Homework 4.1
Due: April 30, 2021

Beverton-Holt stock-recruitment curve
Derive an exact, closed-form solution for the Beverton-Holt difference equation,

$$N_{t+1} = \frac{R_0 N_t}{1 + \left[\left(R_0 - 1\right)/K\right] N_t}$$

Show that you can define R_0 so that your solution matches the solution for the logistic differential equation.
Hint: Show that the substitution

$$u_t \equiv \frac{1}{N_t}$$

gives rise to a linear difference equation.

Solution.

Using the substitution and rearranging

$$\begin{aligned}
\frac{1}{u_{t+1}} &= \frac{R_0 \frac{1}{u_t}}{1 + \frac{R_0 - 1}{K} \frac{1}{u_t}} \\
u_{t+1} &= \frac{1}{R_0} \left(u_t + \frac{R_0 - 1}{K} \right)
\end{aligned}$$

Assuming $N_0 = \frac{1}{u_0}$, we can iterate to find u_t

$$\begin{aligned}
u_1 &= \frac{1}{R_0} \left(u_0 + \frac{R_0 - 1}{K} \right) \\
u_2 &= \frac{1}{R_0} \left(\frac{1}{R_0} \left(u_0 + \frac{R_0 - 1}{K} \right) + \frac{R_0 - 1}{K} \right) = (\frac{1}{R_0})^2 \left(u_0 + \frac{R_0 - 1}{K} \right) + \frac{1}{R_0} \left(\frac{R_0 - 1}{K} \right) \\
u_t &= (\frac{1}{R_0})^t \left(u_0 + \frac{R_0 - 1}{K} \right) + \left(\frac{1}{R_0} + (\frac{1}{R_0})^2 + \cdots + (\frac{1}{R_0})^{t-1} \right) \frac{R_0 - 1}{K} \\
N_t &= \frac{1}{(\frac{1}{R_0})^t \left(\frac{1}{N_0} + \frac{R_0 - 1}{K} \right) + \frac{1 - R_0^{-t+1}}{R_0 - 1} \frac{R_0 - 1}{K}} \\
&= \frac{K}{R_0^{-t}(\frac{K}{N_0} + R_0 - 1) + 1 - R_0^{-t+1}} \\
&= \frac{K}{1 + R_0^{-t}(\frac{K}{N_0} - 1)}
\end{aligned}$$

Let $R_0 = e^r$, then the solution becomes

$$N_t = \frac{K}{1 + (\frac{K}{N_0} - 1)e^{-rt}}$$

which is the solution of the logistic differential equation.