**Homework 3**
Due: May 7, 2021

1. It is natural to ask if a neighborhood of $z = 0$ can be in the absolute stability region $S$ for a LMM. You will show that this cannot be the case. Consider a consistent and zero-stable LMM

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j}).$$

Recall the characteristic polynomial $\pi(\xi; z) = \rho(\xi) - z\sigma(\xi)$. Show:

- Consistency implies that $\pi(1; 0) = 0$.
- Stability implies that $\rho'(1) \neq 0$.
- Suppose $\xi = 1+\eta(z)$ for $z$ near zero so that $\pi(\xi; z) = \pi(1+\eta(z); z) = 0$. Compute $\eta'(0)$. Why does this imply that there must be an interval $(0, \epsilon]$ for some small $\epsilon > 0$ that does not lie in the absolute stability region $S$.

**Solution.**

Consistency requires that

$$\sum_{j=0}^{r} \alpha_j = 0, \quad \sum_{j=0}^{r} j\alpha_j = \sum_{j=0}^{r} \beta_j$$

Recall that $\pi(\xi; z)$ has the form

$$\pi(\xi; z) = \sum_{j=0}^{r} (\alpha_j - z\beta_j)\xi^j$$

Then

$$\pi(1; 0) = \sum_{j=0}^{r} \alpha_j$$

Thus it must be true that $\pi(1; 0) = 0$.

Stability requires that $\pi(\xi; z)$ satisfies the root condition (6.34). Observe that

$$\rho'(1) = \sum_{j=0}^{r} j\alpha_j = \sigma(1)$$

If $\rho'(1) = \sigma(1) = 0$, then $\pi(1; z) = 0$, but this implies that this LMM is not stable for any $z$.
$\pi(1 + \eta(z); z)$ has the form

$$\pi(1 + \eta(z); z) = \rho(1 + \eta(z)) - z\sigma(1 + \eta(z)) = 0$$

Differentiating with respect to $z$

$$\eta'(z)\rho'(1 + \eta(z)) - z\eta'(z)\sigma'(1 + \eta(z)) - \sigma(1 + \eta(z)) = 0$$

Setting $z = 0$

$$\eta'(0)\rho'(1 + \eta(0)) - \sigma(1 + \eta(0)) = 0$$
$$\eta'(0) = \frac{\sigma(1 + \eta(0))}{\rho'(1 + \eta(0))} = 1$$

This implies that $\lim_{x\to 0} \frac{\eta(x)}{x} = 1$. Then there exists a $\delta > 0$ such that if $x \in (0, \delta)$, for every $c > 0$

$$\left| \frac{\eta(x)}{x} - 1 \right| < c$$
$$(1 - c)x < \eta(x) < (1 + c)x$$

Choose $\epsilon = \min(1, \delta)$ and $c = 0.1$. If $x \in (0, \epsilon)$

$$0 < 0.9x < \eta(x) < 1.1x < 1.1$$
$$0 < \eta(x) < 1.1$$

Then $1 < \xi = 1 + \eta(x) < 2.1$, clearly this is not entirely in the stability region.

2. Recall the test problem

$$v'''(t) + v'(t)v(t) - \frac{\beta_1 + \beta_2 + \beta_3}{3} v'(t) = 0,$$

where $\beta_1 < \beta_2 < \beta_3$. It follows that

$$v(t) = \beta_2 + (\beta_3 - \beta_2)\text{cn}^2 \left( \sqrt{\frac{\beta_3 - \beta_1}{12}} t, \sqrt{\frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}} \right)$$

is a solution where $\text{cn}(x, k)$ is the Jacobi cosine function and $k$ is the elliptic modulus. Some notations use $\text{cn}(x, m)$ where $m = k^2$. The corresponding initial conditions are

$$v(0) = \beta_3, v'(0) = 0, v''(0) = -\frac{(\beta_3 - \beta_1)(\beta_3 - \beta_2)}{6}.$$

Write the equation as a system and compute the Jacobian as a function of $t$. For $\beta_1 = -1, \beta_2 = 1, \beta_3 = 2$, based on an analysis of the Jacobian, suggest methods to solve the problem. Repeat for $\beta_1 = 0, \beta_2 = 1, \beta_3 = 10$.

Note: It will help to plot each eigenvalue of the Jacobian as a function of $t$. If you sort the eigenvalues by their imaginary parts, things are bit clearer.

**Solution.**

We write the equation as a system

$$\begin{aligned} u_1'(t) &= v'(t) = u_2(t) \\ u_2'(t) &= v''(t) = u_3(t) \\ u_3'(t) &= \tfrac{\beta_1+\beta_2+\beta_3}{3} u_2(t) - u_2(t)u_1(t) \end{aligned}$$

Let $c = \frac{\beta_1+\beta_2+\beta_3}{3}$, then

$$f(u) = \begin{bmatrix} u_2 \\ u_3 \\ u_2 (c - u_1) \end{bmatrix}$$

The Jacobian is

$$D_u f(u) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -u_2 & c - u_1 & 0 \end{bmatrix}$$

We plot the real parts and imaginary parts of the eigenvalues in Figure 1. Ideally, we want to choose a method such that the eigenvalues are always in the stability region.

For $\beta_1 = -1, \beta_2 = 1, \beta_3 = 2$, Re$\lambda \in (-0.5, 0.5)$ and Im$\lambda \in (-1.5, 1.5)$. I would suggest backward Euler since its stability region covers most of the points except for Re $> 0$. For $\beta_1 = 0, \beta_2 = 1, \beta_3 = 10$, Re$\lambda \in (-2, 2)$ and Im$\lambda \in (-3, 3)$. I would still suggest backward Euler since only a small number of points does not lie in the stability region. If we want higher accuracy, then trapezoidal or Adams-Moulton 2-step method would be good choices, but their stability regions only cover half of the points thus are less stable.

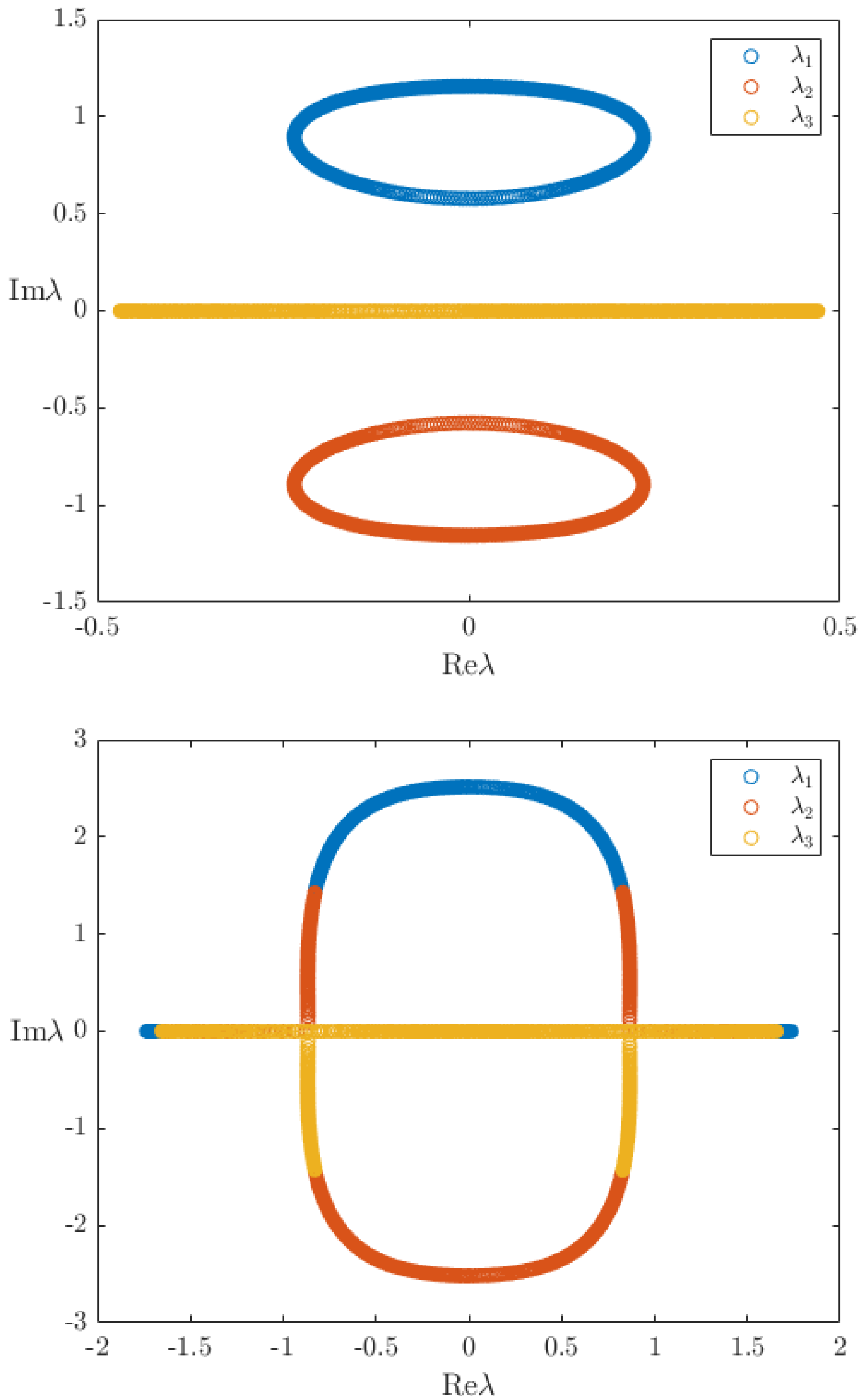


Figure 1: Eigenvalues of the Jacobian.

3. - Plot the absolute stability region for the TR-BDF2 method (8.6).
   - By analyzing $R(z)$, show that the method is both A-stable and L-stable. Hint: To show A-stability, show that $|R(z)| \leq 1$ on the imaginary axis and explain why this is enough.

**Solution.**

Recall (7.19) that TR-BDF2 has

$$R(z) = \frac{1 + \frac{5}{12}z}{1 - \frac{7}{12}z + \frac{1}{12}z^2}$$

On the imaginary axis we have

$$\begin{aligned}
|R(z)|^2 &= \frac{5yi + 12}{(yi-3)(yi-4)} \frac{-5yi + 12}{(-yi-3)(-yi-4)} \\
&= \frac{144 + 25y^2}{(9+y^2)(16+y^2)} \\
&= \frac{144 + 25y^2}{144 + 25y^2 + y^4}
\end{aligned}$$

The denominator is greater or equal to the numerator for any $y$, then clearly $|R(z)| \leq 1$. Since $R(z)$ has no poles in the left-half plane, which is open and connected, by the maximum modulus principle $|R(z)|$ is maximized on the boundary of the left-half plane, i.e. the imaginary axis. Then $R(z)$ is A-stable.
To show L-stability, take $z \to \infty$, then by L'Hopital's rule

$$\lim_{z\to\infty} |R(z)| = \lim_{z\to\infty} \left| \frac{1 + \frac{5}{12}z}{1 - \frac{7}{12}z + \frac{1}{12}z^2} \right| = \lim_{z\to\infty} \left| \frac{\frac{5}{12}}{\frac{1}{6}z - \frac{7}{12}} \right| = 0$$

To plot the stability region we sample $|R(z)|$ on a fine grid and plot the contour equal to 1. This can be done easily using the `contourf` command in MATLAB. We implemented this as `plot_stability.m`.

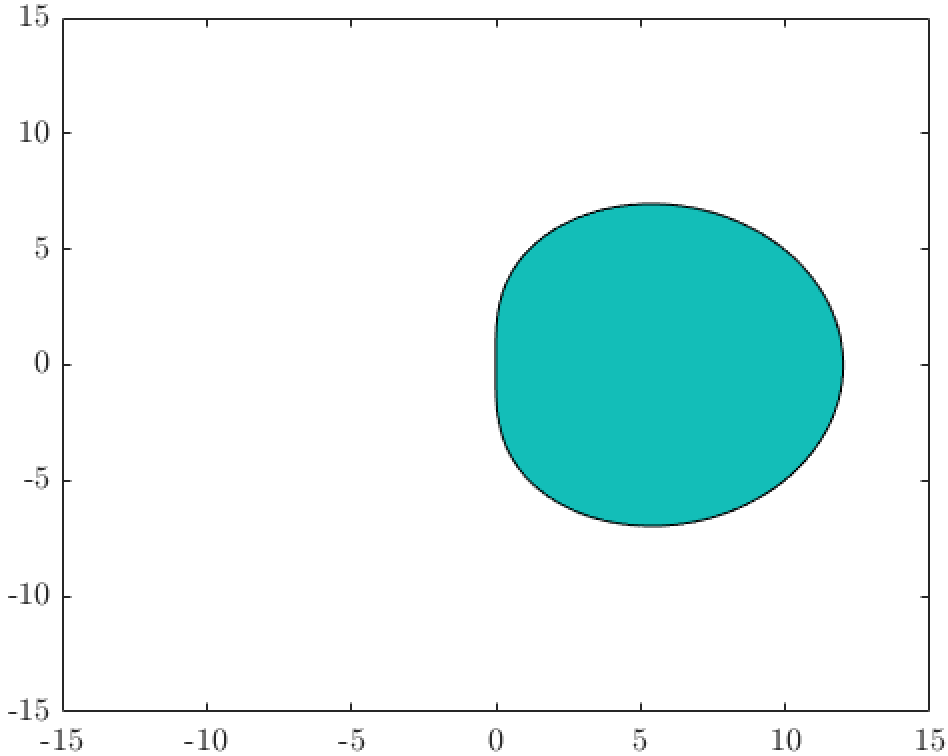


Figure 2: Stability region (white) of the TR-BDF2 method.

4. Let $g(x) = 0$ represent a system of $s$ nonlinear equations in $s$ unknowns, so $x \in \mathbb{R}^s$ and $g : \mathbb{R}^s \to \mathbb{R}^s$. A vector $\bar{x} \in \mathbb{R}^s$ is a *fixed point* of $g(x)$ if

$$\bar{x} = g(\bar{x}). \tag{1}$$

One way to attempt to compute $\bar{x}$ is with *fixed point iteration*: from some starting guess $x^0$, compute

$$x^{j+1} = g(x^j) \tag{2}$$

for $j = 0,\ 1,\ \ldots$.

(a) Show that if there exists a norm $\|\cdot\|$ such that $g(x)$ is Lipschitz continuous with constant $L < 1$ in a neighborhood of $\bar{x}$, then fixed point iteration converges from any starting value in this neighborhood. **Hint:** Subtract equation (1) from (2).

(b) Suppose $g(x)$ is differentiable and let $D_x g(x)$ be the $s \times s$ Jacobian matrix. Show that if the condition of part (a) holds then $\rho(D_x g(\bar{x})) < 1$, where $\rho(A)$ denotes the spectral radius of a matrix.

(c) Consider a predictor-corrector method (see Section 5.9.4) consisting of forward Euler as the predictor and backward Euler as the corrector, and suppose we make $N$ correction iterations, i.e., we set

$$\begin{aligned}
&\hat{U}^0 = U^n + kf(U^n) \\
&\text{for } j = 0,\ 1,\ \ldots,\ N-1 \\
&\qquad \hat{U}^{j+1} = U^n + kf(\hat{U}^j) \\
&\qquad \text{end} \\
&U^{n+1} = \hat{U}^N.
\end{aligned}$$

Note that this can be interpreted as a fixed point iteration for solving the nonlinear equation

$$U^{n+1} = U^n + kf(U^{n+1})$$

of the backward Euler method. Since the backward Euler method is implicit and has a stability region that includes the entire left half plane, as shown in Figure 7.1(b), one might hope that this predictor-corrector method also has a large stability region.

Plot the stability region $S_N$ of this method for $N = 2,\ 5,\ 10,\ 20,\ 50$ and observe that in fact the stability region does not grow much in size.

(d) Using the result of part (b), show that the fixed point iteration being used in the predictor-corrector method of part (c) can only be expected to converge if $|k\lambda| < 1$ for all eigenvalues $\lambda$ of the Jacobian matrix $f'(u)$.

(e) Based on the result of part (d) and the shape of the stability region of Backward Euler, what do you expect the stability region $S_N$ of part (c) to converge to as $N \to \infty$?

**Solution.**

(a) Suppose there exists an $L < 1$ such that

$$\|g(x^j) - g(\bar{x}))\| \leq L\|x^j - \bar{x}\| \tag{3}$$

Subtracting equation (1) from (2)

$$g(x^j) - g(\bar{x}) = x^{j+1} - \bar{x}$$

Plugging into (3)

$$\begin{aligned} \|x^{j+1} - \bar{x}\| &\leq L\|x^j - \bar{x}\| \\ \|x^j - \bar{x}\| &\leq L^j\|x^0 - \bar{x}\| \end{aligned}$$

Thus $x^0$ converges to $\bar{x}$ as $j \to \infty$.

(b) We can rewrite (3) as

$$\frac{\|g(x^j) - g(\bar{x}))\|}{\|x^j - \bar{x}\|} < 1$$

Taking $x^j \to \bar{x}$ and realizing that $\rho(A) \leq \|A\|$

$$\|D_x g(\bar{x})\| \geq \rho(D_x g(\bar{x})) < 1$$

(c) Let us derive $R(z)$ for this method

$$\begin{aligned} \hat{U}^0 &= U^n + kf(U^n) = (1+z)U^n \\ \hat{U}^1 &= U^n + kf(\hat{U}^0) = (1+z+z^2)U^n \\ &\vdots \\ \hat{U}^N &= U^n + kf(\hat{U}^{N-1}) = (1+z+z^2+\cdots+z^{N+1})U^n \\ U^{n+1} = \hat{U}^N &= \frac{1-z^{N+1}}{1-z}U^n \end{aligned}$$

See Figure 3.

(d) By part (b), convergence implies that

$$\begin{aligned} \rho(D_u kf(u)) &< 1 \\ k\rho(f'(u)) &< 1 \end{aligned}$$

Since $\rho(f'(u)) = \max_i \lambda_i(f'(u))$, then $|k\lambda| < 1$.

(e) See Figure 3.

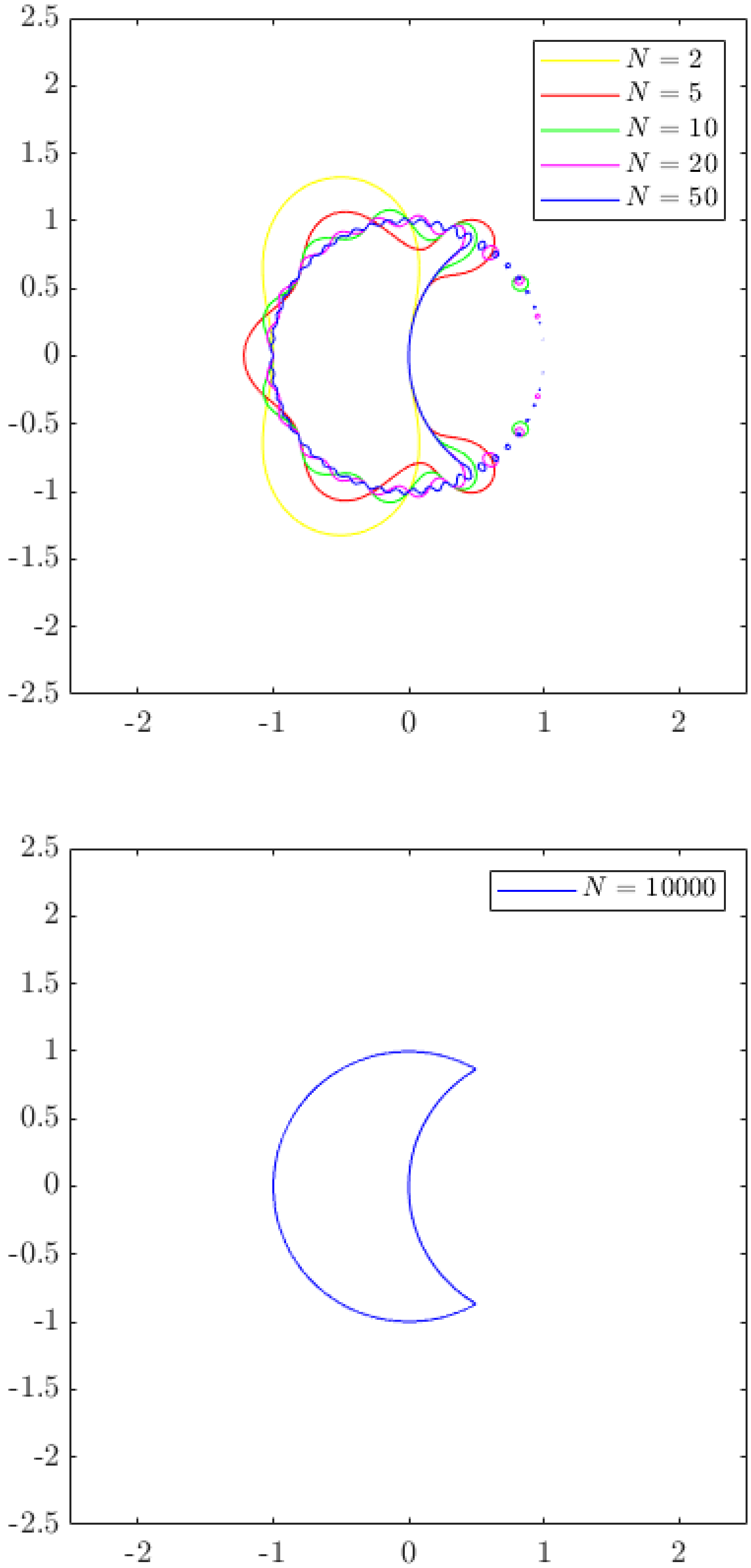


Figure 3: Stability region of the predictor-corrector method.

5. Consider the matrix $M_r = I - rT$ where $T$ is the $m \times m$ matrix.

$$T = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}$$

and $r \geq 0$. Find the largest value of $c$ such that $M_r$ is invertible for all $r \in [0, c)$.

**Solution.**

We derived in the lecture that $T$ has eigenvalues

$$\lambda_j = -2 + 2\cos\left(\frac{j\pi}{m+1}\right)$$

Then the eigenvalues of $M_r$ are

$$\mu_j = 1 + 2r - 2r\cos\left(\frac{j\pi}{m+1}\right), \quad j = 1, \ldots, m$$

Observing that $\mu_j \in [1, 1+4r]$, then 0 will never be an eigenvalue of $M_r$. Thus the largest $c$ is $\infty$.