Homework 1
Due: Friday, October 9, 2020

Question 1. Show that if matrix $\mathbf{A}$ is triangular and unitary, then it is diagonal

Let $\mathbf{A}$ be an unitary and lower triangular matrix with size $n \times n$. The (i, j) element of $\mathbf{A}^*$ takes the form

$$(\mathbf{A}^*)_{ij} = \overline{a}_{ji} \tag{1}$$

and the (i, j) element of the product $\mathbf{AA}^*$ takes the form

$$(\mathbf{AA}^*)_{ij} = \sum_{k=1}^{n} a_{ik}\overline{a}_{kj} \tag{2}$$

Consider a special case where $i = 1$ and $j = 2$, the above reduces to

$$(\mathbf{AA}^*)_{12} = \sum_{k=1}^{n} a_{1k}\overline{a}_{k2} \tag{3}$$

When $k > 1$, $a_{1k} = 0$ since $\mathbf{A}$ is lower triangular, so the expression can be further reduced to

$$(\mathbf{AA}^*)_{12} = a_{11}\overline{a}_{12} \tag{4}$$

Since $\mathbf{A}$ is unitary, we know that $(\mathbf{AA}^*)_{ij} = 0$ if $i \neq j$ and $(\mathbf{AA}^*)_{ij} = 1$ if $i = j$. Since $|a_{11}|^2 = 1$, it must be true that $\overline{a}_{12} = 0 = a_{21}$. Repeat this process for the case of $i = 1$ and $j = m$ ($m \neq 1$), we can show that $a_{m1} = 0$. Repeat again for the case of $i = n$ and $j = m$ ($m \neq n$), we can show that $a_{mn} = 0$, which is indeed diagonal.
If $\mathbf{A}$ is upper triangular, the proof will be almost identical, but this time we would show that $a_{1m} = 0$ where $m \neq 1$ and repeat the same process as before.

Question 2. Consider that the matrices $\mathbf{A} \in \mathbb{C}^{n\times m}$ and $\mathbf{B} \in \mathbb{C}^{n\times m}$ are Hermitian (self-adjoint)

- Prove that all eigenvalues λ_k of $\mathbf{A}$ are real

  If $\mathbf{A}$ is Hermitian, it is also square, meaning that $m = n$. Consider an eigenvector $\mathbf{x} \in \mathbb{C}^{m\times 1}$, we have

  $$\mathbf{Ax} = \lambda\mathbf{x} \tag{5}$$

  where λ is an eigenvalue. Multiplying by $\mathbf{x}^*$

  $$\mathbf{x}^*\mathbf{Ax} = \lambda\mathbf{x}^*\mathbf{x} \tag{6}$$

  Now taking the conjugate transpose of equation 5, we get

  $$\mathbf{x}^*\mathbf{A}^* = \lambda^*\mathbf{x}^* \tag{7}$$

Multiplying by $\mathbf{x}$

$$\mathbf{x}^*\mathbf{A}^*\mathbf{x} = \lambda^*\mathbf{x}^*\mathbf{x} \tag{8}$$

Since $\mathbf{A}$ is Hermitian, $\mathbf{A} = \mathbf{A}^*$. The above becomes

$$\mathbf{x}^*\mathbf{A}\mathbf{x} = \lambda^*\mathbf{x}^*\mathbf{x} \tag{9}$$

Comparing equation 6 and 9, the left hand sides are identical, thus we have

$$\lambda^*\mathbf{x}^*\mathbf{x} = \lambda\mathbf{x}^*\mathbf{x} \tag{10}$$

$$\lambda^* = \lambda \tag{11}$$

Therefore all eigenvalues must be real.

- Prove that if x_k is the k th eigenvector, then eigenvectors with distinct eigenvalues are orthogonal

  Consider two distinct eigenvalues λ_1 and λ_2, with corresponding eigenvectors $\mathbf{x_1}$ and $\mathbf{x_2}$. For λ_1

$$\mathbf{A}\mathbf{x_1} = \boldsymbol{\lambda_1}\mathbf{x_1} \tag{12}$$

  Taking the conjugate transpose

$$\mathbf{x_1^*}\mathbf{A}^* = \boldsymbol{\lambda_1}\mathbf{x_1^*} \tag{13}$$

  Multiplying by $\mathbf{x_2}$ and using the fact that $\mathbf{A}$ is Hermitian

$$\mathbf{x_1^*}\mathbf{A}\mathbf{x_2} = \boldsymbol{\lambda_1}\mathbf{x_1^*}\mathbf{x_2} \tag{14}$$

  For λ_2, we can find a similar expression

$$\mathbf{x_1^*}\mathbf{A}\mathbf{x_2} = \boldsymbol{\lambda_2}\mathbf{x_1^*}\mathbf{x_2} \tag{15}$$

  Comparing equation 14 and 15, we get

$$\boldsymbol{\lambda_1}\mathbf{x_1^*}\mathbf{x_2} = \boldsymbol{\lambda_2}\mathbf{x_1^*}\mathbf{x_2} \tag{16}$$

$$\mathbf{x_1^*}\mathbf{x_2}(\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}) = 0 \tag{17}$$

  Since λ_1 and λ_2 are distinct, $\mathbf{x_1^*}\mathbf{x_2} = 0$ and $\mathbf{x_1}$ and $\mathbf{x_2}$ are orthogonal.

- Prove the sum of two Hermitian matrices is Hermitian

  For the sum to be Hermitian, we want show that $(\mathbf{A}+\mathbf{B})^* = \mathbf{A}+\mathbf{B}$. Notice that

$$(\mathbf{A}+\mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^* \tag{18}$$

  Since $\mathbf{A}$ and $\mathbf{B}$ are Hermitian, we complete the proof.

- Prove the inverse of an invertible Hermitian matrix is Hermitian as well

  Since $\mathbf{A}$ is Hermitian and invertible, we have

  $$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^*\mathbf{A}^{-1} \tag{19}$$

  Multiply by $(\mathbf{A}^{-1})^*$

  $$(\mathbf{A}^{-1}))^*)\mathbf{A}\mathbf{A}^{-1} = (\mathbf{A}^{-1})^*\mathbf{A}^*\mathbf{A}^{-1} \tag{20}$$
  $$(\mathbf{A}^{-1})^* = (\mathbf{A}^*)^{-1}\mathbf{A}^*\mathbf{A}^{-1} \tag{21}$$
  $$(\mathbf{A}^{-1})^* = \mathbf{A}^{-1} \tag{22}$$

- Prove the product of two Hermitian matrices is Hermitian if and only if $\mathbf{AB} = \mathbf{BA}$.

  ($\rightarrow$) Given $\mathbf{AB} = (\mathbf{AB})^*$, we want to show that $\mathbf{AB} = \mathbf{BA}$

  $$\mathbf{AB} = (\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^* = \mathbf{BA} \tag{23}$$

  ($\leftarrow$) Given $\mathbf{AB} = \mathbf{BA}$, we want to show $\mathbf{AB} = (\mathbf{AB})^*$

  $$(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^* = \mathbf{BA} = \mathbf{AB} \tag{24}$$

Question 3. Consider the matrix. $\mathbf{U} \in \mathrm{C}^{n\times m}$ which is unitary

- Prove that the matrix is diagonalizable

  By Schur decomposition, we can write $\mathbf{U}$ as

  $$\mathbf{U} = \mathbf{QAQ}^* \tag{25}$$

  where $\mathbf{Q}$ is unitary and $\mathbf{A}$ is upper triangular. For $\mathbf{U}$ to be diagonalizable, we want $\mathbf{A}$ to be diagonal. From **Question 1**, we know that triangular unitary matrices are diagonal, thus we only need to show that $\mathbf{A}$ is unitary. Taking the conjugate transpose

  $$\mathbf{U}^* = \mathbf{QA}^*\mathbf{Q}^* \tag{26}$$

  Multiplying $\mathbf{U}$ and $\mathbf{U}^*$

  $$\mathbf{UU}^* = \mathbf{QA}^*\mathbf{Q}^*\mathbf{QAQ}^* \tag{27}$$
  $$\mathbf{I} = \mathbf{QA}^*\mathbf{AQ}^* \tag{28}$$
  $$\mathbf{Q}^*\mathbf{IQ} = \mathbf{I} = \mathbf{A}^*\mathbf{A} \tag{29}$$

  This proves $\mathbf{A}$ is unitary.

- Prove that the inverse is $\mathbf{U}^{-1} = \mathbf{U}^*$

  Since $\mathbf{U}$ is unitary, it follows that $\mathbf{U}\mathbf{U}^* = \mathbf{I}$. By definition of the inverse, $\mathbf{U}^{\mathbf{-1}} = \mathbf{U}^*$.

- Prove it is isometric with respect to the ℓ_2 norm, i.e. $\|\mathbf{U}\mathbf{x}\| = \|\mathbf{x}\|$.

$$\begin{aligned} \|\mathbf{U}\mathbf{x}\| &= \sqrt{(\mathbf{U}\mathbf{x})^*\mathbf{U}\mathbf{x}} & (30) \\ &= \sqrt{\mathbf{x}^*\mathbf{U}^*\mathbf{U}\mathbf{x}} & (31) \\ &= \sqrt{\mathbf{x}^*\mathbf{x}} = \|\mathbf{x}\| & (32) \end{aligned}$$

- Prove that all eigenvalues have modulus unity

  Suppose λ is an eigenvalue, we have

$$\mathbf{U}\mathbf{x} = \lambda\mathbf{x} \tag{33}$$

  Multiplying by $\mathbf{U}^*$

$$\begin{aligned} \mathbf{U}^*\mathbf{U}\mathbf{x} &= \lambda\mathbf{U}^*\mathbf{x} & (34) \\ \mathbf{x} &= \lambda\mathbf{U}\mathbf{x} & (35) \end{aligned}$$

  Substitute equation 33 into the above

$$\begin{aligned} \mathbf{x} &= \lambda \cdot \lambda\mathbf{x} & (36) \\ \lambda^2 &= 1 & (37) \\ |\lambda| &= 1 & (38) \end{aligned}$$