Homework 8
Due: March 19, 2021

1. Consider the Optical Parametric Oscillator as given in Lecture 23 of the notes (Pages 99-102)

   (a) Assuming slow time $\tau = \epsilon^2 t$ and slow space $\xi = \epsilon x$, derive the Fisher-Kolmogorov equation for the slow evolution of the instability (the expression after Eq. (518))

   (b) Derive the Swift-Hohenberg type expression which is given by Eq. (519) with the scalings detailed in the notes.

Solution.

(a) The optical parametric oscillator (OPO) is governed the dimensionless coupled equations

$$U_t = \frac{i}{2}U_{xx} + VU^* - (1 + i\Delta_1)\,U \tag{1}$$
$$V_t = \frac{i}{2}\rho V_{xx} - U^2 - (\alpha + i\Delta_2)\,V + S \tag{2}$$

where U is the signal, V is the pump field, Δ_1 and Δ_2 are the cavity detuning parameters, ρ is the diffraction ratio between signal and pump fields, α is the pump-to-signal loss ratio, and S is the external pumping term. The steady-state solution is given by

$$U = 0$$
$$V = \frac{S}{\alpha + i\Delta_2}$$

We can find the critical pumping strength using a linear stability analysis

$$S_c = (\alpha + i\Delta_2)\,(1 + i\Delta_1)$$

We want to investigate of the behavior of OPO near $S = S_c$. By defining the slow scales

$$\tau = \epsilon^2 t$$
$$T = \epsilon^4 t$$
$$\xi = \epsilon x$$
$$X = \epsilon^2 x$$

We expand around the stead-state solution

$$U = 0 + \epsilon u(\tau, T, \xi, x)$$
$$V = \tfrac{S}{\alpha+i\Delta_2} + \epsilon^2 v(\tau, T, \xi, x)$$
$$S - S_c = \epsilon^2 C + \epsilon^3 C_1 + \cdots$$

where C and C_j are constants. This changes the derivatives to

$$\begin{aligned} U_t &= \epsilon^3 u_\tau + \epsilon^5 u_T \\ U_{xx} &= \epsilon^3 u_{\xi\xi} + \epsilon^5 u_{XX} \\ V_t &= \epsilon^4 v_\tau + \epsilon^6 v_T \\ V_{xx} &= \epsilon^4 v_{\xi\xi} + \epsilon^6 v_{XX} \end{aligned}$$

Substituting into (1) and (2) yields

$$(1+i\Delta_1)(u-u^*) = \epsilon^2\left[\frac{i}{2}u_{\xi\xi} - u_\tau + vu^* + \frac{C}{\alpha+i\Delta_2}u^*\right] + \epsilon^4\left[\frac{i}{2}u_{XX} - u_T\right] \tag{3}$$

$$(\alpha+i\Delta_2)\, v = -u^2 + \epsilon^2\left[\frac{i}{2}\rho v_{\xi\xi} - v_\tau\right] + \epsilon^4\left[\frac{i}{2}\rho v_{XX} - v_T\right] \tag{4}$$

Rearranging for v and u^*

$$\begin{aligned} v &= -\tfrac{u^2}{\alpha+i\Delta_2} + \tfrac{\epsilon^2}{\alpha+i\Delta_2}\left[\tfrac{i}{2}\rho v_{\xi\xi} - v_\tau\right] + O\left(\epsilon^4\right) \\ u^* &= u - \tfrac{\epsilon^2}{1+i\Delta_1}\left[\tfrac{i}{2}u_{\xi\xi} - u_\tau + vu^* + \tfrac{C}{\alpha+i\Delta_2}u^*\right] + O\left(\epsilon^4\right) \end{aligned}$$

Solving iteratively for vu^*

$$vu^* = -\frac{1}{\alpha+i\Delta_2}|u|^2 u + \frac{\epsilon^2}{(\alpha+i\Delta_2)^2}\left(u\left(u^2\right)_\tau - \frac{i}{2}\rho u\left(u^2\right)_{\xi\xi}\right) + O\left(\epsilon^4\right)$$

Substituting into (3)

$$\begin{aligned} R = \epsilon^2 &\left[\frac{i}{2}u_{\xi\xi} - u_\tau - \frac{|u|^2 u}{\alpha+i\Delta_2} + \frac{C}{\alpha+i\Delta_2}u^*\right] + \epsilon^4\left[\frac{i}{2}u_{XX} - u_T\right. \\ &\left. + \frac{1}{(\alpha+i\Delta_2)^2}\left(u\left(u^2\right)_\tau - \frac{i}{2}\rho u\left(u^2\right)_{\xi\xi}\right)\right] + O\left(\epsilon^6\right) \end{aligned}$$

By Fredholm alternative theorem, R must be orthogonal to the null space of the adjoint operator associated with the leading order behavior of (3). This gives the solvability condition

$$(1-i\Delta_1)\, R + (1+i\Delta_1)\, R^* = 0 \tag{5}$$

Define

$$\begin{aligned} u &= \left(\left(\alpha^2+\Delta_2^2\right)/\left(\Delta_1\Delta_2 - \alpha\right)\right)^{1/2}\varphi = A\varphi \\ \xi &= \left(\Delta_1/2\right)^{1/2}\zeta \\ \gamma &= -|C|\left(1+\Delta_1^2\right)/S_c \end{aligned}$$

then the derivatives become

$$\begin{aligned} u_\tau &= A\varphi_\tau \\ u_{\xi\xi} &= (2/\Delta_1)A^2\varphi_{\zeta\zeta} \\ (u^2)_\tau &= A^2(\varphi^2)_\tau \\ (u^2)_{\xi\xi} &= (2/\Delta_1)A^4(\varphi^2)_{\zeta\zeta} \end{aligned}$$

Applying (5)

$$\frac{2\epsilon^2\left(\alpha^2A^2(\varphi_{\zeta\zeta}-\varphi_\tau)+|\varphi|^2(\Delta_1\Delta_2\varphi-\alpha\varphi)+\alpha C\varphi+\Delta_2(A^2\Delta_2(\varphi_{\zeta\zeta}-\varphi_\tau)-C\Delta_1\varphi)\right)}{\alpha^2+\Delta_2^2}=0$$

$$\varphi_{\zeta\zeta}-\varphi_\tau\mp|\varphi|^3\pm\frac{|C|(\alpha-\Delta_1\Delta_2)}{\alpha^2+\Delta_2^2}\varphi=0$$

We thereby derive the Fisher-Kolmogorov equation (although I got a different γ...)

$$\varphi_\tau-\varphi_{\zeta\zeta}\pm\varphi^3\mp\gamma\varphi=0$$

where we assume $\Delta_1=O(1)$ and $\alpha-\Delta_1\Delta_2=O(1)$. We also assume $\Delta_1>0$ for the equation to be well-posed.

(b) The Swift-Hohenberg is derived when the slow scales ξ and τ are neglected and the slow scales are captured by T and X. Define

$$\begin{aligned} \alpha &= \epsilon^2\kappa+\Delta_1\Delta_2 \\ \tau &= T/\left(2\left(\alpha^2+\Delta_2^2\right)\right) \\ \zeta &= X/\sqrt{\Delta_1\left(\alpha^2+\Delta_2^2\right)} \\ \varphi &= u \\ a &= \kappa C-C^2K \\ b &= C-\kappa+CK \\ K &= \left[\left(\alpha^2-\Delta_2^2\right)\left(1-\Delta_1^2\right)-4\Delta_1\Delta_2\right]/\left[\left(1+\Delta_1^2\right)\left(\alpha^2+\Delta_2^2\right)\right] \end{aligned}$$

then the derivatives become

$$\begin{aligned} u_T &= \varphi_\tau/(2\left(\alpha^2+\Delta_2^2\right)) \\ u_{XX} &= \varphi_{\zeta\zeta}/(\Delta_1\left(\alpha^2+\Delta_2^2\right)) \end{aligned}$$

Applying (5)

$$\frac{\epsilon^2}{\alpha^2+\Delta_2^2}\Bigg[\epsilon^2(\Delta_1(\alpha^2+\Delta_2^2))\left[\varphi_{\zeta\zeta}/(\Delta_1(\alpha^2+\Delta_2^2)-\varphi_\tau/(2(\alpha^2+\Delta_2^2))\right]$$

$$
\begin{aligned}
&- 2\varphi|\varphi|^2(\alpha - \Delta_1\Delta_2) + 2C(\alpha - \Delta_1\Delta_2)\varphi\Bigg] = 0 \\
&\epsilon^2\left[\varphi_{\zeta\zeta} - \varphi_\tau\frac{\Delta_1(\alpha^2+\Delta_2^2)}{2(\alpha^2+\Delta_2^2)}\right] - 2\varphi|\varphi|^2(\alpha - \Delta_1\Delta_2) + 2C(\alpha - \Delta_1\Delta_2)\varphi = 0 \\
&\varphi_{\zeta\zeta} - \frac{\Delta_1}{2}\varphi_\tau - 2\kappa\varphi|\varphi|^2 + 2\kappa C\varphi = 0 \\
&\varphi_{\zeta\zeta} - \frac{\Delta_1}{2}\varphi_\tau - 2\varphi^3(C + CK - b) + 2\varphi(a + C^2K) = 0
\end{aligned}
$$

This is the best form I get. I tried to do this in many different ways, but the algebra never worked out. I am particularly not sure where the fifth power term comes from... Finally, the Swift-Hohenberg equation should take the form

$$\varphi_\tau - \varphi_{\zeta\zeta} + a\varphi + b\varphi^3 - \varphi^5 = 0$$