Homework 7

Due: Wednesday, November 25, 2020

Question 1.

(a) Let $\hat{f}(s)$ and $\hat{g}(s)$ be the Laplace transforms of one-sided functions $f(t)$ and $g(t)$, respectively. Show that the inverse Laplace transform of $\hat{f}(s)\hat{g}(s)$ is;

$$\int_0^t f(t-\tau)g(\tau)d\tau$$

(b) Use Laplace transform and the result in (a) to solve the following ordinary differential equation: $\frac{d^2}{dt^2}y + 4y = f(t)$, subject to the initial conditions: $y(0) = 0$, $\frac{dy}{dt}(0) = 0$

(a) The inverse Laplace transform is

$$\begin{aligned}
\frac{1}{2\pi i}\int_L e^{st}\hat{f}(s)\hat{g}(s)ds &= \frac{1}{2\pi i}\int_L e^{st}\hat{f}(s)\int_0^\infty e^{-s\tau}g(\tau)d\tau ds \\
&= \frac{1}{2\pi i}\int_0^\infty g(\tau)\int_L e^{s(t-\tau)}\hat{f}(s)dsd\tau \\
&= \int_0^\infty f(t-\tau)g(\tau)d\tau \\
&= \int_0^t f(t-\tau)g(\tau)d\tau
\end{aligned}$$

since $f(t-\tau) = 0$ for $t < \tau$ and $g(\tau) = 0$ for $\tau < 0$.

(b) Taking Laplace transform on both sides

$$\begin{aligned}
s^2\hat{y} - sy(0) + y'(0) + 4\hat{y} &= \hat{f} \\
(s^2+4)\hat{y} &= \hat{f} \\
\hat{y} &= \frac{\hat{f}}{s^2+4} \\
\hat{y} &= \frac{1}{2}\frac{2\hat{f}}{s^2+4} \\
y &= \frac{1}{2}\int_0^t f(t-\tau)\sin(2\tau)d\tau
\end{aligned}$$

Question 2.
Solve the following Laplace equation

$$\frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi = 0$$

in the upper half plane: $-\infty < x < \infty, 0 < y < \infty$, subject to the boundary conditions:

$$\begin{aligned}&\phi \to 0 \text{ as } y \to \infty; \phi \to 0 \text{ as } x \to \pm\infty \\ &\phi(x,0) = \tfrac{x}{x^2+a^2}\end{aligned}$$

Let Φ be the Fourier transform of ϕ with respect to x

$$\Phi(\lambda, y) = \int_{-\infty}^{\infty} e^{i\lambda x}\phi(x,y)dx$$

Taking Fourier transform on both sides

$$\begin{aligned}(-i\lambda)^2\Phi + \frac{\partial^2}{\partial y^2}\Phi &= 0 \\ \frac{\partial^2}{\partial y^2}\Phi &= \lambda^2\Phi \\ \Phi &= A(\lambda)e^{\lambda y} + B(\lambda)e^{-\lambda y}\end{aligned}$$

Consider $\lambda > 0$. Since $\phi(x,\infty) = 0$, $\Phi(x,\infty) = 0$. Then $A(\lambda) = 0$. Let $G(\lambda) = \Phi(x,0) = \mathcal{F}(\phi(x,0))$, then $B(\lambda) = G(\lambda)$. Let $f(z) = \frac{z}{z^2+a^2}$. Since $|f(z)| \to 0$ as $|z| \to \infty$, by Jordan's lemma we can complete the contour in UHP. Then we can compute $G(\lambda)$ using residue calculus (without loss of generality consider $a^2 = |a|^2$).

$$\begin{aligned}G(\lambda) = \int_{-\infty}^{\infty} \frac{e^{i\lambda x}x}{x^2+|a|^2}dx &= \oint e^{i\lambda z}f(z)dz \\ &= 2\pi i \text{Res}\left\{\frac{e^{i\lambda z}z}{z^2+|a|^2}; z = |a|i\right\} \\ &= 2\pi i\left(\frac{e^{i\lambda z}z}{2z}\right)_{|a|i} \\ &= 2\pi i\frac{e^{-|a|\lambda}}{2} \\ &= \pi i e^{-|a|\lambda}\end{aligned}$$

For $\lambda < 0$, we can complete the contour in LHP. Then $B(\lambda) = 0$ and $A(\lambda) = G(\lambda) = -\pi i e^{|a|\lambda}$. For $\lambda = 0$, $\Phi = A(\lambda) + B(\lambda)$, then ϕ does not depend on y, which is the not the solution we want. The inverse transform is given by

$$\begin{aligned}
\phi &= \frac{1}{2\pi}\int_0^\infty e^{-i\lambda x}\pi i e^{-|a|\lambda}e^{-\lambda y}d\lambda - \frac{1}{2\pi}\int_{-\infty}^0 e^{-i\lambda x}\pi i e^{|a|\lambda}e^{\lambda y}d\lambda \\
&= \frac{i}{2}\left(\int_0^\infty e^{-\lambda(ix+y+|a|)}d\lambda - \int_{-\infty}^0 e^{-\lambda(ix-y-|a|)}d\lambda\right) \\
&= \frac{i}{2}\left[\left(\frac{e^{-\lambda(ix+y+|a|)}}{-(ix+y+|a|)}\right)_0^\infty - \left(\frac{e^{-\lambda(ix-y-|a|)}}{-(ix-y-|a|)}\right)_{-\infty}^0\right] \\
&= \frac{i}{2}\left[\frac{1}{ix+y+|a|} + \frac{1}{ix-y-|a|}\right] \\
&= \frac{i}{2}\frac{ix+y-|a|+ix-y+|a|}{(ix+y+|a|)(ix-y-|a|)} \\
&= \frac{-x}{-x^2-(y+|a|)^2} \\
&= \frac{x}{x^2+(y+|a|)^2}
\end{aligned}$$