Homework 3 Extra Problem
Due: February 3, 2021

1. (Finite elements) Use the Galerkin finite element method with continuous piecewise linear basis functions to solve the problem

$$-\frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right) = f(x), \quad 0 \le x \le 1,$$

$$u(0) = 0, \quad u(1) = 0.$$

   (a) Derive the matrix equation that you will need to solve for this problem.

   (b) Write a code to solve this set of equations. You can test your code on a problem where you know the solution by choosing a function $u(x)$ that satisfies the boundary conditions and determining what $f(x)$ must be in order for $u(x)$ to satisfy the differential equation. Try $u(x) = x(1-x)$. Then $f(x) = 2(3x^2 - x + 1)$.

   (c) Try several different values for the mesh size h. Based on your results, what would you say is the order of accuracy of the Galerkin method with continuous piecewise linear basis functions?

   (d) Now try a nonuniform mesh spacing, say, $x_i = (i/(m+1))^2$, $i = 0, 1, \ldots, m+1$. Do you see the same order of accuracy, if h is defined as the maximum mesh spacing, $\max_i(x_{i+1} - x_i)$?

   (e) Suppose the boundary conditions were $u(0) = a$, $u(1) = b$. Show how you would represent the approximate solution $\hat{u}(x)$ as a linear combination of hat functions and how the matrix equation in part (a) would change.

Solution.

(a) Taking an inner product with $\varphi(x) \in S$ on both sides

$$-\int_0^1 \frac{d}{dx}\left((1+x^2)\frac{du}{dx}\right)\varphi(x)dx = \int_0^1 f(x)\varphi(x)dx$$

Integrating by parts on LHS

$$-(1+x^2)u'(x)\varphi(x)\Big|_0^1 + \int_0^1 (1+x^2)u'(x)\varphi'(x)dx = \int_0^1 f(x)\varphi(x)dx$$

The first term is 0 since $\varphi(1) = \varphi(0) = 0$. Let us now consider an approximate solution $\hat{u}(x) = \sum_{j=1}^m c_j\varphi_j(x)$. We can also write $\varphi(x) = \sum_{i=1}^m d_i\varphi_i(x)$. It is possible to choose $c_1 \ldots c_m$ such that

$$\int_0^1 (1+x^2)\sum_{j=1}^m c_j\varphi_j'(x)\varphi_i'(x)dx = \int_0^1 f(x)\varphi_i(x)dx \quad i = 1, \ldots, m$$

Taking out the constant

$$\sum_{j=1}^{m} c_j \int_0^1 (1+x^2)\varphi_j'(x)\varphi_i'(x)dx = \int_0^1 f(x)\varphi_i(x)dx$$

Writing this as a matrix equation, we have $A\mathbf{c} = \mathbf{f}$ where

$$a_{ij} = \int_0^1 (1+x^2)\varphi_j'(x)\varphi_i'(x)dx$$

(b)

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f=inline('2*(3*x.^2-x+1)');
P=inline('x+x^3/3');
n=10;
m=n-1;
x=linspace(0,1,m+2);
x=x';
d=x(2:m+2)-x(1:m+1);
d2=d.^2;
h=max(d);
xmid=0.5*(x(1:m+1)+x(2:m+2));
dmid=xmid-x(1:m+1);
a=zeros(m+2,m+2);
for i=2:m+1
    a(i,i)=(P(x(i))-P(x(i-1)))/d2(i-1)+(P(x(i+1))-P(x(i)))/d2(i);
    a(i,i+1)=-(P(x(i+1))-P(x(i)))/d2(i);
    a(i+1,i)=a(i,i+1);
end
A=a(2:m+1,2:m+1);
fmid = f(xmid);
b=zeros(m,1);
for i=1:m
    b(i)=fmid(i)*dmid(i)+fmid(i+1)*dmid(i+1);
end
u_approx=A\b;
u_true=x.*(1-x);
err=max(abs(u_true(2:m+1) - u_approx)) % infinite norm
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(c)

h	Error
1e-01	7.8120e-04
1e-02	7.8686e-06
1e-03	7.8686e-08
1e-04	7.8206e-10

Table 1: The errors for different mesh size h. It is evident that the order of accuracy is 2.

(d)

h	Error
0.19	3.2693e-03
0.0199	3.3146e-05
0.01999	3.3149e-07
0.001999	3.3148e-09

Table 2: The errors for non-uniform mesh size. It is evident that the order of accuracy is 2.

(e) Since $u(0)$ and $u(1)$ are no longer equal to zero, we need to define two hat functions

$$\varphi_0 = \begin{cases} \frac{x_1-x}{x_1} & x \in [0, x_1] \\ 0 & \text{otherwise} \end{cases} \quad \varphi_{m+1} = \begin{cases} \frac{x-x_m}{1-x_m} & x \in [x_m, 1] \\ 0 & \text{otherwise} \end{cases}$$

Then the first and m-th equations become

$$a\int_0^1 (1+x^2)\varphi_0'(x)\varphi_1'(x)dx + \sum_{j=1}^m c_j \int_0^1 (1+x^2)\varphi_j'(x)\varphi_1'(x)dx = \int_0^1 f(x)\varphi_1(x)dx$$
$$b\int_0^1 (1+x^2)\varphi_{m+1}'(x)\varphi_m'(x)dx + \sum_{j=1}^m c_j \int_0^1 (1+x^2)\varphi_j'(x)\varphi_m'(x)dx = \int_0^1 f(x)\varphi_m(x)dx$$

Thus we only need to change RHS of the matrix equation by subtracting the a and b terms in the first and m-th equations.