Homework 8

Due: Friday, December 11, 2020

Question 1. Patients arrive at an emergency room as a Poisson process with intensity λ. The time to treat each patient is an independent exponential random variable with parameter μ. Let $\mathrm{X} = (\mathrm{X}_t)_{t\geq 0}$ be the number of patients in the system (either being treated or waiting). Write down the generator of X. Show that X has an invariant distribution π if and only if $\lambda < \mu$. Find π. What is the total expected time (waiting + treatment) a patient waits when the system is in its invariant distribution?

$$\mathbf{G} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \dots \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 & \dots \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 & \dots \\ 0 & 0 & \mu & -(\mu+\lambda) & \lambda & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

($\longrightarrow$) If X has an invariant distribution π, then $\pi\mathbf{G} = 0$. We have

$$\begin{aligned} -\lambda\pi(0) + \mu\pi(1) &= 0 \\ \lambda\pi(n-1) - (\mu+\lambda)\pi(n) + \mu\pi(n+1) &= 0 \end{aligned}$$

Then $\pi(n) = (\frac{\lambda}{\mu})^n\pi(0)$. We must have $\sum_{n=0}^{\infty}\pi(n) = 1 = \pi(0)\sum_{n=0}^{\infty}(\frac{\lambda}{\mu})^n$, then $\sum_{n=0}^{\infty}(\frac{\lambda}{\mu})^n < \infty \to \frac{\lambda}{\mu} < 1$. Thus $\lambda < \mu$.

($\longleftarrow$) If $\lambda < \mu$, then $\sum_{n=0}^{\infty}(\frac{\lambda}{\mu})^n < \infty$. We can define $\pi(0)$ such that $\pi(0)\sum_{n=0}^{\infty}(\frac{\lambda}{\mu})^n = 1$ and let $\pi(n) = (\frac{\lambda}{\mu})^n\pi(0)$. Then $\sum_{n=0}^{\infty}\pi(n) = 1$ and $\pi\mathbf{G} = 0$ are both satisfied, thus π must be a stationary distribution.

$\pi(0)\sum_{n=0}^{\infty}(\frac{\lambda}{\mu})^n = 1 \to \frac{\pi(0)}{1-\lambda/\mu} = 1 \to \pi(0) = 1 - \frac{\lambda}{\mu}$. Then $\pi(n) = (1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})^n$.

At steady state, $E\mathrm{X} = \sum_{n=0}^{\infty} n\pi(n) = (1-\frac{\lambda}{\mu})\sum_{n=0}^{\infty} n(\frac{\lambda}{\mu})^n = (1-\frac{\lambda}{\mu})\frac{\lambda/\mu}{(1-\lambda/\mu)^2} = \frac{\lambda}{\mu-\lambda}$. By Little's law

$$L = \lambda W$$

where L is expected number of patients at steady state, λ is the rate of arrival and W is the expected processing time. Then $W = \frac{1}{\mu-\lambda}$.

Question 2. Let $\mathrm{X} = (\mathrm{X}_t)_{t\geq 0}$ be a Markov chain with state space $\mathrm{S} = \{0, 1, 2, \dots\}$ and with a generator G whose ith row has entries

$$g_{i,i-1} = i\mu, \quad g_{i,i} = -i\mu - \lambda, \quad g_{i,i+1} = \lambda$$

with all other entries being zero (the zeroth row has only two entries: $g_{0,0}$ and $g_{0,1}$). Assume $\mathrm{X}_0 = j$. Find $\mathrm{G}_{\mathrm{X_t}}(s) := \mathrm{E}s^{\mathrm{X}_t}$. What is the distribution of X_t as $t \to \infty$?

By Kolmogorov forward equation

$$\begin{aligned}\frac{dp_t(i,0)}{dt} &= -\lambda p_t(i,0) + \mu p_t(i,1)\\ \frac{dp_t(i,j)}{dt} &= \lambda p_t(i,j-1) - (j\mu+\lambda)p_t(i,j) + (j+1)\mu p_t(i,j+1) \quad j \geq 1\end{aligned}$$

Multiplying the j-th equation by s^j and summing all equations

$$\begin{aligned}\frac{dG_{X_t}}{dt} &= \lambda s G_{X_t} - \mu s \frac{dG_{X_t}}{ds} - \lambda G_{X_t} + \mu \frac{dG_{X_t}}{ds}\\ &= \lambda(s-1)G_{X_t} - \mu(s-1)\frac{dG_{X_t}}{ds}\end{aligned}$$

where $G_{X_t} = \sum_{j=0}^{\infty} s^j p_t(i,j)$. Solving the above PDE with the initial condition $G_{X_0}(s) = s^j$ in Mathematica yields

$$G_{X_t}(s) = [(s-1)e^{-\mu t} + 1]^j e^{\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})}$$

As $t \to \infty$, $G_{X_t}(s) \to e^{\frac{\lambda}{\mu}(s-1)}$, then

$$P(X_t = k) = \frac{1}{k!}G^{(k)}(0) = \frac{\lambda^k}{k!\mu^k}e^{-\frac{\lambda}{\mu}}$$

Question 3. Let N be a time-inhomogeneous Poisson process with intensity function $\lambda(t)$. That is, the probability of a jumps of size one in the time interval $(t, t+\mathrm{d}t)$ is $\lambda(t)\mathrm{d}t$ and the probability of two jumps in that interval of time is $\mathcal{O}\left(\mathrm{d}t^2\right)$. Write down the Kolmogorov forward and backward equations of N and solve them. Let $\mathrm{N}_0 = 0$ and let τ_1 be the time of the first jump of N. If $\lambda(t) = c/(1+t)$ show that $\mathbb{E}\tau_1 < \infty$ if and only if $c > 1$.

The generator G of N has entries

$$g_{i,i} = -\lambda(t), \quad g_{i,i+1} = \lambda(t), \quad g_{i,j} = 0 \text{ otherwise}$$

Then the Kolmorogov equations can be written as:

$$\begin{aligned}\tfrac{d}{dt}\mathbf{P}_t &= \mathbf{P}_t\mathbf{G}\\ \tfrac{d}{dt}\mathbf{P}_t &= \mathbf{G}\mathbf{P}_t\end{aligned}$$

Since N is a Poisson process, $p_t(i,j) = 0$ for $i > j$. Consider $p_t(i,i)$ and $p_t(i,i+1)$

$$\begin{aligned}\frac{dp_t(i,i)}{dt} &= -\lambda(t)p_t(i,i)\\ \frac{dp_t(i,i+1)}{dt} &= -\lambda(t)p_t(i,i+1) + \lambda(t)p_t(i,i)\end{aligned}$$

With the initial condition $\mathbf{P_0} = \mathbf{I}$, the first equation has solution $p_t(i,i) = e^{-\int_0^t \lambda(\tau)d\tau}$. Substituting into the second equation and using integrating factor

$$\begin{aligned}
\frac{dp_t(i,i+1)}{dt} + \lambda(t)p_t(i,i+1) &= \lambda(t)e^{-\int_0^t \lambda(\tau)d\tau} \\
\frac{d}{dt}(p_t(i,i+1)e^{\int_0^t \lambda(\tau)d\tau}) &= \lambda(t) \\
p_t(i,i+1) &= \int_0^t \lambda(\tau)d\tau e^{-\int_0^t \lambda(\tau)d\tau}
\end{aligned}$$

Similarly we can solve for $p_t(i,i+2), p_t(i,i+3), \cdots$ iteratively, yielding

$$p_t(i,j) = \frac{(\int_0^t \lambda(\tau)d\tau)^{j-i}}{(j-i)!}e^{-\int_0^t \lambda(\tau)d\tau} \text{ for } i \leq j$$

Then $p_{\tau_1}(0,1)$ can be expressed as

$$\begin{aligned}
p_{\tau_1}(0,1) &= \int_0^{\tau_1} \lambda(t)dte^{-\int_0^{\tau_1} \lambda(t)dt} \\
&= c\ln(\tau_1+1)e^{\ln(\tau_1+1)^{-c}} \\
&= c\ln(\tau_1+1)(\tau_1+1)^{-c} \\
E\tau_1 &= c\int_0^{\infty} \ln(\tau_1+1)(\tau_1+1)^{-c}d\tau_1 \\
&= \left[\frac{\ln(\tau_1+1)(\tau_1+1)^{1-c}}{1-c}\right]_0^{\infty} - \int_0^{\infty} \frac{(\tau_1+1)^{1-c}}{(t+1)(1-c)}d\tau_1
\end{aligned}$$

($\longrightarrow$) If $E\tau_1 < \infty$, then $1-c<0$, thus $c>1$.
($\longleftarrow$) If $c>1$, then the integrand vanishes at infinity, thus $E\tau_1 < \infty$.

Question 4. Let N be a Poisson process with a random intensity Λ which is equal to λ_1 with probability p and λ_2 with probability $1-p$. Find $\mathrm{G}_{\mathrm{N}_t}(s) = \mathrm{E}s^{\mathrm{N}_t}$. What is the mean and variance of N_t?

Let $m = p\lambda_1 + (1-p)\lambda_2$, then the generator G of N has entries

$$g_{i,i} = -m, \quad g_{i,i+1} = m, \quad g_{i,j} = 0 \text{ otherwise}$$

By Kolmogorov forward equation

$$\begin{aligned}
\frac{dp_t(i,0)}{dt} &= -mp_t(i,0) \\
\frac{dp_t(i,j)}{dt} &= -mp_t(i,j) + mp_t(i,j-1) \quad j \geq 1
\end{aligned}$$

Multiplying the j-th equation by s^j and summing all equations

$$\frac{dG_{N_t}}{dt} = -mG_{N_t} + msG_{N_t}$$

where $G_{N_t} = \sum_{j=0}^{\infty} s^j p_t(i,j)$. With the initial condition $N_0 = 0 \to G_0 = 1$, this equation can be solved trivially with $G_{N_t} = e^{m(s-1)t}$. Then $EN_t = G'(1) = mt$ and $\mathrm{Var}N_t = G''(1) + G'(1) - (G'(1))^2 = mt$.