Homework 7

Due: Wednesday, December 2, 2020

Question 1. A six-sided die is rolled repeatedly. Which of the following are Markov chains? For those that are, find the one-step transition matrix. (a) X_n is the largest number rolled up to the nth roll. (b) X_n is the number of sixes rolled in the first n rolls. (c) At time n, X_n is the time since the last six was rolled. (d) At time n, X_n is the time until the next six is rolled.

(a) $\begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 1/3 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 1/2 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 2/3 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ (b) $p(i,j) = \begin{cases} \frac{5}{6} & i = j \\ \frac{1}{6} & j = i+1 \\ 0 & \text{otherwise} \end{cases}$ $i, j \in \{0, 1, 2 \cdots, n\}$

(c) $p(i,j) = \begin{cases} \frac{1}{6} & j = 0 \\ \frac{5}{6} & j = i+1 \\ 0 & \text{otherwise} \end{cases}$ $i, j \in \mathbb{N}_0$ (d) $p(i,j) = \begin{cases} \frac{5^j}{6^{j+1}} & i = 0 \\ 1 & j = i-1 \\ 0 & \text{otherwise} \end{cases}$ $i, j \in \mathbb{N}_0$

Question 2. Let $\mathrm{Y}_n = \mathrm{X}_{2n}$. Compute the transition matrix for Y when (a) X is a simple random walk (i.e., X increases by one with probability p and decreases by 1 with probability q and (b) X is a branching process where G is the generating function of the number of offspring from each individual.

(a) $p(i,j) = \begin{cases} 2pq & i = j \\ p^2 & j = i+2 \\ q^2 & j = i-2 \\ 0 & \text{otherwise} \end{cases}$ $i, j \in \mathbb{Z}$

(b) $p(i,j) = \begin{cases} 1 & i = j = 0 \\ 0 & i = 0, j \neq 0 \\ \text{see below} & \text{otherwise} \end{cases}$ $i, j \in \mathbb{N}_0$

$p(i,j) = P(\mathrm{Y}_{n+1} = j | \mathrm{Y}_n = i) = P(\mathrm{X}_{2n+2} = j | \mathrm{X}_{2n} = i) = P(\mathrm{X}_{n+2} = j | \mathrm{X}_n = i) = \sum_{k=0}^{\infty} P(\mathrm{X}_{n+2} = j | \mathrm{X}_{n+1} = k) P(\mathrm{X}_{n+1} = k | \mathrm{X}_n = i) = \sum_{k=0}^{\infty} \frac{1}{j!k!} ((G(0))^k)^{(j)} ((G(0))^i)^{(k)}$, where (j) denotes the j-th derivative.

Question 3. Let X be a Markov chain with state space S and absorbing state k (i.e., $p(k,j) = 0$ for all $j \in \mathrm{S}$). Suppose $j \to k$ for all $j \in \mathrm{S}$. Show that all states other than k are transient.

Suppose there exists a state $i \in S$ other k that is persistent. Since $i \to k$, $p_{n-1}(i,k) > 0$ for some $n \geq 2$. Then $p_n(i,i) = \sum_k p_{n-1}(i,k) p(k,i) = 0$ for all $i \in S$, which contradicts the definition of persistent states. Then all $i \in S$ other k must be transient.

Question 4. Suppose two distinct states i, j satisfy

$$\mathbb{P}(\tau_j < \tau_i \mid \mathrm{X}_0 = i) = \mathbb{P}(\tau_i < \tau_j \mid \mathrm{X}_0 = j)$$

where $\tau_j := \inf\{n \geq 1 : \mathrm{X}_n = j\}$. Show that, if $\mathrm{X}_0 = i$, the expected number of visits to j prior to re-visiting i is one.

Let $p = \mathbb{P}(\tau_j < \tau_i \mid \mathrm{X}_0 = i) = \mathbb{P}(\tau_i < \tau_j \mid \mathrm{X}_0 = j)$, N denote the number of visits to j before revisiting i and τ_j^n denote the time of n visits to j.

$$\begin{aligned}
\mathbb{P}(N = n) &= \mathbb{P}(\tau_j^n < \tau_i < \tau_j^{n+1} | X_0 = i) \\
&= \mathbb{P}(\tau_i < \tau_j^{n+1} | \tau_j^n < \tau_i, X_0 = i)\mathbb{P}(\tau_j^n < \tau_i | X_0 = i) \\
&= \mathbb{P}(\tau_i < \tau_j | X_0 = j)\mathbb{P}(\tau_j < \tau_i | X_0 = i)\mathbb{P}^{n-1}(\tau_j < \tau_i | X_0 = j) \\
&= p^2(1-p)^{n-1}
\end{aligned}$$

Then $EN = \sum_{n=1}^{\infty} p^2 n(1-p)^{n-1} = p^2 \sum_{n=1}^{\infty} \binom{(n-1)+2-1}{2-1}(1-p)^{n-1} = p^2 \frac{1}{(1-(1-p))^2} = 1$, where we have used $\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n = \frac{1}{(1-x)^k}$.

Question 5. Let X be a Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{pmatrix}, \quad p \in (0,1)$$

Find the invariant distribution $\boldsymbol{\pi}$ and the mean-recurrence times $\overline{\tau}_j$ for $j = 1, 2, 3$.

Since $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$, then

$$\begin{aligned}
\pi(1) &= \pi(1)(1-2p) + \pi(2)p \\
\pi(2) &= \pi(1)2p + \pi(2)(1-2p) + \pi(3)2p \\
\pi(3) &= \pi(2)p + \pi(3)(1-2p)
\end{aligned}$$

We also have $\pi(1) + \pi(2) + \pi(3) = 1$. Solving the equations yields $\pi(1) = \frac{1}{4}, \pi(2) = \frac{1}{2}, \pi(3) = \frac{1}{4}$. By Theorem 1 in lecture 18, $\overline{\tau}_j = 1/\pi(j)$. Then $\overline{\tau}_1 = 4, \overline{\tau}_2 = 2, \overline{\tau}_3 = 4$.