Homework 2
Due: April 23, 2021

1. Consider

$$v'''(t) + v'(t)v(t) - \frac{\beta_1 + \beta_2 + \beta_3}{3} v'(t) = 0,$$

where $\beta_1 < \beta_2 < \beta_3$. It follows that

$$v(t) = \beta_2 + (\beta_3 - \beta_2)\mathrm{cn}^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12}}t, \sqrt{\frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}}\right)$$

is a solution where $\mathrm{cn}(x, k)$ is the Jacobi cosine function and k is the elliptic modulus. Some notations use $\mathrm{cn}(x, m)$ where $m = k^2$. The corresponding initial conditions are

$$v(0) = \beta_3, v'(0) = 0, v''(0) = -\frac{(\beta_3 - \beta_1)(\beta_3 - \beta_2)}{6}.$$

Derive a third-order Runge-Kutta method and verify the order of accuracy on this problem using the methodology in Lecture 6 & 7 — produce a plot and and a table.

Solution.

Recall that a r-stage explicit RK method can have order at most r, we will try to derive a three-stage third-order RK method. This has the general form

$$\begin{aligned}
Y_1 &= U^n + k\sum_{j=1}^{3} a_{1j} f\left(Y_j, t_n + c_j k\right) \\
Y_2 &= U^n + k\sum_{j=1}^{3} a_{2j} f\left(Y_j, t_n + c_j k\right) \\
Y_3 &= U^n + k\sum_{j=1}^{3} a_{3j} f\left(Y_j, t_n + c_j k\right) \\
U^{n+1} &= U^n + k\sum_{i=1}^{3} b_i f\left(Y_i, t_n + c_i k\right)
\end{aligned}$$

Let $t = t_n$ and undo the approximation

$$u(t + k) = u(t) + k\sum_{i=1}^{3} b_i f\left(y_i, t + c_i k\right)$$

Taylor expand $u(t+k)$ and rearrange

$$u'(t) + \frac{k}{2}u''(t) + \frac{k^2}{6}u'''(t) + O(k^3) = \sum_{i=1}^{3} b_i f\left(y_i, t + c_i k\right) \tag{1}$$

Taylor expand $f(y_i, t + c_i k)$ around $(u(t), t)$ and let $f = f(u(t), t)$

$$\begin{aligned} f\left(y_i, t + c_i k\right) &= f + f_u \left(k \sum_{j=1}^{3} a_{ij} f(y_j, t + c_j k) \right) + f_t c_i k \\ &\quad + \frac{1}{2} f_{uu} \left(k \sum_{j=1}^{3} a_{ij} f(y_j, t + c_j k) \right)^2 + \frac{1}{2} f_{tt} (c_i k)^2 + f_{ut} c_i k \left(k \sum_{j=1}^{3} a_{ij} f(y_j, t + c_j k) \right) \end{aligned}$$

Note that

$$\begin{aligned} y_j &= u(t) + O(k) \\ t + c_j k &= t + O(k) \end{aligned}$$

We can rewrite $f(y_i, t + c_i k)$ as

$$f(y_i, t + c_i k) = f + k f_t c_i + \frac{1}{2} k^2 f_{tt} c_i^2 + k^2 f_{ut} f c_i \sum_{j=1}^{3} a_{ij} + k f_u f \sum_{j=1}^{3} a_{ij} + \frac{1}{2} k^2 f_{uu} f^2 (\sum_{j=1}^{3} a_{ij})^2 + O(k^3)$$

Plugging this into (1). Matching the coefficients gives the conditions

$$\begin{aligned} \sum_{i=1}^{3} b_i &= 1 \\ \sum_{i=1}^{3} b_i c_i &= \frac{1}{2} \\ \sum_{i=1}^{3} b_i c_i^2 &= \frac{1}{3} \\ \sum_{i=1}^{3} \sum_{j=1}^{3} b_i a_{ij} c_i &= \frac{1}{6} \end{aligned}$$

This is a system of nonlinear equations with 15 unknowns. For an explicit method, the elements on and above the diagonal in the a_{ij} portion of the Butcher tableau is 0. This eliminates 6 unknowns. We also know that $c_1 = 0$ for all RK methods. Thus we are left with 8 unknowns. Solving with initial guess $c_2 = 1/3$ yields

$$\begin{aligned} a_{21} &= 1/3,\ a_{31} = 0,\ a_{32} = 2/3 \\ b_1 &= 1/4,\ b_2 = 0,\ b_3 = 3/4 \\ c_2 &= 1/3,\ c_3 = 2/3 \end{aligned}$$

Finally, the RK method is

$$U^{n+1} = U^n + \frac{k}{4}(f(Y_1, t_n) + 3f(Y_3, t_n + 2k/3))$$

We implement this method in MATLAB as `jacobi_cos.m`. We observe that the error reduction ratio is approximately $2^3 = 8$, which indicates that our method is indeed third order.

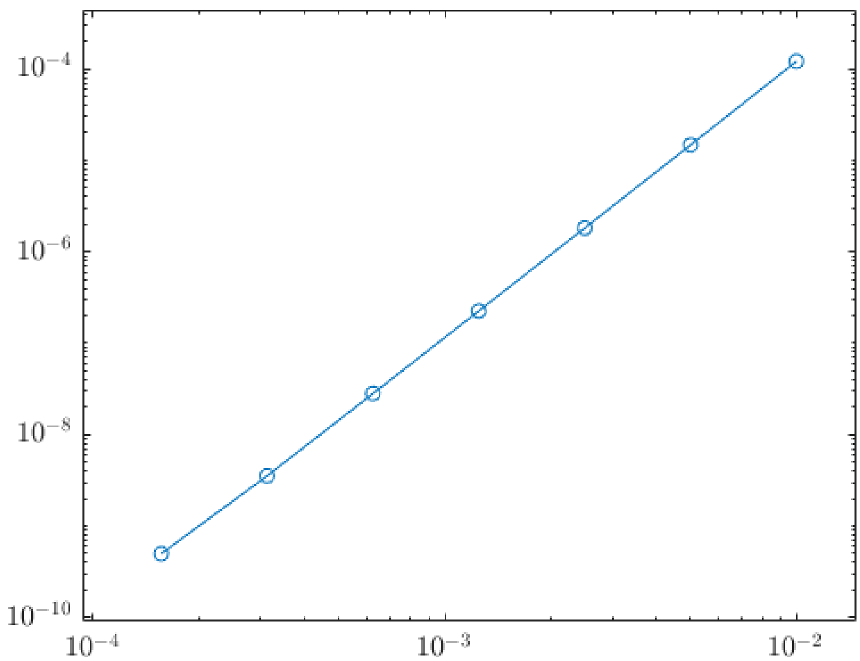

Figure 1: Error of third order RK against step size

k	3rd RK
0.005000	8.2161
0.002500	8.1097
0.001250	8.0556
0.000625	8.0311
0.000313	7.9561
0.000156	7.0955

Table 1: Error reduction ratio of third order RK

2. Which of the following Linear Multistep Methods are convergent? For the ones that are not, are they inconsistent, or not zero-stable, or both?

   (a) $U^{n+3} = U^{n+1} + 2kf(U^n)$,
   (b) $U^{n+2} = \frac{1}{2}U^{n+1} + \frac{1}{2}U^n + 2kf(U^{n+1})$,
   (c) $U^{n+1} = U^n$,
   (d) $U^{n+4} = U^n + \frac{4}{3}k(f(U^{n+3}) + f(U^{n+2}) + f(U^{n+1}))$,
   (e) $U^{n+3} = -U^{n+2} + U^{n+1} + U^n + 2k(f(U^{n+2}) + f(U^{n+1}))$.

Solution.

Recall the general form of LMMs

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j}, t_{n+j})$$

Consistency requires that

$$\sum_{j=0}^{r} \alpha_j = 0, \quad \sum_{j=0}^{r} j\alpha_j = \sum_{j=0}^{r} \beta_j \tag{2}$$

(a) We find the coefficients as

$$\alpha_0 = 0,\ \alpha_1 = -1,\ \alpha_2 = 0,\ \alpha_3 = 1$$
$$\beta_0 = 2,\ \beta_1 = 0,\ \beta_2 = 0,\ \beta_3 = 0$$

Clearly (1) is satisfied, thus this method is consistent. The characteristic polynomial of this method is $z^3 - z = 0$, which has roots $z = 0, \pm 1$. Since all roots are distinct and have absolute values ≤ 1, this method is zero-stable. Thus it is convergent.

(b) We find the coefficients as

$$\alpha_0 = -\frac{1}{2},\ \alpha_1 = -\frac{1}{2},\ \alpha_2 = 1$$
$$\beta_0 = 0,\ \beta_1 = 2,\ \beta_2 = 0$$

(1) is not satisfied because $\alpha_1 + 2\alpha_2 = \frac{3}{2} \neq 2$, thus this method is inconsistent. The characteristic polynomial of this method is $z^2 - \frac{z}{2} - \frac{1}{2} = 0$, which has roots $z = -\frac{1}{2}, 1$. Since all roots are distinct and have absolute values ≤ 1, this method is zero-stable. Thus it is not convergent.

(c) Clearly this method is inconsistent because f does not appear anywhere. The characteristic polynomial of this method is $z-1=0$, which has a root $z=1$. Since it has absolute values ≤ 1, this method is zero-stable. Thus it is not convergent.

(d) We find the coefficients as

$$\alpha_0=-1,\ \alpha_1=0,\ \alpha_2=0,\ \alpha_3=0,\ \alpha_4=1$$
$$\beta_0=0,\ \beta_1=\frac{4}{3},\ \beta_2=\frac{4}{3},\ \beta_3=\frac{4}{3},\ \beta_4=0$$

Clearly (1) is satisfied, thus this method is consistent. The characteristic polynomial of this method is $z^4-1=0$, which has roots $z=\pm 1,\pm i$. Since all roots are distinct and have absolute values ≤ 1, this method is zero-stable. Thus it is convergent.

(e) We find the coefficients as

$$\alpha_0=-1,\ \alpha_1=-1,\ \alpha_2=1,\ \alpha_3=1$$
$$\beta_0=0,\ \beta_1=2,\ \beta_2=2,\ \beta_3=0$$

Clearly (1) is satisfied, thus this method is consistent. The characteristic polynomial of this method is $z^3+z^2-z-1=0$, which has repeated roots $z=-1$ and a simple root $z=1$. Since the repeated roots have absolute values equal to 1, this method is not zero-stable. Thus it is not convergent.

3. For the one-step method (6.17), with Ψ given in (6.18), show that the Lipschitz constant is $L' = L + \frac{k}{2}L^2$ where L is the Lipschitz constant for f.

Solution.

Let

$$\begin{aligned} a &= u + \frac{1}{2}kf(u) \\ b &= v + \frac{1}{2}kf(v) \end{aligned}$$

By Lipschitz condition and triangular inequality

$$\begin{aligned} |f(a) - f(b)| &\leq L|a - b| \\ |\Psi(u) - \Psi(v)| &\leq L|u - v + \frac{1}{2}k(f(u) - f(v))| \\ &\leq L(|u - v| + \frac{1}{2}k|f(u) - f(v)|) \\ &\leq L(|u - v| + \frac{1}{2}kL|u - v|) \\ &= (L + \frac{1}{2}kL^2)|u - v| \end{aligned}$$

4. The Fibonacci numbers

   (a) Determine the general solution to the linear difference equation $U^{n+2} = U^{n+1} + U^n$.

   (b) Determine the solution to this difference equation with the starting values $U^0 = 1$, $U^1 = 1$. Use this to determine U^{30}. (Note, these are the *Fibonacci numbers*, which of course should all be integers.)

   (c) Show that for large n the ratio of successive Fibonacci numbers U^n/U^{n-1} approaches the "golden ratio" $\phi \approx 1.618034$.

Solution.

(a) The characteristic polynomial of this equation is $z^2 - z - 1 = 0$, which has roots $z = \frac{1\pm\sqrt{5}}{2}$. Thus the general solution is

$$U^n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

(b) $U_0 = 1$ gives $c_1 + c_2 = 1$. $U_1 = 1$ gives $c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2} = 1$. Solving yields $c_1 = \frac{5+\sqrt{5}}{10}$, $c_2 = \frac{5-\sqrt{5}}{10}$. Then

$$U^{30} = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^{30} + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^{30} = 1346269$$

(c) Let n be large and realizing that $|\frac{1-\sqrt{5}}{2}| < 1$

$$\begin{aligned}
\frac{U^n}{U^{n-1}} &= \frac{c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n}{c_1\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + c_2\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}} \\
&= \frac{\frac{1+\sqrt{5}}{2}\left(c_1\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \frac{2}{1+\sqrt{5}}\right)}{c_1\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \frac{2}{1-\sqrt{5}}} \\
&\approx \frac{1+\sqrt{5}}{2} = 1.618034
\end{aligned}$$

5. Explicit solution of leapfrog: Consider the difference equation

$$\begin{aligned} U^{n+1} + U^{n-1} &= 2xU^n, \quad n \geq 0, \\ U^0 = 1, \quad U^{-1} &= 0. \end{aligned}$$

Provided that $-1 \leq x \leq 1$ perform the following steps:

(a) Argue that this can be replaced with

$$\begin{aligned} U^{n+1} + U^{n-1} &= (e^{i\theta} + e^{-i\theta})U^n, \quad n \geq 0, \\ U^0 = 1, \quad U^{-1} &= 0, \quad \theta \in \mathbb{R}. \end{aligned}$$

(b) Define $V^n = U^n - e^{i\theta}U^{n-1}$ and find a simpler recurrence relation for V^n. Solve it.

(c) With V^n known, $V^n = U^n - e^{i\theta}U^{n-1}$ gives an inhomogeneous recurrence relation for U^n. Find a formula for U^n. For which value(s) of $x \in [-1, 1]$ is U^n largest?

Note: In this problem you can make the ansatz $U^n = \lambda^n$ and find a quadratic equation to determine two possible values for λ, say, λ_1, λ_2. Then the general solution is a sum of these, $U^n = c_1\lambda_1^n + c_2\lambda_2^n$. This approach will give another representation of the same solution. You are encouraged to do so and derive this remarkable identity!

Solution.

(a) We have $e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$, observing that $-1 \leq \cos\theta \leq 1$.

(b) We start with some simple relations

$$\begin{aligned} V^{n-1} &= U^{n-1} - e^{i\theta}U^{n-2} \\ U^n &= (e^{i\theta} + e^{-i\theta})U^{n-1} - U^{n-2} \end{aligned}$$

Multiplying the first relation by $e^{-i\theta}$, have this in mind

$$e^{-i\theta}V^{n-1} = e^{-i\theta}U^{n-1} - U^{n-2}$$

Plugging the second relation into V^n

$$V^n = (e^{i\theta} + e^{-i\theta})U^{n-1} - U^{n-2} - e^{i\theta}U^{n-1} = e^{-i\theta}V^{n-1}$$

The characteristic polynomial of this is $1 - e^{-i\theta}z^{-1} = 0$ which has a root $z = e^{-i\theta}$. The solution takes the form $V^n = ce^{-in\theta}$. The conditions $U^0 = 1,\ U^{-1} = 0$ give $V^0 = 1$. Solving yields $c = 1$. Thus $V^n = e^{-in\theta}$.

(c) We have the inhomogeneous relations

$$
\begin{aligned}
e^{-in\theta} &= U^n - e^{i\theta}U^{n-1} \\
e^{-i(n+2)\theta} &= U^{n+2} - e^{i\theta}U^{n+1}
\end{aligned}
$$

Let $e^{-i\theta} = r$, the homogeneous problem

$$\lambda^2 - r^{-1}\lambda = 0$$

has solution $\lambda_1 = r^{-1}$ and $\lambda_2 = 0$. Thus $U_{\mathrm{h}}^n = c_1 r^{-n}$. For the inhomogeneous problem, we assume that the solution takes the form $\lambda = ar^n$. Plugging into the difference equation

$$ar^n = r^{-1}ar^{n-1} + r^n$$

Solving yields $a = \frac{1}{1-r^{-2}}$. Thus the particular solution is $U_{\mathrm{p}}^n = c_2\frac{r^n}{1-r^{-2}}$. The general solution is

$$U^n = c_1 r^{-n} + c_2\frac{r^n}{1-r^{-2}}$$

Using $U^0 = 1$ and $U^{-1} = 0$ gives $c_1 = \frac{-1}{r^2-1}$ and $c_2 = 1$, thus

$$
\begin{aligned}
U^n &= \frac{r^{n+1} - r^{-(n+1)}}{r - r^{-1}} \\
&= \frac{-2i\sin(n+1)\theta}{-2i\sin\theta} \\
&= \frac{\sin(n+1)\theta}{\sin\theta}
\end{aligned}
$$

By L'Hopital's Rule,

$$\lim_{\theta\to 0}\frac{\sin(n+1)\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{(n+1)\cos(n+1)\theta}{\cos\theta} = n+1$$

This the largest value of U^n, taken at $x = \cos(0) = 1$.

6. Any r-stage Runge-Kutta method applied to $u' = \lambda u$ will give an expression of the form
$$U^{n+1} = R(z)U^n$$
where $z = \lambda k$ and $R(z)$ is a rational function, a ratio of polynomials in z each having degree at most r. For an explicit method $R(z)$ will simply be a polynomial of degree r and for an implicit method it will be a more general rational function.

   Since $u(t_{n+1}) = e^z u(t_n)$ for this problem, we expect that a pth order accurate method will give a function $R(z)$ satisfying
$$R(z) = e^z + \mathrm{O}(z^{p+1}) \quad \text{as } z \to 0.$$
This indicates that the one-step error is $\mathrm{O}(z^{p+1})$ on this problem, as expected for a pth order accurate method.

   The explicit Runge-Kutta method of Example 5.13 is fourth order accurate in general, so in particular it should exhibit this accuracy when applied to $u'(t) = \lambda u(t)$. Show that in fact when applied to this problem the method becomes $U^{n+1} = R(z)U^n$ where $R(z)$ is a polynomial of degree 4, and that this polynomial agrees with the Taylor expansion of e^z through $O(z^4)$ terms.

   We will see that this function $R(z)$ is also important in the study of absolute stability of a one-step method.

   Solution.

   We apply a fourth-order Runge-Kutta method to this problem
$$\begin{aligned}
Y_1 &= U^n \\
Y_2 &= (1 + \frac{z}{2})U^n \\
Y_3 &= (\frac{z^2}{4} + \frac{z}{2} + 1)U^n \\
Y_4 &= (\frac{z^3}{4} + \frac{z^2}{2} + z + 1)U^n \\
U^{n+1} &= U^n + \frac{k}{6}\left[f(Y_1) + 2f(Y_2) + 2f(Y_3) + f(Y_4)\right] \\
&= U^n + \frac{z}{6}[Y_1 + 2Y_2 + 2Y_3 + Y_4] \\
&= U^n + \frac{z}{6}U^n[1 + 2(1 + \frac{z}{2}) + 2(\frac{z^2}{4} + \frac{z}{2} + 1) + \frac{z^3}{4} + \frac{z^2}{2} + z + 1] \\
&= U^n + \frac{z}{6}(6 + 3z + z^2 + \frac{z^3}{4})U^n \\
&= (1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24})U^n
\end{aligned}$$
Clearly, $R(z)$ is the Taylor expansion of e^z through $O(z^4)$.

7. Determine the function $R(z)$ described in the previous exercise for the TR-BDF2 method given in (5.37). Note that this can be simplified to the form (8.6), which is given only for the autonomous case but that suffices for $u'(t) = \lambda u(t)$. (You might want to convince yourself these are the same method).

   Confirm that $R(z)$ agrees with e^z to the expected order.

   Note that for this implicit method $R(z)$ will be a rational function, so you will have to expand the denominator in a Taylor series, or use the Neumann series

$$1/(1-\epsilon) = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \cdots .$$

Solution.

We apply a TR-BDF2 method to this problem

$$\begin{aligned} Y_1 &= U^n \\ Y_2 &= U^n + \frac{z}{4}(Y_1 + Y_2) \\ Y_3 &= U^n + \frac{z}{3}(Y_1 + Y_2 + Y_3) \\ U^{n+1} &= Y_3 \end{aligned}$$

Rearranging

$$\begin{aligned} Y_2 &= \frac{U^n + \frac{z}{4}U^n}{1 - \frac{z}{4}} \\ Y_3 = U^{n+1} &= \frac{U^n + \frac{z}{3}(U^n + \frac{U^n + \frac{z}{4}U^n}{1-\frac{z}{4}})}{1 - \frac{z}{3}} \end{aligned}$$

Simplifying

$$\begin{aligned} U^{n+1} = \frac{5z+12}{(z-4)(z-3)}U^n &= \left(\frac{27}{3-z} - \frac{32}{4-z}\right)U^n \\ &= \left(9\left(\frac{1}{1-\frac{z}{3}}\right) - 8\left(\frac{1}{1-\frac{z}{4}}\right)\right)U^n \\ &= (9(1 + \frac{z}{3} + \frac{z^2}{9} + \cdots) - 8(1 + \frac{z}{4} + \frac{z^2}{16} + \cdots))U^n \\ &= (1 + z + \frac{z^2}{2} + \cdots)U^n \end{aligned}$$

Clearly this matches e^z through $O(z^2)$.