Homework 4
Due: February 12, 2021

1. Consider the weakly nonlinear oscillator:

$$\frac{d^2y}{dt^2} + y + \epsilon y^5 = 0$$

with $y(0) = 0$ and $y'(0) = A > 0$ and with $0 < \epsilon \ll 1$
(a) Use a regular perturbation expansion and calculate the first two terms.
(b) Determine at what time the approximation of part (a) fails to hold.
(c) Use a Poincare-Lindstedt expansion and determine the first two terms and frequency corrections.
(d) For $\epsilon = 0.1$, plot the numerical solution (from MATLAB), the regular expansion solution, and the Poincare-Lindstedt solution for $0 \leq t \leq 20$.

Solution.

(a) Using the expansion

$$y = y_0 + \epsilon y_1 + \cdots$$

Collecting terms in powers of ϵ

$$\begin{array}{ll} O(1) & y_0'' + y_0 = 0 \quad y_0(0) = 0, \;\; y_0'(0) = A \\ O(\epsilon) & y_1'' + y_1 = -y_0^5 \quad y_1(0) = 0, \;\; y_1'(0) = 0 \end{array}$$

The leading order solution is

$$y_0 = A\sin(t)$$

Note that $\sin^5(t)$ is

$$\sin^5(t) = \frac{1}{16}(10\sin(t) - 5\sin(3t) + \sin(5t))$$

We see immediately the Fredholm-Alternative theorem cannot be satisfied since $\sin(t)$ is in the null space of the operator, but we can still find a solution for y_1. Inserting the trig identity into the equation at $O(\epsilon)$ and solving in Mathematica

$$y_1 = \frac{A^5}{384}(120t\cos(t) - 80\sin(t) - 15\sin(3t) + \sin(5t))$$

Thus the regular perturbation gives the solution

$$y = A\sin(t) + \frac{\epsilon A^5}{384}\left[120t\cos(t) - 80\sin(t) - 15\sin(3t) + \sin(5t)\right]$$

(b) We observe in y_1 a secular term $t\cos(t)$, which grows without bound as $t \to \infty$.

(c) Using the Poincare-Lindstedt expansion

$$\tau = \omega t$$
$$\omega = \omega_0 + \epsilon\omega_1 + \cdots$$

The equation in terms of τ is

$$\omega^2 y'' + y + \epsilon y^5 = 0 \quad y(0) = 0, \quad \omega y'(0) = A$$

Expanding y gives

$$\begin{aligned} O(1) &\quad \omega_0^2 y_0'' + y_0 = 0 \quad y_0(0) = 0, \quad y_0'(0) = A \\ O(\epsilon) &\quad \omega_0^2 y_1'' + y_1 = -2\omega_0\omega_1 y_0'' - y_0^5 \quad y_1(0) = 0, \quad y_1'(0) = 0 \end{aligned}$$

The leading order solution is

$$y_0 = A\sin(\tau/\omega_0)$$

Without loss of generality, let $\omega_0 = 1$. The Fredholm-Alternative theorem requires that

$$\langle 2\omega_0\omega_1 y_0'' + y_0^5, \sin(\tau)\rangle = 0$$

Using the trig identity in (a) gives

$$(-2\omega_1 A\sin(\tau) + \frac{10A^5}{16})\sin(\tau) = 0$$
$$\omega_1 = \frac{5A^4}{16}$$

The equation at $O(\epsilon)$ then becomes

$$y_1'' + y_1 = \frac{A^5}{16}(5\sin(3\tau) - \sin(5\tau))$$

Solving in Mathematica

$$y_1 = \frac{A^5}{384}(40\sin(\tau) - 15\sin(3\tau) + \sin(5\tau))$$

Thus the perturbed solution with frequency shift is

$$y = A\sin((1 + \frac{5\epsilon A^4}{16})t) + \frac{\epsilon A^5}{384}\left[40\sin((1 + \frac{5\epsilon A^4}{16})t) - 15\sin(3(1 + \frac{5\epsilon A^4}{16})t) + \sin(5(1 + \frac{5\epsilon A^4}{16})t)\right]$$

(d) For simplicity we assume $A = 1$.

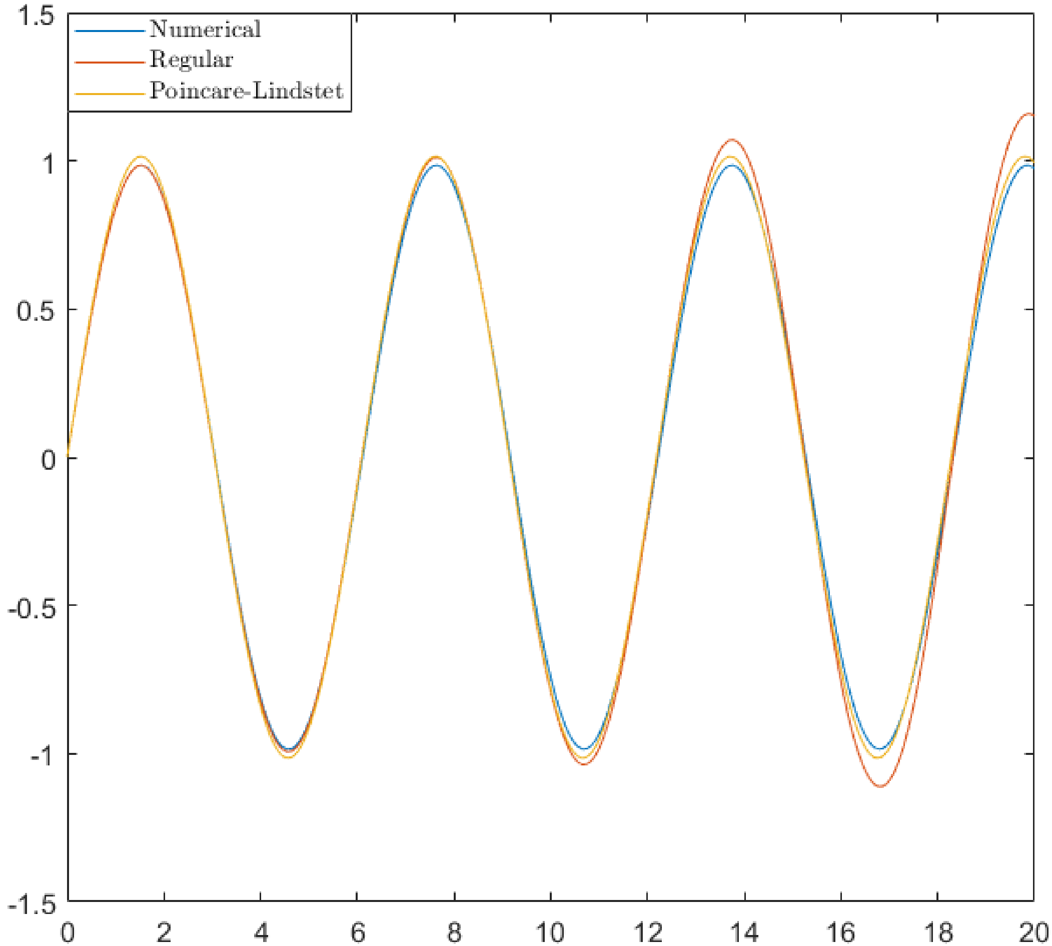

Figure 1: The weakly nonlinear oscillator solved by `bvp4c`, regular expansion and Poincare-Lindstedt method with $\epsilon = 0.1$. It is evident that the regular expansion solution diverges as $t \to \infty$.

2. Consider Rayleigh's equation:

$$\frac{d^2y}{dt^2}+y+\epsilon\left[-\frac{dy}{dt}+\frac{1}{3}\left(\frac{dy}{dt}\right)^3\right]=0$$

which has only one periodic solution called a "limit cycle" ($0<\epsilon\ll 1$). Given

$$y(0)=0$$

and

$$\frac{dy(0)}{dt}=A$$

(a) Use a multiple scale expansion to calculate the leading order behavior.
(b) Use a Poincare-Lindsted expansion and an expansion of $A=A_0+\epsilon A_1+\cdots$ to calculate the leading-order solution and the first non-trivial frequency shift for the limit cycle.
(c) For $\epsilon=0.01,0.1,0.2$ and 0.3, plot the numerical solution and the multiple scale expansion for $0\le t\le 40$ and for various values of A for your multiple scale solution. Also plot the limit cycle solution calculated from part (b).
(d) Calculate the error $E(t)=|y_{\text{numerical}}(t)-y_{\text{approximation}}(t)|$ as a function of time ($0\le t\le 40$) using $\epsilon=0.01,0.1,0.2$ and 0.3.

Solution.

(a) Define a slow time scale $\tau=\epsilon t$ and let $y(t)\to y(t,\tau)$. The modified equation is

$$y_{tt}+2\epsilon y_t+\epsilon^2 y_{\tau\tau}+y+\epsilon\left[-(y_t+\epsilon y_\tau)+\frac{1}{3}(y_t+\epsilon y_\tau)^3\right]=0$$

with boundary theorems

$$y(0)=0,\quad y_t(0)\to y_t(0)+\epsilon y_\tau(0)=A$$

Consider the expansion $y=y_0+\epsilon y_1+\cdots$. This gives the set of equations

$$\begin{aligned}
&O(1)\quad y_{0tt}+y_0=0\quad y_0(0,0)=0,\ \ y_{0t}(0,0)=A\\
&O(\epsilon)\quad y_{1tt}+y_1=-2y_{0t\tau}+y_{0t}-\frac{1}{3}y_{0t}^3\quad y_1(0,0)=0,\ \ y_{1t}(0,0)=-y_{0\tau}(0,0)
\end{aligned}$$

The leading order equation has solution

$$y_0=B(\tau)\cos(t)+C(\tau)\sin(t)\quad B(0)=0,\ \ C(0)=A$$

Substituting into RHS of the equation at $O(\epsilon)$

$$\begin{aligned}
\text{RHS}&=-2(-B_\tau\sin(t)+C_\tau\cos(t))+(-B\sin(t)+C\cos(t))-\frac{1}{3}(-B\sin(t)+C\cos(t))^3\\
&=\frac{1}{4}\left(-B^2C-C^3+4C-8C_\tau\right)\cos(t)+\frac{1}{4}\left(B^3+BC^2-4B+8B_\tau\right)\cos(t)+O(\cos(3t))+O(\sin(3t))
\end{aligned}$$

The Fredholm-Alternative theorem requires that

$$\begin{aligned} C^3 + B^2C - 4C + 8C_\tau = 0 \\ B^3 + BC^2 - 4B + 8B_\tau = 0 \end{aligned}$$

Multiplying the first equation by C, the second by B and summing

$$8(CC_\tau + BB_\tau) - 4(B^2 + C^2) + (B^2 + C^2)^2 = 0$$

Letting $D = B^2 + C^2$

$$D_\tau - D + \frac{1}{4}D^2 = 0$$

Solving this equation yields with initial condition $D(0) = B(0)^2 + C(0)^2 = A^2$

$$D = \frac{4A^2e^\tau}{A^2e^\tau - A^2 + 4}$$

Substituting $C^2 = D - B^2$ into $B^3 + BC^2 - 4B + 8B_\tau = 0$=0

$$\begin{aligned} B_\tau + \frac{B^3}{8} + \frac{B(D - B^2)}{8} - \frac{B}{2} = 0 \\ B_\tau + \frac{BD}{8} - \frac{B}{2} = 0 \end{aligned}$$

Solving this with $B(0) = 0$ yields

$$B(\tau) = 0$$

Then $C = D^{1/2}$, i.e.

$$C = \left(\frac{4A^2e^\tau}{A^2e^\tau - A^2 + 4}\right)^{1/2}$$

Thus the leading order solution is

$$y = \left(\frac{4A^2e^\tau}{A^2e^\tau - A^2 + 4}\right)^{1/2} \sin(t) + O(\epsilon)$$

(b) Using the Poincare-Lindstedt expansion

$$\begin{aligned} \tau &= \omega t \\ \omega &= \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots \end{aligned}$$

The equation in terms of τ is

$$\omega^2y'' + y + \epsilon\left[-\omega y' + \frac{1}{3}(\omega y')^3\right] = 0 \quad y(0) = 0, \quad \omega y'(0) = A_0 + \epsilon A_1 + \epsilon^2 A_2$$

Expanding y gives

$$O(1) \quad \omega_0^2 y_0'' + y_0 = 0 \quad y_0(0) = 0, \;\; y_0'(0) = A_0$$

$$O(\epsilon) \quad \omega_0^2 y_1'' + y_1 = -2\omega_0\omega_1 y_0'' + \omega_0 y_0' - \frac{1}{3}\omega_0^3 y_0'^3 = -F_1 \quad y_1(0) = 0, \;\; y_1'(0) = A_1$$

The leading order solution is

$$y_0 = A_0 \sin(\tau/\omega_0)$$

Without loss of generality, let $\omega_0 = 1$. At $O(\epsilon)$, the Fredholm-Alternative theorem requires that $\langle F_1, \cos(\tau)\rangle = 0$ and $\langle F_1, \sin(\tau)\rangle = 0$, i.e.

$$\langle -2\omega_1 A_0 \sin(\tau) - A_0 \cos(\tau) + \frac{1}{3}A_0^3 \cos^3(\tau), \cos(\tau)\rangle = 0$$
$$\langle -2\omega_1 A_0 \sin(\tau) - A_0 \cos(\tau) + \frac{1}{3}A_0^3 \cos^3(\tau), \sin(\tau)\rangle = 0$$

We need to remove $\sin(\tau)$ and $\cos(\tau)$ terms since they are in the null space. Using the identity $\cos^3(\tau) = \frac{3\cos(\tau)+\cos(3\tau)}{4}$, we require that

$$2\omega_1 A_0 = 0$$
$$-A_0 + \frac{1}{4}A_0^3 = 0$$

Thus $\omega_1 = 0$ and $A_0 = 2$. The $O(\epsilon)$ equation becomes

$$y_1'' + y_1 = -\frac{2}{3}\cos(3\tau) \quad y_1(0) = 0, \;\; y_1'(0) = A_1$$

Solving gives

$$y_1 = \frac{1}{6}\sin(\tau)(6A_1 - \sin(2\tau))$$

We can write the evolution equation at the next order as

$$O(\epsilon^2) \quad y_2'' + y_2 = -2\omega_2 y_0'' + y_1' - y_0'^2 y_1' = -F_2 \quad y_2(0) = 0, \;\; y_1'(0) = A_2$$

Again we require the orthogonality condition $\langle F_2, \cos(\tau)\rangle = 0$ and $\langle F_2, \sin(\tau)\rangle = 0$. Writing out F_2 explicitly

$$F_2 = A_1\cos(\tau) + 2A_1\cos(2\tau)\cos(\tau) - 4\omega_2\sin(\tau) + \frac{\sin(\tau)}{12}$$
$$-\frac{1}{4}\sin(3\tau) + \frac{1}{6}\sin(\tau)\cos(2\tau) - \frac{1}{2}\sin(3\tau)\cos(2\tau)$$

Thus $\omega_2 = \frac{1}{48}$ and $A_1 = 0$. The limit cycle solution is

$$y = 2\sin((1 + \frac{\epsilon^2}{48})t) + O(\epsilon)$$

(c-d) See Figure 2 through 5.

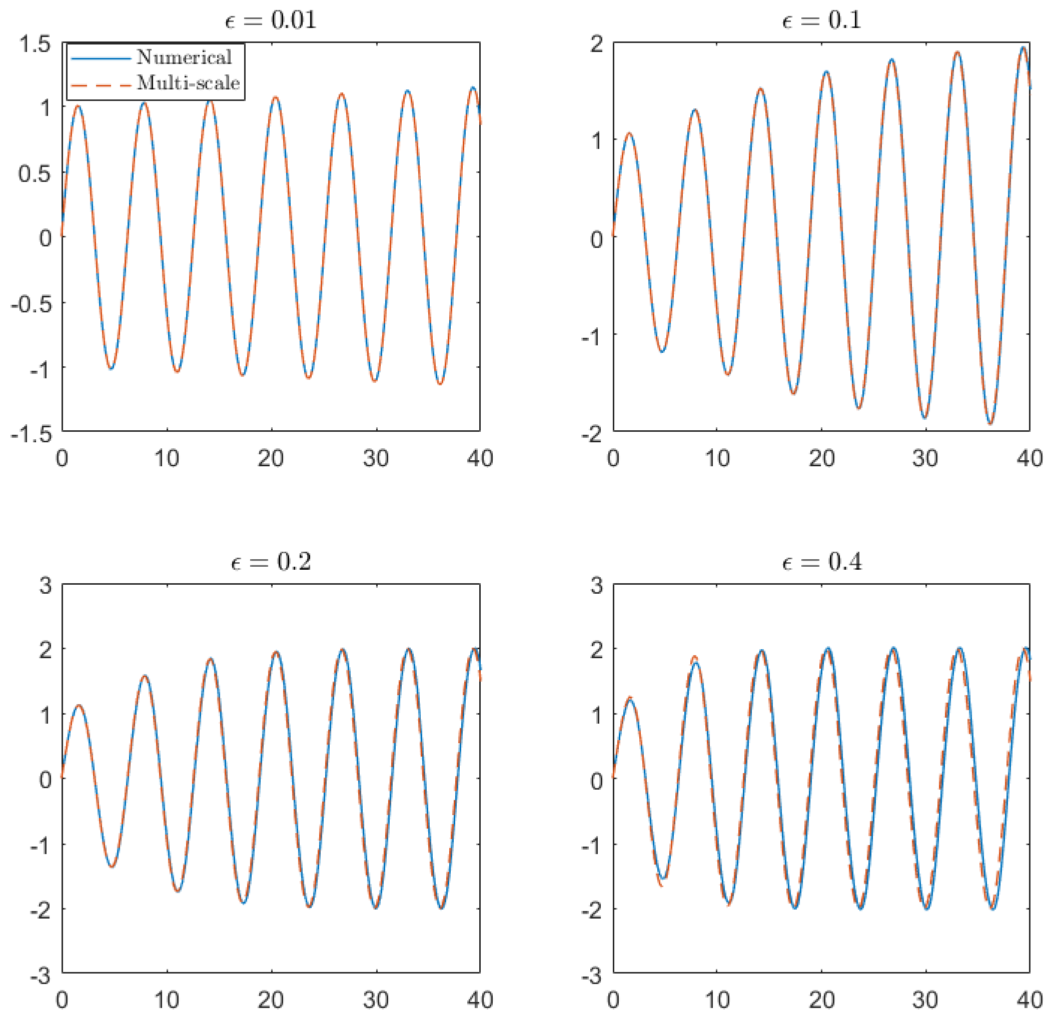

Figure 2: Numerical solution computed by `bvp4c` compared to the multi-scale expansion solution.

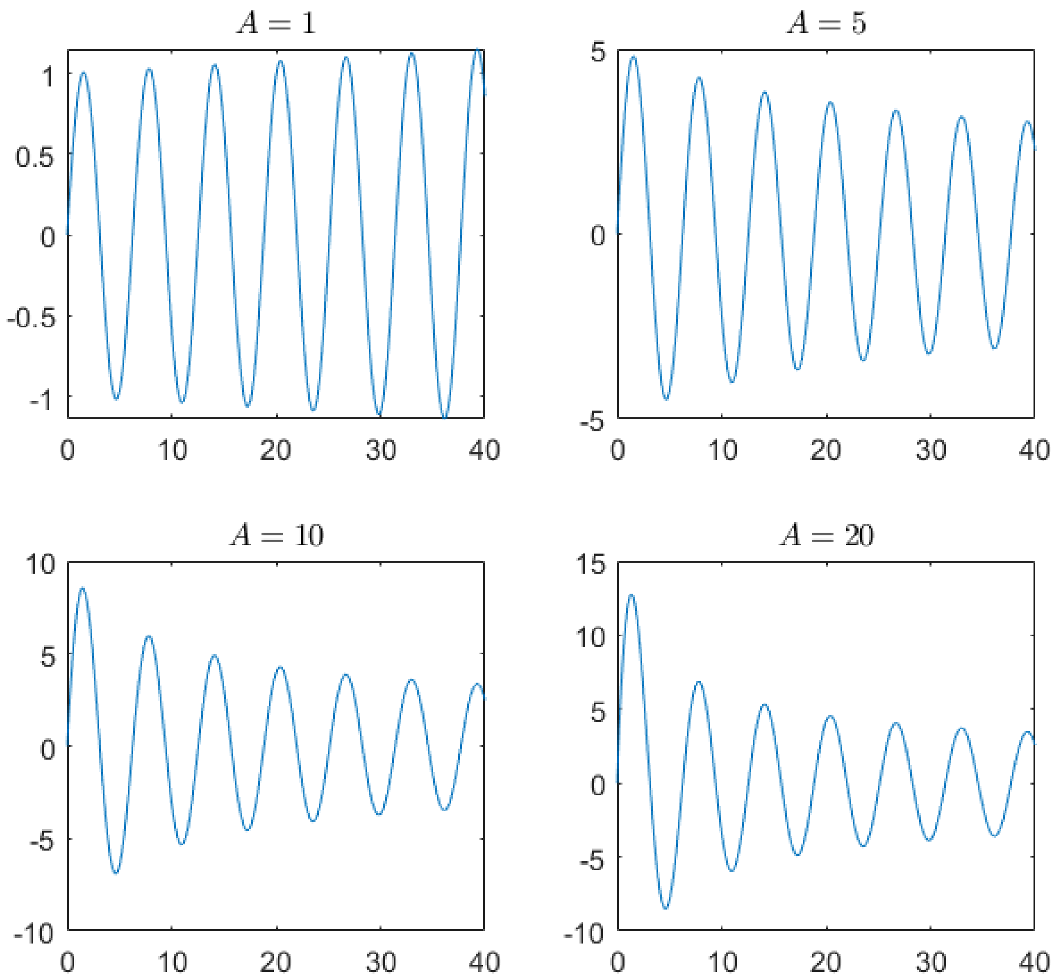

Figure 3: The multi-scale expansion solution calculated for different A. The limit cycle behaviour is clearly observed.

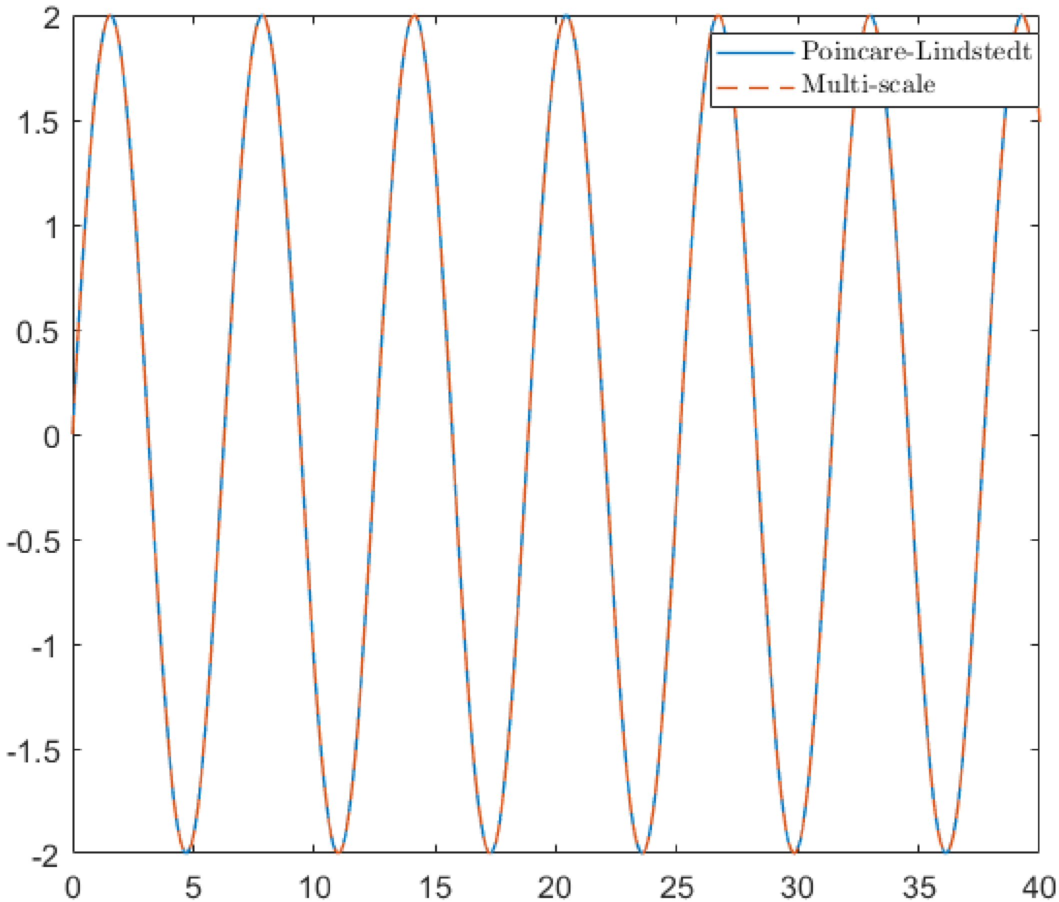

Figure 4: The limit cycle solution calculated by the Poincare-Lindstedt method. It is almost identical to the multi-scale expansion solution at an initial amplitude of 2.

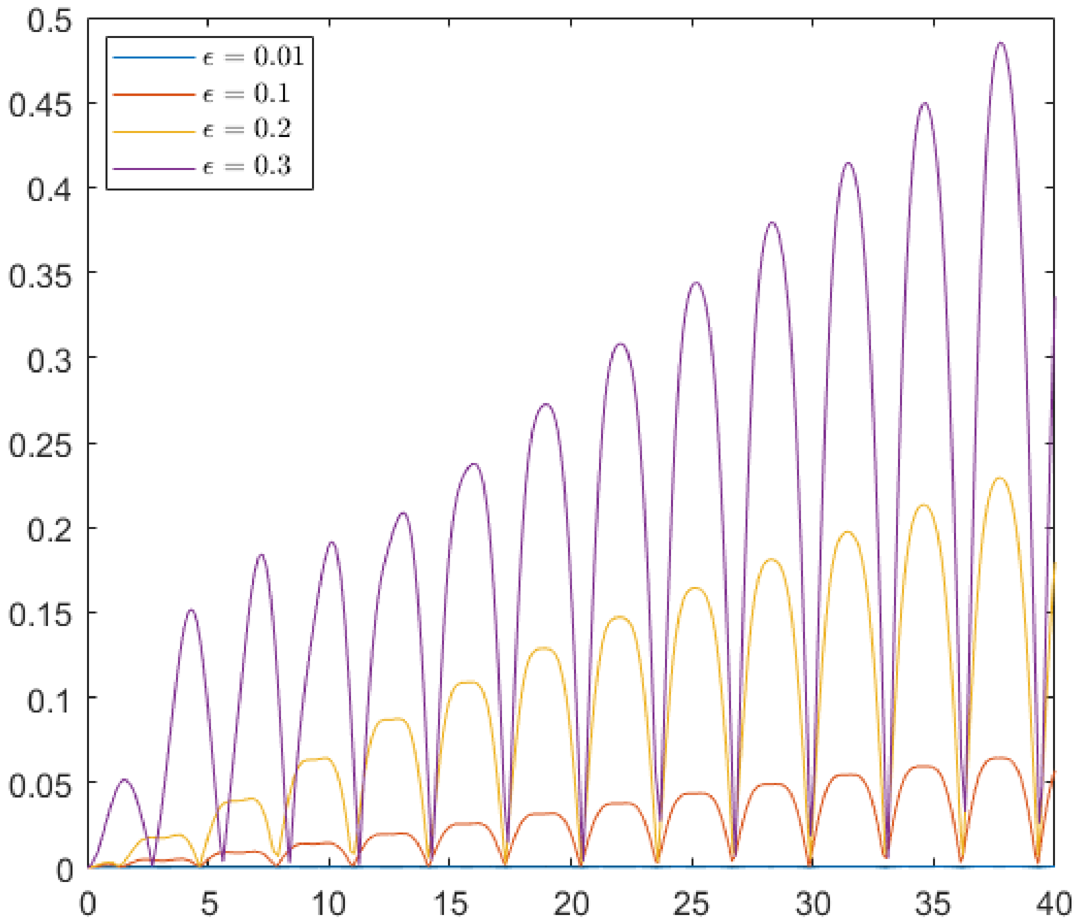

Figure 5: The absolute error in amplitude for different ϵ. It is evident that the error increases with ϵ and grows over time.