Homework 6

Due: Wednesday, November 18, 2020

Question 1. (AF 4.1.1) . Evaluate the integrals

$$\frac{1}{2\pi i}\oint_C f(z)dz$$

where C is the unit circle centered at the origin with $f(z)$ given below. (a) $\frac{z+1}{2z^3-3z^2-2z}$ (b) $\frac{\cosh(1/z)}{z}$ (c) $\frac{e^{-\cosh z}}{4z^2+\pi^2}$ (d) $\frac{\ln(z+2)}{2z+1}$, $-\pi < \arg(z+2) \le \pi$ (e) $\frac{(z+1/z)}{z(2z-1/2z)}$

(a) $\frac{z+1}{2z^3-3z^2-2z} = \frac{z+1}{z(2z+1)(z-2)}$, simple poles at 0 and $-\frac{1}{2}$ in C.

$$\begin{aligned}\frac{1}{2\pi i}\oint_C f(z)dz &= \sum_j \operatorname{Res}\{f(z);z_j\} \\ &= \left(\frac{z+1}{6z^2-6z-2}\right)_0 + \left(\frac{z+1}{6z^2-6z-2}\right)_{-\frac{1}{2}} \\ &= -\frac{1}{2}+\frac{1}{5} \\ &= -\frac{3}{10}\end{aligned}$$

(b) Consider the pole at infinity

$$\begin{aligned}\frac{1}{2\pi i}\oint_C f(z)dz &= \operatorname{Res}\{f(z);z=\infty\} \\ &= \operatorname{Res}\left\{\frac{\cosh(t)}{t};t=0\right\} \\ &= (\cosh(t))_0 \\ &= 1\end{aligned}$$

(c) $\frac{e^{-\cosh z}}{4z^2+\pi^2} = \frac{e^{-\cosh z}}{(2z+i\pi)(2z-i\pi)}$, simple poles at $\pm\frac{i\pi}{2}$, which are not in C. Thus the integral is 0 by Cauchy's theorem.

(d)

$$\begin{aligned}\frac{1}{2\pi i}\oint_C f(z)dz &= \operatorname{Res}\left\{f(z);z=-\frac{1}{2}\right\} \\ &= \left(\frac{\ln(z+2)}{2}\right)_{-\frac{1}{2}} \\ &= \frac{1}{2}\ln\frac{3}{2}\end{aligned}$$

(e) Consider the pole at infinity

$$\begin{aligned}
\frac{1}{2\pi i}\oint_C f(z)dz &= \operatorname{Res}\{f(z); z=\infty\} \\
&= \operatorname{Res}\left\{\frac{1}{t^2}\frac{\frac{1}{t}+t}{\frac{1}{t}(\frac{2}{t}-\frac{t}{2})}; t=0\right\} \\
&= \operatorname{Res}\left\{\frac{2(t+t^3)}{t^2(4-t^2)}; t=0\right\} \\
&= \lim_{t\to 0}\frac{d}{dz}\left(\frac{2(t+t^3)}{4-t^2}\right) \\
&= \lim_{t\to 0}\frac{(4-t^2)(2+6t^2)+2t(2t+2t^3)}{(4-t^2)^2} \\
&= \frac{1}{2}
\end{aligned}$$

Question 2. Show that $\int_0^\infty \frac{\sin x}{x(x^2+1)}dx = \frac{\pi}{2}\left(1-\frac{1}{e}\right)$

Since the integrand is even, we have

$$\begin{aligned} I = \int_0^\infty \frac{\sin x}{x\left(x^2+1\right)}dx &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x\left(x^2+1\right)}dx \\ &= \frac{1}{2}\mathrm{Im}\int_{-\infty}^{\infty} \frac{e^{ix}}{x\left(x^2+1\right)}dx \end{aligned}$$

Let $f(z) = \frac{1}{z(z^2+1)}$. Since $|f(z)| \to 0$ as $|z| \to \infty$, by Jordan's lemma we can complete the contour in UHP

$$\begin{aligned} I &= \frac{1}{2}\mathrm{Im}\oint \frac{e^{iz}}{z\left(z^2+1\right)}dz \\ &= \frac{1}{2}\mathrm{Im}2\pi i\mathrm{Res}\left\{\frac{e^{iz}}{z\left(z^2+1\right)}; z=i\right\} + \pi i\mathrm{Res}\left\{\frac{e^{iz}}{z\left(z^2+1\right)}; z=0\right\} \\ &= \frac{1}{2}\mathrm{Im}2\pi i\left(\frac{e^{iz}}{3z^2+1}\right)_i + \pi i\left(\frac{e^{iz}}{3z^2+1}\right)_0 \\ &= \frac{1}{2}\mathrm{Im}2\pi i(-\frac{1}{2e}) + \pi i \\ &= \frac{\pi}{2}\left(1-\frac{1}{e}\right) \end{aligned}$$

Question 3. Consider the function

$$f(z) = \ln\left(z^2 - 1\right)$$

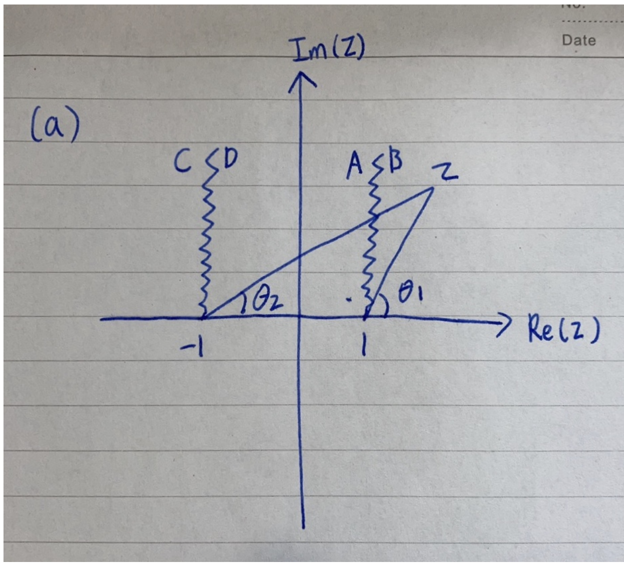
made single-valued by restricting the angles in the following ways, with $z_1 \equiv z - 1 = r_1 e^{i\theta_1}, z_2 \equiv z + 1 = r_2 e^{i\theta_2}$ (a) $-\frac{3\pi}{2} < \theta_1 \le \frac{\pi}{2}, -\frac{3\pi}{2} < \theta_2 \le \frac{\pi}{2}$ (b) $0 < \theta_1 \le 2\pi, 0 < \theta_2 \le 2\pi$ (c) $-\pi < \theta_1 \le \pi, 0 < \theta_2 \le 2\pi$ Find where the branch cuts are for each case by locating where the function is discontinuous. Use the AB tests and show your results.

We have $f(z) = \ln(z^2 - 1) = \ln r_1 r_2 e^{i(\theta_1+\theta_2)} = \ln r_1 r_2 + i(\theta_1 + \theta_2)$. We draw our AB points in Figure 1.

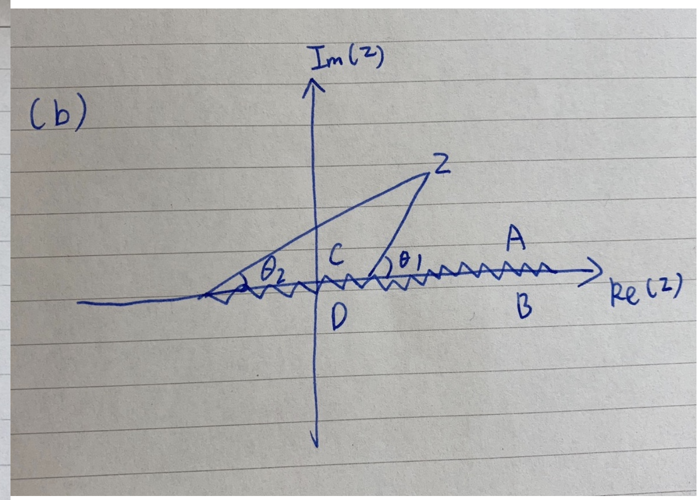
(a) At A, $\theta_1 = -\frac{3\pi}{2}$. At B, $\theta_1 = \frac{\pi}{2}$. θ_2 is the same at A and B. $f(A) = \ln r_1 r_2 + i(\theta_2 - \frac{3\pi}{2})$, $f(B) = \ln r_1 r_2 + i(\theta_2 + \frac{\pi}{2})$, then $f(z)$ is discontinuous across $z = 1$. At C, $\theta_2 = -\frac{3\pi}{2}$. At D, $\theta_2 = \frac{\pi}{2}$. θ_1 is the same at C and D. $f(C) = \ln r_1 r_2 + i(\theta_1 - \frac{3\pi}{2})$, $f(D) = \ln r_1 r_2 + i(\theta_2 + \frac{\pi}{2})$, then $f(z)$ is discontinuous across $z = -1$.

(b) At A, $\theta_1 = \theta_2 = 0$. At B, $\theta_1 = \theta_2 = 2\pi$. $f(A) = \ln r_1 r_2$, $f(B) = \ln r_1 r_2 + 4\pi i$, then $f(z)$ is discontinuous across $z > 1$. At C, $\theta_1 = \pi$, $\theta_2 = 0$. At D, $\theta_1 = \pi$, $\theta_2 = 2\pi$. $f(C) = \ln r_1 r_2 + \pi i$, $f(D) = \ln r_1 r_2 + 3\pi i$, then $f(z)$ is discontinuous across $-1 < z < 1$.

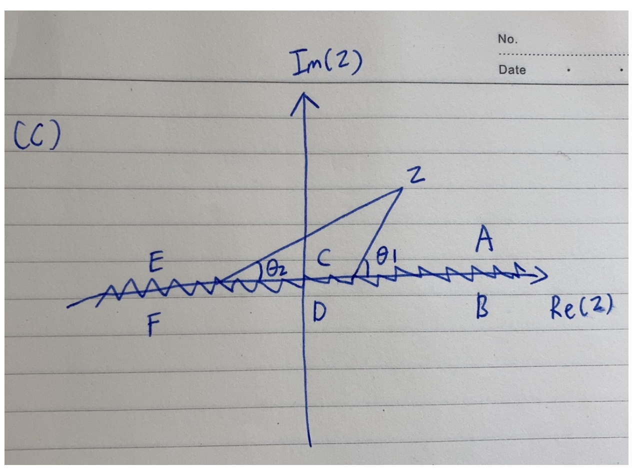
(c) At A, $\theta_1 = \theta_2 = 0$. At B, $\theta_1 = 0$, $\theta_2 = 2\pi$. $f(A) = \ln r_1 r_2$, $f(B) = \ln r_1 r_2 + 2\pi i$, then $f(z)$ is discontinuous across $z > 1$. At C, $\theta_1 = \pi$, $\theta_2 = 0$. At D, $\theta_1 = -\pi$, $\theta_2 = 2\pi$. $f(C) = \ln r_1 r_2 + \pi i = f(D)$, then $f(z)$ is continuous across $-1 < z < 1$. At E, $\theta_1 = \pi$, $\theta_2 = \pi$. At F, $\theta_1 = -\pi$, $\theta_2 = \pi$. $f(E) = \ln r_1 r_2 + 2\pi i$, $f(F) = \ln r_1 r_2$, then $f(z)$ is discontinuous across $z < -1$.



(a)



(b)



(c)

Figure 1: Branch cuts of $f(z)$ in Question 3