Homework 2
Due: Friday, October 16, 2020

Yale Faces B: Download two data sets (ORIGINAL IMAGE and CROPPED IMAGES) Your job is to perform an SVD analysis of these data sets. Please start with the cropped images and perform the following analysis.
1. Do an SVD analysis of the images (where each image is reshaped into a column vector and each column is a new image).
2. What is the interpretation of the $\mathbf{U}$, $\mathbf{\Sigma}$ and $\mathbf{V}$ matrices? (Plot the first few reshaped columns of $\mathbf{U}$)
3. What does the singular value spectrum look like and how many modes are necessary for good image reconstructions using the PCA basis? (i.e. what is the rank r of the face space?)
4. Compare the difference between the cropped (and aligned) versus uncropped images in terms of singular value decay and reconstruction capabilities.

Answer to 1. First we read every image, reshape them into column vectors and put the vectors into the data matrix `D`.

```
% Cropped Images
Z = 'CroppedYale';
Y = dir(fullfile(Z,'*'));
N = setdiff({Y([Y.isdir]).name},{'.','..'});
D=[];
for i = 1:numel(N)
    W = dir(fullfile(Z,N{i},'*'));
    X = {W(¬[W.isdir]).name};
    for j = 1:numel(X)
        F = fullfile(Z,N{i},X{j});
        E=imresize(double(imread(F)),[192 168]);
        G=reshape(E,[32256,1]);
        D=[D G];
    end
end

trainD=D(:,1:2304);
[U,S,V]=svd(trainD,'econ');
```

Since there are 38 people in total, let us use the faces of first 36 people as a training set, i.e. `trainD=D(:,1:2304)`. Then we perform SVD on `trainD`. We plot some eigenfaces and the singular value spectrum in Figure 1.

Answer to 2. $\mathbf{U}$ is the eigenvectors of the covariance matrix $\mathbf{AA}^*$ (in our code $\mathbf{A}$=`D`). In the example of Yale Faces, it is the 'eigenface' space, which contains the most common features of faces among all faces. $\mathbf{\Sigma}$ contains singular values which correspond to the eigenfaces. Larger singular values means the features are more common. $\mathbf{V}$ is the eigenvectors of the covariance matrix $\mathbf{A}^*\mathbf{A}$.

Answer to 3. Now let us use our eigenface space $\mathbf{U}$ to reconstruct the faces of the remaining two people. This is done by the projection onto the eigenvectors $\mathbf{x_{rec}} = \mathbf{UU}^*\mathbf{x_{test}}$.

```
X=D(:,2305); % First face of the 37th person
subplot(2,4,1);imagesc(reshape(X,[192 168])),colormap gray
i=1;
for r=[25 50 100 200 400 800 1600] % Testing for different r values
Xrec = U(:,1:r)*(U(:,1:r)'*X);
i=i+1;
subplot(2,4,i);imagesc(reshape(Xrec,[192 168])),colormap gray
end
```

We observe from Figure 2 that at $r = 200$ the reconstructed face looks pretty similar to the original face and starts to converge after $r = 400$.

Answer to 4. There are 15 people in the uncropped images, thus we use the first 13 people as a training set. We obtain Figure 3 by doing the same SVD analysis. We observe that the reconstructed uncropped images fail to converge to the original images. This makes sense because the database of the uncropped images (165 images) is much smaller than the cropped one (2432 images), thus we would expect the latter to contain more face features that can better approximate new samples. It is also evident that the singular values of uncropped images decay more rapidly than the cropped ones, implying that after the first few eigenfaces of the uncropped images, the rest of the features are merely noises, which are not particularly useful for face reconstrunction. This is obvious in Figure 3, because with increasing rank of the face space, the reconstructed images still look noisy.

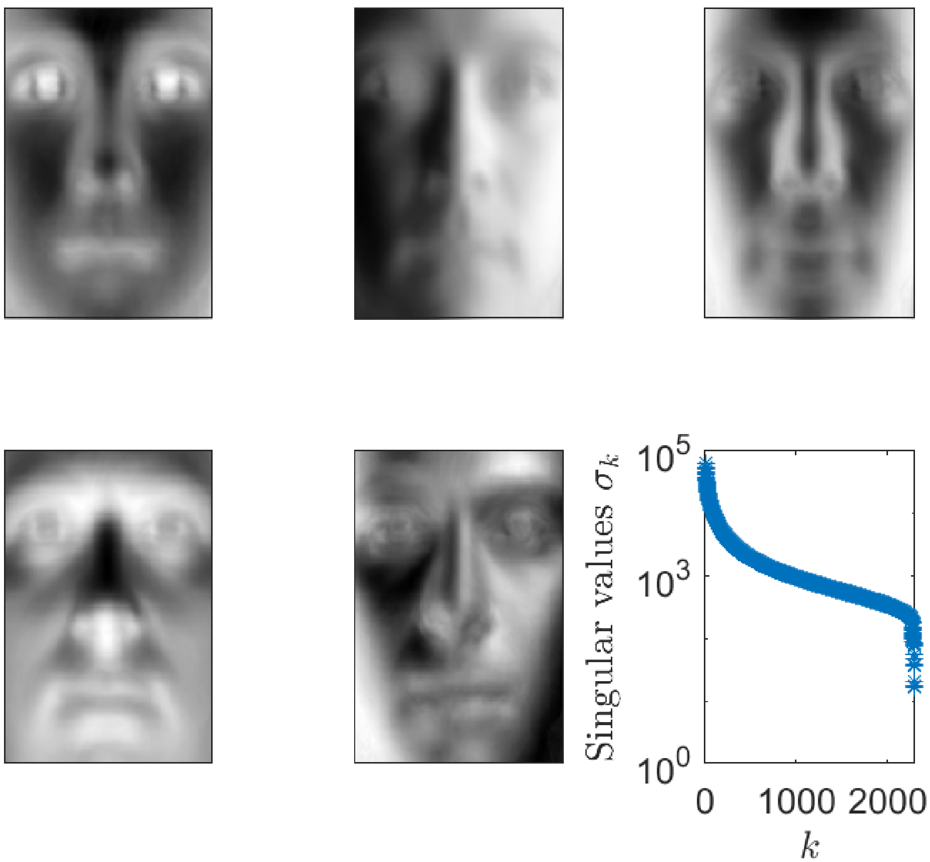
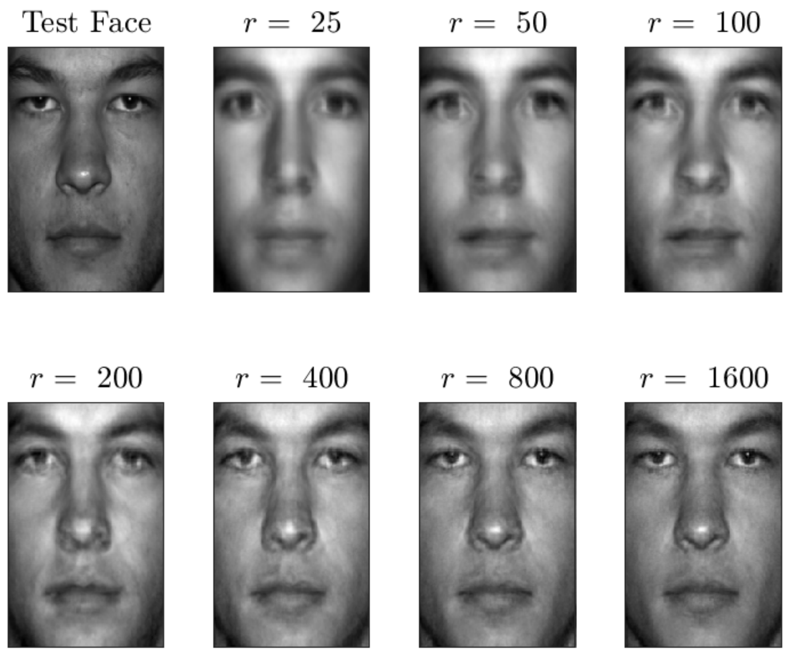
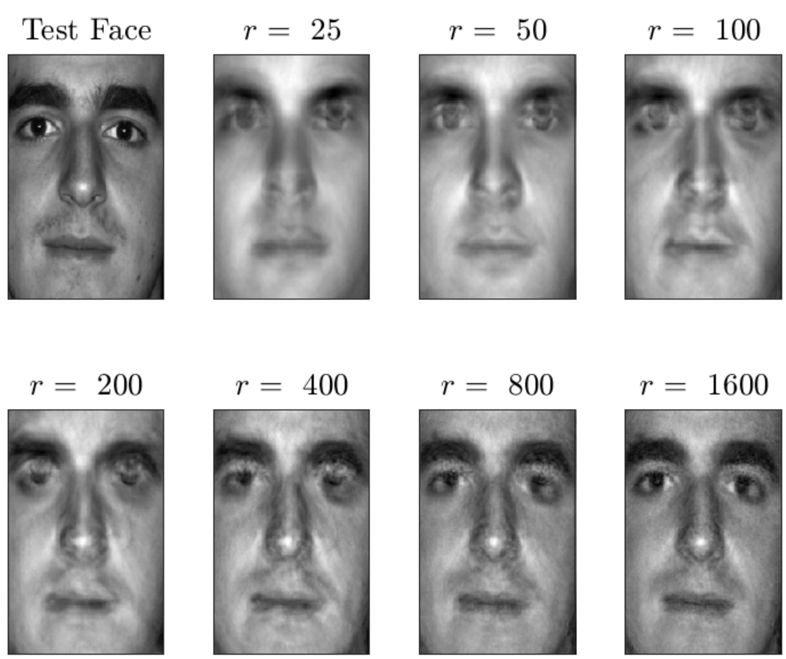


Figure 1: First five eigenfaces and the singular value spectrum of cropped images

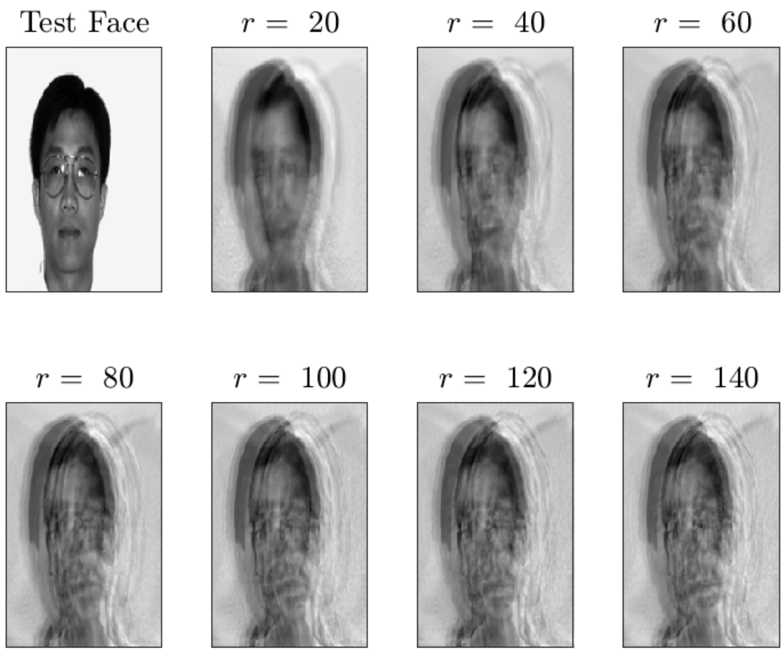


(a)

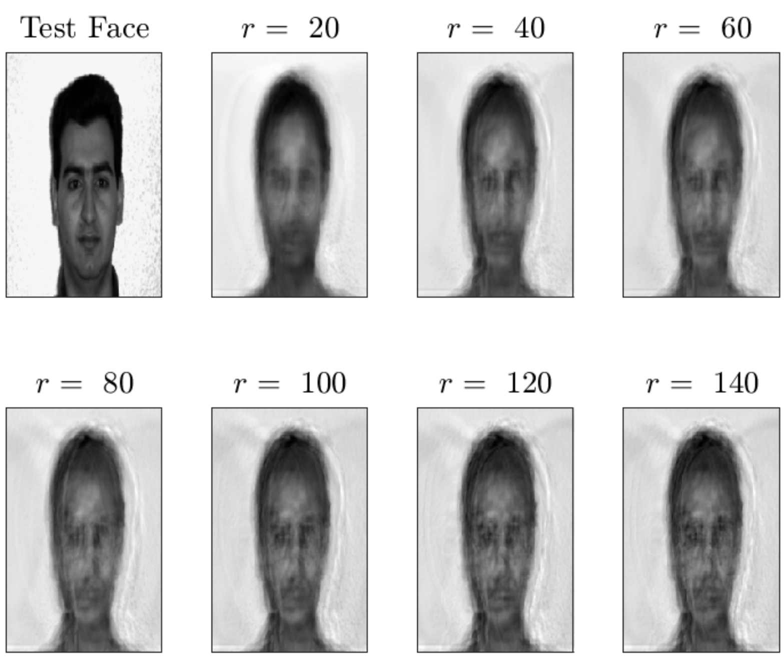


(b)

Figure 2: Approximations of cropped faces in the eigenface space with increasing rank (a) Testing the first face of the 37th person (b) Testing the first face of the 38th person



(a)



(b)

Figure 3: Approximations of uncropped faces in the eigenface space with increasing rank (a) Testing the first face of the 14th person (b) Testing the first face of the 15th person

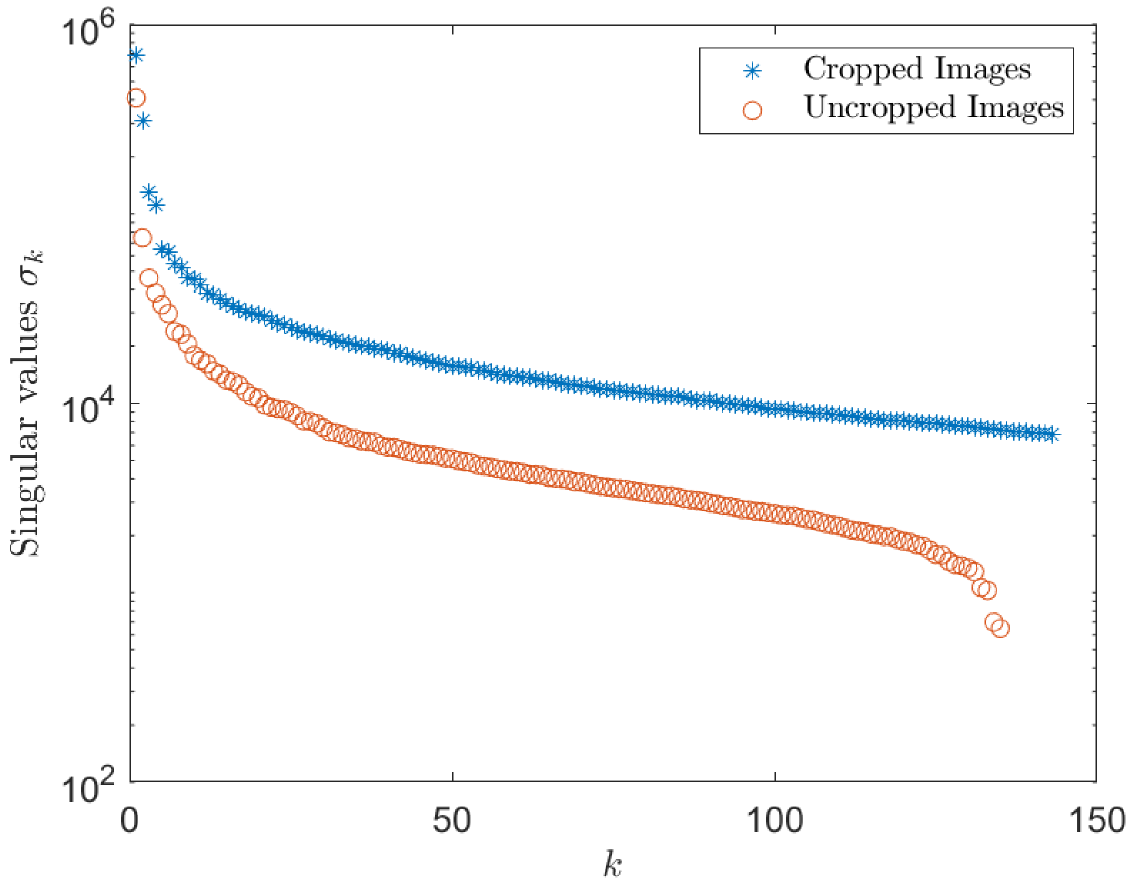


Figure 4: Singular value spectra of cropped images versus uncropped images

Theorems: Show that for a matrix $\mathbf{A}$
1. The nonzero singular values of $\mathbf{A}$ are the square roots of the nonzero eigenvalue of $\mathbf{AA}^*$ and $\mathbf{A}^*\mathbf{A}$

The SVD of $\mathbf{A}$ takes the form

$$\mathbf{A} = \mathbf{U\Sigma V}^*$$

where $\mathbf{U}$, $\mathbf{V}$ are unitary matrices and $\mathbf{\Sigma}$ is a rectangular matrix with non-negative real numbers on the diagonal. Then $\mathbf{A}^* = \mathbf{V\Sigma U}^*$. Multiplying $\mathbf{A}$ and $\mathbf{A}^*$

$$\mathbf{AA}^* = \mathbf{U}\begin{bmatrix} \hat{\mathbf{\Sigma}} \\ \mathbf{0} \end{bmatrix}\mathbf{V}^*\mathbf{V}\begin{bmatrix} \hat{\mathbf{\Sigma}} & \mathbf{0} \end{bmatrix}\mathbf{U}^* = \mathbf{U}\begin{bmatrix} \hat{\mathbf{\Sigma}}^{\mathbf{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^*$$

where $\hat{\mathbf{\Sigma}}$ is the reduced singular value matrix. Multiplying both sides by $\mathbf{U}$

$$\mathbf{AA}^*\mathbf{U} = \mathbf{U}\begin{bmatrix} \hat{\mathbf{\Sigma}}^{\mathbf{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^*\mathbf{U} = \mathbf{U}\begin{bmatrix} \hat{\mathbf{\Sigma}}^{\mathbf{2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Multiplying $\mathbf{A}^*$ and $\mathbf{A}$

$$\mathbf{A}^*\mathbf{A} = \mathbf{V}\begin{bmatrix} \hat{\mathbf{\Sigma}} & \mathbf{0} \end{bmatrix}\mathbf{U}^*\mathbf{U}\begin{bmatrix} \hat{\mathbf{\Sigma}} \\ \mathbf{0} \end{bmatrix}\mathbf{V}^* = \mathbf{V}\hat{\mathbf{\Sigma}}^{\mathbf{2}}\mathbf{V}^*$$

Multiplying both sides by $\mathbf{V}$

$$\mathbf{A}^*\mathbf{AV} = \mathbf{V}\hat{\mathbf{\Sigma}}^{\mathbf{2}}\mathbf{V}^*\mathbf{V} = \mathbf{V}\hat{\mathbf{\Sigma}}^{\mathbf{2}}$$

This implies that nonzero singular values of $\mathbf{A}$ are the square roots of the eigenvalues of $\mathbf{AA}^*$ and $\mathbf{A}^*\mathbf{A}$.

2. If $\mathbf{A} = \mathbf{A}^*$, then the singular values are the absolute values of the eigenvalues of $\mathbf{A}$

If $\mathbf{A} = \mathbf{A}^*$, $\mathbf{A}$ is Hermitian, implying its eigenvectors are orthogonal and eigenvalues are real. Then we have

$$\mathbf{A} = \mathbf{Q\Lambda Q}^*$$

where $\mathbf{Q}$ is unitary and $\mathbf{\Lambda}$ is a diagonal matrix with diagonal entries equal to λ_i (eigenvalues of $\mathbf{A}$), ordered from largest to smallest (it is possible that the entries are ordered differently, in which case $\mathbf{Q}$ will be different). An equivalent form of the above is

$$\mathbf{A} = \mathbf{Q}|\mathbf{\Lambda}|\mathbf{sign}(\mathbf{\Lambda})\mathbf{Q}^* \tag{1}$$

where $\mathbf{sign}(\mathbf{\Lambda})$ is a diagonal matrix whose diagonal entries equal to the signs of λ_i. Since $\mathbf{Q}$ is unitary, $\mathbf{sign}(\mathbf{\Lambda})\mathbf{Q}^*$ is also unitary and Equation 1 is an SVD of $\mathbf{A}$. Therefore $\mathbf{\Sigma} = |\mathbf{\Lambda}|$ as desired.

3. Given that the determinant of a matrix $\mathbf{U}$ is unity, show $|\det(\mathbf{A})| = \Pi_{j=1}^{m}\sigma_j$

$$\begin{aligned}
|\det(\mathbf{A})| &= |\det(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*)| \\
&= |\det(\mathbf{U})\det(\boldsymbol{\Sigma})\det(\mathbf{V}^*)| \\
&= |1 \cdot \det(\boldsymbol{\Sigma}) \cdot 1| \\
&= \det(\boldsymbol{\Sigma}) \\
&= \Pi_{j=1}^{m}\sigma_j
\end{aligned}$$

Appendix

```
close all;
% Cropped Images
Z = 'CroppedYale';
Y = dir(fullfile(Z,'*'));
N = setdiff({Y([Y.isdir]).name},{'.','..'});
D=[];
for i = 1:numel(N)
    W = dir(fullfile(Z,N{i},'*'));
    X = {W(¬[W.isdir]).name};
    for j = 1:numel(X)
        F = fullfile(Z,N{i},X{j});
        E=imresize(double(imread(F)),[192 168]);
        G=reshape(E,[32256,1]);
        D=[D G];
    end
end

trainD=D(:,1:2304);
[U,S,V]=svd(trainD,'econ');


figure(1)
subplot(2,3,6);semilogy(diag(S),'*')
ylim([1e+0 1e+5])
yticks([1e+0 1e+3 1e+5])
xlabel('$k$','interpreter','latex')
ylabel('Singular values $\sigma_k$','interpreter','latex')
set(gca,'Fontsize',[12])
for r=1:5 % Testing for different r values
    subplot(2,3,r);facel=reshape(U(:,r),[192 ...
       168]);imagesc(facel),colormap gray;
    set(gca,'XTick',[], 'YTick', [])
end
saveas(gcf,'eigenface.png')

figure(2)
X=D(:,2305); % First face of the 37th person
subplot(2,4,1);imagesc(reshape(X,[192 168])),colormap gray
title('Test Face','interpreter','latex')
set(gca,'XTick',[], 'YTick', [])
set(gca,'Fontsize',[12])
i=1;
for r=[25 50 100 200 400 800 1600] % Testing for different r values
    Xrec = U(:,1:r)*(U(:,1:r)'*X);
    i=i+1;
    subplot(2,4,i);imagesc(reshape(Xrec,[192 168])),colormap gray
    title(['$r=¬$',num2str(r)],'interpreter','latex')
```

```
    set(gca,'XTick',[], 'YTick', [])
    set(gca,'Fontsize',[12])
end
saveas(gcf,'testface1.png')

figure(3)
X=D(:,2369); % First face of the 38th person
subplot(2,4,1);imagesc(reshape(X,[192 168])),colormap gray
title('Test Face','interpreter','latex')
set(gca,'XTick',[], 'YTick', [])
set(gca,'Fontsize',[12])
i=1;
for r=[25 50 100 200 400 800 1600]
    Xrec = U(:,1:r)*(U(:,1:r)'*X);
    i=i+1;
    subplot(2,4,i);imagesc(reshape(Xrec,[192 168])),colormap gray
    title(['$r=¬$',num2str(r)],'interpreter','latex')
    set(gca,'XTick',[], 'YTick', [])
    set(gca,'Fontsize',[12])
end
saveas(gcf,'testface2.png')


% Uncropped Images
Z = 'yalefaces';
W = dir(fullfile(Z,'*'));
X = {W(¬[W.isdir]).name};
D1=[];
for i = 1:numel(X)
    F = fullfile(Z,X{i});
    E=imresize(double(imread(F)),[192 168]);
    G=reshape(E,[32256,1]);
    D1=[D1 G];
end

trainD1=D1(:,1:143);
[U1,S1,V1]=svd(trainD1,'econ');

figure(4)
X=D1(:,144);
subplot(2,4,1);imagesc(reshape(X,[192 168])),colormap gray
title('Test Face','interpreter','latex')
set(gca,'XTick',[], 'YTick', [])
set(gca,'Fontsize',[12])
i=1;
for r=[20 40 60 80 100 120 140]
    Xrec = U1(:,1:r)*(U1(:,1:r)'*X);
    i=i+1;
    subplot(2,4,i);imagesc(reshape(Xrec,[192 168])),colormap gray
    title(['$r=¬$',num2str(r)],'interpreter','latex')
    set(gca,'XTick',[], 'YTick', [])
```

```
    set(gca,'Fontsize',[12])
end
saveas(gcf,'testface3.png')


figure(5)
X=D1(:,155);
subplot(2,4,1);imagesc(reshape(X,[192 168])),colormap gray
title('Test Face','interpreter','latex')
set(gca,'XTick',[], 'YTick', [])
set(gca,'Fontsize',[12])
i=1;
for r=[20 40 60 80 100 120 140]
    Xrec = U1(:,1:r)*(U1(:,1:r)'*X);
    i=i+1;
    subplot(2,4,i);imagesc(reshape(Xrec,[192 168])),colormap gray
    title(['$r=¬$',num2str(r)],'interpreter','latex')
    set(gca,'XTick',[], 'YTick', [])
    set(gca,'Fontsize',[12])
end
saveas(gcf,'testface4.png')

figure(6)
semilogy(diag(S(1:143,1:143)),'*')
ylim([1e+2 1e+6])
yticks([1e+2 1e+4 1e+6])
hold on
semilogy(diag(S1),'o')
xlabel('$k$','interpreter','latex')
ylabel('Singular values $\sigma_k$','interpreter','latex')
legend('Cropped Images','Uncropped Images','interpreter','latex')
set(gca,'TickLabelInterpreter', 'tex');
set(gca,'Fontsize',[12])
saveas(gcf,'decay.png')
```