Homework 5
Due: February 26, 2021

1. Consider the singular equation:

$$\epsilon y'' + (1+x)^2 y' + y = 0$$

with $y(0) = y(1) = 1$ and with $0 < \epsilon \ll 1$
(a) Obtain a uniform approximation which is valid to $O(\epsilon)$, i.e. determine the leading order behavior and first correction.
(b) Show that assuming the boundary layer to be at $x = 1$ is inconsistent. (hint: use the stretched inner variable $\xi = (1-x)/\epsilon$)
(c) Plot the uniform solution for $\epsilon = 0.01, 0.05, 0.1, 0.2$.

Solution.

Using the regular perturbation expansion for the outer region ($\delta \ll x \leq 1$)

$$y = y_0 + \epsilon y_1 + \cdots$$

This gives a set of equations

$$\begin{array}{lll} O(1) & (1+x)^2 y_0' + y_0 = 0 & y_0(1) = 1 \\ O(\epsilon) & (1+x)^2 y_1' + y_1 = -y_0'' & y_1(1) = 0 \end{array}$$

The leading order equation can be solved by separation of variables, yielding

$$y_0 = e^{-\frac{1}{2} + \frac{1}{1+x}}$$

Substituting into the second equation

$$(1+x)^2 y_1' + y_1 = -e^{-\frac{1}{2}+\frac{1}{1+x}}(2(1+x)^{-3} + (1+x)^{-4})$$

Solving in Mathematica yields

$$y_1 = e^{\frac{1}{1+x} - \frac{1}{2}} \left(10(5x+7)(1+x)^{-5} - \frac{3}{80} \right)$$

Consider the inner region ($0 \leq x < \delta \ll 1$). Since $b(x) = (1+x)^2 > 0$ in this problem, we believe that the boundary layer is at $x = 0$. Let $\xi = \frac{x}{\epsilon}$. This changes the derivatives to

$$y' = y_\xi \xi_x = \frac{1}{\epsilon} y_\xi$$
$$y'' = \frac{1}{\epsilon^2} y_{\xi\xi}$$

The new equation is

$$y_{\xi\xi} + (1+\epsilon\xi)^2 y_\xi + \epsilon y = 0 \quad y(x=0) = y(\epsilon\xi = 0) = 0$$

Again using the regular perturbation expansion yields

$$\begin{aligned} O(1) &\quad y_{0\xi\xi} + y_{0\xi} = 0 \quad y_0(0) = 1 \\ O(\epsilon) &\quad y_{1\xi\xi} + y_{1\xi} = -2\xi y_{0\xi} - y_0 \quad y_1(0) = 0 \end{aligned}$$

Solving the leading order equation yields

$$y_0 = Ae^{-\xi} + 1 - A$$

Substituting into the second equation

$$y_{1\xi\xi} + y_{1\xi} = Ae^{-\xi}(2\xi - 1) - 1 + A$$

Solving in Mathematica yields

$$y_1 = -Ae^{-\xi}\left(\xi^2 - e^\xi\xi + \xi + 2\right) - \xi$$

Matching the two regions requires that

$$\lim_{x\to 0} y_{\text{out}} = e^{\frac{1}{2}} = 1 - A = \lim_{\xi\to\infty} y_{\text{in}} = y_{\text{match}}$$

Thus $A = 1 - e^{1/2}$. The uniform solution is

$$y_{\text{unif}} = y_{\text{in}} + y_{\text{out}} - y_{\text{match}} = (1 - e^{\frac{1}{2}})e^{-\frac{x}{\epsilon}} + e^{-\frac{1}{2} + \frac{1}{1+x}}$$

(b) Expanding y in the outer region (which is the inner region before) gives

$$O(1) \quad (1+x)^2 y_0'' + y_0 = 0 \quad y_0(0) = 1$$

Solving yields

$$y_0 = e^{-1+\frac{1}{1+x}}$$

Consider the inner region and let $\xi = \frac{1-x}{\epsilon}$. This changes the derivatives to

$$\begin{aligned} y' &= y_\xi \xi_x = -\frac{1}{\epsilon} y_\xi \\ y'' &= \frac{1}{\epsilon^2} y_{\xi\xi} \end{aligned}$$

The new equation is

$$y_{\xi\xi} - (2-\epsilon\xi)^2 y_\xi + \epsilon y = 0 \quad y(x=1) = y(\xi = 0) = 1$$

Expanding y yields

$$O(1) \quad y_{0\xi\xi} - 4y_{0\xi} = 0 \quad y_0(0) = 1$$

Solving yields

$$y_0 = Ae^{4\xi} + 1 - A$$

For the inner solution not to blow up we require that $A = 0$. Then $y_0 = 1$. Matching the two regions

$$\lim_{x\to 1} y_{\text{out}} = e^{-\frac{1}{2}} \neq 1 = \lim_{\xi\to -\infty} y_{\text{in}}$$

where inconsistency is clearly observed. Note that we do not need to match y_1.

(c) See Figure 1.

2. Consider the singular equation:

$$\epsilon y'' - x^2 y' - y = 0$$

with $y(0) = y(1) = 1$ and with $0 < \epsilon \ll 1$
(a) With the method of dominant balance, show that there are three distinguished limits: $\delta = \epsilon^{1/2}$, $\delta = \epsilon$, and $\delta = 1$ (the outer problem). Write down each of the problems in the various distinguished limits.
(b) Obtain the leading order uniform approximation (hint: there are boundary layers at $x = 0$ and $x = 1$)
(c) Plot the uniform solution for $\epsilon = 0.01, 0.05, 0.1, 0.2$.

Solution.

(a) Since $b(x) < 0$ everywhere except at zero, we would expect a boundary layer at $x = 1$. There may also be a boundary layer at $x = 0$ since $b(0) = 0$. Near $x = 0$, let $\xi = \frac{x}{\delta}$ and the governing equation becomes

$$\epsilon y_{\xi\xi} - \delta^3 \xi^2 y_\xi - \delta^2 y = 0$$

Since $O(\delta^3) \ll O(\delta^2)$, the dominant balance is

$$\epsilon y_{\xi\xi} - \delta^2 y = 0$$

In order to balance these two terms we let $\delta = \epsilon^{1/2}$. If we let $\delta = 1$, we get

$$\epsilon y_{\xi\xi} - \xi^2 y_\xi - y = 0$$

which is the original outer problem. Near $x = 1$, let $\xi = \frac{1-x}{\delta}$ and the governing equation becomes

$$\epsilon y_{\xi\xi} - \delta(1 - \delta\xi)^2 y_\xi - \delta^2 y = 0$$

Since $O(\delta^2) \ll O(\delta)$, the dominant balance is

$$\epsilon y_{\xi\xi} + \delta y_\xi = 0$$

Thus we require that $\delta = \epsilon$.

(b) Expanding y in the outer problem gives

$$O(1) \quad -x^2 y_0' - y_0 = 0$$

Solving yields

$$y_0 = Ce^{\frac{1}{x}}$$

In order for $\lim_{x\to 0} y\text{out}$ to be bounded, we require that $C = 0$, thus $y_{\text{out}} = 0$. Expanding y in the inner problem near $x = 0$ and let $\xi = \frac{x}{\epsilon^{1/2}}$.

$$y_{0\xi\xi} - y_0 = 0 \quad y_0(0) = 1$$

Solving yields

$$y_0 = Ae^{-\xi} + Be^{-\xi}$$

In order for $\lim_{\xi\to\infty} y\text{in}$ to be bounded, we require that $B = 0$. Imposing the boundary condition $y_0(0) = 1$ gives

$$y_0 = e^{-\xi}$$

Expanding y in the inner problem near $x = 1$ and let $\xi = \frac{1-x}{\epsilon}$.

$$y_{0\xi\xi} + y_{0\xi} = 0 \quad y_0(1) = 1$$

Solving yields

$$y_0 = Ae^{-\xi} + 1 - A$$

Matching the three regions requires that

$$\lim_{x\to 0} y_{\text{out}} = \lim_{x\to\infty} y_{\text{in,left}} = 0$$
$$\lim_{x\to 1} y_{\text{out}} = \lim_{x\to\infty} y_{\text{in,right}} = 0$$

Thus $A = 1$. The uniform solution is

$$y_{\text{in,left}} + y_{\text{in,right}} + y_{\text{out}} - y_{\text{match,left}} - y_{\text{match,right}} = e^{-\frac{x}{\epsilon^{1/2}}} + e^{-\frac{1-x}{\epsilon}}$$

(c) See Figure 1.

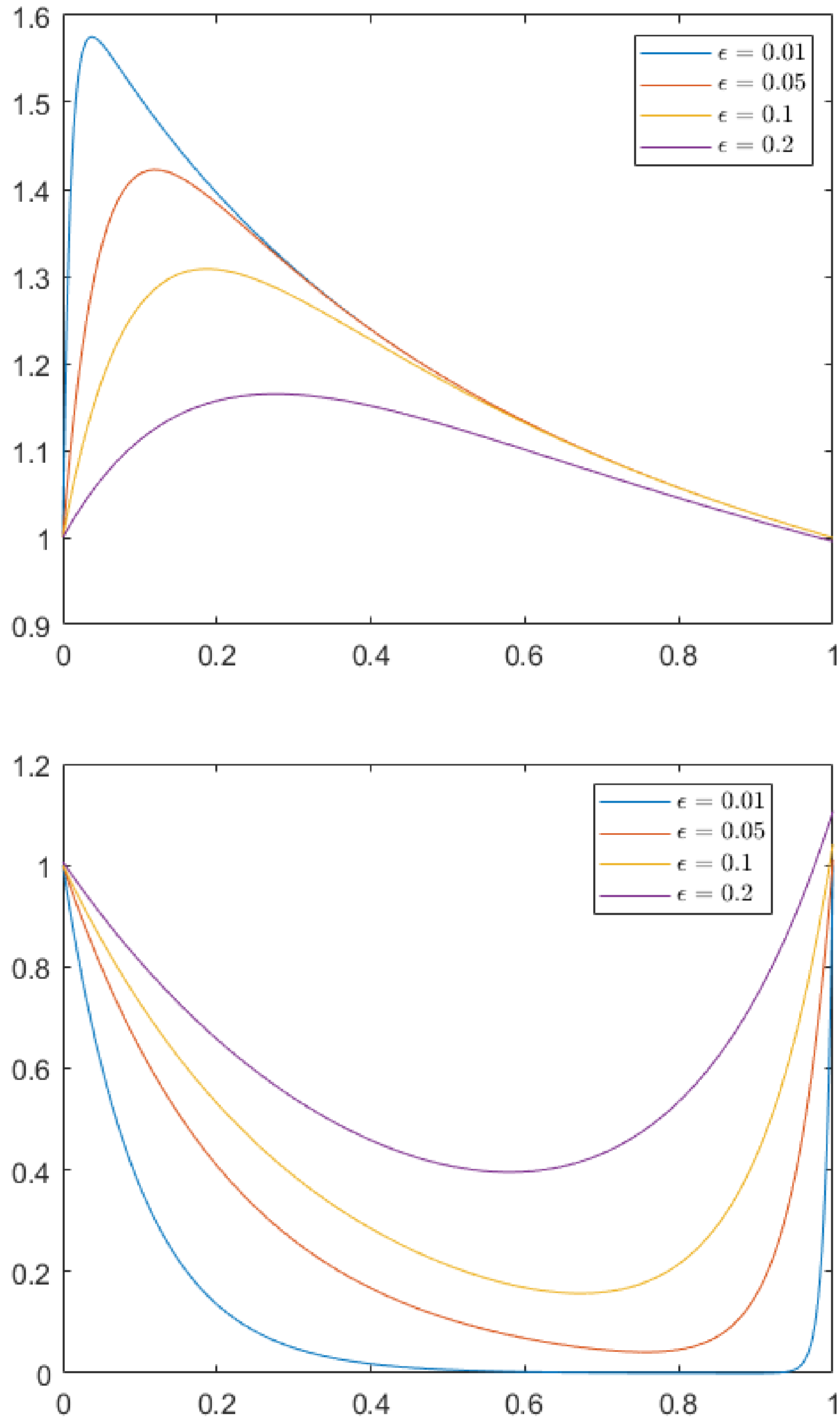

Figure 1: Boundary layer solutions to Q1 and Q2.