Homework 1.2
Due: April 9, 2021

Exact solution of the logistic equation
Show that the logistic equation,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

with the initial condition

$$N(0) = N_0$$

has the solution

$$N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right)e^{-rt}}$$

(a) treating the logistic equation as a separable equation, and
(b) treating the logistic equation as a Bernoulli equation.
Make sure that your integrations and solutions make good sense for both $N < K$ and $N > K$

Solution.

(a) Separating the variables and integrating

$$\begin{aligned}
\int \frac{K}{N(K-N)}dN &= \int r dt \\
\int \frac{1}{N} + \frac{1}{K-N}dN &= rt + C \\
\ln|N| - \ln|K-N| &= rt + C \\
\ln\left|\frac{N}{K-N}\right| &= rt + C \\
\frac{N}{K-N} &= e^{rt}e^{C} \\
N &= \frac{K}{1 + e^{-C}e^{-rt}}
\end{aligned}$$

Using the initial condition yields

$$e^C = \frac{N_0}{K - N_0} = \frac{1}{\frac{K}{N_0} - 1}$$

Plugging into N

$$N(t) = \frac{K}{1 + (\frac{K}{N_0} - 1)e^{-rt}}$$

(b) We rearrange the logistic equation into Bernoulli form

$$\frac{dN}{dt} - rN = -\frac{r}{K}N^2$$
$$\frac{1}{N^2}\frac{dN}{dt} - \frac{r}{N} = -\frac{r}{K}$$

Using the substitution $u = N^{-1}$

$$\frac{du}{dt} + ru = \frac{r}{K}$$
$$\frac{du}{dt} = -r(u - \frac{1}{K})$$

Let $v = u - \frac{1}{K}$, then $\frac{dv}{dt} = \frac{du}{dt}$

$$\frac{dv}{dt} = -rv$$
$$v = Ae^{-rt}$$
$$u = Ae^{-rt} + \frac{1}{K}$$
$$N = \frac{1}{Ae^{-rt} + \frac{1}{K}}$$

Using the initial condition yields

$$A = \frac{1}{N_0} - \frac{1}{K}$$

Plugging into N

$$N(t) = \frac{K}{1 + (\frac{K}{N_0} - 1)e^{-rt}}$$