Homework 3
Due: Wednesday, October 28, 2020

Question 1. (AF 4.2.1 c, d)
(c) Evaluate

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)(x^2+b^2)}, \quad a^2, b^2 > 0$$

Since the integrand is even, we can consider the new integral

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+a^2)(x^2+b^2)}$$

Consider $|z(f(z)|$, where $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$

$$|zf(z)| = \left|\frac{z}{(z^2+a^2)(z^2+b^2)}\right|$$

Clearly $|z(f(z)| \to 0$ as $|z| \to \infty$, thus we can complete the contour in the upper half complex plane

$$I = \frac{1}{2}\oint_C f(z)\mathrm{d}z = \pi i \sum_j \mathrm{Res}\{f(z); z_j\}$$

where C is a closed upper semicircle. Note $f(z)$ has simple poles at $z = \pm|a|i$ and $z = \pm|b|i$, but the upper semicircle only encloses $|a|i$ and $|b|i$. If $a \neq b$,

$$\begin{aligned}
I &= \pi i \left(\mathrm{Res}\{f(z); |a|i\} + \mathrm{Res}\{f(z); |b|i\}\right) \\
&= \pi i \left(\left[\frac{1}{4z^3+2(a^2+b^2)z}\right]_{|a|i} + \left[\frac{1}{4z^3+2(a^2+b^2)z}\right]_{|b|i}\right) \\
&= \pi i \left(\frac{1}{4(|a|i)^3+2(a^2+b^2)|a|i} + \frac{1}{4(|b|i)^3+2(a^2+b^2)|b|i}\right) \\
&= \pi i \left(\frac{1}{2|a|i(b^2-a^2)} + \frac{1}{2|b|i(a^2-b^2)}\right) \\
&= \pi i \left(\frac{|b|-|a|}{2|ab|i(b^2-a^2)}\right) \\
&= \frac{\pi}{2|ab|(|a|+|b|)}
\end{aligned}$$

If $a = b$, then $f(z) = \frac{1}{(z^2+a^2)^2}$ has double poles at $\pm|a|i$. Since $|z(f(z)| \to 0$ as $|z| \to \infty$, thus we can complete the contour in the upper half complex plane and note that only the pole at

$|a|i$ is enclosed by the contour

$$
\begin{aligned}
I = \frac{1}{2}\oint_C f(z)\mathrm{d}z &= \pi i \mathrm{Res}\{f(z); |a|i\} \\
&= \pi i \lim_{z\to|a|i} \frac{\mathrm{d}}{\mathrm{d}z}[f(z)(z-|a|i)^2] \\
&= \pi i \lim_{z\to|a|i} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{(z-|a|i)^2}{(z^2+a^2)^2}\right] \\
&= \pi i \lim_{z\to|a|i} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{(z-|a|i)^2}{((z+|a|i)(z-|ai|))^2}\right] \\
&= \pi i \lim_{z\to|a|i} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{(z+|a|i)^2}\right] \\
&= \pi i \lim_{z\to|a|i} \left[\frac{-2}{(z+|a|i)^3}\right] \\
&= \pi i \left[\frac{-2}{(2|a|i)^3}\right] \\
&= \frac{\pi}{4|a|^3}
\end{aligned}
$$

(d) Evaluate

$$\int_0^\infty \frac{dx}{x^6+1}$$

Let $f(z) = \frac{1}{z^6+1}$. Since $|z(f(z)| \to 0$ as $|z| \to \infty$, we can complete the contour in the upper half complex plane. Since the integrand is also even, we get

$$I = \frac{1}{2}\oint_C f(z)\mathrm{d}z = \pi i \sum_j \mathrm{Res}\{f(z); z_j \text{ in the upper half plane}\}$$

where $f(z)$ has simple poles located at $z_k = e^{i(\pi+2k\pi)/6}$ for $k = 0, 1, 2$. Then

$$\begin{aligned}
I &= \pi i \sum_{k=0}^{2} \mathrm{Res}\left\{\frac{1}{z^6+1}; z_k\right\} \\
&= \pi i \sum_{k=0}^{2} \left(\frac{1}{6z^5}\right)_{z_k} \\
&= \pi i \sum_{k=0}^{2} \frac{1}{6z_k^5} \\
&= \frac{\pi i}{6} e^{-5\pi i/6}(1 + e^{-5\pi i/3} + e^{-10\pi i/3}) \\
&= \frac{\pi i}{6}(-\frac{\sqrt{3}}{2} - \frac{i}{2})(1 + \frac{\sqrt{3}}{2}i + \frac{1}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2}) \\
&= \frac{-\pi i}{12}(\sqrt{3}+i)(\sqrt{3}i+1) \\
&= \frac{\pi}{3}
\end{aligned}$$

Question 2. (AF 4.2.2 a, b, h) Evaluate the following integrals:
(a)

$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+a^2}\mathrm{d}x; \quad a^2>0$$

We have

$$I=\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2+a^2}\mathrm{d}x = \mathrm{Im}\int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2+a^2}\mathrm{d}z$$

We want to complete the contour, thus we need to show that the integral over the upper semicircle C_{R+} is 0 as $R\to\infty$. Let $f(z)=\frac{z}{z^2+a^2}$ and parameterize C_{R+} by $z=Re^{i\theta}$, then $f(z)=\frac{Re^{i\theta}}{R^2e^{2i\theta}+a^2}$, thus

$$\begin{aligned}
|f(z)| &= \frac{R}{|R^2(\cos 2\theta + i\sin 2\theta)+a^2|}\\
&= \frac{R}{\sqrt{(R^2\cos 2\theta + a^2)^2 + (R^2\sin 2\theta)^2}}\\
&= \frac{R}{\sqrt{R^4\cos^2 2\theta + a^4 + 2a^2R^2\cos 2\theta + R^4\sin^2 2\theta}}\\
&= \frac{R}{\sqrt{R^4+a^4+2a^2R^2\cos 2\theta}}\\
&\le \frac{R}{\sqrt{R^4+a^4-2a^2R^2}} = \frac{R}{|R^2-a^2|}
\end{aligned}$$

Since $\frac{R}{|R^2-a^2|}\to 0$ as $R\to\infty$, $|f(z)|\to 0$ as well. By Jordan's lemma, we have $\int_{C_{R+}} f(z)e^{iz}\mathrm{d}z = 0$. Thus we can complete the contour in the upper half plane. Note $f(z)$ has singularites at $z=\pm|a|i$, but only $|a|i$ is enclosed by the contour

$$\begin{aligned}
I &= \mathrm{Im}\oint_C \frac{ze^{iz}}{z^2+a^2}\mathrm{d}z\\
&= \mathrm{Im}\left(2\pi i \mathrm{Res}\left\{\frac{ze^{iz}}{z^2+a^2}; |a|i\right\}\right)\\
&= \mathrm{Im}\left(2\pi i \left(\frac{ze^{iz}}{2z}\right)_{|a|i}\right)\\
&= \mathrm{Im}\left(2\pi i \frac{|a|ie^{-|a|}}{2|a|i}\right)\\
&= \pi e^{-|a|}
\end{aligned}$$

(b)

$$\int_{-\infty}^{\infty} \frac{\cos(kx)dx}{(x^2+a^2)(x^2+b^2)}; \quad a^2, b^2, k > 0$$

We use a similar method as in **Question 2(a)**. Then the integral becomes

$$I = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)} \mathrm{d}z$$

Let $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$ and parameterize C_{R+} by $z = Re^{i\theta}$, then

$$|f(z)| = \frac{1}{|R^2e^{2i\theta}+a^2||R^2e^{2i\theta}+b^2|}$$

We have shown in **Question 2(a)** that $\frac{R}{|R^2e^{2i\theta}+a^2|} \to 0$ as $R \to \infty$. Then it must be true that $\frac{1}{R}\frac{R}{|R^2e^{2i\theta}+a^2|} = \frac{1}{|R^2e^{2i\theta}+a^2|} \to 0$ as $R \to \infty$. Replace a with b, we know that this is true for $\frac{1}{|R^2e^{2i\theta}+b^2|}$ as well. Thus we can conclude $|f(z)| \to 0$ as $R \to \infty$. Since $k > 0$, by Jordan's lemma, we can complete the contour in the upper half complex plane

$$\begin{aligned} I &= \mathrm{Re} \oint_C \frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)} \mathrm{d}z \\ &= \mathrm{Re} \left(2\pi i \sum_j \mathrm{Res} \left\{ \frac{e^{ikz}}{(z^2+a^2)(z^2+b^2)}; z_j \right\} \right) \end{aligned}$$

This looks very similar to the residue we computed in **Question 1(c)** except the numerator is now e^{ikz}, thus for $a \neq b$

$$\begin{aligned} I &= \mathrm{Re} \left(2\pi i \left(\frac{e^{-|a|}}{2|a|i(b^2-a^2)} + \frac{e^{-|b|}}{2|b|i(a^2-b^2)} \right) \right) \\ &= \mathrm{Re} \left(\frac{2\pi i(|b|e^{-|a|} - |a|e^{-|b|})}{2|ab|i(b^2-a^2)} \right) \\ &= \frac{\pi(|b|e^{-|a|} - |a|e^{-|b|})}{|ab|(b^2-a^2)} \end{aligned}$$

For $a = b$, we consider the double pole at $|a|i$, then

$$\begin{aligned}
I &= \mathrm{Re}\left(2\pi i \mathrm{Res}\left\{\frac{e^{ikz}}{(z^2+a^2)^2}; |a|i\right\}\right) \\
&= \mathrm{Re}\left(2\pi i \lim_{z\to|a|i} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{e^{ikz}(z-|a|i)^2}{(z^2+a^2)^2}\right]\right) \\
&= \mathrm{Re}\left(2\pi i \lim_{z\to|a|i} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{e^{ikz}}{(z+|a|i)^2}\right]\right) \\
&= \mathrm{Re}\left(2\pi i \lim_{z\to|a|i} i\frac{(z+|a|i)^2 ike^{ikz} - 2(z+|a|i)e^{ikz}}{(z+|a|i)^4}\right) \\
&= \mathrm{Re}\left(2\pi i \lim_{z\to|a|i} i\frac{(z+|a|i)ike^{ikz} - 2e^{ikz}}{(z+|a|i)^3}\right) \\
&= \mathrm{Re}\left(2\pi i\frac{(2|a|i)ike^{ik|a|i} - 2e^{ik|a|i}}{(2|a|i)^3}\right) \\
&= \mathrm{Re}\left(2\pi i\frac{e^{-k|a|}(-2|a|k-2)}{-8|a|^3 i}\right) \\
&= \frac{\pi e^{-k|a|}(|a|k+1)}{2|a|^3}
\end{aligned}$$

(h)

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{(5-3\sin\theta)^2}$$

We can convert this integral into a contour integral around a unit circle C. Parameterize C by $z=e^{i\theta}$, then $d\theta = dz/iz$. Note $\sin\theta = \frac{1}{2i}(z-\frac{1}{z})$

$$\begin{aligned}
I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{(5-3\sin\theta)^2} &= \oint_C \frac{\mathrm{d}z/iz}{(5-3\frac{1}{2i}(z-\frac{1}{z}))^2} \\
&= \oint_C \frac{\mathrm{d}z/iz}{(5-\frac{3}{2i}(z-\frac{1}{z}))^2} \\
&= \oint_C \frac{-iz\mathrm{d}z}{(5z-\frac{3}{2i}(z^2-1))^2} \\
&= \oint_C \frac{-iz\mathrm{d}z}{25z^2-\frac{9}{4}(z^2-1)^2+15iz(z^2-1)} \\
&= \oint_C \frac{iz\mathrm{d}z}{\frac{9}{4}[(z^2-1)^2-\frac{20}{3}iz(z^2-1)-\frac{100}{9}z^2]} \\
&= \oint_C \frac{iz\mathrm{d}z}{\frac{9}{4}(z^2-1-\frac{10}{3}iz)^2} \\
&= \oint_C \frac{4iz\mathrm{d}z}{9(z-3i)^2(z-\frac{i}{3})^2}
\end{aligned}$$

Thus the function has double poles at $z=3i$ and $z=\frac{i}{3}$, but only $\frac{i}{3}$ is enclosed by the contour

$$\begin{aligned}
I &= 2\pi i \mathrm{Res}\left\{\frac{4iz}{9(z-3i)^2(z-\frac{i}{3})^2}; \frac{i}{3}\right\} \\
&= 2\pi i \lim_{z\to\frac{i}{3}} \frac{\mathrm{d}}{\mathrm{dz}}\left[\frac{4iz}{9(z-3i)^2}\right] \\
&= -\frac{8}{9}\pi \lim_{z\to\frac{i}{3}} \frac{\mathrm{d}}{\mathrm{dz}}\left[\frac{z}{(z-3i)^2}\right] \\
&= -\frac{8}{9}\pi \lim_{z\to\frac{i}{3}} \left[\frac{(z-3i)^2-2z(z-3i)}{(z-3i)^4}\right] \\
&= -\frac{8}{9}\pi \lim_{z\to\frac{i}{3}} \left[\frac{z-3i-2z}{(z-3i)^3}\right] \\
&= \frac{5\pi}{32}
\end{aligned}$$

Question 3. (AF 4.2.7) Use a sector contour with radius R, as in Figure 4.2.6 , centered at the origin with angle $0 \leq \theta \leq \frac{2\pi}{5}$ to flind, for $a > 0$,

$$\int_0^\infty \frac{\mathrm{d}x}{x^5 + a^5} = \frac{\pi}{5a^4 \sin(\pi/5)}$$

We note that $(xe^{2\pi i/5})^5 = x^5$, thus we can use a sector contour $C_R : Re^{i\theta}$ where $\theta \in [0, \frac{2\pi}{5}]$. We have

$$\begin{aligned} I_1 = \oint_C \frac{\mathrm{d}z}{z^5 + a^5} &= (\int_{C_L} + \int_{C_x} + \int_{C_R}) \frac{\mathrm{d}z}{z^5 + a^5} \\ &= 2\pi i \sum_j \operatorname{Res}\left\{ \frac{1}{z^5 + a^5}; z_j \right\} \end{aligned}$$

The only pole enclosed by this sector is $z_0 = ae^{i\pi/5}$, then

$$I_1 = 2\pi i \operatorname{Res}\left\{ \frac{1}{z^5 + a^5}; z_0 \right\} = 2\pi i \left(\frac{1}{5z^4} \right)_{z_0} = \frac{2\pi i e^{-4\pi i/5}}{5a^4}$$

By **Theorem 4.2.1** in AF, the integral over C_R tends to 0. For the integral over C_L, we parameterize C_L by $z = e^{2\pi i/5} r$ and get

$$\int_{C_L} \frac{\mathrm{d}z}{z^5 + a^5} = \int_R^0 \frac{e^{2\pi i/5} \mathrm{d}r}{r^5 + a^5} = -Ie^{2\pi i/5}$$

where $I = \int_0^R \frac{\mathrm{d}r}{r^5 + a^5}$. Since the integral over C_x is just I, we have

$$\begin{aligned} I(1 - e^{2\pi i/5}) &= \frac{2\pi i e^{-4\pi i/5}}{5a^4} \\ I &= \frac{2\pi i e^{-4\pi i/5}}{5a^4(1 - e^{2\pi i/5})} \\ &= \frac{\pi}{5a^4} \frac{2ie^{-\pi i/5}}{e - e^{2\pi i/5}} \\ &= \frac{\pi}{5a^4} \frac{2ie^{-\pi i}}{e^{-\pi i/5} - e^{\pi i/5}} \\ &= \frac{\pi}{5a^4 \sin(\pi/5)} \end{aligned}$$

Question 4. A function that is analytic for all $z \in \mathbb{C}$ is called entire. (a) Show that any bounded entire function is necessarily constant. (b) Suppose $f(z)$ is an entire function, not necessarily bounded, but such that $\text{Im}(f(z)) \leq 0$. Show that $f(z)$ is necessarily constant.

(a) Let $f(z)$ be entire and bounded. Since $f(z)$ is bounded, we have

$$|f(z)| \leq M$$

where M is a constant. Cauchy's integral formula states that

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_{C_R} \frac{f(z)dz}{(z-z_0)^{n+1}}$$

where the radius of the circle is $R = |z - z_0|$. Taking the absolute value on both sides

$$\begin{aligned} |f^n(z_0)| &= \left| \frac{n!}{2\pi i} \oint_{C_R} \frac{f(z)dz}{(z-z_0)^{n+1}} \right| \\ &= \frac{n!}{2\pi} \frac{2\pi R |f(z)|}{R^{n+1}} \\ &\leq \frac{n! M}{R^n} \end{aligned}$$

Then $|f'(z_0)| \leq \frac{M}{R}$. Since $f(z)$ is entire, we let $R \to \infty$, then $|f'(z_0)| = 0$. Thus $f(z)$ is a constant.

(b) Let $g(z) = e^{-if(z)}$. Since $f(z)$ is entire, $g(z)$ is also entire. We have

$$\begin{aligned} |g(z)| &= |e^{-if(z)}| \\ &= \left| e^{-i\text{Re}f(z) + \text{Im}f(z)} \right| \\ &= e^{\text{Im}f(z)} \\ &\leq 1 \end{aligned}$$

Since $g(z)$ is entire and bounded, it is a constant and $f(z)$ has to be a constant too.