Homework 4

Due: Wednesday 4, November, 2020

Question 1. Let $\Omega = \{a, b, c, d\}$ and let $\mathcal{F} = 2^{\Omega}$. We define a probability measure P as follows:

$$P(a) = 1/6, \ \ , P(b) = 1/3, \ \ P(c) = 1/4, \ \ P(d) = 1/4.$$

Next, define three random variables:

$$X(a) = 1, \ \ X(b) = 1, \ \ X(c) = -1, \ \ X(d) = -1,$$
$$Y(a) = 1, \ \ Y(b) = -1, \ \ Y(c) = 1, \ \ X(d) = -1,$$

and $Z = X + Y$. (a) List the sets in $\sigma(X)$. (b) Calculate $E(Y|X)$. (c) Calculate $E(Z|X)$.

(a) $\sigma(X) = \{X \in B | B \in \mathcal{B}\} = \sigma\{\{a, b\}, \{c, d\}\} = \{\{a, b\}, \{c, d\}, \Omega, \emptyset\}$

(b) $E(Y|X) = \frac{E(Y|\Omega_i)}{P(\Omega_i)}$ where $\Omega_1 = \{a, b\}$ and $\Omega_2 = \{c, d\}$
On Ω_1, $E(Y|X) = \frac{1\cdot\frac{1}{6} - 1\cdot\frac{1}{3}}{\frac{1}{6}+\frac{1}{3}} = -\frac{1}{3}$. On Ω_2, $E(Y|X) = \frac{1\cdot\frac{1}{4} - 1\cdot\frac{1}{4}}{\frac{1}{4}\cdot 2} = 0$.

$$\begin{pmatrix} \omega \\ E(Y|X) \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ \frac{-1}{3} & \frac{-1}{3} & 0 & 0 \end{pmatrix}$$

(c) Since $Z = X + Y$, we have $Z(a) = 2$, $Z(b) = Z(c) = 0$, $Z(d) = -2$. Then on Ω_1, $E(Z|X) = \frac{2\cdot\frac{1}{6} - 0\cdot\frac{1}{3}}{\frac{1}{6}+\frac{1}{3}} = -\frac{2}{3}$. On Ω_2, $E(Y|X) = \frac{0\cdot\frac{1}{4} - 2\cdot\frac{1}{4}}{\frac{1}{4}\cdot 2} = -1$.

$$\begin{pmatrix} \omega \\ E(Z|X) \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ \frac{2}{3} & \frac{2}{3} & -1 & -1 \end{pmatrix}$$

Question 2. (a) Prove that $E(E(X|\mathcal{F})) = EX$. (b) Show that if $\mathcal{G} \subset \mathcal{F}$ and $EX^2 < \infty$ then

$$E(\{X - E(X|\mathcal{F})\}^2) + E(\{E(X|\mathcal{F}) - E(X|\mathcal{G})\}^2) = E(\{X - E(X|\mathcal{G})\}^2)$$

(a) Using property (ii) of conditional expectation, we have $\int_A X dP = \int_A E(X|\mathcal{F}) dP$ for all $A \in \mathcal{F}$. Let $A = \Omega$, we have $E(E(X|\mathcal{F})) = EX$.

(b) Let $Z \in L^2(\mathcal{F})$, then by **Theorem 4.1.4** in Durrett,

$$ZE(X|\mathcal{F}) = E(ZX|\mathcal{F})$$

Taking expectations and using result in (a)

$$E[ZE(X|\mathcal{F})] = E[E(ZX|\mathcal{F})] = E(ZX)$$

then $E[Z(X - E(X|\mathcal{F}))] = 0$. Let $Y = E(X|\mathcal{G})$, since $\mathcal{G} \subset \mathcal{F}$, $Y = E(X|\mathcal{G}) \in \mathcal{G} \subset \mathcal{F}$. Since $EX^2 < \infty$, then $E(X|\mathcal{F}), Y \in L^2(\mathcal{F})$, then $E(X|\mathcal{F}) - Y \in L^2(\mathcal{F})$. Let $Z = E(X|\mathcal{F}) - Y$, then

$$\begin{aligned} E(X - Y)^2 &= E[X - E(X|\mathcal{F}) + Z]^2 \\ &= E[X - E(X|\mathcal{F})]^2 + EZ^2 + 2E[Z(X - E(X|\mathcal{F}))] \\ &= E[X - E(X|\mathcal{F})]^2 + EZ^2 \end{aligned}$$

We complete the proof by replacing Y and Z.

Question 3. An important special case of the previous result (2b) occurs when $\mathcal{G} = \{\emptyset, \Omega\}$. Let $\text{var}(X|\mathcal{F}) = E(X^2|\mathcal{F}) - E(X|\mathcal{F})^2$. Show that

$$\text{var}(X) = E(\text{var}(X|\mathcal{F})) + \text{var}(E(X|\mathcal{F})).$$

Taking expectation on $\text{var}(X|\mathcal{F})$

$$\begin{aligned} E(\text{var}(X|\mathcal{F})) &= E[E(X^2|\mathcal{F})] - E[E(X|\mathcal{F})]^2 \\ &= E(X^2) - E[E(X|\mathcal{F})^2] \end{aligned}$$

by the result in 2(a). Since $\text{var}(Y) = E(Y^2) - (EY)^2$, we have

$$\begin{aligned} \text{var}(E(X|\mathcal{F})) &= E[E(X|\mathcal{F})]^2 - [E[E(X|\mathcal{F}))]^2 \\ &= E[E(X|\mathcal{F})]^2 - (EX)^2 \end{aligned}$$

again 2(a). Then

$$\begin{aligned} E(\text{var}(X|\mathcal{F})) + \text{var}(E(X|\mathcal{F})) &= E(X^2) - E[E(X|\mathcal{F})]^2 + E[E(X|\mathcal{F})]^2 - (EX)^2 \\ &= E(X^2) - (EX)^2 \\ &= \text{var}(X) \end{aligned}$$

Question 4. Let $Y_1, Y_2, ...$ be i.i.d. (independent and identically distributed) random variables with mean μ and variance σ^2, N an independent positive integer valued random variable with $EN^2 < \infty$ and $X = Y_1 + ... + Y_N$. Show that $\text{var}(X) = \sigma^2 EN + \mu^2 \text{var}(N)$. (To understand and help remember the formula, think about the two special cases in which N or Y is constant.)

By definition $\text{var}(X) = E(X^2) - (EX)^2$, let us consider $E(X^2)$ and $(EX)^2$ respectively.

$$
\begin{aligned}
E(X^2) = E\left[\left(\sum_{i=1}^{N} Y_i\right)^2\right] &= \sum_{n=1}^{\infty} E\left[\left(\sum_{i=1}^{N} Y_i\right)^2 |N = n\right] P(N = n) \\
&= \sum_{n=1}^{\infty} E\left[\left(\sum_{i=1}^{n} Y_i\right)^2\right] P(N = n) \\
&= \sum_{n=1}^{\infty} E\left[\sum_{i=1}^{n} Y_i^2 + 2 \sum_{1 \le i < j \le n} Y_i Y_j\right] P(N = n) \\
&= \sum_{n=1}^{\infty} [n(\mu^2 + \sigma^2) + n(n-1)\mu^2] P(N = n) \\
&= \sum_{n=1}^{\infty} n\sigma^2 P(N = n) + \sum_{n=1}^{\infty} n^2 \mu^2 P(N = n) \\
&= \sigma^2 EN + \mu^2 E(N^2)
\end{aligned}
$$

$$
\begin{aligned}
EX &= \sum_{n=1}^{\infty} E\left[\sum_{i=1}^{n} Y_i\right] P(N = n) \\
&= \sum_{n=1}^{\infty} n\mu P(N = n) \\
&= \mu EN
\end{aligned}
$$

$$
\begin{aligned}
\text{var}(X) &= \sigma^2 EN + \mu^2 E(N^2) - \mu^2 (EN)^2 \\
&= \sigma^2 EN + \mu^2 \text{var}(N)
\end{aligned}
$$