Homework 6
Due: Friday, November 20, 2020

Question 1. Let $X \sim Binomial(n, U)$, where $U \sim Uniform((0,1))$. What is the probability generating function $G_X(s)$ of X? What is $P(X = k)$ for $k \in \{0, 1, 2, ..., n\}$?

$G_X(s) = \int_0^1 \sum_{k=0}^n \binom{n}{k} U^k (1-U)^{n-k} s^k dU = \int_0^1 (Us + (1-U))^n dU$
Let $t = Us + 1 - U$, then $G_X(s) = \int_s^1 \frac{t^n}{1-s} dt = \frac{1-s^{n+1}}{(n+1)(1-s)} = \frac{1}{n+1} \sum_{k=0}^n s^k$
$P(X = k) = \frac{G_X^{(k)}(0)}{k!} = \frac{k!}{(n+1)k!} = \frac{1}{n+1}$

Question 2. Consider a branching process with immigration

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1},$$

where the (ξ_i^{n+1}) are iid with common distribution ξ, the (Y_n) are iid with common distribution Y and the (ξ_i^{n+1}) and (Y_{n+1}) are independent. What is $G_{Z_{n+1}}(s)$ in terms of $G_{Z_n}(s)$, $G_\xi(s)$ and $G_Y(s)$? Write $G_{Z_2}(s)$ explicitly in terms of $G_\xi(s)$ and $G_Y(s)$.

$G_{Z_{n+1}}(s) = Es^{Z_{n+1}} = Es^{\sum_{i=1}^{Z_n} \xi_i^{n+1} + Y_{n+1}} = Es^{Y_{n+1}} Es^{\sum_{i=1}^{Z_n} \xi_i^{n+1}} = G_Y(s) G_{Z_n}(G_\xi(s))$
$G_{Z_2}(s) = G_Y(s) G_{Z_1}(G_\xi(s)) = G_Y(s) E((G_\xi(s)^{Z_1})) = G_Y(s) E((G_\xi(s)^{\xi+Y}) = G_Y(s)(G_\xi(G_\xi(s)) + G_Y(G_\xi(s)))$

Question 3. (a) Let X be exponentially distributed with parameter λ. Show by elementary integration (not complex integration) that $E(e^{itX}) = \lambda/(\lambda - it)$. (b) Find the characteristic function of the density function $f(x) = \frac{1}{2} e^{-|x|}$ for $x \in \mathbb{R}$.

(a) $Ee^{itX} = E(\cos(tX)) + iE(\sin(tX)) = \int_0^\infty \cos(tx) \lambda e^{-\lambda x} dx + i \int_0^\infty \sin(tx) \lambda e^{-\lambda x} dx$
$= \frac{\lambda^2}{\lambda^2+t^2} + \frac{i\lambda t}{\lambda^2+t^2} = \frac{\lambda(\lambda+it)}{(\lambda+it)(\lambda-it)} = \frac{\lambda}{\lambda-it}$, where

$$\begin{aligned}
I = \int_0^\infty \cos(tx) e^{-\lambda x} dx &= \left[\frac{1}{t} \sin(tx) e^{-\lambda x}\right]_0^\infty + \frac{\lambda}{t} \int_0^\infty \sin(tx) e^{-\lambda x} dx \\
&= \frac{\lambda}{t} \left(\left[\frac{-1}{t} e^{-\lambda x} \cos(tx)\right]_0^\infty - \frac{\lambda}{t} I \right) \\
(1 + \frac{\lambda^2}{t^2}) I &= \frac{\lambda}{t^2} \\
I &= \frac{\lambda}{\lambda^2 + t^2}
\end{aligned}$$

$$
\begin{aligned}
J = \int_0^\infty \sin(tx)e^{-\lambda x}dx &= \left[-\frac{1}{\lambda}\sin(tx)e^{-\lambda x}\right]_0^\infty + \frac{t}{\lambda}\int_0^\infty \cos(tx)e^{-\lambda x}dx \\
&= \frac{t}{\lambda}\left(\left[\frac{1}{\lambda}e^{-\lambda x}\cos(tx)\right]_0^\infty - \frac{t}{\lambda}J\right) \\
(1+\frac{t^2}{\lambda^2})J &= \frac{t}{\lambda^2} \\
J &= \frac{t}{\lambda^2+t^2}
\end{aligned}
$$

(b) $\phi(t) = \frac{1}{2}\int_{-\infty}^\infty e^{itx}e^{-|x|}dx = \frac{1}{2}\int_0^\infty e^{itx}e^{-x}dx + \frac{1}{2}\int_{-\infty}^0 e^{itx}e^{x}dx = \frac{1}{2(1-it)} - \frac{1}{2(1+it)} = \frac{it}{1+t^2}$

Question 4. A coin is tossed repeatedly, with heads turning up with probability p on each toss. Let N be the minimum number of tosses required to obtain k heads. Show that, as $p \to 0$, the distribution function of $2Np$ converges to that of a gamma distribution. Note that, if $X \sim \Gamma(\lambda, r)$ then

$$f_X(x) = \frac{1}{\Gamma(r)}\lambda^r x^{r-1}e^{-\lambda x}\,1_{x\geq 0}.$$

Let n denote the number of tails in N tosses, then

$$
\begin{aligned}
G_N(s) &= \sum_{n=0}^\infty P(N=k+n)s^{n+k} \\
&= \sum_{n=0}^\infty \binom{k+n-1}{n}p^k(1-p)^n s^{n+k} \\
&= p^k s^k(1-(1-p)s)^{-k}
\end{aligned}
$$

where we have used $\sum_{n=0}^\infty \binom{k+n-1}{n}x^n = (1-x)^{-k}$, proved as follows

$$
\begin{aligned}
(1-x)^{-k} &= \sum_{n=0}^\infty \binom{-k}{n}(-1)^n x^n \\
&= \sum_{n=0}^\infty \frac{(-k)(-k-1)\cdots(-k-n+1)}{n!}(-1)^n x^n \\
&= \sum_{n=0}^\infty (-1)^n\frac{k(k+1)\cdots(k+n-1)}{n!}(-1)^n x^n \\
&= \sum_{n=0}^\infty \binom{k+n-1}{n}x^n
\end{aligned}
$$

Let $Y = 2Np$, then

$$\phi_Y(t) = \phi_N(2pt) = G_N(e^{2ipt}) = p^k e^{2ipkt}(1-(1-p)e^{2ipkt})^{-k} = \left(\frac{p}{p-1+e^{-2ipt}}\right)^k$$

Taking $p \to 0$ and using L'Hospital's rule, we have

$$\lim_{p\to 0}\left(\frac{p}{p-1+e^{-2ipt}}\right)^k = \lim_{p\to 0}\left(\frac{1}{1-2ite^{-2ipt}}\right)^k = (1-2it)^{-k}$$

This is the characteristic function of a Gamma distribution with $\lambda = \frac{1}{2}$ and $r = k$, as explained here.