Homework 3

Due: May 26, 2021

1. The Fundamental Problem for the wave equation in two-dimensions is given by

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}G - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)G = \delta(t-\tau)\delta(x-\xi)\delta(y-\eta)\\ &G \to 0 \text{ as } r \to \infty, \text{ where } r^2 = (x-\xi)^2 + (y-\eta)^2\\ &G \equiv 0 \text{ for } 0 < t < \tau.\end{aligned}$$

The solution is $G = \frac{1}{2\pi}\frac{H(t-\tau-r)}{\sqrt{(t-\tau)^2-r^2}}$, where H is the Heaviside function.
(a) Derive this solution using Fourier transform in x and y. Hint: In the inverse transform, use polar coordinates to get $G = \frac{1}{2\pi}\int_0^\infty J_0(kr)\sin(k(t-\tau))dk$. Then use integral tables.
(b) Derive this solution using Laplace transform in t. Hint: First show that the Laplace transform of G is $\tilde{G} = \frac{1}{2\pi}K_0(sr)e^{-s\tau}$, where K is the modified Bessel function of the second kind. Then use Laplace transform tables.

Solution.

(a) Using Fourier transform in x and y

$$U(\mathbf{k}, t) = \int_{-\infty}^{\infty} Ge^{i(k_1x+k_2y)}dxdy$$

where $\mathbf{k} = (k_1, k_2)$. The equation becomes

$$\frac{\partial^2}{\partial t^2}U + k^2U = \delta(t-\tau)\mathcal{F}\mathcal{F}[\delta(x-\xi)\delta(y-\eta)]$$

For $t > \tau$, we solve the homogeneous problem to get

$$U(\mathbf{k}, t) = A(\mathbf{k})\sin(k(t-\tau)) + B(\mathbf{k})\cos(k(t-\tau))$$

Matching the solution across $t = \tau$ yields $B = 0$. We showed in homework 2 that at $t = \tau$

$$\frac{\partial G}{\partial t} = \delta(x-\xi)\delta(y-\eta)$$

which is, after Fourier transform

$$\begin{aligned}\frac{\partial U}{\partial t} &= kA = \mathcal{F}\mathcal{F}[\delta(x-\xi)\delta(y-\eta)]\\ A &= \frac{1}{k}e^{i(k_1\xi+k_2\eta)}\end{aligned}$$

Thus the transformed solution is

$$U(\mathbf{k},t) = \frac{1}{k}e^{i(k_1\xi+k_2\eta)}\sin(k(t-\tau))$$

Inverse transforming and letting $\mathbf{r} = (x-\xi, y-\eta)$

$$\begin{aligned}
G(\mathbf{r},t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{k}e^{-i(k_1(x-\xi)+k_2(y-\eta))}\sin(k(t-\tau))dk_1dk_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{k}e^{-i\mathbf{k}\cdot\mathbf{r}}\sin(k(t-\tau))d\mathbf{k} \\
&= \int_{-\pi}^{\pi}\int_{0}^{\infty}\sin(k(t-\tau))e^{-ikr\cos\theta}dkd\theta \\
&= \frac{1}{2\pi}\int_0^\infty J_0(kr)\sin(k(t-\tau))dk
\end{aligned}$$

where $J_0(z) = \frac{1}{\pi}\int_0^\pi e^{-iz\cos\theta}d\theta$. Using the integral table (page 99 of Bateman Manuscript)

$$G(\mathbf{r},t) = \begin{cases} 0 & 0<t<r \\ \frac{1}{2\pi}((t-\tau)^2-r^2)^{-1/2} & t>r \end{cases}$$

Rewriting as a Heaviside function

$$G(\mathbf{r},t) = \frac{1}{2\pi}\frac{H(t-\tau-r)}{\sqrt{(t-\tau)^2-r^2}}$$

(b) Using Laplace transform in t

$$V(\mathbf{r},s) = \int_0^\infty Ge^{-st}dt$$

The equation becomes

$$s^2V - \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)V = \mathcal{L}[\delta(t-\tau)]\delta(x-\xi)\delta(y-\eta)$$

Converting to polar coordinates and assuming V has no dependence on the angle

$$s^2V - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = e^{-s\tau}\delta_2(\mathbf{r})$$

For $r>0$, the homogeneous problem is

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}-s^2V = 0$$

Using the substitution $z = rs$, the equation becomes

$$z^2\frac{\partial^2 V}{\partial z^2} + z\frac{\partial V}{\partial z} - z^2 V = 0$$

This is the modified Bessel equation of order zero, which has solution

$$V(z) = CI_0(z) + DK_0(z)$$

Since I_0 is exponentially growing, the boundary condition implies that $C = 0$. To get D, we match the solution at $r = 0$. Integrating over a circular disk A with radius ϵ

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\epsilon} s^2 V - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) dr d\theta &= \iint_A e^{-s\tau}\delta_2(\mathbf{r})dA \\
-2\pi D\left[r\frac{\partial K_0}{\partial r}\right]_0^{\epsilon} &= e^{-s\tau} \\
2\pi Drs\frac{1}{rs} &= e^{-s\tau} \\
D &= \frac{1}{2\pi}e^{-s\tau}
\end{aligned}$$

where $K_0(z) \sim -\ln(z)$. Thus the transformed solution is

$$V = \frac{1}{2\pi}K_0(rs)e^{-s\tau}$$

Inverse transforming (page 277 of Bateman Manuscript)

$$G(\mathbf{r}, t) = \mathcal{L}^{-1}[V] = \begin{cases} 0 & 0 < t < r \\ \frac{1}{2\pi}((t-\tau)^2 - r^2)^{-1/2} & t > r \end{cases}$$

Rewriting as a Heaviside function

$$G(\mathbf{r}, t) = \frac{1}{2\pi}\frac{H(t-\tau-r)}{\sqrt{(t-\tau)^2 - r^2}}$$

2. Solve using Laplace transform in t the following well-posed initial value problem:

$$
\begin{aligned}
\text{PDE}: &\ \frac{\partial^2}{\partial t^2}u - c^2\frac{\partial^2}{\partial x^2}u = q(x)e^{i\omega_0 t}, \quad -\infty < x < \infty, t > 0\\
\text{BC}: &\ u(x,t) \to 0 \text{ as } x \to \pm\infty,\ t > 0\\
&\ u(x,0) = 0, \quad -\infty < x < \infty\\
\text{IC}: &\ \frac{\partial}{\partial t}u(x,t)|_{t=0} = 0,\ \infty < x < \infty
\end{aligned}
$$

$q(x)$ is a localized function (vanishes at infinities). ω_0 is a real given forcing frequency. Be careful in stating the condition on the transform variable (s or ω) for the validity of the transform. Note that you do not have the Sommerfeld radiation condition (you actually do not need it).

Solution.

Define Laplace transform in t as

$$
\mathcal{L}[u(x,t)] = U(x,\omega) = \int_0^\infty u(x,t)e^{i\omega t}dt
$$

Consider transforming the forcing term

$$
\begin{aligned}
\mathcal{L}[q(x)e^{i\omega_0 t}] &= q(x)\int_0^\infty e^{i(\omega+\omega_0)t}dt\\
&= \frac{q(x)}{i(\omega+\omega_0)}\ e^{i(\omega+\omega_0)t}\Big|_0^\infty\\
&= -\frac{q(x)}{i(\omega+\omega_0)}
\end{aligned}
$$

given $\mathrm{Im}(\omega) > 0$. The transformed equation is

$$
\frac{\partial^2}{\partial x^2}U + k^2 U = \frac{q(x)}{ic^3(k+k_0)}
$$

where $k = \frac{\omega}{c}$ and $k_0 = \frac{\omega_0}{c}$. To solve this equation, we first get the homogeneous solution and use variation of parameters to get the particular solution. The homogeneous solution is

$$
\begin{gathered}
U_\mathrm{h} = AU_1 + BU_2\\
U_1 = e^{ikx}, \quad U_2 = e^{-ikx}
\end{gathered}
$$

From variation of parameters

$$
\begin{aligned}
U_\mathrm{p} &= \frac{1}{ic^3(k+k_0)}\left[-U_1\int_0^x \frac{U_2(s)q(s)}{W(U_1,U_2)}ds + U_2\int_0^x \frac{U_1(s)q(s)}{W(U_1,U_2)}ds\right]\\
&= \frac{1}{2kc^3(k+k_0)}\left[-e^{ikx}\int_0^x e^{-iks}q(s)ds + e^{-ikx}\int_0^x e^{iks}q(s)ds\right]
\end{aligned}
$$

Thus the general solution is

$$U = \left(A - \frac{1}{2kc^3(k+k_0)}\int_0^x e^{-iks}q(s)ds\right)e^{ikx} + \left(B + \frac{1}{2kc^3(k+k_0)}\int_0^x e^{iks}q(s)ds\right)e^{-ikx}$$

Applying the boundary conditions and realizing $\text{Im}(k) > 0$

$$\begin{aligned} A &= \frac{1}{2kc^3(k+k_0)}\int_{-\infty}^0 e^{-iks}q(s)ds \\ B &= \frac{1}{2kc^3(k+k_0)}\int_0^\infty e^{iks}q(s)ds \end{aligned}$$

Plugging into U

$$U(x,\omega) = \frac{1}{2c\omega(\omega+\omega_0)}\int_{-\infty}^\infty q(s)e^{i\omega|x-s|/c}ds$$

Inverse transforming

$$u(x,t) = \frac{1}{4\pi c}\int_0^\infty\int_{-\infty+i\alpha}^{\infty+i\alpha}\frac{q(s)}{\omega(\omega+\omega_0)}e^{-i\omega(t-|x-s|/c)}dsd\omega$$

where $\alpha > 0$ is required by causality. For $t < |x-s|/c$, we close in the plane above $\text{Im}(\omega) = \alpha$. Since the poles are at $\omega = 0$ and $\omega = -\omega_0$, $u(x,t) = 0$. For $t > |x-s|/c$, we close in the plane below $\text{Im}(\omega) = \alpha$

$$\begin{aligned} u(x,t) &= \frac{i}{2c}\int_0^\infty \text{Res}\left[\frac{1}{\omega(\omega+\omega_0)}e^{-i\omega(t-|x-s|/c)}\right]q(s)ds \\ &= \frac{i}{2c}\int_0^\infty\left(\frac{1}{\omega_0} - \frac{1}{\omega_0}e^{i\omega_0(t-|x-s|/c)}\right)q(s)ds \end{aligned}$$

Rewriting $u(x,t)$ as Heavside function

$$u(x,t) = \frac{i}{2c\omega_0}\int_0^\infty H(t-|x-s|/c)\left(1-e^{i\omega_0(t-|x-s|/c)}\right)q(s)ds$$