Homework 2

Due: May 5, 2021

1. Use the Fourier transform method to solve the 2-D Laplace equation in the upper plane for the bounded solution:

$$\nabla^2 u = 0 \text{ in } y > 0, -\infty < x < \infty$$
$$u(x,0) = f(x), \quad -\infty < x < \infty$$

Solution.

We Fourier transform in x since it is an infinite domain

$$U(\omega, y) = \int_{-\infty}^{\infty} u(x,y) e^{i\omega x} dx$$

Then the transformed equation is

$$(-i\omega)^2 U + \frac{\partial^2}{\partial y^2} U = 0$$
$$\frac{\partial^2}{\partial y^2} U = \omega^2 U$$

This equation has a general solution

$$U = A(\omega) e^{\omega y} + B(\omega) e^{-\omega y}$$

For physical solution we require that $U \to 0$ as $y \to \infty$. The solution has to also satisfy the boundary condition $U(\omega, 0) = F(\omega)$, then

$$U = F(\omega) e^{-|\omega| y}$$

Using the inverse Fourier transform

$$\begin{aligned} u(x,y) &= \mathcal{F}^{-1}[U(\omega, y)] \\ &= \mathcal{F}^{-1}[F(\omega) e^{-|\omega| y}] \end{aligned}$$

Let $G(\omega) = e^{-\omega y}$, by the convolution theorem

$$\mathcal{F}^{-1}[FG] = \mathcal{F}^{-1}[F] * \mathcal{F}^{-1}[G] = f * \mathcal{F}^{-1}[G]$$

$\mathcal{F}^{-1}[G]$ is

$$\begin{aligned} \mathcal{F}^{-1}[G] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|\omega| y - i\omega x} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{0} e^{\omega(y - ix)} d\omega + \frac{1}{2\pi} \int_{0}^{\infty} e^{-\omega(y+ix)} d\omega \\ &= \frac{y}{\pi(x^2 + y^2)} \end{aligned}$$

The solution is then

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x-\xi)^2 + y^2} d\xi$$

2. Green's function of the 1-D heat equation in a semi-infinite domain, $G(x, t; \xi, \tau)$, is defined by:

$$\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) G = \delta(x-\xi)\delta(t-\tau), \quad 0 < x, \xi < \infty, t > 0, \tau > 0$$

subject to zero initial condition: $G = 0$ at $t = 0$. The boundary condition is either $(a) : G = 0$ at $x = 0$ and $x \to \infty$, or (b): $\frac{\partial}{\partial x} G = 0$ at $x = 0$ and $x \to \infty$. The solution in a semi-infinite domain can be constructed from the solution in the infinite domain by adding or subtracting another source located at $x = -\xi$, so that the contributions cancel at $x = 0$ for (a), or the contributions are symmetric about $x = 0$. Find the Green's function defined above for boundary condition (a). Then repeat the problem for boundary condition (b).

Solution.

(a) We derived in the lecture that this problem is equivalent to the "fundamental problem"

$$\begin{aligned}\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) G &= 0, \quad t > \tau \\ G &= \delta(x-\xi), \quad t > \tau \\ G &= 0, \quad 0 < t < \tau\end{aligned}$$

The advantage of this is that we can obtained the Green's function without the forcing term. This problem turns out to be the same as the drunken sailor problem, which can be solved by many techniques. We will use the Fourier transform in x

$$\hat{G}(\omega) = \mathcal{F}[G(x)]$$

The equation then becomes

$$\frac{\partial}{\partial t}\hat{G} + D\omega^2\hat{G} = 0$$

The solution is

$$\hat{G}(\omega, t) = \hat{G}(\omega, \tau)e^{-D\omega^2(t-\tau)}$$

By convolution theorem and Gaussian integral formula

$$\begin{aligned}G &= G(x, \tau) * \mathcal{F}^{-1}[e^{-D\omega^2(t-\tau)}] \\ &= \int_{-\infty}^{\infty} \delta(x-\xi)\frac{1}{\sqrt{4\pi D(t-\tau)}}e^{-\frac{x^2}{4D(t-\tau)}}dx \\ &= \frac{1}{\sqrt{4\pi D(t-\tau)}}e^{-\frac{(x-\xi)^2}{4D(t-\tau)}}\end{aligned}$$

This solution is in fact for an infinite domain, for a semi-infinite domain and the given boundary conditions, we need to subtract another source at $x = -\xi$, then

$$G = \frac{1}{\sqrt{4\pi D(t-\tau)}} e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} - \frac{1}{\sqrt{4\pi D(t-\tau)}} e^{-\frac{(x+\xi)^2}{4D(t-\tau)}}$$

(b) Now we want $\frac{\partial}{\partial x} G$ to be 0 at the boundaries, so we add another source at $x = -\xi$, then

$$G = \frac{1}{\sqrt{4\pi D(t-\tau)}} e^{-\frac{(x-\xi)^2}{4D(t-\tau)}} + \frac{1}{\sqrt{4\pi D(t-\tau)}} e^{-\frac{(x+\xi)^2}{4D(t-\tau)}}$$

3. Consider the following one-dimension nonhomogeneous wave equation (assume the boundary conditions are such that the solution is integrable):

$$\text{PDE}: \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = \delta(x-\xi)\delta(t-\tau), \quad t>0, \tau>0, -\infty<x<\infty, -\infty<\xi<\infty$$

subject to zero initial conditions: $u=0$ and $\frac{\partial}{\partial t}u = 0$ at $t=0$.
(a) Show that the above problem is the same as the following homogenous problem: PDE: $\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0, t>\tau$ subject to the following "initial condition" at $t=\tau$: $u=0$ at $t=\tau$, and $\frac{\partial}{\partial t}u = \delta(x-\xi)$ at $t=\tau$. And $u\equiv 0$ for $t<\tau$
(b) Solve the problem defined by (a), using Fourier transform.
(c) Use the result in (b) to solve: PDE: $\frac{\partial^2}{\partial t^2}u - c^2\frac{\partial^2}{\partial x^2}u = Q(x,t), \quad -\infty<x<\infty, t>0$
BC: $u(x,t)\to 0$ as $x\to\pm\infty, t>0$
IC: $u(x,0)=0, \quad \frac{\partial}{\partial t}u(x,t=0)=0, -\infty<x<\infty,$
You can leave your solution in (c) in integral form, but the integrand should be as simple as possible.

Solution.

(a) For $0\le t<\tau$, the initial value problem is

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0, \quad t<\tau$$
$$u = \frac{\partial u}{\partial t} = 0, \quad t=0$$

This yields the trivial solution $u=0$. For $t>\tau$, we need to specify an "initial condition", in order to be consistent with the PDE. We know that $\frac{\partial u}{\partial t}$ can no longer be 0 and there must be a discontinuity at $t=\tau$. Integrating with respect to t near τ

$$\int_{\tau^-}^{\tau^+}\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u dt = \delta(x-\xi)$$
$$\int_{\tau^-}^{\tau^+}\frac{\partial^2 u}{\partial t^2} = \left[\frac{\partial u}{\partial t}\right]_{t=\tau^-}^{t=\tau^+}$$
$$\int_{\tau^-}^{\tau^+} -c^2\frac{\partial^2 u}{\partial x^2}dt \to 0 \text{ as } \tau^+\to\tau^-$$

Then it must be true that at $t=\tau$, u is continuous, i.e. $u=0$ and $\frac{\partial u}{\partial t} = \delta(x-\xi)$. This is the homogeneous problem that we want to solve.

(b) Fourier transform in x, the equation becomes

$$U_{tt} + c^2w^2U = 0$$

This has a general solution

$$U(\omega,t) = A(\omega)\sin(c\omega(t-\tau)) + B(\omega)\cos(c\omega(t-\tau))$$

Using the "initial conditions", we determine that $A = \frac{1}{c\omega}\mathcal{F}[\delta(x-\xi)] = \frac{1}{c\omega}e^{i\omega\xi}$ and $B = 0$, then the solution is

$$U(\omega,t) = \frac{1}{c\omega}e^{i\omega\xi}\sin(c\omega(t-\tau))$$

Inverse Fourier transform

$$\begin{aligned} u(x,t) = \mathcal{F}^{-1}[U(\omega,t)] &= \int_{-\infty}^{\infty} \frac{1}{c\omega}e^{i\omega\xi}\sin(c\omega(t-\tau))e^{-i\omega x}d\omega \\ &= \frac{1}{c}\int_{-\infty}^{\infty} \frac{\sin(c\omega(t-\tau))}{\omega}e^{-i\omega(x-\xi)}d\omega \end{aligned}$$

To avoid contour integration, let us do this inverse transform in a different way. Differentiating u with respect to x

$$u'(x,t) = -\frac{i}{c}\int_{-\infty}^{\infty}\sin(c\omega(t-\tau))e^{-i\omega(x-\xi)}d\omega$$

Thus we only need to compute the inverse transform of sine and integrating the result. Observe that sine can be written as

$$\sin(c\omega(t-\tau)) = \frac{1}{2i}(e^{ic\omega(t-\tau)} - e^{-ic\omega(t-\tau)})$$

Then $u'(x,t)$ is

$$\begin{aligned} u'(x,t) &= \frac{1}{2c}\int_{-\infty}^{\infty}(e^{-ic\omega(t-\tau)} - e^{ic\omega(t-\tau)})e^{-i\omega(x-\xi)}d\omega \\ &= \frac{1}{2c}(\delta(x-\xi+c(t-\tau)) - \delta(x-\xi-c(t-\tau))) \end{aligned}$$

Integrating a delta function yields the Heaviside step function, thus

$$u(x,t) = \frac{1}{2c}(H(x-\xi+c(t-\tau)) - H(x-\xi-c(t-\tau)))$$

(c) The solution in (b) is in fact the Green's function for the "fundamental problem" of the wave equation. If we have a forcing term $Q(x,t)$, we can find the solution by integrating the contribution from all points

$$u(x,t) = \int_0^t d\tau\int_{-\infty}^{\infty}G(x,t;\xi,\tau)Q(\xi,\tau)d\xi = \frac{1}{2c}\int_0^t d\tau\int_{x-c(t-\tau)}^{x+c(t-\tau)}Q(\xi,\tau)d\xi$$

Note that τ is integrated up to t since the forcing term is only "switched on" at $t > \tau$.