**Homework 1**

Due: Monday, October 12, 2020

**Question 1.** Describe the probability space for the following experiments: a) a biased coin is tossed three times; b) two balls are drawn without replacement from an urn which originally contained two blue and two red balls.

a) $\Omega =$ {HHH,HHT,HTH,HTT,THH,THT,TTH,TTT}
$\mathcal{F} =$ the power set of $\Omega$
Let the probability of a head be $p$ and tail be $(1-p)$, let $\omega_1 =$ HHH, $\omega_2 =$ HHT, $\cdots$
$P(\omega_1) = p^3$, $P(\omega_2) = P(\omega_3) = P(\omega_5) = p^2(1-p)$, $P(\omega_4) = P(\omega_6) = P(\omega_7) = p(1-p)^2$, $P(\omega_8) = (1-p)^3$, $P(A) = \sum_{\omega \in A} P(\omega)$

b) $\Omega =$ {RR,RB,BR,BB}
$\mathcal{F} =$ the power set of $\Omega$
Let $\omega_1 =$ RR, $\omega_2 =$ RB, $\cdots$
$P(\omega_1) = P(\omega_4) = \frac{1}{6}$, $P(\omega_2) = P(\omega_3) = \frac{1}{3}$, $P(A) = \sum_{\omega \in A} P(\omega)$

**Question 2.** (No translation-invariant random integer). Show that there is no probability measure $P$ on the integers $\mathbb{Z}$ with the discrete $\sigma$-algebra $2^{\mathbb{Z}}$ with the translation-invariance property $P(E+n) = P(E)$ for every event $E \in 2^{\mathbb{Z}}$ and every integer $n$. $E+n$ is obtained by adding $n$ to every element of $E$.

Suppose $P$ exists. Consider picking the number 0 from $\mathbb{Z}$. The property $P(E+n) = P(E)$ implies that $P(0) = P(1) = \cdots = P(Z)$, for all $Z \in \mathbb{Z}$. Since the sequence $-Z, \cdots, -1, 0, 1, \cdots, Z$ (let us call this sequence $Z_i$) is countable with every set disjoint, we have $P(\Omega) = P(\cup_i Z_i) = \sum_i P(Z_i)$ with $P(Z_i) \geq 0$. If $P(Z_i) > 0$, the sum will go to infinity, thus $P(Z_i) = 0$. However, this implies the sum is 0, contradicting the fact that $P(\Omega) = 1$. Hence no such $P$ exists.

**Question 3.** (No translation-invariant random real). Show that there is no probability measure $P$ on the reals $\mathbb{R}$ with the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ with the translation-invariance property $P(E+x) = P(E)$ for every event $E \in \mathcal{B}(\mathbb{R})$ and every real $x$. Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ is the $\sigma$-algebra generated by intervals $(a,b] \subset \mathbb{R}$.

Suppose $P$ exists. Consider picking an interval $E = (0,x]$ from $\mathbb{R}$. The property $P(E+x) = P(E)$ implies that $P((0,x]) = P((x,2x]) = \cdots = P((nx,(n+1)x])$, for $n \in \mathbb{Z}$. Let us call these intervals $E_i$ and we notice that the sequence of $E_i$ are countable with every set disjoint, we have $P(\Omega) = P(\cup_i E_i) = \sum_i P(E_i)$ with $P(E_i) \geq 0$. If $P(E_i) > 0$, the sum will go to infinity, thus $P(E_i) = 0$. However, this implies the sum is 0, contradicting the fact that $P(\Omega) = 1$. Hence no such $P$ exists.

**Question 4.** Let $\Omega = \mathbb{R}$, $\mathcal{F}$ = all subsets of $\mathbb{R}$ so that $A$ or $A^c$ is countable. $P(A) = 0$ in the first case and $P(A) = 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

We need to show that a) $\mathcal{F}$ is a $\sigma$-algebra and b) $P$ is a probability measure.
a) i) $\mathcal{F}$ is closed under complements
Let $A \in \mathcal{F}$. If $A$ is countable, then $(A^c)^c$ is also countable, thus $A^c \in \mathcal{F}$. If $A^c$ is countable, then $A^c \in \mathcal{F}$. This shows $A$ and $A^c$ are both in $\mathcal{F}$.
ii) $\mathcal{F}$ is closed under countable unions
Let $A_i$ be a countable sequence of sets in $\mathcal{F}$ and $A = \cup_i A_i$. If every $A_i$ is countable, then $\cup_i A_i$ is also countable and $A \in \mathcal{F}$. If there is some $A_{i_0}$ which is uncountable, then $A_{i_0}^c$ is countable. $A^c = (\cup_i A_i)^c = \cap_i A_i^c \subset A_{i_0}^c$. Thus $A^c$ is countable and $A \in \mathcal{F}$. This shows $\mathcal{F}$ is a $\sigma$-algebra.
b) i) $P(\emptyset) = 0$ and $P(\Omega) = 1$
Since $\emptyset$ is countable, $P(\emptyset) = 0$ and $P(\Omega) = P(\emptyset^c) = 1$.
ii) $P(\cup_i A_i) = \sum_i P(A_i)$
Let $A_i$ be a countable sequence of disjoint sets in $\mathcal{F}$ and $A = \cup_i A_i$. If every $A_i$ is countable, so is $A$ and $P(A) = \sum_n P(A_i) = 0$. If there is some $A_{i_0}$ which is uncountable, then $A_{i_0}^c$ is countable and $P(A_{i_0}) = 1$. Similarly to a), $A^c$ is countable so $P(A) = 1$. Since $A_i$ is disjoint, every $A_{i \neq i_0}$ is countable, so $P(A_{i \neq i_0}) = 0$. Thus $P(A) = \sum_n P(A_i) = P(A_{i_0}) + \sum_{i \neq i_0} P(A_i) = 1$.
Therefore $(\Omega, \mathcal{F}, P)$ is a probability space.