Homework 3.5
Due: April 23, 2021

Waterfowl Dynamics
A population of B ducks lives on two ponds, one large, one small. Let $N(t)$ be the number of birds on the small pond. You may assume that there are $B - N(t)$ birds on the large pond. Let the probability of a departure from the small pond be proportional to the departure rate r_d, to the interval, and to the number of birds on the small pond,

$$\Pr[N(t+\Delta t) = n-1 \mid N(t) = n] = r_d n \Delta t + o(\Delta t)$$

Similarly, assume that the probability of an arrival onto the small pond is proportional to the arrival rate r_a, to the interval, and to the number of birds off the small pond,

$$\Pr[N(t+\Delta t) = n+1 \mid N(t) = n] = r_a(B-n)\Delta t + o(\Delta t).$$

(a) Derive a system of differential equations for p_n, the probability of n birds on the small pond.
(b) Derive a partial differential equation for the probability generating function.

Solution.

(a) The probability of n birds on the small pond is a sum of three probabilities: one bird departures from an $n+1$ population, one bird arrives into an $n-1$ population, nothing happens for an n population. Writing this as an equation

$$p_n(t+\Delta t) = p_{n+1}(t)r_d(n+1)\Delta t + p_{n-1}(t)r_a(B-n+1)\Delta t + (1-(B-n)r_a\Delta t - nr_d\Delta t)p_n(t) + o(\Delta t)$$

Rearranging and taking $\Delta t \to 0$

$$\frac{dp_n}{dt} = (n+1)r_d p_{n+1} + (B-n+1)r_a p_{n-1} - ((B-n)r_a + nr_d)p_n$$

(b) The probability generating function is

$$F(t,x) = \sum_{n=0}^{\infty} p_n x^n$$

Taking a derivative in t

$$\begin{aligned}\frac{dF}{dt} &= \sum_{n=0}^{\infty} \frac{dp_n}{dt} x^n \\ &= \sum_{n=0}^{\infty} (n+1)r_d p_{n+1}x^n + (B-n+1)r_a p_{n-1}x^n - ((B-n)r_a + nr_d)p_n x^n\end{aligned}$$

$$
\begin{aligned}
&= r_d \sum_{n=0}^{\infty} (n+1) p_{n+1} x^n + B r_a x \sum_{n=0}^{\infty} p_{n-1} x^{n-1} - r_a x^2 \sum_{n=0}^{\infty} (n-1) p_{n-1} x^{n-2} \\
&- B r_a \sum_{n=0}^{\infty} p_n x^n - (r_d - r_a) x \sum_{n=0}^{\infty} n p_n x^{n-1} \\
&= B r_a (x-1) F + (r_d - r_a x^2 - (r_d - r_a) x) \frac{dF}{dx}
\end{aligned}
$$

Thus the PDE is

$$
\frac{dF}{dt} = B r_a (x-1) F + (r_a x + r_d)(1-x) \frac{dF}{dx}
$$