Homework 3.6 (Extra Credit)
Due: April 28, 2021

Birth, Death, and Immigration
Consider a birth and death process with constant immigration,

$$\begin{gathered}P[\Delta N = +1 \mid N(t) = n] = \beta n \Delta t + I\Delta t + o(\Delta t)\\ P[\Delta N = -1 \mid N(t) = n] = \mu n \Delta t + o(\Delta t)\\ P[\Delta N = 0 \mid N(t) = n] = 1 - [(\beta + \mu)n + I]\Delta t + o(\Delta t)\end{gathered}$$

Assume that the mortality rate is greater than the birth rate, $\mu > \beta$.
(a) Derive differential equations for the probabilities.
(b) Derive a partial differential equation for the probability generating function.
(c) Find the probability generating function for the equilibrium probability distribution. To wit, set the time derivative in your partial differential equation equal to zero. Solve the corresponding ordinary differential equation. Choose the generating function so that the corresponding probabilities sum to one.
(d) What is the expected population size at equilibrium?
(e) What is the variance of the equilibrium population sizes?
(f) What is the probability that the population is of size zero at equilibrium?

Solution.

(a) The probabilites follow that

$$p_n(t + \Delta t) = p_{n+1}(t)\mu(n + 1)\Delta t + p_{n-1}(t)(\beta(n - 1) + I)\Delta t + p_n(t)(1 - ((\beta + \mu)n + I)\Delta t) + o(\Delta t)$$

Rearranging and taking $\Delta t \to 0$

$$\frac{dp_n}{dt} = p_{n+1}\mu(n + 1) + p_{n-1}(\beta(n - 1) + I) - p_n((\beta + \mu)n + I)$$

The probability generating function is

$$F(t, x) = \sum_{n=0}^{\infty} p_n x^n$$

Taking a derivative in t

$$\begin{aligned}\frac{dF}{dt} &= \sum_{n=0}^{\infty} \frac{dp_n}{dt} x^n\\ &= \sum_{n=0}^{\infty} p_{n+1}\mu(n + 1)x^n + p_{n-1}(\beta(n - 1) + I)x^n - p_n((\beta + \mu)n + I)x^n\end{aligned}$$

$$
\begin{aligned}
&= \mu \sum_{n=0}^{\infty}(n+1)p_{n+1}x^n + \beta x^2 \sum_{n=0}^{\infty}(n-1)p_{n-1}x^{n-2} + Ix\sum_{n=0}^{\infty}p_{n-1}x^{n-1} \\
&- (\beta+\mu)x\sum_{n=0}^{\infty}np_nx^{n-1} - I\sum_{n=0}^{\infty}np_nx^n \\
&= I(x-1)F + (\mu + \beta x^2 - (\beta+\mu)x)\frac{dF}{dx}
\end{aligned}
$$

Thus the PDE is

$$\frac{dF}{dt} = I(x-1)F + (\beta x - \mu)(x-1)\frac{dF}{dx}$$

(c) Setting $\frac{dF}{dt} = 0$

$$I(x-1)F + (\beta x - \mu)(x-1)\frac{dF}{dx} = 0$$

We try separation of variables

$$
\begin{aligned}
\frac{dF}{F} &= I\frac{1-x}{(\beta x-\mu)(x-1)}dx \\
\ln F &= -\frac{I}{\beta}\ln(\beta x - \mu) + C \\
F &= A(\beta x - \mu)^{-\frac{I}{\beta}}
\end{aligned}
$$

Choosing A such that the probabilities sum to 1

$$\sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty}\frac{F^{(n)}(0)}{n!} = 1$$

Evaluating this in Mathematica gives

$$
\begin{aligned}
A(\mu-\beta)^{-\frac{I}{\beta}} = 1 \\
A = (\mu-\beta)^{\frac{I}{\beta}}
\end{aligned}
$$

Thus $F = (\mu-\beta)^{\frac{I}{\beta}}(\beta x - \mu)^{-\frac{I}{\beta}}$.

(d) $E(N) = F'(1) = (\mu-\beta)^{\frac{I}{\beta}}\beta(\beta x-\mu)^{-\frac{I}{\beta}-1}|_{x=1} = \frac{\beta(-1)^{\frac{I}{\beta}}}{\beta-\mu}$

(e) $Var(N) = F'' + F' - F'^2|_{x=1} = \frac{\beta^2(-1)^{\frac{I}{\beta}}}{(\beta-\mu)^2} + \frac{\beta(-1)^{\frac{I}{\beta}}}{\beta-\mu} - (\frac{\beta(-1)^{\frac{I}{\beta}}}{\beta-\mu})^2 = \frac{\beta(-1)^{\frac{I}{\beta}}\left(\beta\left((-1)^{\frac{I}{\beta}}+2\right)-\mu\right)}{(\beta-\mu)^2}$

(f) $p_0 = F(0) = (\frac{\beta-\mu}{\mu})^{\frac{I}{\beta}}$