Homework 5.3
Due: May 17, 2021

Theta-logistic predator-prey model
At the start of the quarter, we studied the theta-logistic equation

$$\frac{dN}{dT} = rN\left[1 - \left(\frac{N}{K}\right)^{\theta}\right]$$

Consider the predator-prey model

$$\begin{aligned}\frac{dN}{dT} &= rN\left[1 - \left(\frac{N}{K}\right)^{\theta}\right] - \frac{cNP}{a+N}\\ \frac{dP}{dT} &= \frac{bNP}{a+N} - mP\end{aligned}$$

Analyze and discuss the effects of the parameter θ on the occurrence of limit cycles for the above predator-prey model. Explain how your results resolve the biological control paradox.

Solution.

Introducing the change of variables

$$N = ax, \quad P = r\frac{a}{c}y, \quad T = \frac{1}{r}t$$

The model reduces to

$$\begin{aligned}ra\frac{dx}{dt} &= rax\left[1 - \left(\frac{ax}{K}\right)^{\theta}\right] - \frac{ra^2xy}{a+ax}\\ r^2\frac{a}{c}\frac{dy}{dt} &= \frac{bra^2xy}{(a+ax)c} - m\frac{ra}{c}y\end{aligned}$$

Simplifying and letting $\alpha = \frac{m}{b}$, $\beta = \frac{b}{r}$, $\gamma = \frac{K}{a}$

$$\begin{aligned}\frac{dx}{dt} &= x\left[1 - \left(\frac{x}{\gamma}\right)^{\theta}\right] - \frac{xy}{1+x}\\ \frac{dy}{dt} &= \beta\left(\frac{x}{1+x} - \alpha\right)y\end{aligned}$$

Setting $\frac{dx}{dt} = \frac{dy}{dt} = 0$, we obtain the zero-growth isoclines

$$\begin{aligned}&x = 0, \quad y = (1+x)\left[1 - \left(\frac{x}{\gamma}\right)^{\theta}\right]\\ &x = \frac{\alpha}{1-\alpha}, \quad y = 0\end{aligned}$$

The equilibria occur at the intersections of these isoclines

$$
\begin{aligned}
(x_0, y_0) &= (0, 0) \\
(x_1, y_1) &= (\gamma, 0) \\
(x_2, y_2) &= \left[x^*, (1 + x^*)\left[1 - \left(\frac{x^*}{\gamma}\right)^\theta\right]\right]
\end{aligned}
$$

where $x^* = \frac{\alpha}{1-\alpha}$. Define

$$
f(x) = \frac{x}{1+x}, \quad g(x) = (1 + x)\left[1 - \left(\frac{x}{\gamma}\right)^\theta\right]
$$

The model becomes

$$
\begin{aligned}
\frac{dx}{dt} &= f(x)[g(x) - y] \\
\frac{dy}{dt} &= \beta[f(x) - \alpha]y
\end{aligned}
$$

The Jacobian takes the form

$$
J = \begin{bmatrix} f(x)g'(x) + f'(x)g(x) - yf'(x) & -f(x) \\ \beta f'(x)y & \beta[f(x) - \alpha] \end{bmatrix}
$$

We will only consider the third equilibrium since it depends on θ. The Jacobian at this point is

$$
J = \begin{bmatrix} \alpha g'(x^*) & -\alpha \\ \beta f'(x^*) g(x^*) & 0 \end{bmatrix}
$$

and the characteristic equation is

$$
\lambda^2 - \alpha g'(x^*) \lambda + \alpha\beta f'(x^*) g(x^*) = 0
$$

Since the third term is positive, the stability is determined by the sign of $g'(x^*)$. The system is stable for $g'(x^*) < 0$ and unstable for $g'(x^*) > 0$. We plot $g(x)$ for various θ. in Figure 1. It is evident that increasing θ shifts the peak to the right, then the slope will be positive for a wider range of x, causing the system more likely to destabilize. However, we saw in practice that θ is less than 1 for most species, thus the slope is likely to be negative at low populations. We would expect the system to be stable, even if we introduce biological control to suppress the prey population.

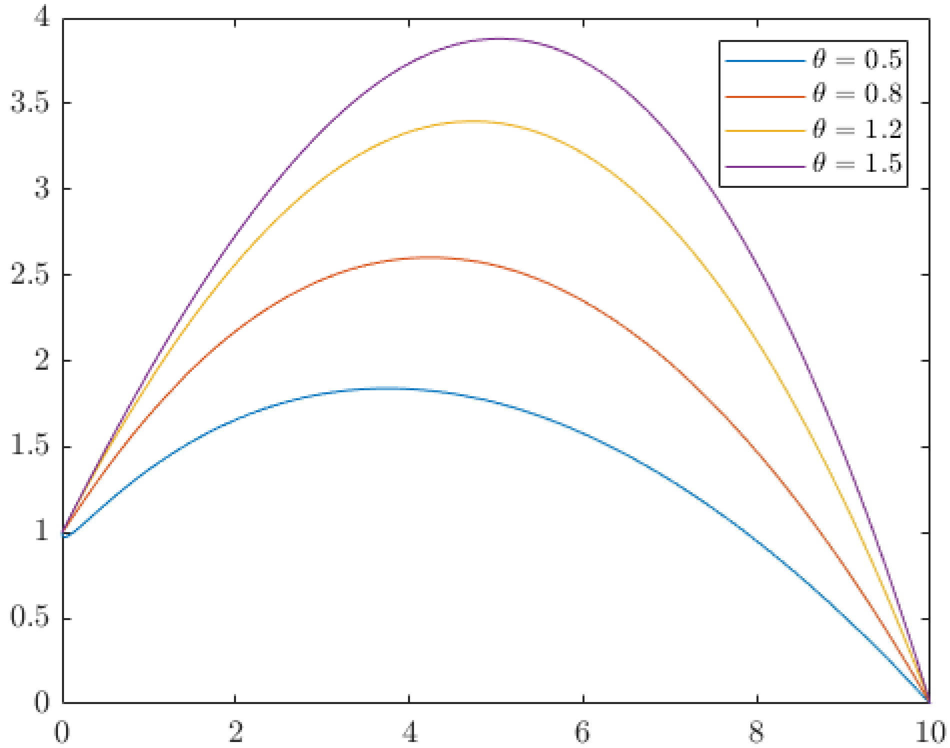

Figure 1: Plot of $g(x)$