Homework 1
Due: April 9, 2021

1. Using the Taylor series representation of the matrix exponential:

   (a) Verify the identities

   $$\frac{\mathrm{d}}{\mathrm{d}\,t}\,\mathrm{e}^{tA} = A\,\mathrm{e}^{tA} = \mathrm{e}^{tA}\,A$$

   for an $n \times n$ matrix A.

   (b) Verify that $u(t) = \mathrm{e}^{tA}\,\eta$ is indeed the solution of the IVP

   $$\begin{cases} u'(t) = Au(t), \\ u(0) = \eta. \end{cases}$$

   Solution.

   (a) $\frac{d}{dt}e^{tA} = \frac{d}{dt}(\sum_{j=0}^{\infty}\frac{(tA)^j}{j!}) = \sum_{j=1}^{\infty}\frac{jt^{j-1}A^j}{j!} = A\sum_{j=1}^{\infty}\frac{(tA)^{j-1}}{(j-1)!} = Ae^{tA} = e^{tA}A$

   (b) The initial condition is satisfied by $u(0) = e^0\eta = \eta$. Also $u'(t) = \frac{d}{dt}e^{tA}\eta = Ae^{tA}\eta = Au(t)$.

2. Construct a system (i.e., needs to be not scalar valued)

$$\begin{cases} u'(t) = f(u(t)), \end{cases}$$

and two choices of initial data $u_0 \neq v_0$ so that two solutions

$$\begin{cases} u'(t) = f(u(t)), \\ u(0) = u_0, \end{cases} \qquad \begin{cases} v'(t) = f(v(t)), \\ v(0) = v_0, \end{cases}$$

satisfy

$$\|u(t) - v(t)\|_2 = \|u(0) - v(0)\|_2 \, \mathrm{e}^{Lt} \tag{1}$$

where L a Lipschitz constant for $f(u)$. Recall that we have shown that for any solution

$$\|u(t) - v(t)\|_2 \leq \|u(0) - v(0)\|_2 \, \mathrm{e}^{Lt} .$$

So, you are tasked with showing that this is sharp. Then show that equality (1) fails to hold for $u'(t) = -f(u(t)), v'(t) = -f(v(t))$ with the same intial conditions.

Solution.

Suppose $f(u(t)) = Lu(t)$, the system is solved trivially with $u(t) = e^{Lt}u_0$ and $v(t) = e^{Lt}v_0$. Clearly, L is a Lipschitz constant for $f(u(t))$. We have

$$\|u(t) - v(t)\|_2 = \|e^{Lt}u_0 - e^{Lt}v_0\|_2 = \|u(0) - v(0)\|_2 e^{Lt}$$

For $u'(t) = -Lu(t)$, the solutions are now $u(t) = e^{-Lt}u_0$ and $v(t) = e^{-Lt}v_0$. Then we would have

$$\|u(t) - v(t)\|_2 = \|u(0) - v(0)\|_2 e^{-Lt}$$

instead of (1).

3. Consider the IVP

$$\begin{cases} u_1'(t) = 2u_1(t), \\ u_2'(t) = 3u_1(t) - u_2(t), \end{cases}$$

with initial conditions specified at time $t = 0$. Solve this problem in two different ways:

(a) Solve the first equation, which only involves u_1, and then insert this function into the second equation to obtain a nonhomogeneous linear equation for u_2. Solve this using (5.8). Check that your solution satisfies the initial conditions and the ODE.

(b) Write the system as $u' = Au$ and compute the matrix exponential using (D.30) to obtain the solution.

Solution.

(a) We get $u_1(t) = e^{2t}u_1(0)$. Plugging into the second equation

$$u_2'(t) = 3e^{2t}u_1(0) - u_2(t)$$

Recall (5.8)

$$u(t) = e^{A(t-t_0)}\eta + \int_0^t e^{A(t-\tau)}g(\tau)d\tau$$

In this case, $A = -1$, $t_0 = 0$, $\eta = u_2(0)$ and $g(t) = 3e^{2t}u_1(0)$. (5.8) then becomes

$$\begin{aligned} u_2(t) &= e^{-t}u_2(0) + 3\int_0^t e^{3\tau-t}d\tau u_1(0) \\ &= e^{-t}u_2(0) + (e^{2t} - e^{-t})u_1(0) \end{aligned}$$

At $t = 0$, $u_2(t) = u_2(0)$ as desired. Differentiating $u_2(t)$

$$\begin{aligned} u_2'(t) &= -e^{-t}u_2(0) + (2e^{2t} + e^{-t})u_1(0) \\ &= 3e^{2t}u_1(0) - e^{-t}u_2(0) - (e^{2t} - e^{-t})u_1(0) \\ &= 3u_1(t) - u_2(t) \end{aligned}$$

(b) We have

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$$

Then the solution is $u(t) = e^{At}u(0)$, where

$$e^{At} = Re^{\Lambda t}R^{-1}$$

by (D.30). We can find R and Λ by diagonalizing A

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad R^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Plugging these into e^{At}

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & 0 \\ e^{2t} - e^{-t} & e^{-t} \end{bmatrix} \end{aligned}$$

Clearly, this is the same as the solution we get in (a).

4. Consider the IVP
$$\begin{cases} u_1'(t) = 2u_1(t), \\ u_2'(t) = 3u_1 + 2u_2(t), \end{cases}$$
with initial conditions specified at time $t = 0$. Solve this problem.

Solution.

Solving the first equation directly, we get $u_1(t) = e^{2t}u_1(0)$. Plugging into the second equation
$$u_2'(t) = 3e^{2t}u_1(0) + 2u_2(t)$$
Recall (5.8)
$$u(t) = e^{A(t-t_0)}\eta + \int_0^t e^{A(t-\tau)}g(\tau)d\tau$$
In this case, $A = 2$, $t_0 = 0$, $\eta = u_2(0)$ and $g(t) = 3e^{2t}u_1(0)$. (5.8) then becomes
$$\begin{aligned} u_2(t) &= e^{2t}u_2(0) + 3e^{2t}\int_0^t d\tau u_1(0) \\ &= e^{2t}u_2(0) + 3e^{2t}tu_1(0) \end{aligned}$$

5. Consider the Lotka–Volterra system[1]

$$\begin{cases} u_1'(t) = \alpha u_1(t) - \beta u_1(t) u_2(t), \\ u_2'(t) = \delta u_1(t) u_2(t) - \gamma u_2(t). \end{cases}$$

For $\alpha = \delta = \gamma = \beta = 1$ and $u_1(0) = 5, u_2(0) = 0.8$ use the forward Euler method to approximate the solution with $k = 0.001$ for $t = 0, 0.001, \ldots, 50$. Plot your approximate solution as a curve in the (u_1, u_2)-plane and plot your approximations of $u_1(t)$ and $u_2(t)$ on the same axes as a function of t. Repeat this with backward Euler. What do you notice about the behavior of the numerical solutions? The most obvious feature is most apparent in the (u_1, u_2)-plane.

Solution.

We implement these methods in MATLAB as `predator_prey.m`.
If we zoom in the phase plane in Figure 1, we observe limit cycles in the solution. For $t = 50$, there are six cycles. We also observe a lag between u_1 and u_2, which makes sense because u_2 is supposed to be the predator.

[1]This is a famous model of predator-prey dynamics.

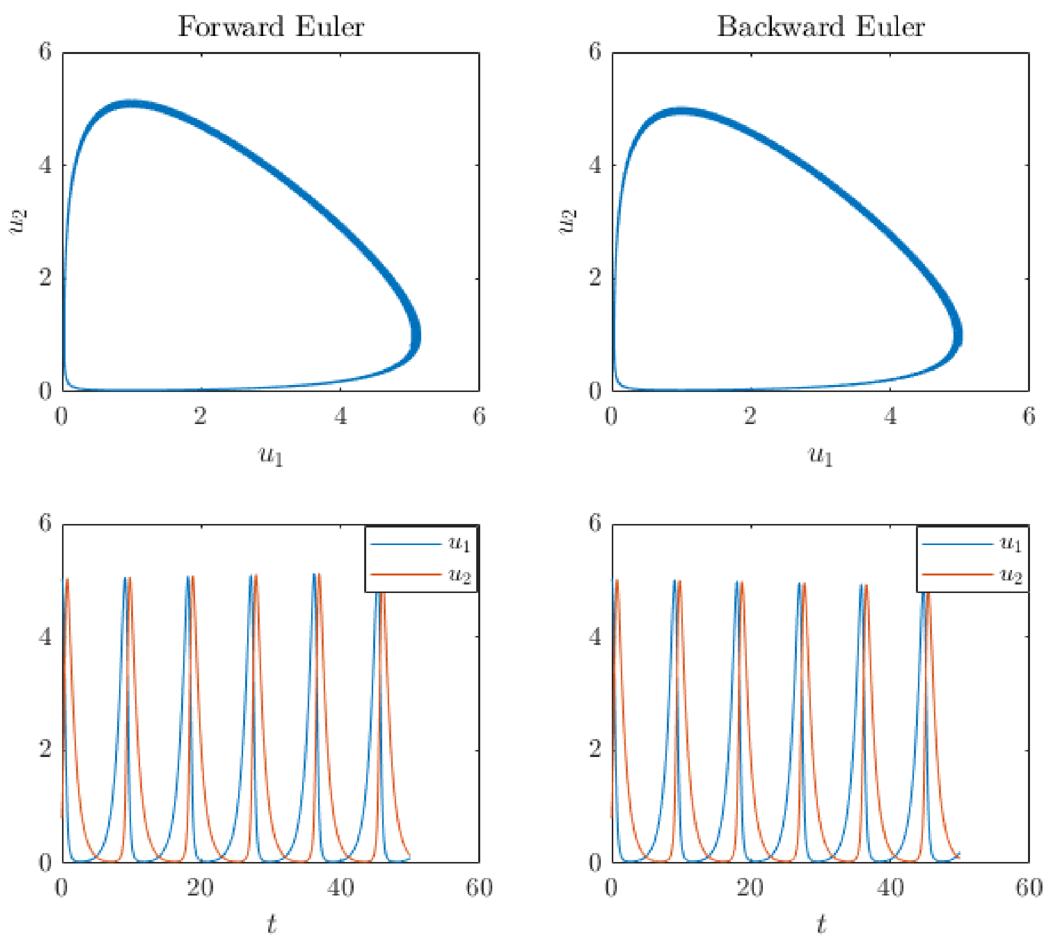

Figure 1: Solutions of the Lotka-Volterra system

6. Determine the coefficients $\beta_0,\ \beta_1,\ \beta_2$ for the third order, 2-step Adams-Moulton method. Do this in two different ways:

   (a) Using the expression for the local truncation error in Section 5.9.1,

   (b) Using the relation

$$u(t_{n+2}) = u(t_{n+1}) + \int_{t_{n+1}}^{t_{n+2}} f(u(s))\,ds.$$

   Interpolate a quadratic polynomial $p(t)$ through the three values $f(U^n),\ f(U^{n+1})$ and $f(U^{n+2})$ and then integrate this polynomial exactly to obtain the formula. The coefficients of the polynomial will depend on the three values $f(U^{n+j})$. It's easiest to use the "Newton form" of the interpolating polynomial and consider the three times $t_n = -k$, $t_{n+1} = 0$, and $t_{n+2} = k$ so that $p(t)$ has the form

$$p(t) = A + B(t+k) + C(t+k)t$$

   where $A,\ B$, and C are the appropriate divided differences based on the data. Then integrate from 0 to k. (The method has the same coefficients at any time, so this is valid.)

Solution.

(a) A 2-step Adams method have $\alpha_2 = 1,\ \alpha_1 = -1,\ \alpha_0 = 0$. Since the method is third order, we want the first four terms in the local truncation error to vanish

$$\sum_{j=0}^{2} j\alpha_j - \beta_j = 0, \quad \sum_{j=0}^{2} \frac{1}{2}j^2\alpha_j - j\beta_j = 0, \quad \sum_{j=0}^{2} \frac{1}{6}j^3\alpha_j - \frac{1}{2}j^2\beta_j = 0$$

Writing this as a linear system

$$\begin{bmatrix} \alpha_1 + 2\alpha_2 \\ \alpha_1 + 4\alpha_2 \\ \alpha_1 + 8\alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 3 & 12 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

Solving yields $\beta_0 = -\frac{1}{12},\ \beta_1 = \frac{2}{3},\ \beta_2 = \frac{5}{12}$.

(b) We consider a Newton polynomial $p(t)$ for three points $f(U^n),\ f(U^{n+1}),\ f(U^{n+2})$ with three times $-k,\ 0,\ k$, which has the form

$$p(t) = f(U^n) + \frac{f(U^{n+1}) - f(U^n)}{k}(t+k) + \frac{f(U^{n+2}) - 2f(U^{n+1}) + f(U^n)}{2k^2}(t+k)t$$

Integrating from 0 to k using Mathematica gives

$$\int_0^k p(t)dt = \frac{k}{12}\left(-f\left(U^n\right) + 8f\left(U^{n+1}\right) + 5f\left(U^{n+2}\right)\right)$$