Homework 1
Due: January 15, 2021

1. Use MATLAB to evaluate the second order accurate approximation

$$u''(x) \approx \frac{u(x+h) + u(x-h) - 2u(x)}{h^2}$$

for $u(x) = \sin x$ and $x = \pi/6$. Try $h = 10^{-1}, 10^{-2}, \ldots, 10^{-16}$, and make a table of values of h, the computed finite difference quotient, and the error. Explain your results.

Solution.

```
uxx=(sin(x+h)+sin(x-h)-2*sin(x))/h^2;
```

h	FDQ	Error
1e-01	-0.4996	4.1653e-04
1e-02	-0.5000	4.1667e-06
1e-03	-0.5000	4.1674e-08
1e-04	-0.5000	3.0387e-09
1e-05	-0.5000	-1.1516e-06
1e-06	-0.4999	6.6572e-05
1e-07	-0.4996	3.9964e-04
1e-08	-1.1102	-0.6102
1e-09	1.1102e+02	1.1152e+02
1e-10	0	0.5000
1e-11	0	0.5000
1e-12	0	0.5000
1e-13	1.1102e+10	1.1102e+10
1e-14	-1.1102e+12	-1.1102e+12
1e-15	0	0.5000
1e-16	-1.1102e+16	-1.1102e+16

When h is relatively large, the error decreases by 10^{-2} as h decreases by 10^{-1}, which is expected for a second order accurate approximation. As h gets smaller, rounding error becomes more dominant and is proportional to $10^{-16}/h^2$. This explains why the error reduces for the first few h and becomes large again. Notice that for some particular values of h, the computed FDQ is exactly 0.

2. Use the formula in the previous exercise with $h = 0.2$, $h = 0.1$, and $h = 0.05$ to approximate $u''(x)$, where $u(x) = \sin x$ and $x = \pi/6$. Use one step of Richardson extrapolation, combining the results from $h = 0.2$ and $h = 0.1$, to obtain a higher order accurate approximation. Do the same with the results from $h = 0.1$ and $h = 0.05$. Finally do a second step of Richardson extrapolation, combining the two previously extrapolated values, to obtain a still higher order accurate approximation. Make a table of the computed results and their errors. What do you think is the order of accuracy after one step of Richardson extrapolation? How about after two?

Solution. Let $\phi_0(h) = \frac{1}{h^2}[u(x+h) + u(x-h) - 2u(x)]$, then

$$u''(x) = \phi_0(h) + Ch^2 + O(h^4)$$

where the odd powers of h cancel out in the Taylor series. Replacing h with $\frac{h}{2}$

$$u''(x) = \phi_0(\frac{h}{2}) + C(\frac{h}{2})^2 + O(h^4)$$

Rearranging to get one step of Richard extrapolation

$$u''(x) = \frac{4\phi_0(\frac{h}{2}) - \phi_0(h)}{3} + O(h^4)$$

Let $\phi_1(h) = \frac{1}{3}(4\phi_0(\frac{h}{2}) - \phi_0(h))$, then

$$u''(x) = \phi_1(h) + Dh^4 + O(h^6)$$

Similarly, the second step of Richard extrapolation is

$$u''(x) = \frac{16\phi_1(\frac{h}{2}) - \phi_1(h)}{15} + O(h^6)$$

h	RE	Error
0.2, 0.1	-0.5000	5.5506e-07
0.1, 0.05	-0.5000	3.4714e-08
2nd step	-0.5000	2.4805e-11

The order of accuracy is 4 after one step and 6 after two.

3. Using Taylor series, derive the error term for the approximation

$$u'(x) \approx \frac{1}{2h}[-3u(x) + 4u(x+h) - u(x+2h)].$$

Solution.

$$\begin{aligned}
\text{RHS} &= \frac{1}{2h}[-3u(x) + 4(u(x) + hu'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x) + O(h^4)) \\
&- (u(x) + 2hu'(x) + \frac{(2h)^2}{2!}u''(x) + \frac{(2h)^3}{3!}u'''(x) + O(h^4))] \\
&= \frac{1}{2h}[2hu'(x) - \frac{2h^3}{3}u'''(x) + O(h^4)] \\
&= u'(x) - \frac{h^2}{3}u'''(x) + O(h^3)
\end{aligned}$$

4. Consider a forward difference approximation for the second derivative of the form

$$u''(x) \approx Au(x) + Bu(x+h) + Cu(x+2h).$$

Use Taylor's theorem to determine the coefficients A, B, and C that give the maximal order of accuracy and determine what this order is.

Solution.

$$\begin{aligned}
\text{RHS} &= Au(x) + B[u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x)] \\
&+ C[u(x) + 2hu'(x) + \frac{(2h)^2}{2}u''(x) + \frac{(2h)^3}{6}u'''(x)] + O(h^4) \\
&= (A+B+C)u(x) + h(B+2C)u'(x) + \frac{h^2}{2}(B+4C)u''(x) + \frac{h^3}{6}(B+8C)u'''(x) + O(h^4)
\end{aligned}$$

Comparing coefficients

$$\begin{aligned}
A + B + C &= 0 \\
B + 4C &= \frac{2}{h^2} \\
B + 2C &= 0
\end{aligned}$$

Then $A = C = \frac{1}{h^2}$ and $B = -\frac{2}{h^2}$. Substituting back to RHS

$$\text{RHS} = u''(x) + hu'''(x) + O(h^4)$$

Thus the maximal order of accuracy is 1.

5. Consider the two-point boundary value problem

$$u'' + 2xu' - x^2u = x^2, \quad u(0) = 1, \quad u(1) = 0.$$

Let $h = 1/4$ and explicitly write out the difference equations, using centered differences for all derivatives.

Solution. By centered difference

$$\frac{u_{j+1} + u_{j-1} - 2u_j}{h^2} + 2x_j\frac{u_{j+1} - u_{j-1}}{2h} - x_j^2u_j = x_j^2$$

Using $h = \frac{1}{m+1} = \frac{1}{4}$, we have $m = 3$, i.e. three grid points.

$$16(u_{j+1} + u_{j-1} - 2u_j) + 4x_j(u_{j+1} - u_{j-1}) - x_j^2u_j = x_j^2 \text{ for } j = 1, 2, 3$$

Writing out explicitly

$$\begin{aligned} 16(u_2 + 1 - 2u_1) + 4x_1(u_2 - 1) - x_1^2u_1 &= x_1^2 \\ 16(u_3 + u_1 - 2u_2) + 4x_2(u_3 - u_1) - x_2^2u_2 &= x_2^2 \\ 16(u_2 - 2u_3) - 4x_3u_2 - x_3^2u_3 &= x_3^2 \end{aligned}$$

6. A rod of length 1 meter has a heat source applied to it and it eventually reaches a steady-state where the temperature is not changing. The conductivity of the rod is a function of position x and is given by $c(x) = 1 + x^2$. The left end of the rod is held at a constant temperature of 1 degree. The right end of the rod is insulated so that no heat flows in or out from that end of the rod. This problem is described by the boundary value problem:

$$\frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right) = f(x), \quad 0 \le x \le 1,$$
$$u(0) = 1, \quad u'(1) = 0.$$

(a) Write down a set of difference equations for this problem. Be sure to show how you do the differencing at the endpoints. [Note: It is better **not** to rewrite $\frac{d}{dx}((1+x^2)\frac{du}{dx})$ as $(1+x^2)u''(x) + 2xu'(x)$; leave the equation in the form above.]

(b) Write a MATLAB code to solve the difference equations. You can test your code on a problem where you know the solution by choosing a function $u(x)$ that satisfies the boundary conditions and determining what $f(x)$ must be in order for $u(x)$ to solve the problem. Try $u(x) = (1-x)^2$. Then $f(x) = 2(3x^2 - 2x + 1)$.

(c) Try several different values for the mesh size h. Based on your results, what would you say is the order of accuracy of your method?

Solution (a). We can use a centered difference approximation

$$\frac{1}{h}[c(x_{j+\frac{1}{2}})u'_{j+\frac{1}{2}} - c(x_{j-\frac{1}{2}})u'_{j-\frac{1}{2}}] = f(x_j) \quad \text{for } j = 1 \ldots m$$

Using centered difference again for the derivative

$$\frac{1}{h^2}[c(x_{j+\frac{1}{2}})(u_{j+1} - u_j) - c(x_{j-\frac{1}{2}})(u_j - u_{j-1})] = f(x_j)$$

For the Neumann boundary condition at $x = 1$, we can use a one-sided approximation

$$\frac{u_{m+1} - u_m}{h} = 0$$

Thus the system of equations that we will solve is

$$\frac{1}{h^2}\begin{bmatrix} h^2 & & & & & & \\ b_0 & a_1 & b_1 & & & & \\ & b_1 & a_2 & b_2 & & & \\ & & b_2 & a_3 & b_3 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & b_{m-2} & a_{m-1} & b_{m-1} \\ & & & & & b_{m-1} & a_m & b_m \\ & & & & & & -h & h \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{m-1} \\ u_m \\ u_{m+1} \end{bmatrix} = \begin{bmatrix} 1 \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) \\ 0 \end{bmatrix}$$

where $b_j = c(x_{j+\frac{1}{2}})$, $a_j = -c(x_{j+\frac{1}{2}}) - c(x_{j-\frac{1}{2}})$.

Solution (b).

```
h=0.1;
m=1/h-1;
x=h:h:1-h;
f=zeros(m+2,1);
u=zeros(m+2,1);
f(1)=1;
f(2:m+1)=2*(3*x.^2-2*x+1);
A=zeros(m+2,m+2);
A(1,1)=h^2;
A(m+2,m+2)=h;
A(m+2,m+1)=-h;
for j=2:m+1
    A(j,j)=-(1+(x(j-1)+h/2)^2)-(1+(x(j-1)-h/2)^2);
    A(j,j-1)=1+(x(j-1)-h/2)^2;
    A(j,j+1)=1+(x(j-1)+h/2)^2;
end
A=A/h^2;
u=A\f;
err=u(m+2);
```

Solution (c). We compute the error of $u(1)$. Since $u(x) = (1-x)^2$, $u(1) = 0$ analytically.

h	Error
1e-01	1.5146e-01
1e-02	1.5651e-02
1e-03	1.5702e-03
1e-04	1.5707e-04

It is evident that our method is first order accurate. This makes sense because the one-sided approximation that we used for the Neumann boundary condition is first order accurate.