = a + bi$ and $z_2 = c + di$, where a, b, c, d are real.

$$\begin{aligned}|z_1||z_2| &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2} \\ |z_1 z_2| &= |(a+bi)(c+di)| \\ &= |(ac - bd) + (ad + bc)i| \\ &= \sqrt{(ac-bd)^2 + (ad+bc)^2} \\ &= \sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2}\end{aligned}$$

Therefore $|z_1||z_2| = |z_1 z_2|$
Question 5. Prove the triangle inequality $\left|\sum_{j=1}^{N} z_j\right| \leq \sum_{j=1}^{N} |z_j|$. What is the condition for equality?
Consider the case of $N = 2$.

$$\begin{aligned}|z_1 + z_2|^2 - (|z_1| + |z_2|)^2 &= (z_1 + z_2)(z_1^* + z_2^*) - |z_1|^2 - |z_2|^2 - 2|z_1 z_2| \\ &= |z_1|^2 + |z_2|^2 + z_1 z_2^* + z_1^* z_2 - |z_1|^2 - |z_2|^2 - 2|z_1 z_2| \\ &= z_1 z_2^* + z_1^* z_2 - 2|z_1 z_2|\end{aligned}$$

We know from **Question 4(e).** that $|wz^* + w^*z| \leq 2|wz|$, therefore $|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$ and $|z_1 + z_2| \leq |z_1| + |z_2|$. Now adding $|z_3|$ to both sides

$$|z_1 + z_2| + |z_3| \leq |z_1| + |z_2| + |z_3|$$

We know

$$|(z_1 + z_2) + z_3| \leq |z_1 + z_2| + |z_3|$$

using the triangular inequality we just proved. Thus

$$|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$$

We can continue adding $z_4, z_5, \cdots z_N$, which gives the desired result.
For the condition of equality, let us again consider the $N = 2$ case.

$$\begin{aligned} |z_1 + z_2|^2 - (|z_1| + |z_2|)^2 &= z_1 z_2^* + z_1^* z_2 - 2|z_1 z_2| \\ &= 2\text{Re}(z_1 z_2^*) - 2|z_1 z_2| \end{aligned}$$

For $|z_1 + z_2| = |z_1| + |z_2|$, we need $\text{Re}(z_1 z_2^*) = |z_1 z_2|$. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, where $r_1, r_2, \theta_1, \theta_2$ are real. The equation becomes

$$r_1 r_2 e^{i(\theta_1 - \theta_2)} = r_1 r_2$$

This is only true if $\theta_1 = \theta_2$. Now adding $|z_3|$ to the equality

$$|z_1 + z_2| + |z_3| = |z_1| + |z_2| + |z_3|$$

We want to find the condition such that $|z_1 + z_2 + z_3| = |z_1 + z_2| + |z_3|$, then in turn $|z_1 + z_2 + z_3| = |z_1| + |z_2| + |z_3|$ as desired. Using what we just proved (treating $z_1 + z_2$ as z_1' and z_3 as z_2'), we know z_3 must have the same argument as $z_1 + z_2$, i.e. $\theta_3 = \theta_2 = \theta_1$. We can continue using this method to show that $\theta_N = \cdots = \theta_1$. Therefore the condition for the equality is that every z_j has the same argument.