Homework 1.3
Due: April 12, 2021

The Gompertz (1825) equation
Consider a population that grows with an intrinsic rate of growth that decays exponentially:

$$\frac{dN}{dt} = r_o e^{-\alpha t} N$$
$$N(0) = N_0$$

(a) Solve this *nonautonomous* ordinary differential equation.
(b) Sketch typical solution curves.
(c) What is the "carrying capacity" for this model? In particular, compute

$$K = \lim_{t\to\infty} N(t)$$

How does this carrying capacity differ from that of the logistic equation? (Look for something peculiar.)
(d) Where (for what N) is the inflection point for your solution? How does this differ from the logistic equation?
(e) Show that the Gompertz equation can be rewritten as

$$\frac{dN}{dt} = \alpha N \ln\left(\frac{K}{N}\right)$$

where K is the carrying capacity.

Solution.

(a) We try a separation of variables

$$\int \frac{1}{N} dN = \int r_0 e^{-\alpha t} dt$$
$$\ln N = \frac{r_0}{\alpha} e^{-\alpha t} + c$$
$$N = e^c e^{-\frac{r_0}{\alpha}\alpha t}$$

Using $N(0) = N_0$

$$N_0 = e^c e^{-\frac{r_0}{\alpha}}$$
$$e^c = N_0 e^{\frac{r_0}{\alpha}}$$

Plugging in e^c

$$N = N_0 e^{\frac{r_0}{\alpha}(1-e^{-\alpha t})}$$

(b) For simplicity we assume $r_0 = \alpha = 1$.

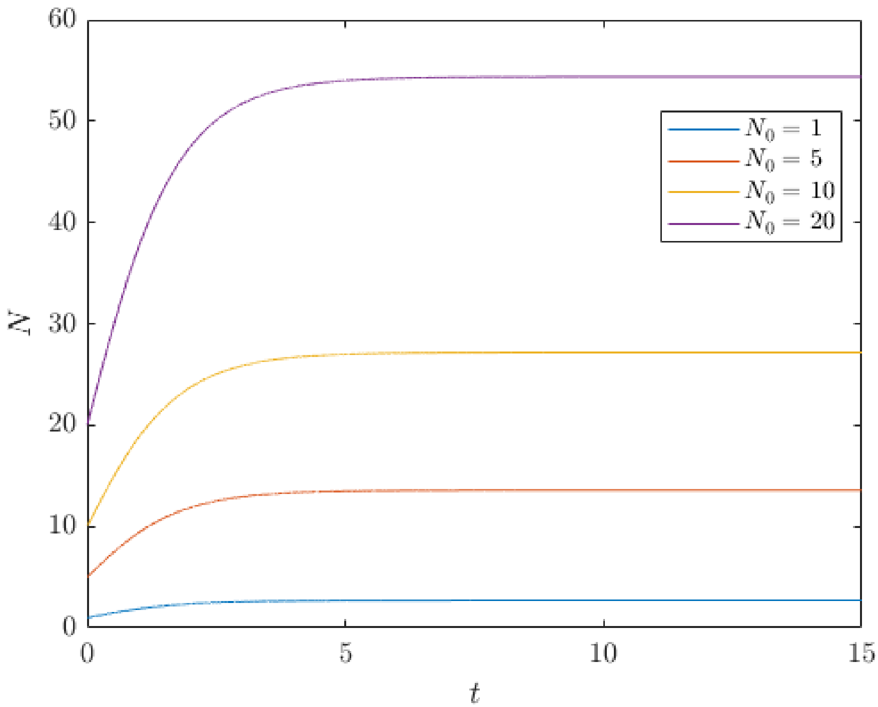

Figure 1: Solutions of the Gompertz equation

(c) Taking $t \to \infty$, then

$$N \to N_0 e^{\frac{r_0}{\alpha}} = K$$

We observe that K is proportional to N_0. This means that the final population will be $O(N_0)$. Whereas in the logistic equation, K does not have this dependence on initial population.

(d) Taking derivative on both sides of the equation and setting to zero

$$\begin{aligned} \frac{d^2N}{dt^2} = -\alpha r_0 e^{-\alpha t} N + r_0 e^{-\alpha t} \frac{dN}{dt} &= 0 \\ \alpha N &= \frac{dN}{dt} \\ \alpha N &= r_0 e^{-\alpha t} N \\ \alpha &= r_0 e^{-\alpha t} \end{aligned}$$

Plugging into N

$$N = N_0 e^{\frac{r_0}{\alpha}(1-\frac{\alpha}{r_0})} = \frac{K}{e}$$

whereas the logistic equation has an inflection point at $N = \frac{K}{2}$.

(e) We take out a factor of α in the equation

$$\begin{aligned}
\frac{dN}{dt} &= \alpha N \frac{r_0}{\alpha} e^{-\alpha t} \\
&= \alpha N \ln(e^{\frac{r_0}{\alpha}} e^{-\alpha t}) \\
&= \alpha N \ln\left(\frac{1}{e^{-\frac{r_0}{\alpha}} e^{-\alpha t}}\right) \\
&= \alpha N \ln\left(\frac{N_0 e^{\frac{r_0}{\alpha}}}{N_0 e^{\frac{r_0}{\alpha}(1-e^{-\alpha t})}}\right) \\
&= \alpha N \ln\left(\frac{K}{N}\right)
\end{aligned}$$