Homework 7
Due: March 12, 2021

1. Consider the inverted pendulum dynamics:

$$y'' + (\delta + \epsilon \cos \omega t) \sin y = 0$$

(a) Perform a Floquet analysis (computationally) of the pendulum with continuous forcing $\cos \omega t$.
(b) Evaluate for what values of δ, ϵ and ω the pendulum is stabilized.

Solution.

(a) We perform a Floquet analysis for both the downward pendulum and inverted pendulum. Consider the linear approximations

$$\begin{aligned} y &\approx 0 \quad \sin(y) \approx y - \frac{y^3}{3!} + \cdots \\ y &\approx \pi \quad \sin(y + \pi) = -\sin(y) \approx -y + \frac{y^3}{3!} + \cdots \end{aligned}$$

Plugging into the differential equation and shifting t by $\frac{\pi}{2}$, we get Eq. 612-613 in Kutz

$$\begin{aligned} \text{Downward, linear} \quad & y'' + (\delta + \epsilon \sin(\omega t))y = 0 \\ \text{Downward, nonlinear} \quad & y'' + (\delta + \epsilon \sin(\omega t)) \sin y = 0 \\ \text{Inverted, linear} \quad & y'' - (\delta + \epsilon \sin(\omega t))y = 0 \\ \text{Inverted, nonlinear} \quad & y'' - (\delta + \epsilon \sin(\omega t)) \sin y = 0 \end{aligned}$$

By imposing the initial conditions

$$\begin{aligned} y_1(0) = 1, \quad y_1'(0) = 0 \\ y_2(0) = 0, \quad y_2'(0) = 1 \end{aligned}$$

We can solve for y_1 and y_2, which are linearly-independent fundamental solutions of the differential equation. The Floquet discriminant is then computed as

$$\Gamma = y_1(T) + y_2'(T), \quad T = \frac{2\pi}{\omega}$$

These equations are solved numerically for various δ, ϵ and ω, with `ode45` in MATLAB.

(b) See Figure 1. The top four figures show the cases of $\delta > \epsilon$ ($\delta = 0.01, \epsilon = 0.001$) and bottom four figures show the cases of $\delta < \epsilon$ ($\delta = 0.001, \epsilon = 0.01$). For $\delta > \epsilon$, the down pendulum is always stabilized and the inverted pendulum can be stabilized for high frequencies. For $\delta < \epsilon$, both pendulums can be destabilized for low frequencies, while they are both stabilized for high frequencies.

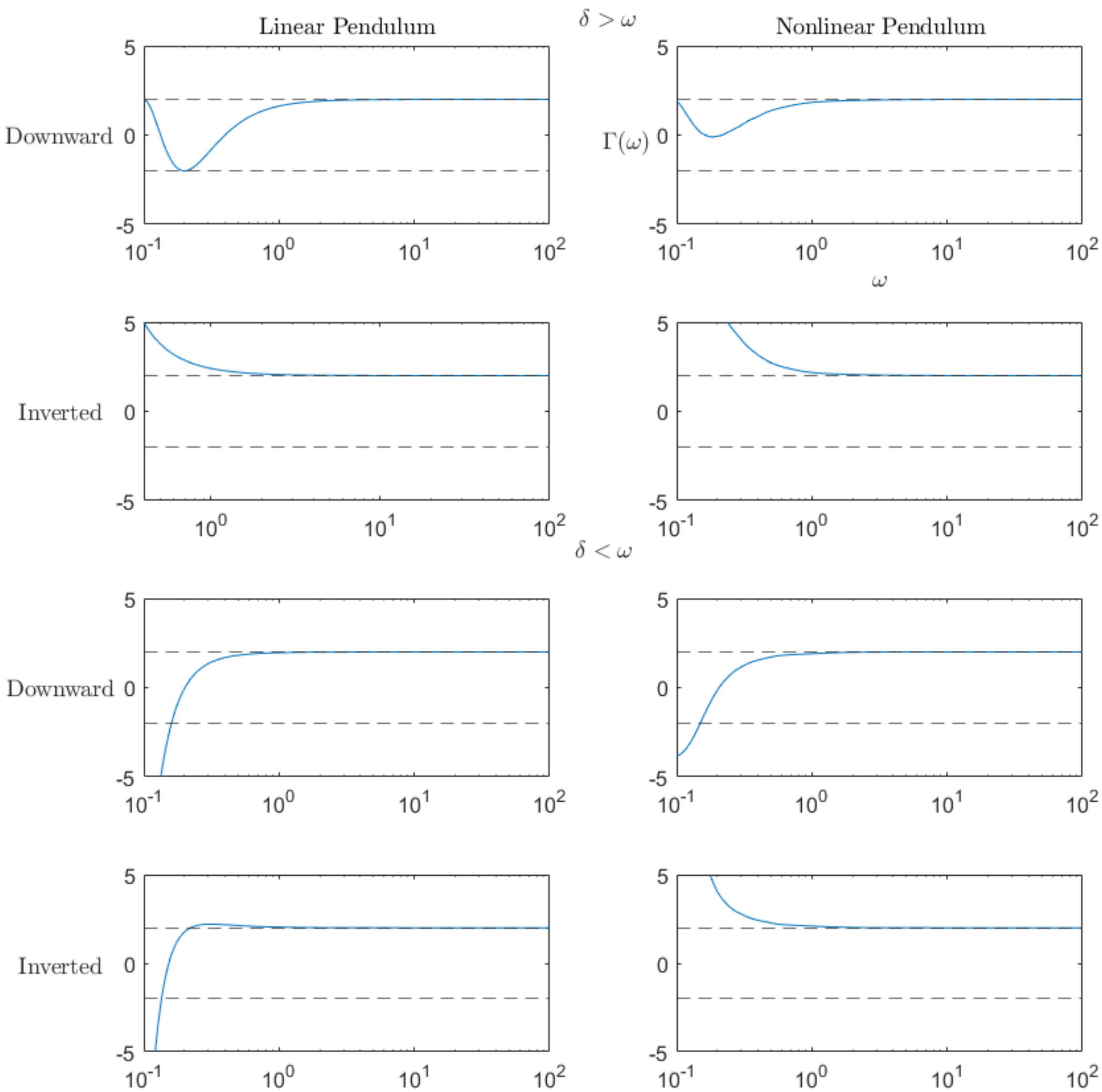

Figure 1: Floquet analysis for the inverted pendulum equations. The dashed lines mark the stablity range ($\Gamma = \pm 2$).

`floquet` takes a function, δ and ϵ as inputs and computes the Floquet discriminant `F`.

```
function [T,F]=floquet(fun,∆,epsilon)
T=linspace(0.01*2*pi,10*2*pi,100);
tspan=T;
bc_x1=[1;0];
bc_x2=[0;1];
F=zeros(length(T),1);
for i=1:length(T)
    [¬,y] = ode45(@(t,y) fun(t,y,∆,epsilon,2*pi/T(i)),tspan,bc_x1);
    x1=y(i,1);
    [¬,y] = ode45(@(t,y) fun(t,y,∆,epsilon,2*pi/T(i)),tspan,bc_x2);
    dx2=y(i,2);
    F(i)=x1+dx2;
end
```

For instance, the linear inverted pendulum equation

```
function dydx = inv_pen_li(t,y,∆,epsilon,omega)
dydx = [y(2)
       (∆+epsilon*sin(omega*t))*y(1)];
end
```