Homework 6
Due: March 5, 2021

1. Consider the singular equation:

$$\epsilon y'' + (1+x)^2 y' + y = 0$$

with $y(0) = y(1) = 1$ and with $0 < \epsilon \ll 1$
(a) Obtain the leading order uniform solution using the WKB method.
(b) Plot the uniform solution for $\epsilon = 0.01, 0.05, 0.1, 0.2$.

Solution.

(a) The WKB method assumes the solution takes the form

$$y = \exp\left(\frac{S_0(x) + \epsilon S_1(x) + \cdots}{\epsilon}\right)$$

Computing the derivatives

$$\begin{aligned} y' &= \left(\frac{S_{0x}(x) + \epsilon S_{1x}(x) + \cdots}{\epsilon}\right) \exp\left(\frac{S_0(x) + \epsilon S_1(x) + \cdots}{\epsilon}\right) \\ y'' &= \left(\frac{S_{0x}^2(x)}{\epsilon^2} + \frac{2S_{0x}(x)S_{1x}(x)}{\epsilon} + \cdots + \frac{S_{0xx}}{\epsilon} + \cdots\right) \exp\left(\frac{S_0(x) + \epsilon S_1(x) + \cdots}{\epsilon}\right) \end{aligned}$$

Substituting into the singular equation

$$\epsilon\left(\frac{S_{0x}^2(x)}{\epsilon^2} + \frac{2S_{0x}(x)S_{1x}(x)}{\epsilon} + \frac{S_{0xx}}{\epsilon}\right) + (1+x)^2\left(\frac{S_{0x}(x) + \epsilon S_{1x}(x)}{\epsilon}\right) + 1 = 0$$

Collecting powers of epsilon gives the hierarchy of equations

$$\begin{aligned} O(1/\epsilon) &\quad S_{0x}^2(x) + (1+x)^2 S_{0x}(x) = 0 \\ O(1) &\quad S_{0xx} + 2S_{0x}(x)S_{1x}(x) + (1+x)^2 S_{1x}(x) + 1 = 0 \end{aligned}$$

The first equation gives $S_{0x} = 0$ or $S_{0x} = -(1+x)^2$. Substituing into the leading order equation

$$\begin{aligned} S_{0x} &= 0 & (1+x)^2 S_{1x}(x) + 1 &= 0 \\ S_{0x} &= -(1+x)^2 & -2(1+x) - (1+x)^2 S_{1x}(x) + 1 &= 0 \end{aligned}$$

This gives the solution

$$\begin{aligned} S_{0x} &= 0 & S_1(x) &= -\int_0^x \frac{1}{(1+\xi)^2} d\xi = \frac{1}{1+x} \\ S_{0x} &= -(1+x)^2 & S_1(x) &= \int_0^x \frac{1}{(1+\xi)^2} d\xi - \ln(1+x)^2 = -\frac{1}{1+x} - 2\ln(1+x) \end{aligned}$$

Then solution takes the form

$$y = C_1 \exp\left(\frac{1}{1+x}\right) + \frac{C_2}{(1+x)^2}\exp\left(-\frac{(1+x)^3}{3\epsilon} - \frac{1}{1+x}\right)$$

Plugging in the boundary conditions yields

$$C_1 = -C_2 e^{-\frac{1}{3\epsilon}-2} + e^{-1}$$
$$C_2 = -\frac{4\left(e^{1/2}-1\right)e^{\frac{8}{3\epsilon}+1}}{4e^{\frac{7}{3\epsilon}} - e}$$

Then we can simplify C_1 as

$$C_1 = \frac{1 - 4e^{\frac{7}{3\epsilon}-\frac{1}{2}}}{e - 4e^{\frac{7}{3\epsilon}}}$$

Putting everything together

$$y = \frac{1 - 4e^{\frac{7}{3\epsilon}-\frac{1}{2}}}{e - 4e^{\frac{7}{3\epsilon}}}\exp\left(\frac{1}{1+x}\right) - \frac{4\left(e^{1/2}-1\right)e^{\frac{8}{3\epsilon}+1}}{(4e^{\frac{7}{3\epsilon}} - e)(1+x)^2}\exp\left(-\frac{(1+x)^3}{3\epsilon} - \frac{1}{1+x}\right)$$

(b) We note that this solution is identical to Q1 in the previous homework.

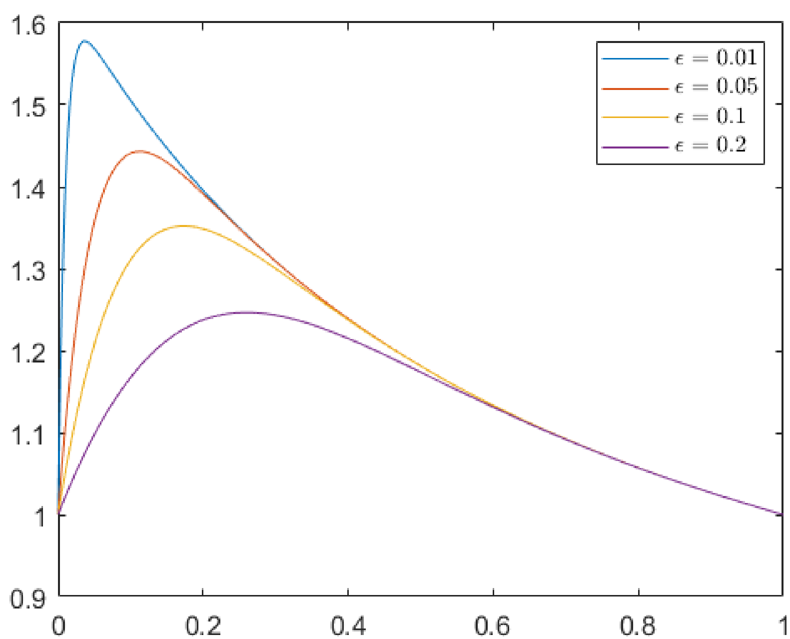

Figure 1: WKB solution of the singular equation