Homework 4
Due: June 2, 2021

1. Orthogonality of Bessel functions: Let

$$I_{jk} = \frac{1}{a^2}\int_0^a J_p\left(z_{pj}(r/a)\right) J_p\left(z_{pk}(r/a)\right) r dr$$

where z_{pj} is the j-th zero of $J_p(z)$. Show that $I_{jk} = 0$ if $j \neq k$, but $I_{jk} =$ a positive constant if $j = k$.

Solution.

For convenience, let $\phi_j = J_p\left(z_{pj}(r/a)\right)$ and $\phi_k = J_p\left(z_{pk}(r/a)\right)$. Let λ_j and λ_k be the eigenvalues of ϕ_j and ϕ_k, respectively. They must satisfy the Liouville form of the Bessel equation

$$(r\phi_j')' + (\lambda_j r - r)\phi_j = 0$$
$$(r\phi_k')' + (\lambda_k r - r)\phi_k = 0$$

Multiplying the top by ϕ_k and the bottom by ϕ_k and subtracting

$$\phi_k(r\phi_j')' - \phi_j(r\phi_k')' = a^2\frac{d}{dr}[\phi_k(r\phi_j') - \phi_j(r\phi_k')] = (\lambda_k - \lambda_j)r\phi_j\phi_k$$

Integrating both sides from 0 to a

$$(\lambda_k - \lambda_j)\frac{1}{a^2}\int_0^a r\phi_j\phi_k dr = [\phi_k(r\phi_j') - \phi_j(r\phi_k')]_0^a = 0$$

since $r = 0$ at the left boundary and $r = a$ gives $\phi_j = \phi_k = 0$. If $\lambda_k \neq \lambda_j$, i.e. $j \neq k$, then $I_{jk} = 0$. If $j = k$, then $I_{jk} = \frac{1}{a^2}\int_0^a r|\phi_j|^2 dr > 0$ given that $r > 0$.

2. For the 1-dimensional heat equation for conduction in a copper rod:

$$\frac{\partial}{\partial t}u = \alpha^2 \frac{\partial^2}{\partial x^2}u,\ 0 < x < L,\ t > 0$$
$$u(0,t) = 0 = u(L,t)$$
$$u(x,0) = f(x),\ 0 < x < L$$

(a) Solve using separation of variables. Show that the time-dependence for the n-th standing mode is $\exp\left(-n^2\left(t/t_e\right)\right)$, where $t_e \equiv (L/(\pi\alpha))^2$, about 1 hour for a 2m-copper rod.
(b) For $t > t_e$, find the mode that dominates the solution, and thus write down the approximate solution (in time and in space). Describe in words how an initial condition, which may not look like a sine wave, becomes the sine shape with the largest wavelength fitting in between the boundaries.

Solution.

(a) Assume the solution has the form

$$u(x,t) = T(t)X(x)$$

Substituting into the equation

$$XT_t = \alpha^2 T X_{xx}$$

Dividing both sides by $\alpha^2 XT$

$$\frac{T_t}{\alpha^2 T} = \frac{X_{xx}}{X} = -\lambda^2$$

where λ is a constant. Then we get two ODEs

$$X_{xx} = -\lambda^2 X$$
$$T_t = -\alpha^2\lambda^2 T$$

Consider the first ODE, we have a general solution

$$X = A\sin(\lambda x) + B\cos(\lambda x)$$

Since $X(0) = 0$, $B = 0$. Since $X(L) = 0$, $\lambda = \frac{n\pi}{L}$. Thus the n-th mode in space is

$$X_n = A_n \sin(\frac{n\pi}{L}x)$$

Consider the second ODE, the n-th mode in time is

$$T_n = C_n e^{-\alpha^2(\frac{n\pi}{L})^2 t} = Ce^{-n^2(t/t_e)}$$

Combining the solutions (C_n gets absorbed into A_n)

$$u_n(x,t) = A_n e^{-\alpha^2 \lambda_n^2 t} \sin(\frac{n\pi}{L}x)$$

Writing the solution as a series

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2(t/t_e)} \sin(\frac{n\pi}{L}x)$$

Applying the initial condition

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{L}x)$$

This is a Fourier sine series, where A_n is determined by

$$A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx$$

(b) The dominant mode is the one that decays most slowly, i.e. $n = 1$. Thus the approximate solution is

$$u_1 = A_1 e^{-t/t_e} \sin(\frac{\pi}{L}x)$$

which has a sine shape with wavelength equal to L.

3. Consider the following nonhomogeneous system:

$$\frac{\partial^2}{\partial t^2}u = \frac{\partial^2}{\partial x^2}u + 1,\ 0 < x < 1$$
$$u(0,t) = 0 = u(1,t)$$
$$u(x,0) = 0,\ \frac{\partial}{\partial t}u(x,0) = 0$$

Solve this problem in two ways:
(a) by eigenfunction expansion; that is, expand the solution in the form of an infinite sum of eigenfunctions (in space) of the homogeneous system with unknown coefficient (which is a function of time) in front of each eigenfunction. Do the same for the forcing term, "1". Then solve an ODE in time.
(b) by first finding the steady state solution to the nonhomogeneous equation and then the transient solution; the latter is the difference between the true solution and the steady state solution and should satisfy a homogeneous equation.

Solution.

(a) The eigenfunctions in space have the form (similar to Problem 2)

$$X_n(x) = \sin(n\pi x)$$

Thus we use a Fourier sine series expansion of the solution

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$$

and the forcing term

$$1 = \sum_{n=1}^{\infty} f_n \sin(n\pi x)$$

where f_n is computed by

$$f_n = 2\int_0^1 \sin(n\pi x)dx = \frac{2}{n\pi}(1-\cos(n\pi))$$

Plugging everything into the PDE

$$T_n'' + (n\pi)^2 T_n = \frac{2}{n\pi}(1-\cos(n\pi))$$

This has solution

$$T_n = A\sin(n\pi t) + B\cos(n\pi t) + \frac{2}{(n\pi)^3}(1-\cos(n\pi))$$

Applying initial conditions yields

$$A = 0, \quad B = \frac{2}{(n\pi)^3}(\cos(n\pi) - 1)$$

Thus the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{(n\pi)^3}(\cos(n\pi) - 1)\sin(n\pi x)(\cos(n\pi t) - 1)$$

(b) The steady solution satisfies

$$0 = \frac{\partial^2}{\partial x^2}u_{\text{steady}} + 1$$

which yields

$$u_{\text{steady}} = -\frac{x^2}{2} + bx + c$$

Applying boundary conditions yields $b = \frac{1}{2}$ and $c = 0$. Then

$$u_{\text{steady}} = -\frac{1}{2}(x^2 - x)$$

The transient solution is given by

$$u_{\text{transient}}(x,t) = u(x,t) - u_{\text{steady}}(x)$$

It satisfies a homogeneous PDE

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}u_{\text{transient}} &= \frac{\partial^2}{\partial x^2}u_{\text{transient}}, \;\; 0 < x < 1 \\
u_{\text{transient}}(0,t) &= 0 = u_{\text{transient}}(1,t) \\
u_{\text{transient}}(x,0) &= \frac{1}{2}(x^2 - x), \;\; \frac{\partial}{\partial t}u_{\text{transient}}(x,0) = 0
\end{aligned}$$

The general solution has the form

$$u_{\text{transient}}(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\cos(n\pi t)$$

where A_n is given by

$$A_n = \int_0^1 (x^2 - x)\sin(n\pi x)dx$$

Evaluating in Mathematica yields

$$A_n = \frac{2}{(n\pi)^3}(\cos(n\pi) - 1)$$

Thus the transient solution is

$$u_{\text{transient}}(x,t) = \sum_{n=1}^{\infty} \frac{2}{(n\pi)^3}(\cos(n\pi) - 1)\sin(n\pi x)\cos(n\pi t)$$

Adding $u_{\text{steady}}(x) = -\sum_{n=1}^{\infty} A_n \sin(n\pi x)$, we have

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{(n\pi)^3}(\cos(n\pi) - 1)\sin(n\pi x)(\cos(n\pi t) - 1)$$