Homework 4

Due: Monday, November 16, 2020

Question 1. Eigenvalues and multiplicity:
(a) Consider the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Determine its eigenvalues and eigenvectors and the algebraic and geometric multiplicity of each.

Let us call this matrix $\mathbf{A}$. Since $\det(\mathbf{A} - \lambda\mathbf{I}) = (2-\lambda)^3$, then the eigenvalue is 2 with algebraic multiplicity of 3. Note $\mathbf{A} = 2\mathbf{I}$, $\mathbf{A}\mathbf{x} = 2\mathbf{x}$ for all $\mathbf{x} \in \mathbb{C}^3$. Then we can choose any vector to be the eigenvector of $\mathbf{A}$. Since we can find at maximum three linearly independent eigenvectors $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$, the geometric multiplicity is 3.

(b) Consider the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Determine its eigenvalues and eigenvectors and the algebraic and geometric multiplicity of each.

Let us call this matrix $\mathbf{B}$. Since $\det(\mathbf{B} - \lambda\mathbf{I}) = (2-\lambda)^3$, then the eigenvalue is 2 with algebraic multiplicity of 3. Consider $\mathbf{B}\mathbf{x} = \mathbf{2x}$, where $\mathbf{x} \in \mathbb{C}^3$ is an eigenvector of $\mathbf{B}$. Then it must be true that $2x_1 + x_2 = 2x_1$ and $2x_2 + x_3 = 2x_2$. Solving these equations yields $x_2 = x_3 = 0$. Thus the eigenvector of $\mathbf{B}$ is a scalar multiple of $\mathbf{e_1}$ with geometric multiplicity 1.

Question 2. Eigen Decomposition: For each of the following statements, prove that it is true or give an example to show it is false. Here $\mathbf{A} \in \mathbb{C}^{m \times m}$ unless otherwise indicated.
(a) If λ is an eigenvalue of $\mathbf{A}$ and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an eigenvalue of of $\mathbf{A} - \mu\mathbf{I}$

We know $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ is true for some $\mathbf{x}$, then $(\mathbf{A} - \mu\mathbf{I})\mathbf{x} = \mathbf{A}\mathbf{x} - \mu\mathbf{x} = \lambda\mathbf{x} - \mu\mathbf{x} = (\lambda - \mu)\mathbf{x}$.

(b) If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$, then so is $-\lambda$.

$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, 1 is an eigenvalue, but -1 is not.

(c) If $\mathbf{A}$ is real and λ is an eigenvalue of $\mathbf{A}$, then so is $\bar{\lambda}$ (bar denotes complex conjugate).

$\overline{\mathbf{Ax}} = \overline{\lambda \mathbf{x}} \Rightarrow \mathbf{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$, then $\overline{\lambda}$ is an eigenvalue.

(d) If λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{A}$ is nonsingular, then λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$.

$\mathbf{A^{-1}Ax} = \lambda \mathbf{A^{-1}x} \Rightarrow \mathbf{x} = \lambda \mathbf{A^{-1}x} \Rightarrow \lambda^{-1}\mathbf{x} = \mathbf{A^{-1}x}$, then $\lambda^{\mathbf{-1}}$ is an eigenvalue of $\mathbf{A^{-1}}$.

(e) If all the eigenvalues of $\mathbf{A}$ are zero, then $\mathbf{A} = 0$.

$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(f) If $\mathbf{A}$ is Hermitian and λ is an eigenvalue of $\mathbf{A}$, then $|\lambda|$ is a singular value of $\mathbf{A}$.

If $\mathbf{A}$ is Hermitian, its eigenvectors are orthogonal and eigenvalues are real. Then we have

$$\mathbf{A} = \mathbf{Q\Lambda Q}^*$$

where $\mathbf{Q}$ is unitary and $\mathbf{\Lambda}$ is a diagonal matrix with diagonal entries equal to λ_i, ordered from largest to smallest (it is possible that the entries are ordered differently, in which case $\mathbf{Q}$ will be different). An equivalent form of the above is

$$\mathbf{A} = \mathbf{Q}|\mathbf{\Lambda}|\mathbf{sign}(\mathbf{\Lambda})\mathbf{Q}^* \tag{1}$$

where $\mathbf{sign}(\mathbf{\Lambda})$ is a diagonal matrix whose diagonal entries equal to the signs of λ_i. Since $\mathbf{Q}$ is unitary, $\mathbf{sign}(\mathbf{\Lambda})\mathbf{Q}^*$ is also unitary and Equation 1 is an SVD of $\mathbf{A}$. Thus $|\lambda|$ is a singular value of $\mathbf{A}$.

(g) If $\mathbf{A}$ is diagonalizable and all its eigenvalues are equal, then $\mathbf{A}$ is diagonal.

Since $\mathbf{A}$ is diagonalizable, $\mathbf{A} = \mathbf{PDP^{-1}}$ exists for a diagonal matrix $\mathbf{D}$ and an invertible matrix $\mathbf{P}$. Then $\mathbf{A} = \lambda\mathbf{PIP^{-1}} \Rightarrow \mathbf{A} = \lambda\mathbf{I} = \mathbf{D}$.

Question 3. Special Matrices: Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ by tridiagonal and Hermitian, with all its sub- and super-diagonal entries nonzero. Prove that the eigenvalues of $\mathbf{A}$ are distinct (Hint: Show that for any $\lambda \in \mathbb{C}, \mathbf{A} - \lambda\mathbf{I}$ has rank at least $m-1$).

If we delete the first row and last column of $\mathbf{A} - \lambda\mathbf{I}$, we have an upper triangular matrix with rank $m-1$, then the rank of $\mathbf{A} - \lambda\mathbf{I}$ is at least $m-1$, implying the dimension of the nullspace of $\mathbf{A} - \lambda I = 1$. Thus the geometric multiplicity of λ_i is 1. Since $\mathbf{A}$ is Hermitian, it is diagonalizable (**Theorem 24.7** in Trefethen), then the algebraic multiplicity of λ_i is also 1. Thus λ_i are distinct.