Homework 1
Due: April 14, 2021

1. Consider the initial value problem in infinite domain:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = 0$$
$$u(x,0) = f(x), \text{ where } f(x) = \begin{cases} 1, & x \le 0 \\ 1-x, & 0 < x < 1 \\ 0, & x \ge 1 \end{cases}$$

(a) Find where and when a shock first forms.
(b) Solve the problem and sketch or plot the solution before when a shock first forms.
(c) Find the shock speed using the Rankine-Hugoniot condition.
(d) Solve the problem and sketch or plot the solution after the shock has formed.

Solution.

(a) By the method of characteristics

$$\frac{du}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = u$$

Define $x(0) = \xi$, we have $x = \xi + ut$ and the solution

$$u(x,t) = f(\xi) = f(x - ut)$$

If $x - ut \le 0$, $u = 1$ for $x \le t$. If $x - ut \ge 1$, $u = 0$ for $x \ge 1$. If $0 < x - ut < 1$, $u = 1 - (x - ut) \to u = \frac{1-x}{1-t}$ for $t < x < 1$. In summary

$$u(x,t) = \begin{cases} 1, & x \le t \\ \frac{1-x}{1-t}, & t < x < 1 \\ 0, & x \ge 1 \end{cases}$$

Clearly, a shock forms when $t^* = 1$ at $x = 1$, where the characteristics intersect.

(b) See Figure 1.

(c) The shock speed $\frac{dX}{dt} = \frac{1}{2}(u^- + u^+) = \frac{1}{2}(1+0) = \frac{1}{2}$.

(d) For $t > 1$, the shock will travel with speed $\frac{1}{2}$. The curve is given by $x - 1 = \frac{1}{2}(t-1) \to x = \frac{t+1}{2}$

$$u(x,t) = \begin{cases} 1, & x < \frac{t+1}{2} \\ 0, & x > \frac{t+1}{2} \end{cases}$$

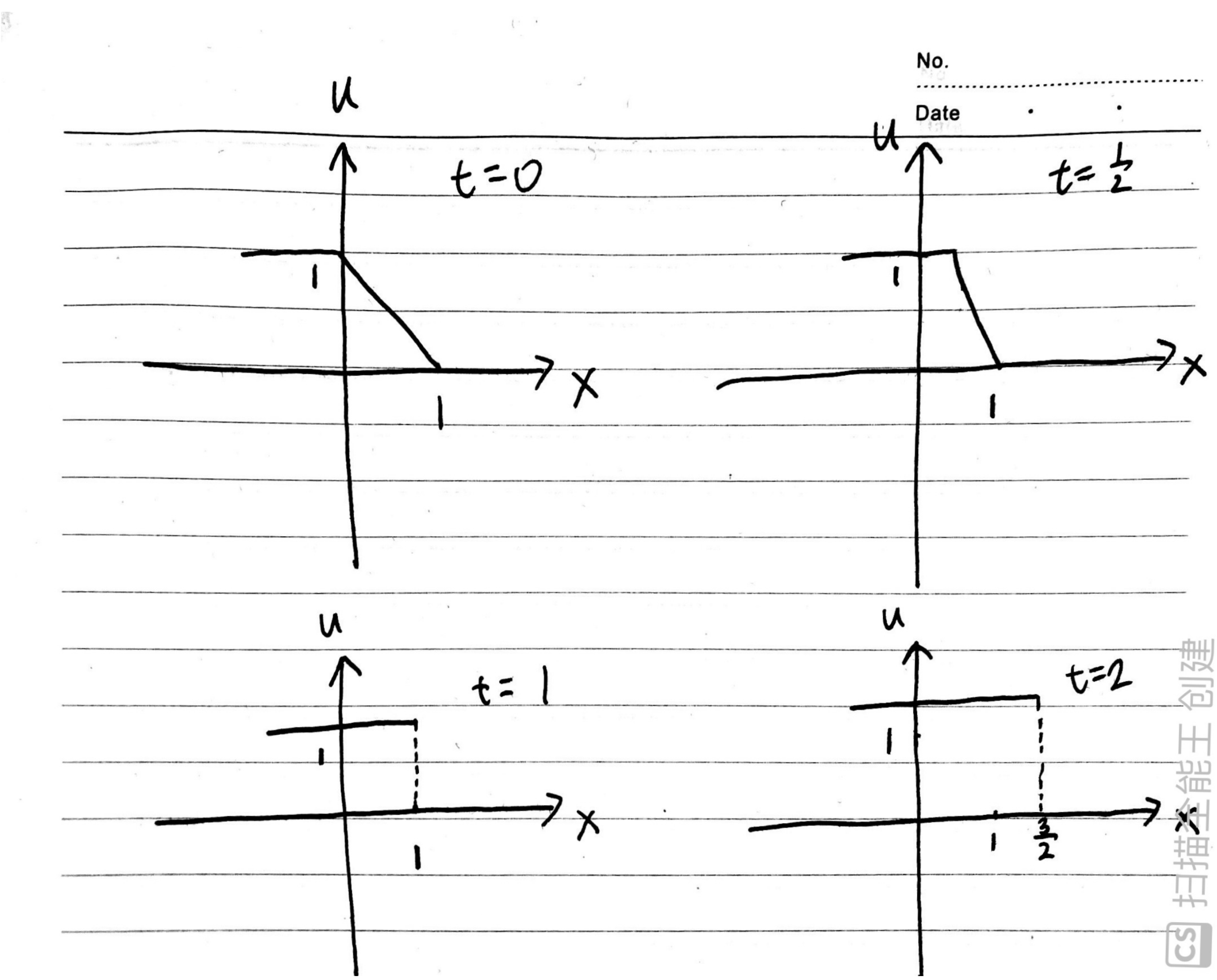

Figure 1: Solution of the Burgers' equation

2. For the Burgers equation, with $0 < \varepsilon \ll 1$:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = \varepsilon\frac{\partial^2}{\partial x^2}u, \quad -\infty < x < \infty, t > 0,$$

subject to the initial condition

$$u(x,0) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$$

The governing equation for the outer solution is obtained by setting $\varepsilon \to 0$ and keeping all the terms $O(1)$. Find the outer solution. Express it analytically if you can.

Solution.

Setting $\epsilon \to 0$, the outer problem is

$$\frac{\partial}{\partial t}u^{\text{out}} + u^{\text{out}}\frac{\partial}{\partial x}u^{\text{out}} = 0$$

This problem can be solved similarly as Problem 1, the solution takes the form

$$u(x,t) = f(x - ut)$$

If $x - ut < 0$, $u = -1$ for $x < -t$. If $x - ut > 0$, $u = 1$ for $x > t$. If $x - ut = 0$, $u = \frac{x}{t}$, which corresponds to a fan region for $-t \leq x \leq t$. In summary

$$u(x,t) = \begin{cases} -1, & x < -t \\ \frac{x}{t}, & -t \leq x \leq t \\ 1, & x > t \end{cases}$$

3. Same as problem 2, except that the initial condition is

$$u(x,0) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x > 0. \end{cases}$$

(a) Find the outer solution. State the region in the x − t plane where the solution is valid before using the Rankine-Hugoniot condition. Do not use this condition yet,
(b) Find the scaling for the shock layer. Obtain the inner equation.
(c) Solve for the inner solution and match to the valid part of the outer solution. Use symmetry arguments to determine the location of the shock.

Solution.

(a) Since the characteristics intersect, we know a shock will form instead of a fan. The outer solution will be similar to Problem 2 with a change of sign

$$u(x,t) = \begin{cases} 1, & x < -t \\ -1, & x > t \end{cases}$$

This solution is valid for

$$\{(x,t) \in \mathbb{R}^2 : x < -t \text{ or } x > t\}$$

(b) Denote the shock speed by $U(t)$. Let $\hat{x} = x - \int_0^t U dt$. This changes the original equation to

$$(u - U)u_{\hat{x}} = \nu u_{\hat{x}\hat{x}}$$

Let $\bar{x} = \frac{\hat{x}}{\epsilon}$, then

$$\frac{1}{\epsilon}(u - U)u_{\bar{x}} = \frac{\nu}{\epsilon^2} u_{\bar{x}\bar{x}}$$

Choosing $\epsilon = \nu$

$$(u^{\text{in}} - U)u^{\text{in}}_{\bar{x}} = u^{\text{in}}_{\bar{x}\bar{x}}$$

which is the inner equation.

(c) Integrating the inner equation

$$\frac{1}{2}u^2 - Uu + C = u_{\bar{x}}$$

Matching the outer solution

$$\bar{x} \to +\infty, \quad u \to u^{\text{out}+}$$
$$\bar{x} \to -\infty, \quad u \to u^{\text{out}-}$$

Noting that as $\bar{x} \to \pm\infty$, $\epsilon \to 0$, then the RHS of the equation tends to 0. Then we have two equations for two unknowns. Solving yields

$$U = \frac{1}{2}(u^{\text{out}-} + u^{\text{out}+})$$
$$C = \frac{1}{2}u^{\text{out}-}u^{\text{out}+}$$

Plugging into the intergrated equation

$$(u - u^{\text{out}+})(u^{\text{out}-} - u) = -2\frac{du}{d\bar{x}}$$

Using a separation of variables, we get

$$\bar{x} = \frac{\hat{x}}{\nu} = \frac{2}{u^{\text{out}-}u^{\text{out}+}} \ln\left(\frac{u^{\text{out}-} - u}{u - u^{\text{out}+}}\right)$$
$$u = u^{\text{out}+} + \frac{u^{\text{out}-} - u^{\text{out}+}}{1 + \exp\left(\frac{(u^{\text{out}-} - u^{\text{out}+})(x - Ut)}{2\nu})\right)}$$

Note that there should be a constant of integration from this, but from symmetry we know the shock has to be at $x = 0$, so the constant is 0. Plugging in the values we get

$$u = -1 + \frac{2}{1 + \exp(\frac{x}{\nu})} = \tanh(-\frac{x}{2\nu})$$

which is the inner solution. We can verify this by plotting it with $\nu = 0.001$. See Figure 2.

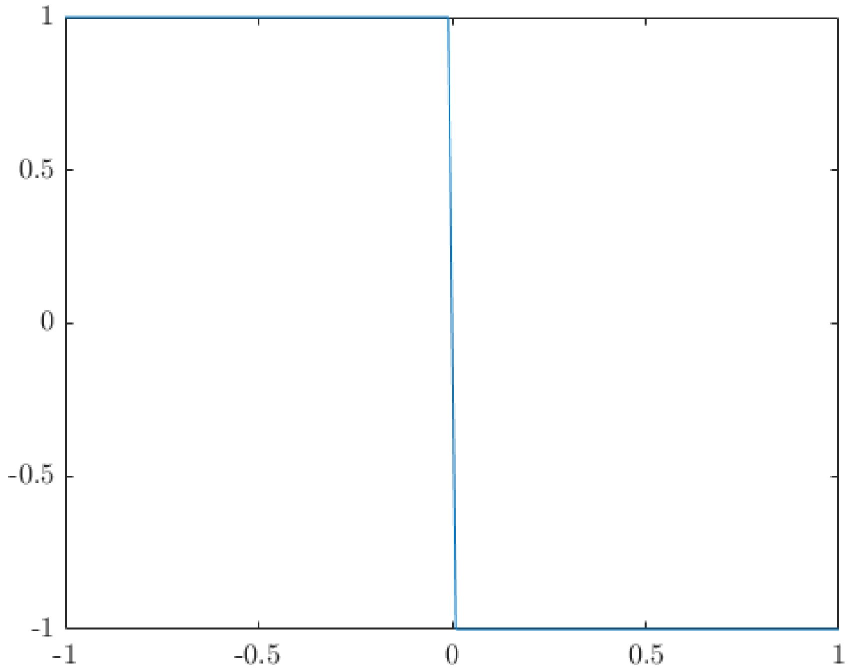

Figure 2: Inner solution of the Burgers' equation