Homework 2

Due: January 22, 2021

1. (Inverse matrix and Green's functions)

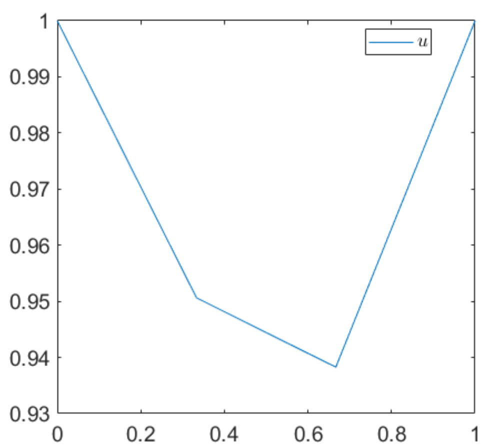
   (a) Write out the 4×4 matrix A from (2.43) for the boundary value problem $u''(x) = f(x)$ with $u(0) = u(1) = 1$ and for $h = 1/3$.

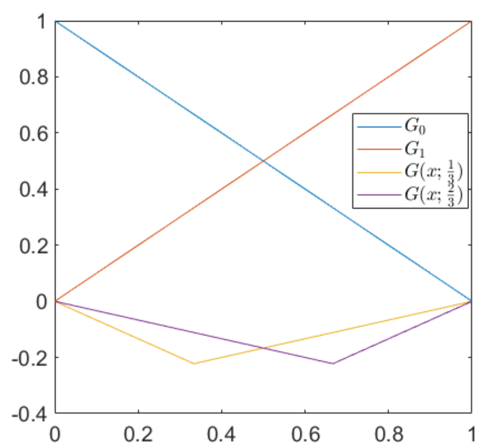
   (b) Write out the 4×4 inverse matrix A^{-1} explicitly for this problem.

   (c) If $f(x) = x$, determine the discrete approximation to the solution of the boundary value problem on this grid and sketch this solution and the four Green's functions from which the solution is obtained. (You may use Matlab or other tools for plotting, or you may plot by hand.)

Solution.

$$A = 9\begin{bmatrix} \frac{1}{9} & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & \frac{1}{9} \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{2}{27} & -\frac{1}{27} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{27} & -\frac{2}{27} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





2. (Another way of analyzing the error using Green's functions) The *composite trapezoid rule* for integration approximates the integral from a to b of a function g by dividing the interval into segments of length h and approximating the integral over each segment by the integral of the linear function that matches g at the endpoints of the segment. (For $g > 0$, this is the area of the trapezoid with height $g(x_j)$ at the left endpoint x_j and height $g(x_{j+1})$ at the right endpoint x_{j+1}.) Letting $h = (b-a)/(m+1)$ and $x_j = a + jh$, $j = 0, 1, \ldots, m, m+1$:

$$\int_a^b g(x)\,dx \approx h\sum_{j=0}^{m} \frac{g(x_j)+g(x_{j+1})}{2} = h\left[\frac{g(x_0)}{2} + \sum_{j=1}^{m} g(x_j) + \frac{g(x_{m+1})}{2}\right].$$

(a) Assuming that g is sufficiently smooth, show that the error in the composite trapezoid rule approximation to the integral is $O(h^2)$. [Hint: Show that the error on each subinterval is $O(h^3)$.]

(b) Recall that the true solution of the boundary value problem $u''(x) = f(x)$, $u(0) = u(1) = 0$ can be written as

$$u(x) = \int_0^1 f(\bar{x})G(x;\bar{x})\,d\bar{x}, \tag{1}$$

where $G(x;\bar{x})$ is the Green's function corresponding to $\bar{x}$. The finite difference approximation u_i to $u(x_i)$, using the centered finite difference scheme in (2.43), is

$$u_i = h\sum_{j=1}^{m} f(x_j)G(x_i;x_j), \quad i = 1, \ldots, m. \tag{2}$$

Show that formula (2) is the trapezoid rule approximation to the integral in (1) when $x = x_i$, and conclude from this that the error in the finite difference approximation is $O(h^2)$ at each node x_i. [Recall: The Green's function $G(x;x_j)$ has a *discontinuous* derivative at $x = x_j$. Why does this not degrade the accuracy of the composite trapezoid rule?]

Solution. (a) Consider the error on each subinterval

$$E_j = h\frac{g(x_j)+g(x_{j+1})}{2} - \int_{x_j}^{x_{j+1}} g(x)dx$$

Let $c = \frac{x_j+x_{j+1}}{2}$, then $c - x_j = x_{j+1} - c = \frac{h}{2}$. Using integration by parts with $u = x - c$ and $dv = g'(x)$

$$\begin{aligned}\int_{x_j}^{x_{j+1}} (x-c)g'(x)dx &= [(x-c)g(x)]_{x_j}^{x_{j+1}} - \int_{x_j}^{x_{j+1}} g(x)dx \\ &= (x_{j+1}-c)g(x_{j+1}) - (x_j - c)g(x_j) - \int_{x_j}^{x_{j+1}} g(x)dx \\ &= E_j\end{aligned}$$

Using integration by parts again with $u = g'(x)$ and $dv = x - c$

$$\begin{aligned}\int_{x_j}^{x_{j+1}} (x-c)g'(x)dx &= \left[\frac{(x-c)^2}{2}g'(x)\right]_{x_j}^{x_{j+1}} - \int_{x_j}^{x_{j+1}} \frac{(x-c)^2}{2}g''(x)dx \\ &= \frac{(x_{j+1}-c)^2}{2}g'(x_{j+1}) - \frac{(x_j-c)^2}{2}g'(x_j) - \int_{x_j}^{x_{j+1}} \frac{(x-c)^2}{2}g''(x)dx \\ &= \frac{1}{2}\int_{x_j}^{x_{j+1}} \left[\left(\frac{h}{2}\right)^2 - (x-c)^2\right] g''(x)dx\end{aligned}$$

Since g is sufficiently smooth, $|g''(x)| \leq M$, then

$$|E_j| \leq \frac{M}{2}\int_{x_j}^{x_{j+1}} \left|\left(\frac{h}{2}\right)^2 - (x-c)^2\right| dx = O(h^3)$$

Then the global error is bounded by

$$|GE| \leq \sum_{j=0}^{m} |E_j| = (m+1)\cdot O(h^3) = \frac{b-a}{h}O(h^3) = O(h^2)$$

(b) Since $f(x_0) = f(x_{m+1}) = 0$

$$\begin{aligned}u_i &= h\sum_{j=1}^{m} f(x_j)G(x_i;x_j) + \frac{f(x_0)G(x_i;x_0) + f(x_{m+1})G(x_i;x_{m+1})}{2}, \quad i = 1,\dots,m. \\ &= h\sum_{j=0}^{m} \frac{f(x_j)G(x_i;x_j) + f(x_{j+1})G(x_i;x_{j+1})}{2}\end{aligned}$$

which is the definition of the trapezoid rule approximation of $u(x)$ when $x = x_i$. Although $G(x;x_j)$ has a discontinuous derivative at $x = x_j$, in part (a) we only used the fact that g'' is bounded in (x_j, x_{j+1}). Thus we can still conclude that for each node i the error is $O(h^2)$.

3. (Green's function with Neumann boundary conditions)

   (a) Determine the Green's functions for the two-point boundary value problem $u''(x) = f(x)$ on $0 < x < 1$ with a Neumann boundary condition at $x = 0$ and a Dirichlet condition at $x = 1$, i.e, find the function $G(x, \bar{x})$ solving

   $$u''(x) = \delta(x - \bar{x}), \quad u'(0) = 0, \quad u(1) = 0$$

   and the functions $G_0(x)$ solving

   $$u''(x) = 0, \quad u'(0) = 1, \quad u(1) = 0$$

   and $G_1(x)$ solving

   $$u''(x) = 0, \quad u'(0) = 0, \quad u(1) = 1.$$

   (b) Using this as guidance, find the general formulas for the elements of the inverse of the matrix in equation (2.54). Write out the 4×4 matrices A and A^{-1} for the case $h = 1/3$.

Solution. (a) For $G(x, \bar{x})$

$$u(x) = \begin{cases} c_1 x + d_1 & 0 < x \le \bar{x} \\ c_2 x + d_2 & \bar{x} \le x < 1 \end{cases}$$

$u(1) = 0 \to d_2 = -c_2$, $u'(0) = 0 \to c_1 = 0$. Since $c_2 - c_1 = 1$, $c_2 = 1$. $G(x, \bar{x})$ is continuous at $x = \bar{x}$, then $d_1 = \bar{x} - 1$. Thus

$$G(x, \bar{x}) = \begin{cases} \bar{x} - 1 & 0 < x \le \bar{x} \\ x - 1 & \bar{x} \le x < 1 \end{cases}$$

For $G_0(x)$ and $G_1(x)$, $u''(x) = 0$ means that $u = ax + b$. Applying BCs yields $G_0(x) = x - 1$ and $G_1(x) = 1$.

(b)

$$A = 9 \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & \frac{1}{9} \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -1 & -\frac{2}{9} & -\frac{1}{9} & 1 \\ -\frac{2}{3} & -\frac{2}{9} & -\frac{1}{9} & 1 \\ -\frac{1}{3} & -\frac{1}{9} & -\frac{1}{9} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. (Solvability condition for Neumann problem) Determine the null space of the matrix A^T, where A is given in equation (2.58), and verify that the condition (2.62) must hold for the linear system to have solutions.

Solution. Reducing A^T to row echelon form

$$A^T = \frac{1}{h^2}\begin{bmatrix} -h & 1 & & & & & \\ h & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & h \\ & & & & & 1 & -h \end{bmatrix} \to \frac{1}{h^2}\begin{bmatrix} -h & 1 & & & & & \\ 0 & -1 & 1 & & & & \\ & 0 & -1 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 0 & -1 & 1 & \\ & & & & 0 & -1 & h \\ & & & & & 0 & 0 \end{bmatrix}$$

Then for the null space $\mathbf{x}$

$$\begin{aligned} -hx_0 + x_1 &= 0 \\ -x_j + x_{j+1} &= 0 \quad j = 1, \dots, m-1. \\ hx_{m+1} - x_m &= 0 \end{aligned}$$

Letting $x_0 = 1$ yields $\mathbf{x} = [1\ h \dots h\ 1]^T$. For this linear system to have solutions we require that $\mathbf{f^T x} = 0$ by Fredholm alternative theorem

$$\begin{aligned} \mathbf{f^T x} = \sigma_0 + \frac{h}{2}(x_0) + \sum_{j=1}^{m} f(x_j) - \sigma_1 + \frac{h}{2}(x_{m+1}) &= 0 \\ \frac{h}{2}(x_0) + \sum_{j=1}^{m} f(x_j) + \frac{h}{2}(x_{m+1}) &= \sigma_1 - \sigma_0 \end{aligned}$$

which is condition (2.62).

5. (Symmetric tridiagonal matrices)

   (a) Consider the **Second approach** described on p. 31 for dealing with a Neumann boundary condition. If we use this technique to approximate the solution to the boundary value problem $u''(x) = f(x)$, $0 \le x \le 1$, $u'(0) = \sigma$, $u(1) = \beta$, then the resulting linear system $A\mathbf{u} = \mathbf{f}$ has the following form:

$$\frac{1}{h^2}\begin{pmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \\ u_m \end{pmatrix} = \begin{pmatrix} \sigma + (h/2)f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{pmatrix}.$$

   Show that the above matrix is similar to a symmetric tridiagonal matrix via a *diagonal* similarity transformation; that is, there is a diagonal matrix D such that DAD^{-1} is symmetric.

   (b) Consider the **Third approach** described on pp. 31-32 for dealing with a Neumann boundary condition. [**Note:** If you have an older edition of the text, there is a typo in the matrix (2.57) on p. 32. There should be a row above what is written there that has entries $\frac{3}{2}h$, $-2h$, and $\frac{1}{2}h$ in columns 1 through 3 and 0's elsewhere. I believe this was corrected in newer editions.] Show that if we use that first equation (given at the bottom of p. 31) to eliminate u_0 and we also eliminate u_{m+1} from the equations by setting it equal to β and modifying the right-hand side vector accordingly, then we obtain an m by m linear system $A\mathbf{u} = \mathbf{f}$, where A is similar to a symmetric tridiagonal matrix via a diagonal similarity transformation.

Solution. (a) We can denote the diagonal entries of such D by $d_{ii},\ i = 0 \ldots m$. Since D is diagonal, D^{-1} is found by replacing the diagonals with its reciprocal.

$$DAD^{-1} = \frac{1}{h^2}\begin{pmatrix} -h & h\frac{d_{00}}{d_{11}} & & & \\ \frac{d_{11}}{d_{00}} & -2 & \frac{d_{11}}{d_{22}} & & \\ & \frac{d_{22}}{d_{11}} & -2 & \frac{d_{22}}{d_{33}} & \\ & & \ddots & \ddots & \frac{d_{m-1,m-1}}{d_{mm}} \\ & & & \frac{d_{mm}}{d_{m-1,m-1}} & -2 \end{pmatrix}$$

For this product to be symmetric, we require that

$$h\frac{d_{00}}{d_{11}} = \frac{d_{11}}{d_{00}}$$
$$\frac{d_{jj}}{d_{j+1,j+1}} = \frac{d_{j+1,j+1}}{d_{jj}} \quad j = 1 \ldots m-1$$

Letting $d_{11} = 1$ yields that $d_{00} = \frac{1}{\sqrt{h}}$ and $d_{jj} = 1,\ j = 1 \ldots m$.

(b) Rearranging the equation at the bottom of p. 31

$$u_0 = \frac{2\sigma h + 4u_1 - u_2}{3}$$

We also have

$$\frac{1}{h^2}(u_0 - 2u_1 + u_2) = f(x_1)$$
$$\frac{1}{h^2}(u_{m-1} - 2u_m + u_{m+1}) = f(x_m)$$

Substituting u_0 and u_{m+1} into the equations

$$\frac{1}{h^2}(-\frac{2}{3}u_1 + \frac{2}{3}u_2) = f(x_1) - \frac{2\sigma}{3h}$$
$$\frac{1}{h^2}(u_{m-1} - 2u_m) = f(x_m) - \frac{\beta}{h^2}$$

Then the modified matrix is

$$\frac{1}{h^2}\begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & & & \\ 1 & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix}$$

Let D be a diagonal matrix where $d_{11} = \sqrt{\frac{3}{2}}$ and all other diagonals equal to 1, then DAD^{-1} is symmetric and tridiagonal.