Homework 5
Due: Friday, November 13, 2020

Question 1. Let X and $Y_0, Y_1, Y_2, ...$ be random variables on a probability space $(\Omega, \mathcal{F}, P)$ and suppose $E|X| < \infty$. Define $\mathcal{F}_n = \sigma(Y_0, Y_1, ..., Y_n)$ and $X_n = E(X|\mathcal{F}_n)$. Show that the sequence $X_0, X_1, X_2, ...$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$.

i) $E|X_n| = E|E(X|\mathcal{F}_n)| \leq E(E(|X||\mathcal{F}_n)) = E|X| < \infty$, where $|E(X|\mathcal{F}_n)| \leq E(|X||\mathcal{F}_n)$ follows from Jensen's inequality
ii) By definition $X_n \in \mathcal{F}_n$ for all n
iii) $E(X_{n+1}|\mathcal{F}_n) = E(E(X|\mathcal{F}_{n+1})|\mathcal{F}_n) = E(X|\mathcal{F}_n) = X_n$ (smaller σ-algebra wins)

Question 2. Let $X_0, X_1, ...$ be i.i.d Bernoulli random variables with parameter p (i.e., $P(X_i = 1) = p, P(X_i = 0) = 1 - p$). Define $S_n = \sum_{i=1}^n X_i$ where $S_0 = 0$. Define

$$Z_n = \left(\frac{1-p}{p}\right)^{2S_n - n}, \quad n = 0, 1, 2, ...$$

Let $\mathcal{F}_n = \sigma(X_0, X_1, ..., X_n)$. Show that Z_n is a martingale with respect to this filtration.

i) For $p < \frac{1}{2}$, $E|Z_n| \leq (\frac{1-p}{p})^n < \infty$. For $p \geq \frac{1}{2}$, $E|Z_n| \leq (\frac{1-p}{p})^{-n} < \infty$. Thus $E|Z_n| < \infty$
ii) Since $X_i \in \mathcal{F}_n$ and Z_n is a function of X_i, then $Z_n \in \mathcal{F}_n$ for all n
iii) $E(Z_{n+1}|\mathcal{F}_n) = E((\frac{1-p}{p})^{2S_{n+1}-(n+1)}|\mathcal{F}_n) = E((\frac{1-p}{p})^{(2S_n - n)+2X_{n+1}-1}|\mathcal{F}_n) = E(Z_n(\frac{1-p}{p})^{2X_{n+1}-1}|\mathcal{F}_n) = Z_n E((\frac{1-p}{p})^{2X_{n+1}-1}) = Z_n(p \cdot \frac{1-p}{p} + (1-p) \cdot \frac{p}{1-p}) = Z_n$

Question 3. Let ξ_i be a sequence of random variables such that the partial sums

$$X_n = \xi_0 + \xi_1 + ... + \xi_n, \quad n \geq 1,$$

determine a martingale. Show that the summands are mutually uncorrelated, i.e. that $E(\xi_i \xi_j) = E(\xi_i)E(\xi_j)$ for $i \neq j$.

Since X_n is a martingale, then

$$\begin{aligned}
E(X_{n+1}|\mathcal{F}_n) &= X_n \\
E(X_n + \xi_{n+1}|\mathcal{F}_n) &= X_n \\
X_n + E(\xi_{n+1}|\mathcal{F}_n) &= X_n \\
E(\xi_{n+1}) &= 0
\end{aligned}$$

It must be true that $E(\xi_i) = 0$ for $i \geq 0$. Then for $i \neq j$, $E(\xi_i \xi_j) = E(E(\xi_i \xi_j|\mathcal{F}_{j-1})) = E(\xi_i E(\xi_j|\mathcal{F}_{j-1})) = E(\xi_i \cdot 0) = 0 = E(\xi_i)E(\xi_j)$

Question 4. Galton and Watson who invented the process that bears their names were interested in the survival of family names. Suppose each family has exactly 3 children but coin flips determine their sex. In the 1800s, only male children kept the family name so following the male offspring leads to a branching process with $p_0 = 1/8, p_1 = 3/8, p_2 = 3/8, p_3 = 1/8$. Compute the probability ρ that the family name will die out when $Z_0 = 1$. What is ρ if we assume that each family has exactly 2 children?

We want to find the solutions of $\phi(\rho) = \sum_{k\geq 0} p_k \rho^k = \rho$, then

$$\begin{aligned}
\phi(\rho) = \frac{1}{8}\rho^3 + \frac{3}{8}(\rho^2 + \rho) + \frac{1}{8}\rho^0 &= \rho \\
\rho^3 + 3\rho^2 - 5\rho + 1 &= 0 \\
(\rho - 1)(\rho^2 + 4\rho - 1) &= 0 \\
\rho = 1, \sqrt{5} - 2, -\sqrt{5} - 2&
\end{aligned}$$

Then the die out probability is $\rho = \sqrt{5} - 2$

If each family has exactly two children, then we have a branching process with $p_0 = \frac{1}{4}, p_1 = \frac{1}{2}, p_2 = \frac{1}{4}$.

$$\begin{aligned}
\phi(\rho) = \frac{1}{4}(\rho^2 + \rho^0) + \frac{1}{2}\rho &= \rho \\
(\rho - 1)^2 &= 0 \\
\rho &= 1
\end{aligned}$$