Homework 3.1
Due: April 21, 2021

Expected value
Derive the expected value,

$$E[N(t)] \equiv \sum_{n=0}^{\infty} n p_n(t) = n_0 e^{\beta t}$$

for the Yule-Furry (simple birth) process.
Hint: You can do this directly or, perhaps most easily, by deriving and solving a simple differential equations for the expected value of your stochastic process.

Solution.

Let $p = e^{-\beta t}$ and have in mind the identity [1]

$$\binom{-\beta - 1}{k} = (-1)^k \binom{k + \beta}{k}$$

and the binomial series[2]

$$(1 + x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

$$\begin{aligned}
E[N(t)] &= \sum_{n=n_0}^{\infty} \frac{n(n-1)!}{(n_0-1)!(n-n_0)!} p^{n_0}(1-p)^{n-n_0} \\
&= n_0 \sum_{n=n_0}^{\infty} \frac{(n-n_0+n_0)!}{n_0!(n-n_0)!} p^{n_0}(1-p)^{n-n_0} \\
&= n_0 p^{n_0} \sum_{n=n_0}^{\infty} \binom{n-n_0+n_0}{n-n_0} (1-p)^{n-n_0} \\
&= n_0 p^{n_0} \sum_{n=n_0}^{\infty} \binom{-n_0-1}{n-n_0} (-1)^{n-n_0}(1-p)^{n-n_0} \\
&= n_0 p^{n_0} \sum_{n=n_0}^{\infty} \binom{-n_0-1}{n-n_0} (p-1)^{n-n_0} \\
&= n_0 p^{n_0} (1+p-1)^{-n_0-1} \\
&= \frac{n_0}{p} \\
&= n_0 e^{\beta t}
\end{aligned}$$

[1] Binonimal series - Wikipedia
[2] See footnote 1.