Homework 2
Due: January 27, 2021

1. Consider the nonhomogeneous problems of Problem 1 and 2 : $\vec{x}' = \mathbf{A}\vec{x} + \vec{g}(t)$.
(a) Let $\vec{x} = \mathbf{M}\vec{y}$ where the columns of $\mathbf{M}$ are the eigenvectors of the above problems.
(b) Write the equations in terms of $\vec{y}$ and multiply through by $\mathbf{M}^{-1}$.
(c) Show the resulting equation is

$$\vec{y}' = \mathbf{D}\vec{y} + \vec{h}(t)$$

where $\mathbf{D} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ is a diagonal matrix whose diagonal elements are the eigenvalues of the problem considered and $\vec{h}(t) = \mathbf{M}^{-1}\vec{g}(t)$
(d) Show that this system is now decoupled so that each component of $\vec{y}$ can be solved independently of the other components.

Solution. Substituting $\vec{x} = \mathbf{M}\vec{y}$ into the equation

$$\begin{aligned}
\mathbf{M}\vec{y}' &= \mathbf{A}\mathbf{M}\vec{y} + \vec{g}(t) \\
\mathbf{M}^{-1}\mathbf{M}\vec{y}' &= \mathbf{M}^{-1}\mathbf{A}\mathbf{M}\vec{y} + \mathbf{M}^{-1}\vec{g}(t) \\
\vec{y}' &= \mathbf{D}\vec{y} + \vec{h}(t)
\end{aligned}$$

Since $\mathbf{D}$ is diagonal, we have a set of independent equations $y_i' = d_{ii}y_i + h_i$ where d_{ii} is the diagonal entry.

2. Given $L = -d^2/dx^2$ find the eigenfunction expansion solution of

$$\frac{d^2y}{dx^2} + 2y = -10\exp(x) \quad y(0) = 0, y'(1) = 0$$

Solution. First rearranging the equation into Sturm-Liouville problem

$$Ly = 2y + 10e^x$$

Thus $\mu = 2, f(x) = 10e^x$. The eigenvalue problem associated with this problem is

$$y'' + \lambda_n y = 0$$

which has a general solution

$$y = c_1 \sin(\sqrt{\lambda_n}x) + c_2 \cos(\sqrt{\lambda_n}x)$$

Since $y(0) = 0$, $c_2 = 0$. Since $y'(1) = 0$, $\cos(\sqrt{\lambda_n}) = 0 \rightarrow \sqrt{\lambda_n} = (n - 1/2)\pi$. Then $y_n = N\sin((n - 1/2)\pi x)$. Consider the normalization

$$\begin{aligned}
\langle y_n, y_n \rangle &= N^2 \int_0^1 \sin^2((n - 1/2)\pi x)dx \\
&= \frac{N^2}{2} \int_0^1 1 - \cos((2n - 1)\pi x)dx \\
&= \frac{N^2}{2} \left[x - \frac{\sin((2n - 1)\pi x)}{(2n - 1)\pi} \right]_0^1 \\
&= \frac{N^2}{2} = 1
\end{aligned}$$

Then $N = \sqrt{2}$ and $y_n = \sqrt{2}\sin((n - 1/2)\pi x)$. Writing the solution as series

$$y = \sum_{n=1}^{\infty} \frac{b_n}{(n - 1/2)^2\pi^2 - 2} \sqrt{2}\sin((n - 1/2)\pi x)$$

where $b_n = \langle f, y_n \rangle$. Evaluating b_n with Mathematica

$$b_n = 10\sqrt{2}\frac{(n - 1/2)\pi + (-1)^{n+1}e}{(n - 1/2)^2\pi^2 + 1}$$

Thus the eigenfunction expansion is

$$y = 20 \sum_{n=1}^{\infty} \frac{[(n - 1/2)\pi + (-1)^{n+1}e]\sin((n - 1/2)\pi x)}{((n - 1/2)^2\pi^2 + 1)((n - 1/2)^2\pi^2 - 2)}$$

3. Given $L = -d^2/dx^2$ find the eigenfunction expansion solution of

$$\frac{d^2y}{dx^2} + 2y = -x \quad y(0) = 0, y(1) + y'(1) = 0$$

Solution. First rearranging the equation into Sturm-Liouville problem

$$Ly = 2y + x$$

Thus $\mu = 2, f(x) = x$. The eigenvalue problem associated with this problem is

$$y'' + \lambda_n y = 0$$

which has a general solution

$$y = c_1 \sin(\sqrt{\lambda_n}x) + c_2 \cos(\sqrt{\lambda_n}x)$$

Since $y(0) = 0$, $c_2 = 0$. Since $y(1) + y'(1) = 0$, $c_1 \sin(\sqrt{\lambda_n}) + c_1\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}) = 0 \to \sin(\sqrt{\lambda_n}) = -\sqrt{\lambda_n}\cos(\sqrt{\lambda_n})$. Then $y_n = N\sin(\sqrt{\lambda_n}x)$. Consider the normalization

$$\begin{aligned}
\langle y_n, y_n \rangle &= N^2 \int_0^1 \sin^2(\sqrt{\lambda_n}x)dx \\
&= \frac{N^2}{2}\int_0^1 1 - \cos(2\sqrt{\lambda_n}x)dx \\
&= \frac{N^2}{2}\left[x - \frac{\sin(2\sqrt{\lambda_n}x)}{2\sqrt{\lambda_n}}\right]_0^1 \\
&= \frac{N^2}{2}\left(1 - \frac{\sin(2\sqrt{\lambda_n})}{2\sqrt{\lambda_n}}\right) \\
&= \frac{N^2}{2}\left(1 - \frac{\sin(\sqrt{\lambda_n})\cos(\sqrt{\lambda_n})}{\sqrt{\lambda_n}}\right) \\
&= \frac{N^2}{2}(1 + \cos^2(\sqrt{\lambda_n})) = 1
\end{aligned}$$

Then $N = (\frac{2}{1+\cos^2(\lambda_n)})^{1/2}$ and $y_n = (\frac{2}{1+\cos^2(\lambda_n)})^{1/2}\sin(\sqrt{\lambda_n}x)$. Writing the solution as series

$$y = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n - 2}\left(\frac{2}{1+\cos^2(\sqrt{\lambda_n})}\right)^{1/2} \sin(\sqrt{\lambda_n}x)$$

where $b_n = \langle f, y_n \rangle$. Evaluating b_n with Mathematica

$$b_n = \left(\frac{2}{1+\cos^2(\lambda_n)}\right)^{1/2}\left(\frac{2\sin(\sqrt{\lambda_n})}{\lambda_n}\right)$$

Thus the eigenfunction expansion is

$$y = 4\sum_{n=1}^{\infty} \frac{\sin(\sqrt{\lambda_n})\sin(\sqrt{\lambda_n}x)}{\lambda_n(\lambda_n - 2)(1+\cos^2(\lambda_n))}$$

4. Consider the Sturm-Liouville eigenvalue problem:

$$Lu = -\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = \lambda\rho(x)u \quad 0 < x < L$$

with the boundary conditions

$$\begin{aligned} \alpha_1 u(0) - \beta_1 u'(0) = 0 \\ \alpha_2 u(L) - \beta_2 u'(L) = 0 \end{aligned}$$

and with $p(x) > 0, \rho(x) > 0$, and $q(x) \geq 0$ and with $p(x), \rho(x), q(x)$ and $p'(x)$ continuous over $0 < x < L$. With the inner product $(\phi, \psi) = \int_0^L \rho(x)\phi(x)\psi^*(x)dx$, show the following:
(a) L is a self-adjoint operator.
(b) Eigenfunctions corresponding to different eigenvalues are orthogonal, i.e. $(u_n, u_m) = 0$.
(c) Eigenvalues are real, non-negative and eigenfunctions may be chosen to be real valued.
(d) Each eigenvalue is simple, i.e. it only has one eigenfunction. (Hint: recall that for each eigenvalue, there can be at most two linearly independent solutions - calculate the Wronskian of these two solutions and see what it implies.)

Solution. **(a)** We can find the adjoint by the relation $\langle v, Lu\rangle = \langle L^\dagger v, u\rangle$. To compute the inner product we use integration by parts twice

$$\begin{aligned} \langle v, Lu\rangle &= \int_0^L -\frac{d}{dx}\left[p(x)\frac{du}{dx}v\right] + q(x)uv \\ &= \left[-p(x)\frac{du}{dx}v\right]_0^L + \int_0^L p(x)\frac{du}{dx}\frac{dv}{dx} + q(x)uvdx \\ &= \left[-p(x)(\frac{du}{dx}v + \frac{dv}{dx}u)\right]_0^L + \int_0^L -\frac{d}{dx}\left[p(x)\frac{dv}{dx}u\right] + q(x)uv \end{aligned}$$

The first term is 0 due to the boundary conditions, thus $L = L^\dagger$.

Solution. **(b)** Suppose u_n and u_m are eigenfunctions with distinct eigenvalues λ_n and λ_m. They both are solutions to the SL problem

$$Lu_n = \lambda_n\rho(x)u_n \quad Lu_m = \lambda_m\rho(x)u_m$$

Multiplying the first equation by u_m and the second equation by u_n

$$\begin{aligned} \lambda_n\langle u_n, u_m\rangle_\rho = \lambda_m\langle u_m, u_n\rangle_\rho \\ (\lambda_n - \lambda_m)\langle u_n, u_m\rangle_\rho = 0 \end{aligned}$$

Since $\lambda_n - \lambda_m \neq 0$ and $\rho(x) > 0$, $\langle u_n, u_m\rangle = 0$.

Solution. **(c)** Suppose u_n is an eigenfunction with eigenvalue λ_n

$$\begin{aligned} \langle Lu_n, u_n\rangle &= \lambda_n\langle u_n, u_n\rangle \\ \langle Lu_n, u_n\rangle &= \langle u_n, Lu_n\rangle = \langle Lu_n, u_n\rangle^* = \lambda_n^*\langle u_n, u_n\rangle \end{aligned}$$

Since $\langle u_n, u_n \rangle \neq 0$, $\lambda_n = \lambda_n^*$, thus λ_n is real. For λ_n to be nonnegative, we need to assume that L is positive semi-definite, i.e. $u_n^* L u_n \geq 0$. Since $u_n^* L u_n = \lambda_n \rho(x) u_n^* u_n$, $\rho(x) > 0$ and $u_n^* u_n > 0$, then $\lambda_n \geq 0$.

Since the SL problem is second order, it has at most two linearly independent eigenfunctions u_1 and u_2, which are complex conjugates. We can write

$$u_1 = u + iv \quad u_2 = u - iv$$

Then we can always choose the eigenfunction as $u_1 + u_2 = 2u$, which is real.

Solution. **(d)** Suppose u_1 and u_2 are two linearly independent eigenfunctions with the same eigenvalue λ. Then the Wronskian can be computed as

$$W[u_1, u_2] = u_1 u_2' - u_1' u_2$$

Consider the Wronskian at $x = 0$, then we have the boundary conditions

$$\alpha u_1(0) - \beta u_1'(0) = 0 \to u_1(0) = \frac{\beta}{\alpha} u_1'(0)$$
$$\alpha u_2(0) - \beta u_2'(0) = 0 \to u_2(0) = \frac{\beta}{\alpha} u_2'(0)$$

Substituting back to W

$$W[u_1(0), u_2(0)] = \frac{\beta}{\alpha} u_1'(0) u_2'(0) - \frac{\beta}{\alpha} u_1'(0) u_2'(0) = 0$$

This implies that u_1 and u_2 are linearly dependent, which contradicts our assumption. Thus each eigenvalue has only one eigenfunction.