Midterm

Due: 10am Seattle time (PST), Saturday, November 7, 2020

Question 1. Prove that the LU decomposition of a matrix $\mathbf{A}$ is unique.

Assume $\mathbf{A}$ can be LU decomposed and invertible. Consider two LU decompositions

$$\begin{aligned}\mathbf{A} &= \mathbf{L_1U_1}\\ \mathbf{A} &= \mathbf{L_2U_2}\end{aligned}$$

Since $\mathbf{A}$ is invertible, $\det(\mathbf{A}) = \det(\mathbf{L_1})\det(\mathbf{U_1}) \neq 0$, $\mathbf{L_1}$, $\mathbf{U_1}$ are also invertible and so are $\mathbf{L_2}$, $\mathbf{U_2}$. Then we have

$$\begin{aligned}\mathbf{L_1U_1} &= \mathbf{L_2U_2}\\ \mathbf{L_1^{-1}L_1U_1U_2^{-1}} &= \mathbf{L_1^{-1}L_2U_2U_2^{-1}}\\ \mathbf{U_1U_2^{-1}} &= \mathbf{L_1^{-1}L_2}\end{aligned}$$

Since $\mathbf{U_1}$ and $\mathbf{U_2}$ are upper triangular, so is $\mathbf{U_1U_2^{-1}}$. Since $\mathbf{L_1}$ and $\mathbf{L_2}$ are unit lower triangular, so is $\mathbf{L_1^{-1}L_2}$. Then $\mathbf{L_1^{-1}L_2}$ must be the identity matrix. Thus $\mathbf{U_1} = \mathbf{U_2}$, then $\mathbf{L_1} = \mathbf{L_2}$ also holds. Then LU decomposition of $\mathbf{A}$ is unique under our assumptions.

Question 2. Show that the largest singular value of a matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$ is given by

$$\sigma_{\max}(\mathbf{A}) = \max_{\mathbf{x}\in\mathbb{R}^n, \mathbf{y}\in\mathbb{R}^m} \frac{\mathbf{y}^T\mathbf{Ax}}{\|\mathbf{x}\|_2\|\mathbf{y}\|_2}$$

Using **Theorem 1.** from **Homework 2**, which says the nonzero singular values of $\mathbf{A}$ are the square roots of nonzero eigenvalues of $\mathbf{A}^*\mathbf{A}$, we have

$$\begin{aligned}\sigma_{\max}(\mathbf{A}) &= \sqrt{\lambda_{\max}(\mathbf{A}^*\mathbf{A})}\\ &= \max_{\mathbf{x}\in\mathbb{R}^n} \sqrt{\frac{\mathbf{x}^T\mathbf{A}^*\mathbf{Ax}}{\mathbf{x}^T\mathbf{x}}}\\ &= \max_{\mathbf{x}\in\mathbb{R}^n} \frac{\|\mathbf{Ax}\|_\mathbf{2}}{\|\mathbf{x}\|_\mathbf{2}}\\ &= \max_{\mathbf{x}\in\mathbb{R}^n} \frac{\|\mathbf{Ax}\|_\mathbf{2}\|\mathbf{y}\|_\mathbf{2}}{\|\mathbf{x}\|_\mathbf{2}\|\mathbf{y}\|_\mathbf{2}}\end{aligned}$$

By Cauchy–Schwarz inequality, we have

$$\begin{aligned}\frac{\mathbf{y}^T\mathbf{Ax}}{\|\mathbf{x}\|_\mathbf{2}\|\mathbf{y}\|_\mathbf{2}} &\leq \frac{\|\mathbf{Ax}\|_\mathbf{2}\|\mathbf{y}\|_\mathbf{2}}{\|\mathbf{x}\|_\mathbf{2}\|\mathbf{y}\|_\mathbf{2}}\\ \max_{\mathbf{x}\in\mathbb{R}^n, \mathbf{y}\in\mathbb{R}^m} \frac{\mathbf{y}^T\mathbf{Ax}}{\|\mathbf{x}\|_2\|\mathbf{y}\|_2} &= \max_{\mathbf{x}\in\mathbb{R}^n} \frac{\|\mathbf{Ax}\|_\mathbf{2}}{\|\mathbf{x}\|_\mathbf{2}} = \sigma_{\max}(\mathbf{A})\end{aligned}$$

Question 3. What are the singular values of an orthogonal projection?

Let $\mathbf{P}$ be an orthogonal projection, $\{\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_m\}$ be an orthonormal basis for $\mathbb{C}^m$ and $\mathbf{Q}$ be an unitary matrix whose ith column is $\mathbf{q}_i$. Let $\mathbf{S}_1 = \{\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n\}$ and $\mathbf{S}_2 = \{\mathbf{q}_{n+1}, \mathbf{q}_{n+2}, \cdots, \mathbf{q}_m\}$, then $\mathbf{S}_1 \perp \mathbf{S}_2$. Consider the columns of $\mathbf{PQ}$, for $i \leq n$, the ith column is $\mathbf{q}_i$; for $i > n$, the ith column is 0. Then the matrix $\mathbf{Q}^*\mathbf{PQ}$ has 1 on the first n diagonal entries and 0 for all other entries. Let $\mathbf{Q}^*\mathbf{PQ} = \mathbf{\Sigma}$, then $\mathbf{P} = \mathbf{Q\Sigma Q}^*$, which is a singular value decomposition of $\mathbf{P}$. Thus the singular values of $\mathbf{P}$ are 1.

Question 4. Show that for a given norm $\kappa(\mathbf{AB}) \leq \kappa(\mathbf{A})\kappa(\mathbf{B})$ and that $\kappa(\alpha\mathbf{A}) = \kappa(\mathbf{A})$ for a given (nonzero) constant α.

Let us consider 2-norms, we have

$$\begin{aligned}
\kappa(\mathbf{AB}) &= \|\mathbf{AB}\|\|(\mathbf{AB})^{\mathbf{-1}}\| \\
&= \|\mathbf{AB}\|\|\mathbf{B}^{\mathbf{-1}}\mathbf{A}^{\mathbf{-1}}\| \\
&\leq \|\mathbf{A}\|\|\mathbf{B}\|\|\mathbf{B}^{\mathbf{-1}}\|\|\mathbf{A}^{\mathbf{-1}}\| \text{ (by 3.14 in Trefethen)} \\
&= \kappa(\mathbf{A})\kappa(\mathbf{B}) \\
\kappa(\alpha\mathbf{A}) &= \|\alpha\mathbf{A}\|\|(\alpha\mathbf{A})^{\mathbf{-1}}\| \\
&= \alpha\alpha^{-1}\|\mathbf{A}\|\|\mathbf{A}^{\mathbf{-1}}\| \\
&= \kappa(\mathbf{A})
\end{aligned}$$

Question 5. Write a python or matlab script that does an LU decomposition (including pivoting)

```
function [P,L,U] = lu1(A)
[m,¬] = size(A);
L=eye(m);
P=L;
for k=1:m-1
    [¬,i]=max(abs(A(k:m,k)));
    i=i+k-1;
    A([k i],k:m)=A([i k],k:m);
    L([k i],1:k-1)=L([i k],1:k-1);
    P([k i],:)=P([i k],:);
    for j=k+1:m
        L(j,k)=A(j,k)/A(k,k);
        A(j,k:m)=A(j,k:m)-L(j,k)*A(k,k:m);
    end
end
U=A;
```