Homework 2

Due: Wednesday, October 21, 2020

Question 1. (AF 2.5.1) Evaluate $\oint_C f(z)\mathrm{d}z$, where C is the unit circle centered at the origin, and $f(z)$ is given by the following:

(a) e^{iz}

Since $f'(z) = ie^{iz}$ exists for every point in C, e^{iz} is analytic in C. By Cauchy's theorem $\oint_c e^{iz}\mathrm{d}z = 0$

(b) e^{z^2}

Since $f'(z) = 2ze^{z^2}$ exists for every point in C, e^{z^2} is analytic in C. By Cauchy's theorem $\oint_c e^{z^2}\mathrm{d}z = 0$

(c) $\frac{1}{z-1/2}$

Let us parameterize C by $z - 1/2 = re^{i\theta}$, $\theta \in [0, 2\pi]$, then $\frac{\mathrm{d}z}{\mathrm{d}\theta} = ire^{i\theta}$. $\oint_C \frac{1}{z-1/2}\mathrm{d}z = \int_0^{2\pi} \frac{ire^{i\theta}\mathrm{d}\theta}{re^{i\theta}} = \int_0^{2\pi} i\mathrm{d}\theta = 2\pi i$

We can show that $\oint_C \frac{1}{z-z_0}\mathrm{d}z = 2\pi i$ using the same method and we will use this result later

(d) $\frac{1}{z^2-4}$

Since the unit circle does not enclose the singularities $z = 2$ or $z = -2$, $\frac{1}{z^2-4}$ is analytic in C and by Cauchy's theorem $\oint_c \frac{1}{z^2-4}\mathrm{d}z = 0$

(e) $\frac{1}{2z^2+1}$

Using partial fractions, we have $\frac{1}{2z^2+1} = \frac{\frac{\sqrt{2}i}{4}}{z+\frac{i}{\sqrt{2}}} - \frac{\frac{\sqrt{2}i}{4}}{z-\frac{i}{\sqrt{2}}}$. Then $\oint_C \frac{1}{2z^2+1}\mathrm{d}z = \frac{\sqrt{2}i}{4}\oint_C \frac{1}{z+\frac{i}{\sqrt{2}}} - \frac{1}{z-\frac{i}{\sqrt{2}}}\mathrm{d}z$. Using the result in (c), we can show that $\oint_C \frac{1}{z+\frac{i}{\sqrt{2}}}\mathrm{d}z = \oint_C \frac{1}{z-\frac{i}{\sqrt{2}}}\mathrm{d}z = 2\pi i$. Thus $\oint_C \frac{1}{2z^2+1}\mathrm{d}z = \frac{\sqrt{2}i}{4} \cdot (2\pi i - 2\pi i) = 0$

(f) $\sqrt{z-4}, 0 \leq \arg(z-4) < 2\pi$

Since $f'(z) = \frac{1}{2}(z-4)^{-\frac{1}{2}}$ exists for every point in C and is single valued, $\sqrt{z-4}$ is analytic in C. By Cauchy's theorem $\oint_C \sqrt{z-4}\mathrm{d}z = 0$

Question 2. (AF 2.5.5) We wish to evaluate the integral $\int_0^\infty e^{ix^2}dx$. Consider the contour $I_R = \oint_{C_{(R)}} e^{iz^2}\mathrm{d}z$ where $C_{(R)}$ is the closed circular sector in the upper half plane with boundary points $0, R$, and $Re^{i\pi/4}$. Show that $I_R = 0$ and that $\lim_{R\to\infty}\int_{C_{1(R)}} e^{iz^2}\mathrm{d}z = 0$ where $C_{1(R)}$ is the line integral along the circular sector from R to $\mathrm{R}e^{i\pi/4}$. (Hint: use $\sin(x) \geq \frac{2x}{\pi}$ on $0 \leq x \leq \pi/2$). Then, breaking up the contour $C_{(R)}$ into three component parts, deduce

$$\lim_{R\to\infty}\left(\int_0^R e^{ix^2}\mathrm{d}x - e^{i\pi/4}\int_0^R e^{-r^2}\mathrm{d}r\right) = 0$$

and from the well-known result of real integration, $\int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$, deduce that $I = e^{i\pi/4}\sqrt{\pi}/2$.

Since $\frac{de^{iz^2}}{\mathrm{d}z} = 2ize^{iz^2}$ exists for every point in $C_{(R)}$, e^{iz^2} is analytic in $C_{(R)}$ and by Cauchy's theorem $I_R = 0$
Let $z = Re^{i\theta}$, then

$$\begin{aligned}\int_{C_{1(R)}} e^{iz^2}\mathrm{d}z &= \int_{C_{1(R)}} e^{iR^2e^{2i\theta}} iRe^{i\theta}\mathrm{d}z\\ &= \int_{C_{1(R)}} iRe^{iR^2(\cos 2\theta + i\sin 2\theta)}e^{i\theta}\mathrm{d}\theta\\ &= \int_{C_{1(R)}} iRe^{iR^2\cos 2\theta}e^{-R^2\sin 2\theta}e^{i\theta}\mathrm{d}\theta\end{aligned}$$

Using the properties of integration, we have

$$\begin{aligned}\left|\int_{C_{1(R)}} e^{iz^2}\mathrm{d}z\right| &\leq \int_{C_{1(R)}} \left|iRe^{iR^2\cos 2\theta}e^{-R^2\sin 2\theta}e^{i\theta}\right|\mathrm{d}\theta\\ &= R\int_{C_{1(R)}} e^{-R^2\sin 2\theta}\left|e^{i(R^2\cos 2\theta+\theta)}\right|\mathrm{d}\theta\\ &= R\int_{C_{1(R)}} e^{-R^2\sin 2\theta}\mathrm{d}\theta\end{aligned}$$

Given $\sin 2\theta \geq \frac{4\theta}{\pi}$ on $0 \leq 2\theta \leq \frac{\pi}{2}$, we have

$$\begin{aligned}R\int_{C_{1(R)}} e^{-R^2\sin 2\theta}\mathrm{d}\theta &\leq R\int_{C_{1(R)}} e^{-R^2\frac{4\theta}{\pi}}\mathrm{d}\theta\\ &= R\frac{-\pi}{4R^2}[e^{\frac{-4R^2\theta}{\pi}}]_0^{\frac{\pi}{4}}\\ &= \frac{-\pi}{4R}[e^{-4R^2} - 1] \to 0 \text{ as } R\to\infty\end{aligned}$$

Thus $\lim_{R\to\infty}\left|\int_{C_{1(R)}} e^{iz^2}\mathrm{d}z\right| = 0$ and $\lim_{R\to\infty}\int_{C_{1(R)}} e^{iz^2}\mathrm{d}z = 0$.
We may break $C_{(R)}$ into three parts, from 0 to R, $C_{1(R)}$, from $Re^{\frac{i\pi}{4}}$ to 0, then

$$\begin{aligned}
\oint_{C_{(R)}} e^{iz^2}\mathrm{d}z &= \int_0^R e^{ix^2}\mathrm{d}x + \int_{C_{1(R)}} e^{iz^2}\mathrm{d}z + e^{\frac{i\pi}{4}}\int_R^0 e^{-r^2}\mathrm{d}r \\
&= \int_0^R e^{ix^2}\mathrm{d}x - \int_0^R e^{-r^2}\mathrm{d}r \\
&= 0
\end{aligned}$$

Taking $R\to\infty$

$$\begin{aligned}
\lim_{R\to\infty}(\int_0^R e^{ix^2}\mathrm{d}x - e^{\frac{i\pi}{4}}\int_0^R e^{-r^2}\mathrm{d}r) = 0 \\
\int_0^\infty e^{ix^2}\mathrm{d}x = e^{\frac{i\pi}{4}}\int_0^\infty e^{-r^2}\mathrm{d}r
\end{aligned}$$

Given $\int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$, we have $\int_0^\infty e^{ix^2}\mathrm{d}x = e^{\frac{i\pi}{4}}\frac{\sqrt{\pi}}{2}$

Question 3. (AF 2.5.6) Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+1}$$

Show how to evaluate this integral by considering

$$\oint_{C_{(R)}} \frac{\mathrm{d}z}{z^2+1}$$

where $C_{(R)}$ is the closed semicircle in the upper half plane with endpoints at $(-R, 0)$ and $(R, 0)$ plus the x axis. Hint: use

$$\frac{1}{z^2+1} = \frac{-1}{2i}\left(\frac{1}{z+i} - \frac{1}{z-i}\right)$$

and show that the integral along the open semicircle in the upper half plane vanishes as $R \to \infty$. Verify your answer by usual integration in real variables.

We can break the $C_{(R)}$ into two parts, from $-R$ to R and the arc along the semicircle, $C_{1(R)} = Re^{i\theta}$, $\theta \in [0, \pi]$.

$$\oint_{C_{(R)}} \frac{\mathrm{d}z}{z^2+1} = \int_{-R}^{R} \frac{\mathrm{d}x}{x^2+1} + \int_0^{\pi} \frac{iRe^{i\theta}}{R^2e^{2i\theta}+1}\mathrm{d}\theta$$

Since the singularity $z = -i$ is outside of $C_{(R)}$, $\frac{1}{z+i}$ is analytic in $C_{(R)}$. Consider the hint

$$\begin{aligned}\oint_{C_{(R)}} \frac{\mathrm{d}z}{z^2+1} &= \frac{-1}{2i}\oint_{C_{(R)}} (\frac{1}{z+i} - \frac{1}{z-i})\mathrm{d}z \\ &= \frac{1}{2i}\oint_{C_{(R)}} \frac{1}{z-i}\mathrm{d}z\end{aligned}$$

We can deform $C_{(R)}$ to a small circle centered around i, then using the result in **Question 1(c)**, we have

$$\oint_{C_{(R)}} \frac{\mathrm{d}z}{z^2+1} = \frac{1}{2i}2\pi i = \pi$$

For the integral along $C_{1(R)}$, by the properties of integration we have

$$\begin{aligned}\left|\int_0^{\pi} \frac{iRe^{i\theta}}{R^2e^{2i\theta}+1}\mathrm{d}\theta\right| &\le \int_0^{\pi} \left|\frac{iRe^{i\theta}}{R^2e^{2i\theta}+1}\right| \mathrm{d}\theta \\ &= \int_0^{\pi} \frac{R}{|R^2e^{2i\theta}+1|}\mathrm{d}\theta\end{aligned}$$

Using the triangular inequality, $|R^2e^{2i\theta}+1|+|-1| \geq |R^2e^{2i\theta}+1-1| = |R^2e^{2i\theta}| = R^2$. Then we have

$$\begin{aligned}\frac{1}{|R^2e^{2i\theta}+1|} &\leq \frac{1}{R^2}\\ \int_0^\pi \frac{R}{|R^2e^{2i\theta}+1|}\mathrm{d}\theta &\leq \int_0^\pi \frac{R}{R^2}\mathrm{d}\theta\\ &= \frac{\pi R}{R^2}\end{aligned}$$

Since $\lim_{R\to\infty}\frac{\pi R}{R^2} = 0$, $\lim_{R\to\infty}\int_0^\pi \frac{R}{|R^2e^{2i\theta}+1|}\mathrm{d}\theta = 0$. Thus

$$\lim_{R\to\infty}\int_0^\pi \frac{iRe^{i\theta}}{R^2e^{2i\theta}+1}\mathrm{d}\theta = 0$$

and therefore

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2+1} = \oint_{C_{(R)}}\frac{\mathrm{d}z}{z^2+1} = \pi$$

The usual integration $\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2+1} = [\arctan(x)]_{-\infty}^{\infty} = \pi$

Question 4. (AF 3.3.5) Let

$$f(z) = e^{\frac{t}{2}(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(t) z^n$$

Show from the definition of Laurent series and using properties of integration that

$$\begin{aligned} J_n(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)} \mathrm{d}\theta \\ &= \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - t\sin\theta) \mathrm{d}\theta \end{aligned}$$

The functions $J_n(t)$ are called Bessel functions, which are well-known special functions in mathematics and physics.

By the definition of Laurent series

$$J_n(t) = \frac{1}{2\pi i} \oint_C \frac{f(z)\mathrm{d}z}{z^{n+1}}$$

Let us parameterize C by $z = e^{i\theta}$, $\theta \in [-\pi, \pi]$. Then we have

$$\begin{aligned} J_n(t) &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(e^{i\theta} - e^{-i\theta})} i e^{i\theta} d\theta}{e^{i(n+1)\theta}} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{\frac{t}{2}(\cos\theta + i\sin\theta - \cos(-\theta) - i\sin(-\theta))}}{e^{in\theta}} \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\frac{t}{2} 2i\sin\theta} e^{-in\theta} \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)} \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(-(n\theta - t\sin\theta)) + i\sin(-(n\theta - t\sin\theta)) \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(n\theta - t\sin\theta) - i\sin(n\theta - t\sin\theta) \mathrm{d}\theta \end{aligned}$$

Since sin is an odd function, its integral from $-\pi$ to π is 0. Since cos is an even function, it is symmetric about the origin and the integral from $-\pi$ to π is double the integral from 0 to π, thus we can rewrite the integral as

$$J_n(t) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - t\sin\theta) \mathrm{d}\theta$$