Homework 4.3
Due: May 3, 2021

Generalized Beverton-Holt curve
Consider a generalized Beverton-Holt curve of the form

$$N_{t+1} = \frac{R_0^\beta N_t}{\{1 + [(R_0 - 1)/K] N_t\}^\beta}$$

for $R_0 > 0$ and $\beta > 0$. Find the fixed points of this mapping. Determine their stability. Determine and sketch the portions of the (R_0, β) first quadrant that correspond to monotonic damping to the carrying capacity, oscillatory damping to the carrying capacity, and instability of the carrying capacity.

Solution.

The fixed points must satisfy

$$N_{t+1} = N_t = N^*$$

Plugging into the equation

$$N^* = \frac{R_0^\beta N^*}{\{1 + [(R_0 - 1)/K] N^*\}^\beta}$$

The solutions are $N^* = 0$ and $N^* = K$, same as the Beverton-Holt stock-recruitment curve. We can determine their stability by investigating

$$\begin{aligned} f'(N^*) &= \frac{\{1 + [(R_0 - 1)/K] N^*\}^\beta R_0^\beta - \beta[(R_0 - 1)/K] \{1 + [(R_0 - 1)/K] N^*\}^{\beta - 1} R_0^\beta N^*}{\{1 + [(R_0 - 1)/K] N^*\}^{2\beta}} \\ &= \frac{R_0^\beta - \beta[(R_0 - 1)/K] \{1 + [(R_0 - 1)/K] N^*\}^{-1} R_0^\beta N^*}{\{1 + [(R_0 - 1)/K] N^*\}^\beta} \end{aligned}$$

At the fixed points we have

$$\begin{aligned} f'(0) &= R_0^\beta \\ f'(K) &= 1 - \beta(1 - \frac{1}{R_0}) \end{aligned}$$

Clearly $|f'(K)| > 1$ for $\beta(1 - \frac{1}{R_0}) < 0$ and $\beta(1 - \frac{1}{R_0}) > 2$ (instability), $0 < f'(K) < 1$ for $0 < \beta(1 - \frac{1}{R_0}) < 1$ (monotonic damping) and $-1 < f'(K) < 0$ for $1 < \beta(1 - \frac{1}{R_0}) < 2$ (oscillatory damping). We plot these curves in Figure 1.

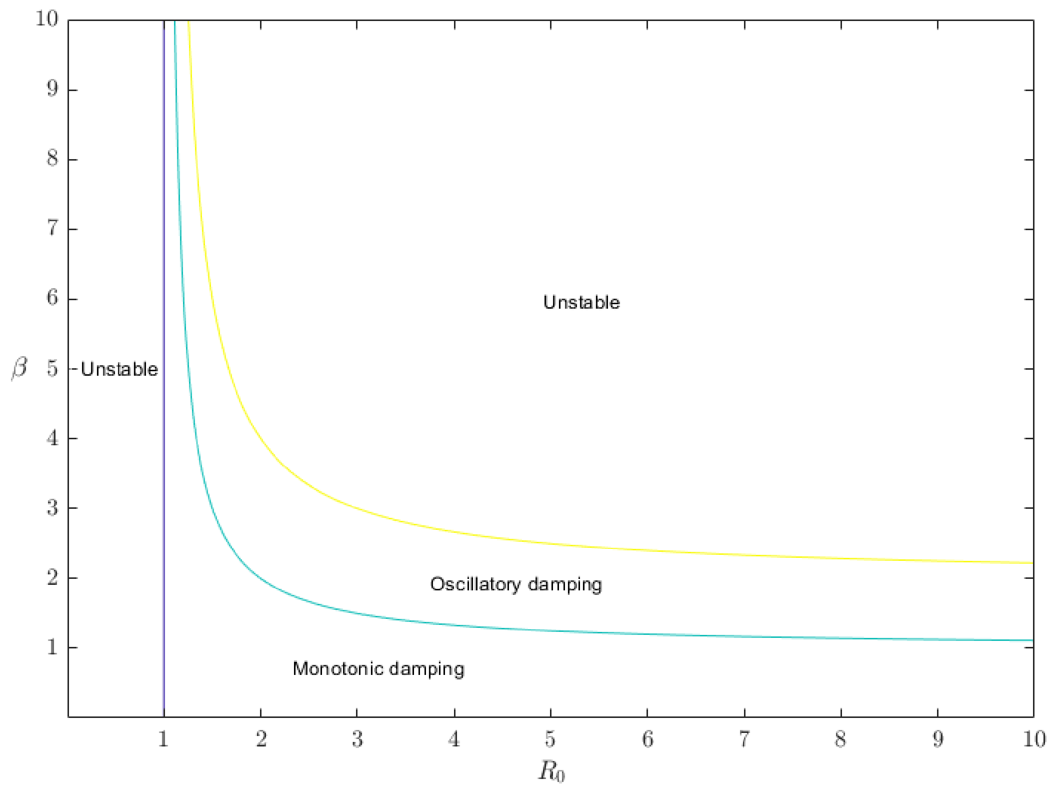

Figure 1: Stability regions of the generalized Beverton-Holt curve