**Homework 2**

Due: Monday, October 19, 2020

**Question 1.** Suppose $X$ and $Y$ are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, then $Z$ is a random variable.

We want to show that

$$Z^{-1}(B) = \{\omega : Z(\omega) \in B\} \in \mathcal{F}$$

for $B$ in an arbitrary measurable space. Note that

$$\begin{aligned} Z^{-1}(B) = \{\omega : Z(\omega) \in B\} &= \{\omega \in A : Z(\omega) \in B\} \cup \{\omega \in A^c : Z(\omega) \in B\} \\ &= \{\omega \in A : X(\omega) \in B\} \cup \{\omega \in A^c : Y(\omega) \in B\} \\ &= (X^{-1}(B) \cap A) \cup (Y^{-1}(B) \cap A^c) \end{aligned}$$

Since $X^{-1}(B),\ Y^{-1}(B),\ A,\ A^c \in \mathcal{F}$ and $\mathcal{F}$ is closed under countable intersections and unions, $Z^{-1}(B) \in \mathcal{F}$.

**Question 2.** Suppose $X$ is a continuous random variable with distribution function $F_X$. Let $g$ be a strictly increasing continuous function. Define $Y = g(X)$. a) What is $F_Y$, the distribution function of $Y$? b) What is $f_Y$, the density function of $Y$?

Since $g$ is strictly increasing, $g^{-1}$ exists.
a) $F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$
b) Since $g$ and $X$ are both continuous, $Y$ is also continuous, then we have $f_Y = F_Y' = F_X'(g^{-1}(y)) = f_X(g^{-1}(y))\frac{dg^{-1}(y)}{dy}$

**Question 3.** Suppose $X$ is a continuous random variable with distribution function $F_X$. Find $F_Y$ where $Y$ is given by a) $X^2$ b) $\sqrt{|X|}$ c) $\sin X$ d) $F_X(X)$.

a) $F_Y(y) = P(Y \le y) = P(X^2 \le y) = \begin{cases} P(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y \ge 0 \\ 0 & y < 0 \end{cases}$

b) $F_Y(y) = P(Y \le y) = P(\sqrt{|X|} \le y) = \begin{cases} P(-y^2 \le X \le y^2) = F_X(y^2) - F_X(-y^2) & y \ge 0 \\ 0 & y < 0 \end{cases}$

c) $F_Y(y) = P(Y \le y) = P(\sin(X) \le y) = \begin{cases} 0 & y < -1 \\ \text{See below} & -1 \le y \le 1 \\ 1 & y > 1 \end{cases}$

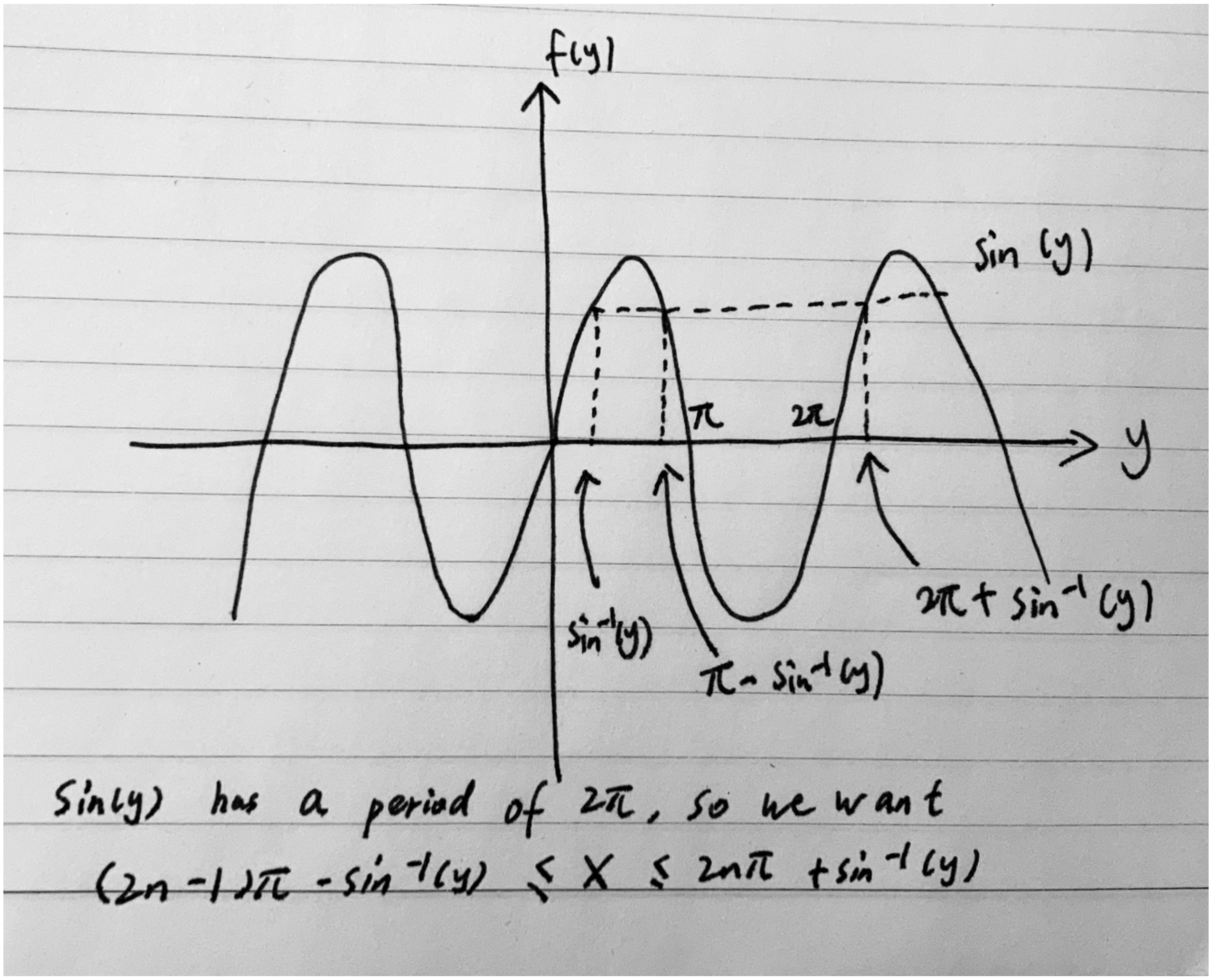


Figure 1: Question 3(c) drawing

When $-1 \leq y \leq 1$ (See Figure 1 for a graph explanation)

$$\begin{aligned} P(\sin(X) \leq y) &= \sum_{n=-\infty}^{+\infty} P((2n-1)\pi - \sin^{-1}(y) \leq X \leq 2n\pi + \sin^{-1}(y)) \\ &= \sum_{n=-\infty}^{+\infty} F_X(2n\pi + \sin^{-1}(y)) - F_X((2n-1)\pi - \sin^{-1}(y)) \end{aligned}$$

d) $F_Y(y) = P(Y \leq y) = P(F_X(X) \leq y) = P(X \leq F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$ when $0 \leq y \leq 1$. $F_Y(y) = 1$ when $y > 1$ and $F_Y(y) = 0$ when $y < 0$.

**Question 4.** Let $X : [0,1] \to \mathbf{R}$ be a function that maps every rational number in the interval $[0,1]$ to 0, and every irrational number to 1. We assume that the probability space where $X$ is defined is $([0,1], \mathcal{B}[0,1], P)$, where $\mathcal{B}[0,1]$ is the Borel $\sigma$-algebra on [0,1], and $P$ is the Lebesgue measure. Is $X$ a random variable (and why)? If it is, what are its distribution function and expectation? Does $X$ have a density function? Is $X$ discrete?

Yes it is. $X$ is an indicator function:

$$X(\omega) = \begin{cases} 0 & \omega \in A \\ 1 & \omega \notin A \end{cases}$$

where $A = \mathbf{Q} \cap [0,1]$. For any Borel set $B$, we have

$$X^{-1}(B) = \{\omega \mid X(\omega) \in B\} = \begin{cases} A & 1 \in B, 0 \notin B \\ A^c & 1 \notin B, 0 \in B \\ [0,\ 1] & 1 \in B, 0 \in B \\ \emptyset & 1 \notin B, 0 \notin B \end{cases}$$

Therefore $X$ is a random variable. Then $EX = \int_{[0,1]} X dP = 0 \cdot P(A) + 1 \cdot P(A^c) = P(A^c) = 1$. Its distribution function is

$$F_X(x) = \begin{cases} 0 & x < 1 \\ 1 & x \geq 1 \end{cases}$$

$X$ does not have a density function. Suppose a density function $f(x)$ exists, we would have

$$P(X = 1) = \lim_{\epsilon \to 0} \int_{1-\epsilon}^{1+\epsilon} f(x) dx = 0$$

But in our example $P(X = 1) = P(A^c) = 1$. Therefore $X$ does not have a density function. $X$ is dicrete because it only takes a finite set of values.