Homework 3

Due: Monday, October 26, 2020

Question 1. Give an example of a probability space $(\Omega, \mathcal{F}, P)$, a random variable X and a function f such that $\sigma(f(X))$ is strictly smaller than $\sigma(X)$ but $\sigma(f(X)) \neq \{\emptyset, \Omega\}$. Give a function g such that $\sigma(g(X)) = \{\emptyset, \Omega\}$. Hint: Look at finite sample spaces with a small number of elements.

Let $\Omega = \{-1, 0, 1\}$, $F = 2^\Omega$, $P(\omega_i) = \frac{1}{3}$, $X(\omega) = \omega$. For any Borel set in a measurable space $\mathcal{S}$,

$$\sigma(X) = \{X \in B : B \in \mathcal{S}\} = 2^\Omega$$

Let $f(X) = |X|$, then

$$f(X(\omega)) = \begin{cases} 1 & \omega \in \{1, -1\} \\ 0 & \omega \in \{0\} \end{cases}$$

Then we have

$$\sigma(f(X)) = \{f(X) \in B : B \in \mathcal{S}\} = \{\Omega, \emptyset, \{1, -1\}, 0\} \subset 2^\Omega$$

Let $g(X) = 0$, then $g(X(\omega)) = 0$ for $\omega \in \{-1, 0, 1\}$. We have

$$\sigma(g(X)) = \{g(X) \in B : B \in \mathcal{S}\} = \{\emptyset, \Omega\}$$

Question 2. Give an example of events A, B, and C, each of probability strictly between 0 and 1, such that $P(A \cap B) = P(A)P(B), P(A \cap C) = P(A)P(C)$, and $P(A \cap B \cap C) = P(A)P(B)P(C)$ but $P(B \cap C) \neq P(B)P(C)$. Are A, B and C independent? Hint: You can let Ω be a set of eight equally likely points.

Consider rolling an 8 sided fair die, then $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $P(\omega_i) = \frac{1}{8}$. Let $A = \{\omega \text{ is even}\}$, $B = \{\omega \geq 4 \text{ and } \omega \neq 8\}$ and $C = \{\omega \leq 4\}$. Then $P(A) = P(B) = P(C) = \frac{1}{2}$. We have

$P(A \cap B) = P(\omega = 4, 6) = \frac{1}{4} = P(A)P(B)$

$P(A \cap C) = P(\omega = 2, 4) = \frac{1}{4} = P(A)P(C)$

$P(A \cap B \cap C) = P(\omega = 4) = \frac{1}{8} = P(A)P(B)P(C)$

$P(B \cap C) = P(\omega = 4) = \frac{1}{8} \neq P(B)P(C)$

A, B, C are not independent because they are not pairwise independent.

Question 3. Let $(\Omega, \mathcal{F}, P)$ be a probability space such that Ω is countable, and $\mathcal{F} = 2^\Omega$. Show that it is impossible for there to exist a countable collection of events $A_1, A_2, ... \in \mathcal{F}$ which are independent, such that $P(A_i) = 1/2$ for each i. Hint: First show that for each $\omega \in \Omega$ and each $n \in \mathbb{N}$, we have $P(\omega) \leq 1/2^n$. Then derive a contradiction.

Consider $n \in \mathbb{N}$ events

$$P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i) = \frac{1}{2^n}$$

Since $P(\bigcap_{i=1}^n A_i) > 0$, $\bigcap_{i=1}^n A_i$ is nonempty and there exists some $\omega \in \Omega$ such that $\{\omega\} \subseteq \bigcap_{i=1}^n A_i$. Thus $P(\omega) \le 1/2^n$. Consider $n \to \infty$, then $P(\omega) \le 0$. Since $P(\omega) \ge 0$ by definition of probability measure, then $P(\omega) = 0$. Thus $P(A_i) = \sum_{\omega_j \in A_i} P(\omega_j)$=0, which contradicts $P(A_i) = \frac{1}{2}$.

Question 4. (a) Let $X \ge 0$ and $Y \ge 0$ be independent random variables with distribution functions F and G. Find the distribution function of XY.

$$\begin{aligned}
F_Z(z) = P(Z \le z) &= P(XY \le z) \\
&= P(XY \le z, X \ge 0) + P(XY \le z, X < 0) \\
&= P(Y \le \frac{z}{X}, X \ge 0) \\
&= \int_0^\infty P(Y \le \frac{z}{x}) \mathrm{d}F(x) \\
&= \int_0^\infty G(\frac{z}{x}) \mathrm{d}F(x) \text{ for } z \ge 0
\end{aligned}$$

$$F_Z(z) = 0 \text{ for } z < 0$$

(b) If $X \ge 0$ and $Y \ge 0$ are independent continuous random variables with density functions f and g, find the density function of XY.

$$F_Z(z) = P(Y \le \frac{z}{X}, X \ge 0) = \int_0^\infty f(x) \int_0^{\frac{z}{x}} g(y) \mathrm{d}y \mathrm{d}x$$

$$f_{XY}(z) = F'_Z(z) = \int_0^\infty f(x) g(\frac{z}{x}) \frac{1}{x} \mathrm{d}x \text{ for } z \ge 0$$

$$f_{XY}(z) = 0 \text{ for } z < 0$$

(c) If X and Y are independent exponentially distributed random variables with parameter λ, find the density function of XY.

Given X and Y are exponential random variables, $f(x) = g(x) = \lambda e^{-\lambda x}$

$$\begin{aligned}
f_{XY}(z) &= \int_0^\infty \lambda e^{-\lambda x} \lambda e^{-\lambda \frac{z}{x}} \frac{1}{x} \mathrm{d}x \\
&= \lambda^2 \int_0^\infty e^{-\lambda(x + \frac{z}{x})} \frac{1}{x} \mathrm{d}x \text{ for } z \ge 0
\end{aligned}$$

$$f_{XY}(z) = 0 \text{ for } z < 0$$