Homework 2.5
Due: April 16, 2021

A Bunch of Bifurcations
Sketch the bifurcation diagram for the following differential equation in which x is the state variable and μ is the bifurcation parameter:

$$\frac{dx}{dt} = x(9 - \mu x)\left(\mu + 2x - x^2\right)\left[(\mu - 10)^2 + (x - 3)^2 - 1\right]$$

Be sure to show the stability of each branch and to identify each bifurcation. Explain how you determined the stability of the various branches.

Solution.

We set RHS of the equation to 0 and find the locations of the equilibria

$$x = 0,\ \frac{9}{\mu},\ 1 \pm \sqrt{1+\mu},\ 3 \pm \sqrt{1 - (\mu - 10)^2}$$

We plot this in Figure 1. We observe three fold bifurcations (one at each end of circle and one at $x = 1$) and two transcritical bifurcations (two intersections). For stability, we compute the derivatives of these branches and see where the derivatives are positive or negative.

$$\begin{aligned}
\text{circle,} \quad & \frac{dx}{dt} = 2(x - 3) \\
\text{parabola,} \quad & \frac{dx}{dt} = 2 - 2x \\
\text{hyperbola,} \quad & \frac{dx}{dt} = -\mu \\
\text{line,} \quad & \frac{dx}{dt} = 1
\end{aligned}$$

For instance, for the circle, $\frac{dx}{dt} < 0$ when $x < 3$, thus the lower branch is asymptotically stable. We can continue this process to determine the stability of other branches. At the intersections, we expect these branches to exchange their stability.

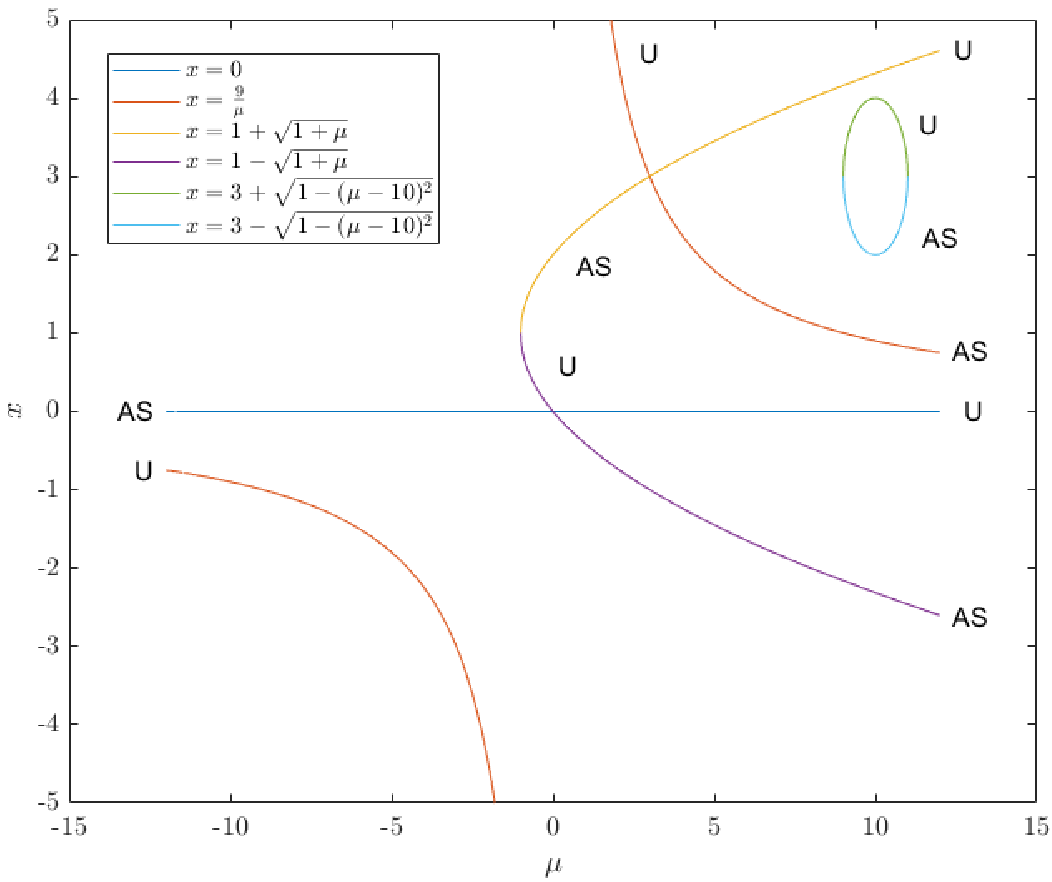

Figure 1: A bunch of bifurcations (U- unstable, AS- asymptotically stable).