Homework 8

Due: Friday, December 4, 2020

Question 1.

(a) Construct the bilinear transformation

$$w(z) = \frac{az+b}{cz+d}$$

that maps the region between the two circles $\left|z - \frac{1}{4}\right| = \frac{1}{4}$ and $\left|z - \frac{1}{2}\right| = \frac{1}{2}$ into an infinite strip bounded by the vertical lines $u = \mathrm{Re}\{w\} = 0$ and $u = \mathrm{Re}\{w\} = 1$. To avoid ambiguity, suppose that the outer circle is mapped to $u = 1$.

(b) Upon finding the appropriate transformation w, carefully show that the image of the inner circle under w is the vertical line $u = 0$, and similarly for the outer circle.

(a) Consider the transformation $a\frac{z-z_1}{z-z_0}$. We want to transform $z = 0$ (which both circles pass through) to $w = \pm\infty$, thus we pick $z_0 = 0$. We want to transform $z = \frac{1}{2}$ to $w = 0$, thus $z_1 = \frac{1}{2}$. We want to transform $z = 1$ to $w = 1$, solving for a yields $a = 2$. Thus our transformation is $\frac{2z-1}{z}$.

(b) Consider the inner circle, we have

$$\begin{aligned} |z - \frac{1}{4}| &= \frac{1}{4} \\ (z - \frac{1}{4})(\bar{z} - \frac{1}{4}) &= \frac{1}{16} \\ |z|^2 - \frac{1}{4}(z + \bar{z}) &= 0 \\ |z|^2 &= \frac{1}{2}x \end{aligned}$$

$w = \frac{(2z-1)\bar{z}}{z\bar{z}} = \frac{2|z|^2-(x-iy)}{|z|^2}$, then $u = \mathrm{Re}(w) = \frac{2|z|^2-x}{|z|^2} = 0$. Similarly for the outer circle we have $|z|^2 = x$, then $u = \frac{|z|^2}{|z|^2} = 1$.

Question 2.
Use the result of Question 1 to find the steady state temperature $T(x,y)$ in the region bounded by the two circles, where the inner circle is maintained at $T = 0°\text{C}$ and the outer circle at $T = 100°\text{C}$. Assume T satisfies the two-dimensional Laplace equation.

Since T in the $w-$plane is bounded by two parallel lines, the heat flow is uniform. Due to the geometry of the flow, T does not depend on $\text{Im}(w)$. We have boundary conditions $T(0,y) = 0$ and $T(1,y) = 100$, then the solution is $T = 100\text{Re}(w)$, $0 \le \text{Re}(w) \le 1$ (this is a Dirichlet problem and the solution is unique by the uniqueness theorem). Since $\text{Re}(w) = \frac{2|z|^2 - x}{|z|^2}$, $T = \frac{200(x^2+y^2)-100x}{x^2+y^2}$.