Homework 5.1
Due: May 14, 2021

Refuge
Consider the differential equations

$$\frac{dN}{dT} = rN - c(N-s)P$$
$$\frac{dP}{dT} = b(N-s)P - mP$$

for a predator P and its prey N. In this model, s of the prey may hide in a refuge and avoid predation.
(a) Nondimensionalize the system.
(b) Determine the zero-growth isoclines of your nondimensionalized system.
(c) Find the equilibria.
(d) Find the stability of each equilibrium using a linearized stability analysis.
(e) Use whatever other techniques you have at your disposal to analyze this system.
(f) Draw typical phase portraits for this system.

Solution.

(a) Let $N = sx$, $P = \frac{r}{c}y$, $T = \frac{1}{r}t$

$$rs\frac{dx}{dt} = rsx - c(sx-s)\frac{r}{c}y$$
$$r\frac{r}{c}\frac{dy}{dt} = b(sx-s)\frac{r}{c}y - m\frac{r}{c}y$$

Simplifying yields

$$\frac{dx}{dt} = x - xy + y$$
$$\frac{dy}{dt} = \frac{bs}{r}(x-1)y - \frac{m}{r}y$$

Let $\alpha = \frac{bs}{r}$, $\beta = \frac{m}{r}$

$$\frac{dx}{dt} = x - xy + y$$
$$\frac{dy}{dt} = [\alpha(x-1) - \beta]y$$

(b) The zero-growth isoclines are found at

$$0 = \frac{dx}{dt} = x - xy + y \Rightarrow x = \frac{y}{y-1},\ y = \frac{x}{x-1}$$
$$0 = \frac{dy}{dt} = [\alpha(x-1) - \beta]y \Rightarrow x = \frac{\beta}{\alpha} + 1,\ y = 0$$

(c) The equilibria are the intersections of the zero-growth isoclines

$$(x_0, y_0) = (0, 0)$$
$$(x_1, y_1) = \left(\frac{\beta}{\alpha} + 1, \frac{\alpha}{\beta} + 1\right)$$

(d) Let $F = \frac{dx}{dt}$, $G = \frac{dy}{dt}$, the Jacobian is

$$J = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-y & 1-x \\ \alpha y & \alpha(x-1)-\beta \end{bmatrix}$$

Evaluating J at $(0, 0)$

$$J = \begin{bmatrix} 1 & 1 \\ 0 & -\alpha-\beta \end{bmatrix}$$

The eigenvalues are 1 and $-\alpha - \beta$. Since one is real and positive and the other is real and negative, we have a saddle point.
Evaluating J at $(\frac{\beta}{\alpha} + 1, \frac{\alpha}{\beta} + 1)$

$$J = \begin{bmatrix} -\frac{\alpha}{\beta} & -\frac{\beta}{\alpha} \\ \frac{\alpha^2}{\beta} & 0 \end{bmatrix}$$

The eigenvalues are found by solving the characteristic equation

$$\lambda^2 + \frac{\alpha}{\beta}\lambda + \alpha = 0$$
$$\lambda = \frac{-\frac{\alpha}{\beta} \pm \sqrt{(\frac{\alpha}{\beta})^2 - 4\alpha}}{2}$$

Since $\sqrt{(\frac{\alpha}{\beta})^2 - 4\alpha} < \frac{\alpha}{\beta}$, the equilibrium is always stable. If $\alpha > 4\beta^2$, we have a node. If $\alpha < 4\beta^2$, we have a focus.

(e) Let $B = \frac{1}{xy}$ and consider

$$BF = \frac{1}{y} - 1 + \frac{1}{x},\ BG = \alpha - \frac{\alpha+\beta}{x}$$

Then

$$\frac{\partial BF}{\partial x} + \frac{\partial BG}{\partial y} = -\frac{1}{x^2} < 0$$

By Bendixson–Dulac negative criterion, there is no closed orbits in the first quadrant.

(f) See Figure 1 and Figure 2.

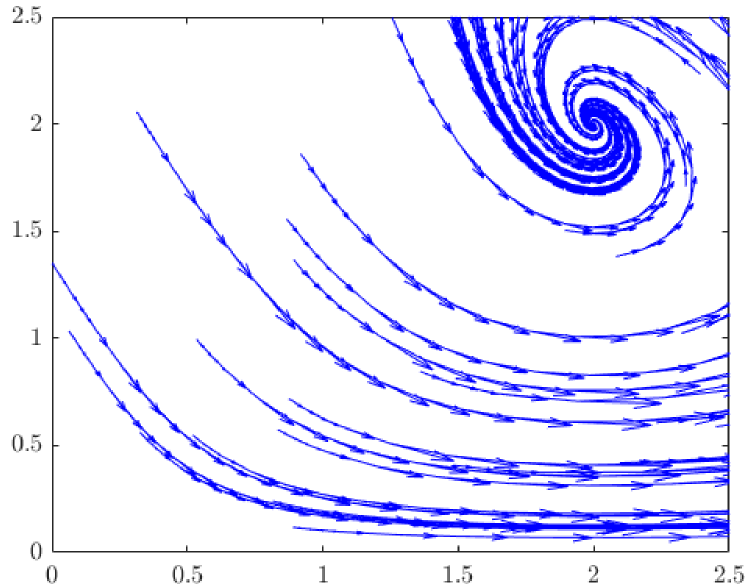

Figure 1: Phase portraits for $\alpha < 4\beta^2$ ($\alpha = 1,\ \beta = 1$). Clearly we have a saddle point at $(0, 0)$ and a stable focus at $(2, 2)$.

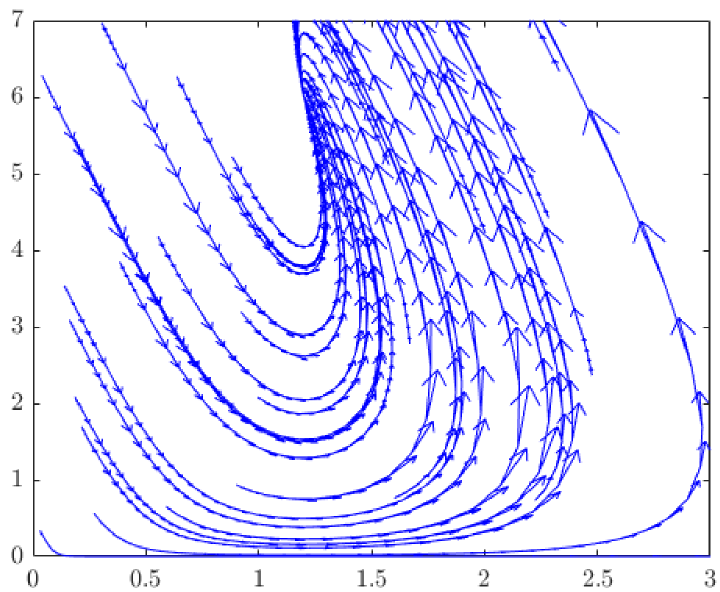

Figure 2: Phase portraits for $\alpha > 4\beta^2$ ($\alpha = 5,\ \beta = 1$). Clearly we have a saddle point at $(0, 0)$ and a stable node at $(1.2, 6)$.