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AMATH 515 Homework Set 0

Due: Wednesday, January 13th, by midnight.

The goal of this homework is to make sure you are comfortable with all prerequisites for this class, to set-up Python and Jupyter Notebook, and to try submitting your work to Gradescope Autograder. The theoretical portion of the homework will be graded based on completeness, and is intended as a primer on calculus and linear algebra.

1. Theory

(1) Submit your write-up to Gradescope. Look for the assignment "Homework 0 – theory".

(2) **Calculus primer**. For a function $f : \mathbb{R}^n \to \mathbb{R}$, we define the *gradient* to be the vector of partial derivatives:

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

and the *Hessian* to be the matrix of second partial derivatives:

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

Compute the gradients and hessians of the following functions, with $x \in \mathbb{R}^4$ in all three examples.

(a) $f(x) = \sin(x_1 + x_2 + x_3 + x_4)$

(b) $f(x) = \|x\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$

(c) $f(x) = \ln(x_1 x_2 x_3 x_4)$.

Solution a.

$$\nabla f(x) = \cos(x_1 + x_2 + x_3 + x_4) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\nabla^2 f(x) = -\sin(x_1 + x_2 + x_3 + x_4)\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Solution b.

$$\nabla f(x) = 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \nabla^2 f(x) = 2\mathbf{I}$$

Solution c.

$$\nabla f(x) = \begin{bmatrix} \frac{1}{x_1} \\ \frac{1}{x_2} \\ \frac{1}{x_3} \\ \frac{1}{x_4} \end{bmatrix} \qquad \nabla^2 f(x) = \begin{bmatrix} -\frac{1}{x_1^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{x_2^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{x_3^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{x_4^2} \end{bmatrix}$$

(3) **Linear algebra primer.**

(a) What are the eigenvalues of the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \pi & 2 & 0 & 0 \\ 64 & -15 & 3 & 0 \\ 321 & 0 & 0 & 5 \end{bmatrix}$$

Solution. We wish to solve $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. Writing out explicitly

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ \pi & 2-\lambda & 0 & 0 \\ 64 & -15 & 3-\lambda & 0 \\ 321 & 0 & 0 & 5-\lambda \end{vmatrix} &= (1-\lambda)\begin{vmatrix} 2-\lambda & 0 & 0 \\ -15 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} \\ &= (1-\lambda)(2-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ 0 & 5-\lambda \end{vmatrix} \\ &= (1-\lambda)(2-\lambda)(3-\lambda)(5-\lambda) = 0 \end{aligned}$$

Thus the eigenvalues are 1, 2, 3, 5.

(b) Write down bases for the range and nullspace of the following matrix, written as the outer product of two vectors:

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Solution. Since the column vectors of A are all $[1\ 0\ 1]^T$, the basis for range(A) is $[1\ 0\ 1]^T$. To find null(A), we reduce A to row echelon form

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For $Ax = 0$, we have $x_1 + x_2 + x_3 = 0$, where x_2 and x_3 are free variables. Thus we can write x as

$$x = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus the nullspace is span($[-1\ 1\ 0]^T, [-1\ 0\ 1]^T$).

(c) Let A be a 10×5 matrix, and b a vector in $\mathbb{R}^{10}$. The notation A^T denotes the *transpose* of A, where the columns of A are rows of A^T. Thus we can write
   - What is the size of A^TA? What is the size of A^Tb?
     5×5, 5×1
   - How many solutions might there be to the system $Ax = b$?
     There might be no solution, a unique solution or infinitely many solutions.
   - How many solutions might there be to the system $A^TAx = A^Tb$?
     There might be no solution, a unique solution or infinitely many solutions.
   - Suppose the columns of A are linearly independent. How many solutions might there be to the system $Ax = b$? To the system $A^TAx = A^Tb$?
     $Ax = b$ has no solution, $A^TAx = A^Tb$ has a unique solution.

2. Practice

(1) Install Anaconda3 distribution. Instruction:
https://www.anaconda.com/products/individual
   - If you've never used Python before – here is an excellent Python introduction: https://www.learnpython.org
   - If you have experience with scientific computing in MATLAB, but you've never tried Python, here is a useful migration guide:
     https://www.enthought.com/white-paper-matlab-to-python-a-migration-guide/

(2) Download "Homework0.ipynb" from Canvas, open it as a Jupyter Notebook, and complete all the tasks there.
   - If you've never used Jupyter Notebooks then take a look at this tutorial: https://www.dataquest.io/blog/jupyter-notebook-tutorial/

(3) Submit your Jupyter Notebook to Gradescope. Look for the assignment ”Homework 0 – coding”. There is no limit for the number of attempts for the coding part this time.