Homework 1
Due: January 20, 2021

Determine the eigenvalues and eigenvectors (real solutions), sketch the behavior and classify the behavior.

1. $\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$
2. $\vec{x}' = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix} \vec{x}$
3. $\vec{x}' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \vec{x}$
4. $\vec{x}' = \begin{pmatrix} 2 & -5/2 \\ 9/5 & -1 \end{pmatrix} \vec{x}$
5. $\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x}$
6. $\vec{x}' = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \vec{x}$
7. $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}$

8. Consider $x' = -(x-y)(1-x-y)$ and $y' = x(2+y)$ and plot the solutions. Verify your qualitative dynamics with MATLAB/Python/fortran.
9. Consider $x' = x - y^2$ and $y' = y - x^2$ and plot the solutions. Verify your qualitative dynamics with MATLAB/Python/fortran.
10. Consider $x' = (2+x)(y-x)$ and $y' = (4-x)(y+x)$ and plot the solutions. Verify your qualitative dynamics with MATLAB/Python/fortran.

Solution 1. Consider $\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda)+5 = 0$. Then $\lambda = \pm i$. For $\lambda = i$, $\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$. For $\lambda = -i$, $\begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix}\mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} 5 \\ 2+i \end{pmatrix}$. Since λ are purely imaginary, the solution is neutrally stable.

Solution 2. Consider $\begin{vmatrix} -1-\lambda & -1 \\ 0 & -0.25-\lambda \end{vmatrix} = (-1-\lambda)(-0.25-\lambda) = 0$. Then $\lambda = -1, -0.25$. For $\lambda = -1$, $\begin{pmatrix} 0 & -1 \\ 0 & 0.75 \end{pmatrix}\mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. For $\lambda = -0.25$, $\begin{pmatrix} -0.75 & -1 \\ 0 & 0 \end{pmatrix}\mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$. Since λ are real, unequal and negative, the solution is a stable node.

Solution 3. Consider $\begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda)+4 = 0$. Then $\lambda = 1$. $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Since $\lambda = 1$ is a double root and there is only one eigenvector, we must find a second generalized eigenvector and the solution is an unstable improper node.

Solution 4. Consider $\begin{vmatrix} 2-\lambda & -5/2 \\ 9/5 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) + \frac{2}{9} = 0$. Then $\lambda = \frac{1}{2} \pm \frac{3}{2}i$. For $\lambda = \frac{1}{2} + \frac{3}{2}i$, $\begin{pmatrix} \frac{3}{2}(i+1) & -5/2 \\ 9/5 & \frac{3}{2}(i-1) \end{pmatrix} \mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} 5 \\ 3(i+1) \end{pmatrix}$. For $\lambda = \frac{1}{2} - \frac{3}{2}i$, $\begin{pmatrix} \frac{3}{2}(1-i) & -5/2 \\ 9/5 & \frac{3}{2}(-1-i) \end{pmatrix} \mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} 5 \\ 3(1-i) \end{pmatrix}$. Since λ are complex and $\mathrm{Re}\lambda > 0$, the solution is an unstable spiral.

Solution 5. Consider $\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 3 = 0$. Then $\lambda = \pm 1$. For $\lambda = 1$, $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda = -1$, $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Since λ are real and opposite sign, the solution is a saddle.

Solution 6. Consider $\begin{vmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 5 = 0$. Then $\lambda = \pm 2$. For $\lambda = 2$, $\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$. For $\lambda = -2$, $\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$. Since λ are real and opposite sign, the solution is a saddle.

Solution 7. Consider $\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 4 = 0$. Then $\lambda = 2, -1$. For $\lambda = 2$, $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. For $\lambda = -1$, $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \mathbf{v} = 0$, then $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Since λ are real and opposite sign, the solution is a saddle.

Solution 8. Consider the critical points by setting $x' = y' = 0$, then

$$\begin{aligned} -(x-y)(1-x-y) &= 0 \\ x(2+y) &= 0 \end{aligned}$$

Solving the equations yields i) $x = y = 0$ ii) $x = y = -2$ iii) $x = 0, y = 1$ iv) $x = 3, y = -2$. We compute the eigenvalues using the methods from lecture 2.

$$\begin{aligned} &F(x,y) = -(x-y)(1-x-y) \quad F_x = 2x-1 \quad F_y = -2y+1 \\ &G(x,y) = x(2+y) \quad G_x = 2+y \quad G_y = x \end{aligned}$$

i) $\begin{vmatrix} -1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = (-1-\lambda)(-\lambda) - 2 = 0$, $\lambda = 1, -2$, i.e. unstable saddle at $(0,0)$.

ii) $\begin{vmatrix} -5-\lambda & -3 \\ 0 & -2-\lambda \end{vmatrix} = (-5-\lambda)(-2-\lambda) = 0$, $\lambda = -5, -2$, i.e. stable node at $(-2,-2)$.

iii) $\begin{vmatrix} -1-\lambda & -1 \\ 3 & -\lambda \end{vmatrix} = (-1-\lambda)(-\lambda)+3=0$, $\lambda = \frac{-1\pm\sqrt{11}}{2}$, i.e. stable spiral at $(0,1)$.

iv) $\begin{vmatrix} 5-\lambda & 5 \\ 0 & 3-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda)=0$, $\lambda = 5, 3$, i.e. unstable node at $(3,-2)$.

Solution 9. Consider the critical points by setting $x' = y' = 0$, then

$$x - y^2 = 0$$
$$y - x^2 = 0$$

Solving the equations yields i) $x = y = 0$ ii) $x = y = 1$. Then

$$F(x,y) = x - y^2 \quad F_x = 1 \quad F_y = -2y$$
$$G(x,y) = y - x^2 \quad G_x = -2x \quad G_y = 1$$

i) $\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)=0$, $\lambda = 1$, i.e. unstable improper node at $(0,0)$.

ii) $\begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)-4=0$, $\lambda = 3, -1$, i.e. unstable saddle at $(1,1)$.

Solution 10. Consider the critical points by setting $x' = y' = 0$, then

$$(2+x)(y-x) = 0$$
$$(4-x)(y+x) = 0$$

Solving the equations yields i) $x = y = 0$ ii) $x = -2, y = 2$ iii) $x = y = 4$. Then

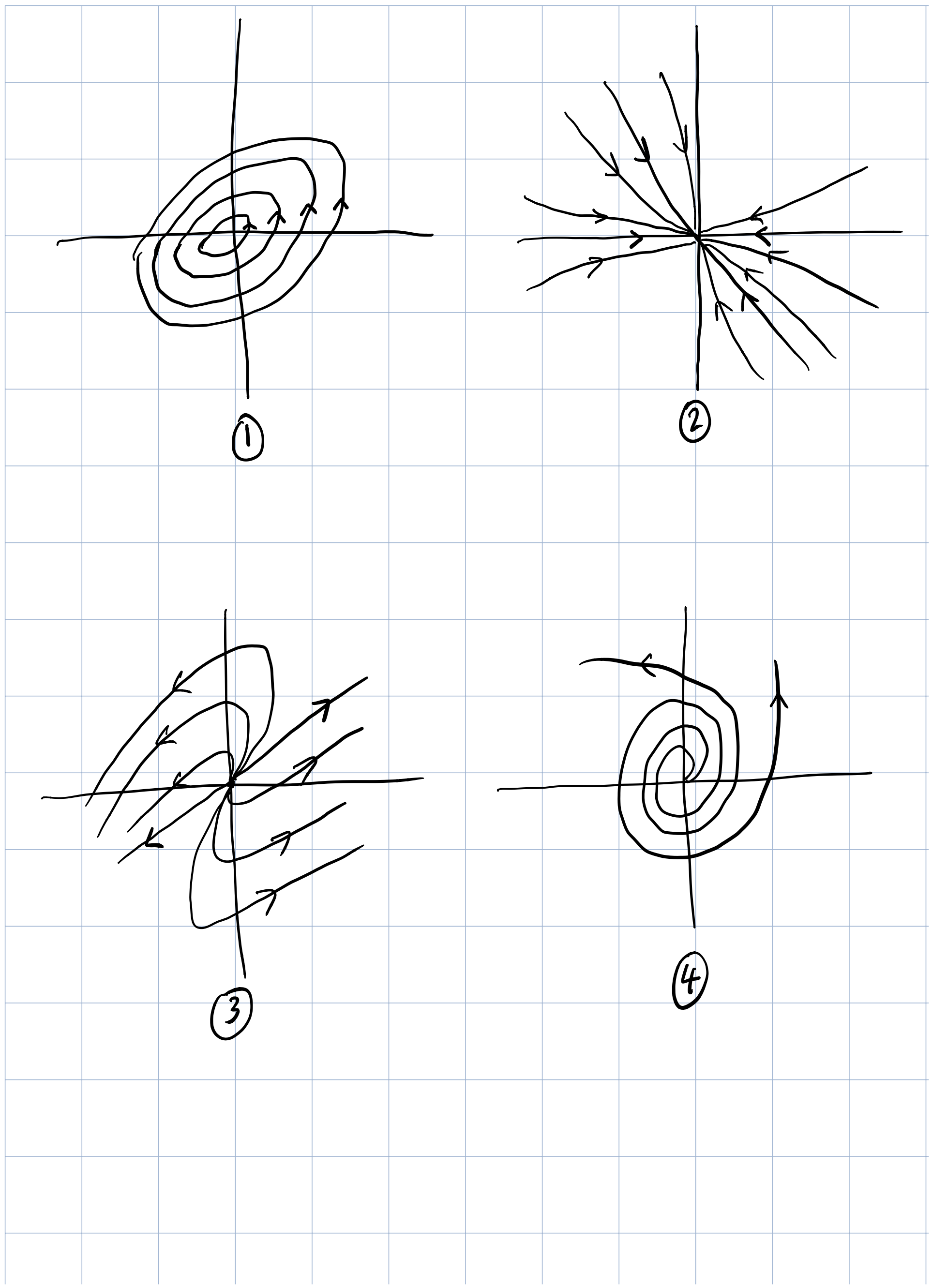
$$F(x,y) = (2+x)(y-x) \quad F_x = -2x + y - 2 \quad F_y = 2 + x$$
$$G(x,y) = (4-x)(y+x) \quad G_x = 4 - y - 2x \quad G_y = 4 - x$$

i) $\begin{vmatrix} -2-\lambda & 2 \\ 4 & 4-\lambda \end{vmatrix} = (-2-\lambda)(4-\lambda)-8=0$, $\lambda = 1\pm\sqrt{17}$, i.e. unstable saddle at $(0,0)$.

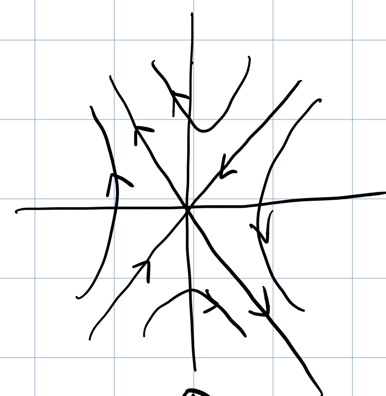
ii) $\begin{vmatrix} 4-\lambda & 0 \\ 6 & 6-\lambda \end{vmatrix} = (4-\lambda)(6-\lambda)=0$, $\lambda = 4, 6$, i.e. unstable node at $(-2,2)$.

iii) $\begin{vmatrix} -6-\lambda & 6 \\ -8 & -\lambda \end{vmatrix} = (-6-\lambda)(-\lambda)+48=0$, $\lambda = -3\pm\sqrt{39}$, i.e. stable spiral at $(4,4)$.

See next pages for figures.

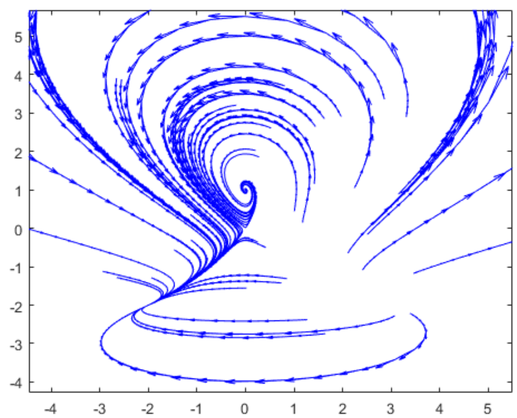


5

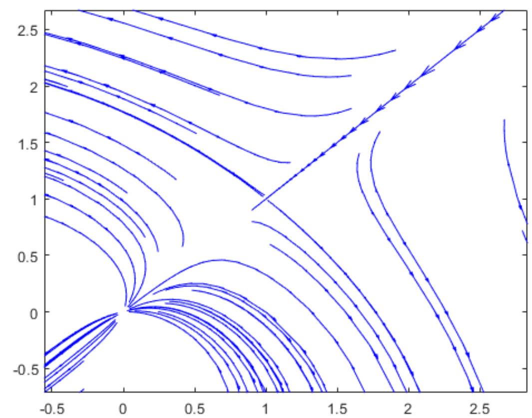


6

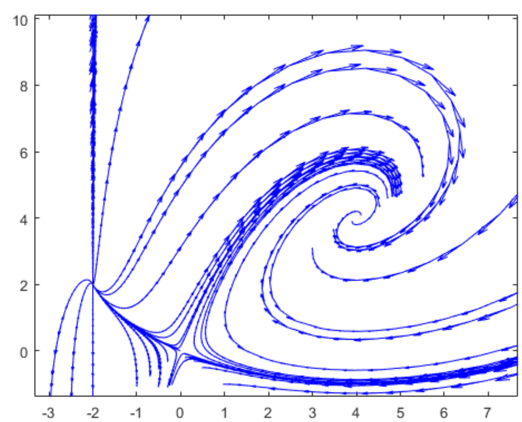
7



(a) Question 8



(b) Question 9



(c) Question 10

Figure 1: Phase planes of the nonlinear dynamics. Note the initial condition are randomly generated.