Homework 4

Due: March 17, 2021

1. Prove the following identity for $\alpha \in \mathbb{R}$:

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2.$$

Solution.

$$\begin{aligned}
\text{LHS} &= \alpha^2 x^T x + 2\alpha(1-\alpha)x^T y + (1-\alpha)^2 y^T y + \alpha(1-\alpha)x^T x + \alpha(1-\alpha)y^T y - 2\alpha(1-\alpha)x^T y \\
&= (\alpha^2 + \alpha - \alpha^2)x^T x + (1 + \alpha^2 - 2\alpha + \alpha - \alpha^2)y^T y \\
&= \alpha\|x\|^2 + (1-\alpha)\|y\|^2
\end{aligned}$$

2. An operator T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all (x, y). For any such nonexpansive operator T, define

$$T_\lambda = (1 - \lambda)I + \lambda T.$$

(a) Show that T_λ and T have the same fixed points.

(b) Use problem 1 to show

$$\|T_\lambda z - \overline{z}\|^2 \leq \|z - \overline{z}\|^2 - \lambda(1 - \lambda)\|z - Tz\|^2.$$

where $\overline{z}$ is any fixed point of T, i.e. $T\overline{z} = \overline{z}$.

Solution.

(a) Suppose w is any fixed point of T

$$\begin{aligned} T_\lambda w &= (1 - \lambda)Iw + \lambda Tw \\ &= (1 - \lambda)w + \lambda w \\ &= w \end{aligned}$$

(b) Since $\overline{z}$ is a fixed point of T, then it is also a fixed point of T_λ

$$\overline{z} = T_\lambda \overline{z} = (1 - \lambda)\overline{z} + \lambda T\overline{z}$$

Plugging into LHS of the inequality

$$\begin{aligned} \|T_\lambda z - \overline{z}\|^2 &= \|(1 - \lambda)z + \lambda Tz - (1 - \lambda)\overline{z} - \lambda T\overline{z}\|^2 \\ &= \|\lambda T(z - \overline{z}) + (1 - \lambda)(z - \overline{z})\|^2 \end{aligned}$$

By problem 1

$$\|\lambda T(z - \overline{z}) + (1 - \lambda)(z - \overline{z})\|^2 = \lambda\|T(z - \overline{z})\|^2 + (1 - \lambda)\|z - \overline{z}\|^2 - \lambda(1 - \lambda)\|z - Tz\|^2$$

Since T is nonexpansive

$$\begin{aligned} \|T(z - \overline{z})\| &\leq \|z - \overline{z}\| \\ \|T(z - \overline{z})\|^2 &\leq \|z - \overline{z}\|^2 \end{aligned}$$

Putting everything together

$$\begin{aligned} \|T_\lambda z - \overline{z}\|^2 &= \lambda\|T(z - \overline{z})\|^2 + (1 - \lambda)\|z - \overline{z}\|^2 - \lambda(1 - \lambda)\|z - Tz\|^2 \\ &\leq \lambda\|z - \overline{z}\|^2 + (1 - \lambda)\|z - \overline{z}\|^2 - \lambda(1 - \lambda)\|z - Tz\|^2 \\ &= \|z - \overline{z}\|^2 - \lambda(1 - \lambda)\|z - Tz\|^2 \end{aligned}$$

3. An operator T is *firmly nonexpansive* when it satisfies

$$\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \le \|x-y\|^2.$$

   (a) Show T is firmly nonexpansive if and only if

$$\langle x-y, Tx-Ty\rangle \ge \|Tx-Ty\|^2.$$

   (b) Show T is firmly nonexpansive if and only if

$$\langle Tx-Ty, (I-T)x-(I-T)y\rangle \ge 0.$$

   (c) Suppose that $S = 2T - I$. Let

$$\mu = \|Tx-Ty\|^2 + \|(I-T)x-(I-T)y\|^2 - \|x-y\|^2$$

   and let

$$\nu = \|Sx-Sy\|^2 - \|x-y\|^2.$$

   Show that $2\mu = \nu$ (you may find it helpful to use problem (1)). Conclude that T is firmly nonexpansive exactly when S is nonexpansive.

Solution.

(a) Notice that

$$\begin{aligned}\|(I-T)x-(I-T)y\|^2 &= \|x-y-(Tx-Ty)\|^2\\ &= \|x-y\|^2 + \|Tx-Ty\|^2 - 2\langle x-y, Tx-Ty\rangle\\ \|Tx-Ty\|^2 + \|(I-T)x-(I-T)y\|^2 &= 2\|Tx-Ty\|^2 + \|x-y\|^2 - 2\langle x-y, Tx-Ty\rangle\end{aligned}$$

($\longrightarrow$) If T is firmly nonexpansive, then

$$\begin{aligned}2\|Tx-Ty\|^2 - 2\langle x-y, Tx-Ty\rangle &\le 0\\ \|Tx-Ty\|^2 &\le \langle x-y, Tx-Ty\rangle\end{aligned}$$

($\longleftarrow$) If $\|Tx-Ty\|^2 \le \langle x-y, Tx-Ty\rangle$, then

$$\|Tx-Ty\|^2 + \|(I-T)x-(I-T)y\|^2 \le \|x-y\|^2$$

(b) Notice that

$$\begin{aligned}\langle Tx-Ty, (I-T)x-(I-T)y\rangle &= \langle Tx-Ty, x-y-(Tx-Ty)\rangle\\ &= \langle x-y, Tx-Ty\rangle - \|Tx-Ty\|^2\end{aligned}$$

By (a), T is nonexpansive if and only if $\langle x-y, Tx-Ty\rangle - \|Tx-Ty\|^2 \ge 0$, i.e.

$$\langle Tx-Ty, (I-T)x-(I-T)y\rangle \ge 0$$

(c) Notice that

$$\begin{aligned}
2\mu &= 2\|Tx - Ty\|^2 + 2\|(I-T)x - (I-T)y\|^2 - 2\|x-y\|^2 \\
&= 4\|Tx - Ty\|^2 + 2\|x-y\|^2 - 4\langle x-y, Tx-Ty\rangle - 2\|x-y\|^2 \\
&= \|2Tx - 2Ty\|^2 - 2\langle x-y, 2Tx-2Ty\rangle \\
&= \|(S+I)x - (S+I)y\|^2 - 2\langle x-y, (S+I)x-(S+I)y\rangle \\
&= \|Sx - Sy + x - y\|^2 - 2\langle x-y, Sx-Sy+x-y\rangle \\
&= \|Sx - Sy\|^2 + \|x-y\|^2 - 2\|x-y\|^2 \\
&= \|Sx - Sy\|^2 - \|x-y\|^2 = \nu
\end{aligned}$$

If S is nonexpansive, then

$$\begin{aligned}
2\mu = \nu = \|Sx - Sy\|^2 - \|x-y\|^2 &\le 0 \\
\mu = \|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 - \|x-y\|^2 &\le 0 \\
\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 &\le \|x-y\|^2
\end{aligned}$$

This proves T is firmly nonexpansive.

Coding Assignment

Please download `515Hw4_Coding.ipynb` and `solvers.py` to complete problem (4) and (5).

(4) Implement an interior point method to solve the problem

$$\min_x \frac{1}{2}\|Ax - b\|^2 \quad \text{s.t.} \quad Cx \le d.$$

Let the user input A, b, C, and d. Test your algorithm using a box constrained problem (where you can apply the prox-gradient method).

(5) Implement a Chambolle-Pock method to solve

$$\min_x \|Ax - b\|_1 + \|x\|_1.$$