Sample AMATH 569 Examination Solutions

1**.** Consider the Laplace's equation in a long rectangular domain:

$$(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})u=0, \quad 0<x<1,\ 0<y<\infty.$$

(a) Using the method of separation of variables find the general solution satisfying

$$u(0,y)=0, \quad u(1,y)=0.$$

Separation of variables: Assume $u(x,y)=X(x)Y(y)$.

$$-\frac{Y''(y)}{Y(y)}=\frac{X''(x)}{X(x)}=\text{constant}\equiv-\lambda^2.$$

$$X(x)=X_n(x)=\sin n\pi x,\ \lambda=\lambda_n=n\pi,\ n=1,2,3,...$$

$$Y(x)=Y_n(x)=A_n\cosh n\pi y+B_n\sinh n\pi y.$$

The general solution is:

$$u(x,y)=\sum_{n=1}^{\infty}\{A_n\cosh n\pi y+B_n\sinh n\pi y\}\sin n\pi x.$$

(b) Find the solution satisfying the boundary conditions in (a) and the following:

$$\frac{\partial}{\partial y}u(x,0)=0,\ u(x,0)=f(x)=e^{-\sqrt{m}}\sin((4m+1)\pi x),\ \text{where } m \text{ is an integer.}$$

To satisfy these "initial conditions",

$B_n=0$, for all n. $A_n=0$ for all n except $n=(4m+1)$, $A_{4m+1}=e^{-\sqrt{m}}$.

The solution is

$$u(x,y)=e^{-\sqrt{m}}\cosh((4m+1)\pi y)\sin((4m+1)\pi x).$$

(c) Is the solution in (b) unique? Is it well posed? Explain your answers.

The solution is unique but is not well-posed. Consider the case of $m\to\infty$. The boundary data $f(x)\to 0$, but the solution for any *y>0* goes to infinity. This means that small changes in boundary data lead to large changes in the solution a small distance away from the boundary.

The uniqueness part can be obtained by assuming that there are two solutions, and so their difference satisfy the same equation but with zero boundary conditions $f(x)=0$. Using the above general solution obtained from separation of variables, we will find that $A_n=0$ and $B_n=0$ for all n. So the difference is identically zero.

2. For the solution $u(r,\theta)$ of the Laplace equation inside a unit circular disk with the value of $u(1,\theta)=|\sin\theta|$, $0\le\theta<2\pi$ specified on the boundary.

(a) Where is the maximum of $u(r,\theta)$ found? Where is the minimum? Explain.

Using the Maximum Principle for Laplace equation, the maximum can occur only at the boundary. So at $r=1, \theta=\pi/2, -\pi/2$. Using the Minimum Principle for the Laplace equation, the minimum can occur only at the boundary. So at $r=1, \theta=0, \pi$.

(b) What is the value of $u(0,\theta)$? Explain.

By the Mean-Value Theorem, the value of the solution to a Laplace equation at the center of a circle is equal to its mean value at the circumference of that circle. So

$$u(0,\theta)=\frac{1}{2\pi}\int_0^{2\pi}|\sin\theta|\,d\theta=\frac{1}{2\pi}\int_0^{\pi}\sin\theta d\theta-\frac{1}{2\pi}\int_{\pi}^{2\pi}\sin\theta d\theta=\frac{2}{\pi}.$$

—

3. (a) Write down the Green's function of the 1-D heat equation in a semi-infinite domain, $G(x,t;\xi,\tau)$, defined by:

$$\left(\frac{\partial}{\partial t}-D\frac{\partial^2}{\partial x^2}\right)G=\delta(x-\xi)\delta(t-\tau),\quad 0<x,\xi<\infty,\quad t>0,\tau>0.$$

subject to zero initial condition: $G=0$ at $t=0$. The boundary condition is: $G=0$ at $x=0$ and $x\to\infty$.

The Green's function for the heat equation in infinite domain is

$$G(x,t;\xi,\tau)=\frac{1}{\sqrt{4\pi D(t-\tau)}}\exp\left\{-\frac{|x-\xi|^2}{4D(t-\tau)}\right\}\text{ for } t>\tau,$$

$$=0\text{ for } t<\tau.$$

For the semi-infinite domain we add a negative image at $x=-\xi$. So the required Green's function satisfying zero boundary condition at x=0 is

$$G(x,t;\xi,\tau)=\frac{1}{\sqrt{4\pi D(t-\tau)}}\left[\exp\left\{-\frac{|x-\xi|^2}{4D(t-\tau)}\right\}-\exp\left\{-\frac{|x+\xi|^2}{4D(t-\tau)}\right\}\right]\text{ for } t>\tau.$$

(b) Using the Green's function method, solve the following problem:

$$\left(\frac{\partial}{\partial t}-D\frac{\partial^2}{\partial x^2}\right)u=\delta(x-10),\quad 0<x<\infty,\quad t>0.$$

subject to zero initial condition: $u=0$ at $t=0$.
The boundary condition is: $u=0$ at $x=0$ and $x\to\infty$. If you have integrals, please be sure to specify the upper and lower limits.

The solution to the inhomogeneous problem is given by:

$$u(x,t)=\int_0^t d\tau\int_0^{\infty}\delta(\xi-10)G(x,t;\xi,\tau)d\xi$$

$$u(x,t)=\int_0^t d\tau \frac{1}{\sqrt{4\pi D(t-\tau)}}[\exp\{-\frac{|x-10|^2}{4D(t-\tau)}\}-\exp\{-\frac{|x+10|^2}{4D(t-\tau)}\}]$$

4. Solve the initial value problem:

PDE: $\frac{\partial^2}{\partial t^2}u-c^2\frac{\partial^2}{\partial x^2}u=\delta(x)e^{i\omega_0 t}, \quad -\infty<x<\infty,\ t>0$

BC: $u(x,t)\to 0$ as $x\to\pm\infty,\ t>0$

IC: $u(x,0)=0,\ -\infty<x<\infty,$

$\frac{\partial}{\partial t}u(x,t)|_{t=0}=0, \quad -\infty<x<\infty$

Same as HW3, problem 3.