Homework 4.4 (Extra Credit)
Due: May 5, 2021

Two-cycles
Determine the two-cycle points of the logistic difference equation, as a function of r, analytically. Determine the range of r where the two-cycle is asymptotically stable. Please do these calculations analytically, by hand, and without the help of a computer.
Hint: The two-fold composition of the logistic difference equation yields a quartic equation, and we are interested in the fixed points of this quartic. Fortunately, the trivial equilibrium and carrying capacity are 'trivial" two-cycles (why?) and may be factored out of the quartic. Also, don't hesitate to use the chain rule in evaluating stability.

Solution.

Recall the logistic difference equation

$$N_{t+1} = (1+r)N_t - \frac{r}{K}N_t^2 = f(N_t)$$

Let N_0 and N_1 be the fixed points, we have

$$\begin{aligned}
N_0 &= f \circ f(N_0) = f(N_1) \\
0 &= (1+r)[(1+r)N_0 - \frac{r}{K}N_0^2] - \frac{r}{K}[(1+r)N_0 - \frac{r}{K}N_0^2]^2 - N_0 \\
0 &= N_0\left((1+r)[(1+r) - \frac{r}{K}N_0] - \frac{r}{K}N_0[(1+r) - \frac{r}{K}N_0]^2 - 1\right) \\
0 &= \frac{N_0}{K^3}\left(2K^3r - 2K^2N_0r + K^3r^2 - 3K^2N_0r^2 + 2KN_0^2r^2 - K^2N_0r^3 + 2KN_0^2r^3 - N_0^3r^3\right) \\
0 &= \frac{rN_0}{K^3}(K - N_0)(r^2N_0^2 - r(r+2)KN_0 + (r+2)K^2)
\end{aligned}$$

Solving yields (neglecting trivial fixed points $N_0 = 0$ and $N_0 = K$)

$$\begin{aligned}
N_0 &= \frac{K(r+2 \pm \sqrt{r^2-4})}{2r} \\
N_1 = f(N_0) &= \frac{(1+r)K(r+2 \pm \sqrt{r^2-4})}{2r} - \frac{r}{K}\frac{K^2(r+2 \pm \sqrt{r^2-4})^2}{4r^2} \\
&= \frac{K(r+2 \mp \sqrt{r^2-4})}{2r}
\end{aligned}$$

The stability is determined from the derivative of the two-fold composition

$$\begin{aligned}
\left.\frac{df \circ f}{dN}\right|_{N_0} &= \left.\frac{df}{dN}\right|_{N_1}\left.\frac{df}{dN}\right|_{N_0} \\
&= (1+r) - \left.\frac{2rN}{K}\right|_{N_1} \cdot (1+r) - \left.\frac{2rN}{K}\right|_{N_0}
\end{aligned}$$

$$
\begin{aligned}
&= (1 + r - r - 2 - \sqrt{r^2 - 4})(1 + r - r - 2 + \sqrt{r^2 - 4}) \\
&= 5 - r^2
\end{aligned}
$$

This two-cycle is stable if

$$|5 - r^2| < 1$$

Thus $2 < r < \sqrt{6}$.