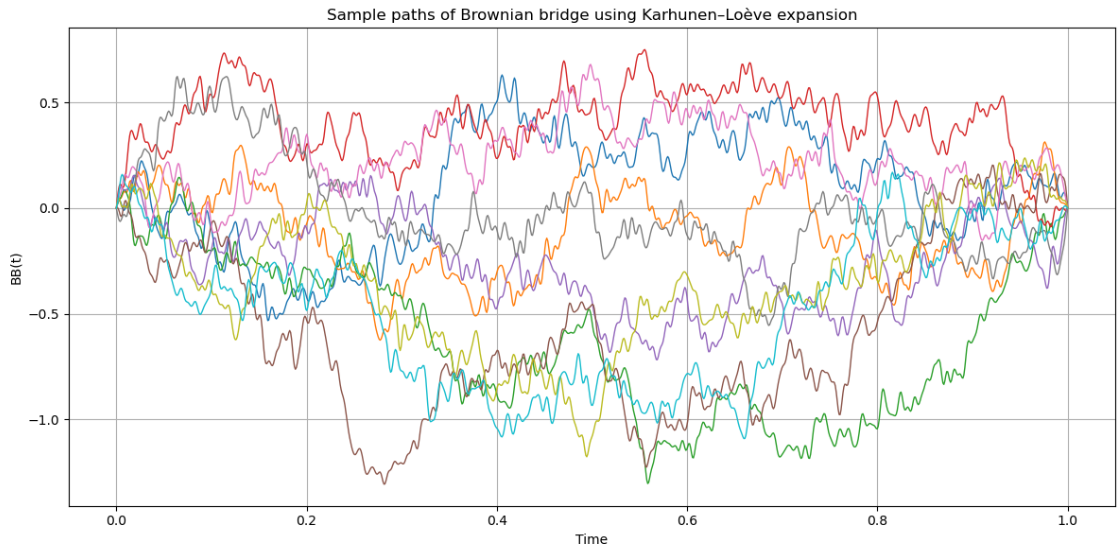
Pstat160b hw final

March 15, 2025

[ ]:
```python
import numpy as np
import matplotlib.pyplot as plt

# 8.24 Parameters
num_paths = 100
T = 1.0
N = 300  # Number of terms in the Karhunen-Loève expansion
M = 1000  # Number of time steps
t = np.linspace(0, T, M)

# Generate random coefficients for the expansion
Z = np.random.randn(num_paths, N)

# Compute the Karhunen-Loève expansion
W = np.zeros((num_paths, M))
for n in range(1, N+1):
    eigenvalue = (1 / ((n - 0.5) * np.pi)) ** 2
    eigenfunction = np.sqrt(2) * np.sin((n - 0.5) * np.pi * t)
    W += np.sqrt(eigenvalue) * Z[:, n-1][:, np.newaxis] * eigenfunction[np.
 ↪newaxis, :]

# Scale by sqrt(T)
W *= np.sqrt(T)

# Plot some sample paths
plt.figure(figsize=(12, 6))
for i in range(min(num_paths, 10)):
    plt.plot(t, W[i], lw=1)
plt.title('Sample paths of Brownian motion using Karhunen-Loève expansion')
plt.xlabel('Time')
plt.ylabel('W(t)')
plt.grid(True)
plt.tight_layout()
plt.show()
```

[2]:
```python
# Simulate Brownian Bridge using the Karhunen-Loève (KKL) expansion
```

```python
# Generate random coefficients for Brownian Bridge
Z_bridge = np.random.randn(num_paths, N)

# Compute the Karhunen-Loève expansion for Brownian Bridge
# The eigenfunctions for Brownian Bridge are sqrt(2) * sin(n * pi * t)
# The eigenvalues are (1 / (n * pi))^2
BB = np.zeros((num_paths, M))
for n in range(1, N+1):
    eigenvalue_bridge = (1 / (n * np.pi)) ** 2
    eigenfunction_bridge = np.sqrt(2) * np.sin(n * np.pi * t)
    BB += np.sqrt(eigenvalue_bridge) * Z_bridge[:, n-1][:, np.newaxis] *␣
↪eigenfunction_bridge[np.newaxis, :]

# Scale by sqrt(T)
BB *= np.sqrt(T)

# Plot some sample paths
plt.figure(figsize=(12, 6))
for i in range(min(num_paths, 10)):
    plt.plot(t, BB[i], lw=1)
plt.title('Sample paths of Brownian bridge using Karhunen-Loève expansion')
plt.xlabel('Time')
plt.ylabel('BB(t)')
plt.grid(True)
plt.tight_layout()
plt.show()
```



[4]:
```python
import numpy as np
from scipy.stats import norm

# 8.39 Parameters
mu = 1.5
sigma = np.sqrt(4)  # Standard deviation
t = 3
threshold = 4
num_simulations = 100000

# Simulate X_3 = mu * t + sigma * sqrt(t) * Z
Z = np.random.randn(num_simulations)
X_t = mu * t + sigma * np.sqrt(t) * Z

# Empirical probability
simulated_prob = np.mean(X_t > threshold)

# Theoretical (exact) probability
mean_Xt = mu * t
std_Xt = sigma * np.sqrt(t)
exact_prob = 1 - norm.cdf(threshold, loc=mean_Xt, scale=std_Xt)

(simulated_prob, exact_prob)

## The simulation results for Exercise 8.39 are: Simulated probability ( 3> 4):␣
↪0.5598, Exact theoretical probability: 0.5574
## The simulation closely matches the exact result, confirming the correctness␣
↪of both the numerical and analytical approach.
```

[4]: (0.55764, 0.5573830427633992)

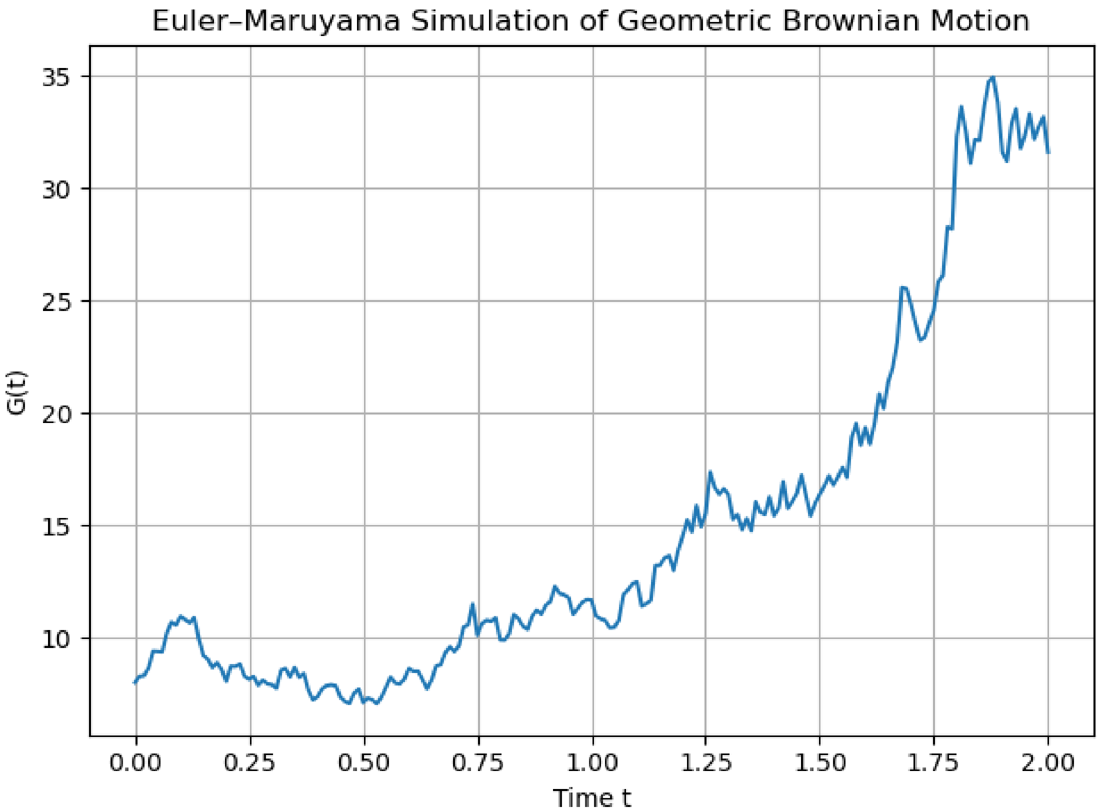
[5]:
```python
import numpy as np
from scipy.stats import norm

# Parameters
S0 = 50  # Initial stock price
mu = -0.85
sigma = np.sqrt(2.4)
T = 2  # Time in years
threshold_price = 40
num_simulations = 100000

# Simulate final stock price under geometric Brownian motion
Z = np.random.randn(num_simulations)
ST = S0 * np.exp((mu - 0.5 * sigma**2) * T + sigma * np.sqrt(T) * Z)

# Empirical probability
```

```python
simulated_prob = np.mean(ST < threshold_price)

# Theoretical (exact) probability using log-normal distribution
log_mean = np.log(S0) + (mu - 0.5 * sigma**2) * T
log_std = sigma * np.sqrt(T)
exact_prob = norm.cdf(np.log(threshold_price), loc=log_mean, scale=log_std)

(simulated_prob, exact_prob)
```

[5]: (0.96088, 0.9615976635485949)

[7]:
```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters
G0 = 8
mu = 1
sigma = np.sqrt(0.25)
T = 2
dt = 0.01
N = int(T / dt)
t = np.linspace(0, T, N + 1)

# Euler-Maruyama simulation of GBM
np.random.seed(42)
G = np.zeros(N + 1)
G[0] = G0
for i in range(1, N + 1):
    dW = np.sqrt(dt) * np.random.randn()
    G[i] = G[i - 1] + mu * G[i - 1] * dt + sigma * G[i - 1] * dW

# Part (a): Plot a sample path
plt.plot(t, G)
plt.title("Euler-Maruyama Simulation of Geometric Brownian Motion")
plt.xlabel("Time t")
plt.ylabel("G(t)")
plt.grid(True)
plt.tight_layout()
plt.show()

# Part (b): Simulate mean and variance of G(2)
num_simulations = 100000
G2_values = np.zeros(num_simulations)

for j in range(num_simulations):
    Gt = G0
    for i in range(N):
```

```
        dW = np.sqrt(dt) * np.random.randn()
        Gt += mu * Gt * dt + sigma * Gt * dW
    G2_values[j] = Gt

simulated_mean = np.mean(G2_values)
simulated_variance = np.var(G2_values)

# Theoretical mean and variance for GBM:
# E[G_T] = G0 * exp(mu * T)
# Var[G_T] = E[G_T]^2 * (exp(sigma^2 * T) - 1)

theoretical_mean = G0 * np.exp(mu * T)
theoretical_variance = (theoretical_mean**2) * (np.exp(sigma**2 * T) - 1)

(simulated_mean, simulated_variance, theoretical_mean, theoretical_variance)
```

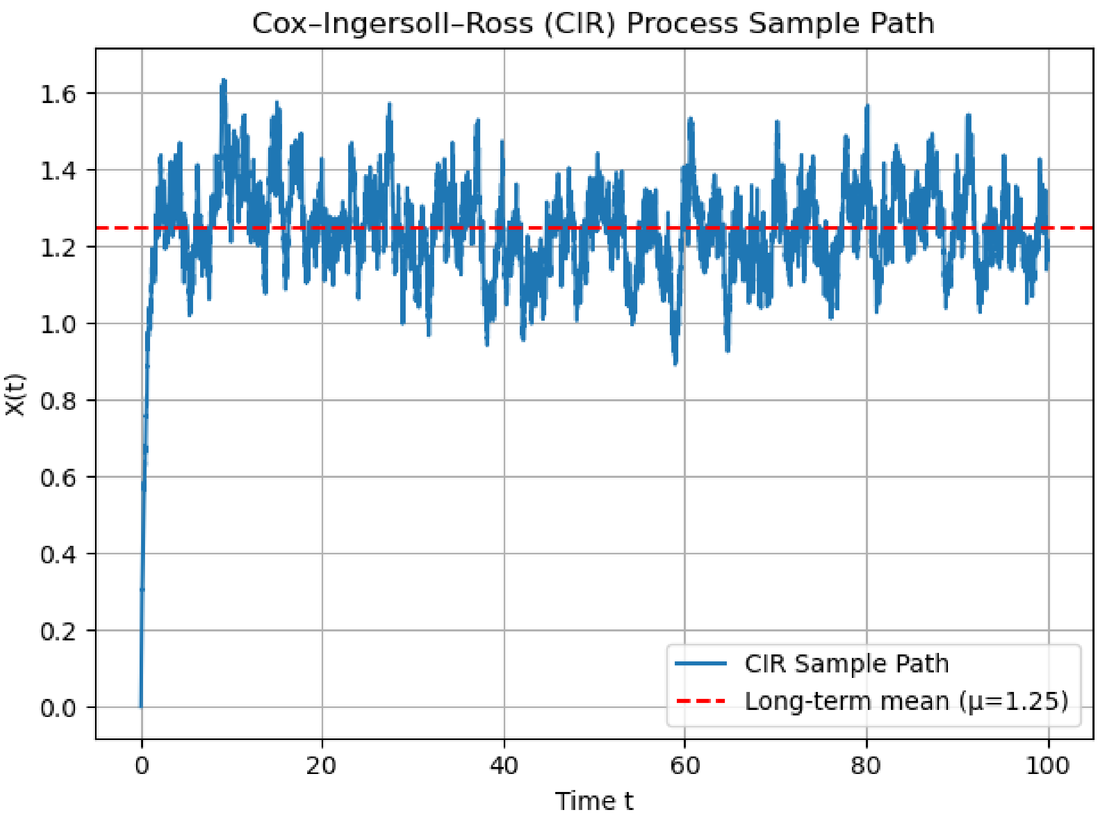


[7]: (58.4349148839813, 2129.8162726797477, 59.1124487914452, 2266.8148011121652)

[ ]:
```
import numpy as np
import matplotlib.pyplot as plt
```

```
# 9.11 Parameters for the CIR process
X0 = 0
mu = 1.25
r = 2
sigma = 0.2
T = 100
dt = 0.01
N = int(T / dt)
time = np.linspace(0, T, N + 1)
num_paths = 1  # for plotting a sample path

# Euler-Maruyama approximation for CIR (non-negativity may be violated slightly)
X = np.zeros((num_paths, N + 1))
X[:, 0] = X0

np.random.seed(42)
for i in range(N):
    dW = np.sqrt(dt) * np.random.randn(num_paths)
    X[:, i + 1] = X[:, i] + r * (mu - X[:, i]) * dt + sigma * np.sqrt(np.maximum(X[:, i], 0)) * dW

# Plot sample path
plt.plot(time, X[0], label="CIR Sample Path")
plt.axhline(mu, color='r', linestyle='--', label='Long-term mean ( =1.25)')
plt.title("Cox-Ingersoll-Ross (CIR) Process Sample Path")
plt.xlabel("Time t")
plt.ylabel("X(t)")
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()

# Simulate many paths to estimate asymptotic mean and variance at t=100
num_simulations = 100000
X_final = np.zeros(num_simulations)
for j in range(num_simulations):
    x = X0
    for i in range(N):
        dW = np.sqrt(dt) * np.random.randn()
        x += r * (mu - x) * dt + sigma * np.sqrt(max(x, 0)) * dW
    X_final[j] = x

simulated_mean = np.mean(X_final)
simulated_variance = np.var(X_final)

# Theoretical long-term mean and variance of CIR process:
theoretical_mean = mu
```

```
theoretical_variance = (sigma ** 2 * mu) / (2 * r)

(simulated_mean, simulated_variance, theoretical_mean, theoretical_variance)
```

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[ ]:
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pstat160b hw3

8.24 Geometric Brownian Motion (GBM)

The stochastic differential equation (SDE) for geometric Brownian motion is:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{1}$$

where:

- μ is the drift,
- σ is the volatility,
- $W(t)$ is a standard Brownian motion.

The solution to this SDE, obtained using Itô's Lemma, is:

$$S(t) = S(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right). \tag{2}$$

Mean of $S(t)$

Taking the expectation: $\mathrm{E[S(t)]} = \mathrm{E}\left[S(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right)\right]$.

Since $W(t) \sim \mathcal{N}(0,t)$, we use the moment-generating function of a normal distribution:

$$E[e^{\sigma W(t)}] = e^{\frac{1}{2}\sigma^2 t}. \tag{3}$$

Thus, we obtain:

$$E[S(t)] = S(0)e^{\mu t}. \tag{4}$$

Variance of $S(t)$

We compute $E[S(t)^2]$: $\mathrm{S(t)}^2 = S(0)^2\exp\left(2(\mu - \frac{1}{2}\sigma^2)t + 2\sigma W(t)\right)$.

Since $2\sigma W(t) \sim \mathcal{N}(0, 4\sigma^2 t)$, using the moment-generating function:

$$E[S(t)^2] = S(0)^2 e^{2\mu t + \sigma^2 t}. \tag{5}$$

Thus, the variance is: $\mathrm{Var(S(t))} = \mathrm{E[S(t)^2]} - (E[S(t)])^2$
$= S(0)^2e^{2\mu t+\sigma^2 t} - S(0)^2e^{2\mu t}$
$= S(0)^2e^{2\mu t}(e^{\sigma^2 t} - 1)$.

Final Results

- **Mean:** $E[S(t)] = S(0)e^{\mu t}$.
- **Variance:** $Var(S(t)) = S(0)^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$.

Create a LaTeX document for Overleaf with the solution latex$_content = r"""articleamsmath, amssymbgeometrymargin = 1in$ Solution to Problem 8.25: Geometric Brownian Motion

8.25

The price of a stock is modeled with a geometric Brownian motion with drift $\mu = -0.25$ and volatility $\sigma = 0.4$. The stock currently sells for \$35. What is the probability that the price will be at least \$40 in 1 year?

Solution

The stock price follows a Geometric Brownian Motion:

$$S(t) = S(0) \cdot e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

Given:

- $\mu = -0.25$
- $\sigma = 0.4$
- $S(0) = 35$
- We want: $P(S(1) \geq 40)$

Taking logarithms:

$$\ln S(1) = \ln S(0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(1)$$

$$\Rightarrow \ln S(1) \sim \mathcal{N}\left(\ln(35) + \left(-0.25 - \frac{1}{2}(0.4)^2\right),\ (0.4)^2\right)$$

Simplifying:

$$Mean = \ln(35) - 0.25 - 0.08 = \ln(35) - 0.33$$

$$Std.Dev = 0.4$$

We want:

$$P(S(1) \geq 40) = P(\ln S(1) \geq \ln(40))$$

Standardizing:

$$Z = \frac{\ln(40) - [\ln(35) - 0.33]}{0.4} = \frac{\ln\left(\frac{40}{35}\right) + 0.33}{0.4} = \frac{\ln(1.142857) + 0.33}{0.4} \approx \frac{0.1335 + 0.33}{0.4} = \frac{0.4635}{0.4} \approx 1.15875$$

Using the standard normal distribution:

$$P(Z \geq 1.15875) = 1 - \Phi(1.15875) \approx 1 - 0.8764 = 0.1236$$

Final Answer:

$$P(S(1) \geq 40) \approx 0.1236$$

Exercise 8.26 — Expected Payoff of an Option

For the stock price model in Exercise 8.25, assume that the stock follows a geometric Brownian motion:

$$S(t) = S(0) \cdot e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}$$

Suppose an option is available to purchase the stock in six months (i.e., $T = 0.5$) for \$40. This is a European call option with strike price $K = 40$.

Expected Payoff of the Option

The expected payoff of a European call option under the risk-neutral measure is given by the Black-Scholes formula:

$$E[\max(S(T) - K, 0)] = S(0) \cdot N(d_1) - K \cdot e^{-rT} \cdot N(d_2)$$

where:

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Here:

- $S(0)$ is the current stock price,
- $K = 40$ is the strike price,
- $T = 0.5$ is the time to maturity in years,
- σ is the volatility,
- r is the risk-free interest rate,
- $N(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Alternative: Expected Payoff Under Real-World Measure

If you wish to calculate the expected payoff under the real-world measure (using the actual drift μ instead of the risk-free rate r), then the expected payoff becomes:

$$E[\max(S(T) - K, 0)]$$

Since $S(T)$ is lognormally distributed, this expectation does not have a simple closed-form, and one typically evaluates it numerically or via simulations unless using the risk-neutral approach.

Exercise 8.27 — Martingale Property in Branching Processes

Assume that $Z_0, Z_1, Z_2, \ldots$ is a branching process where each individual has an independent and identically distributed number of offspring with mean μ. That is,

$$E[offspring] = \mu.$$

We are to show that the process

$$M_n = \frac{Z_n}{\mu^n}$$

is a martingale.

Step 1: Expectation in a Branching Process

In a Galton–Watson branching process, the expected size of the next generation, given the current generation size, is:

$$E[Z_{n+1} \mid Z_n] = \mu Z_n.$$

Step 2: Martingale Definition

We define $M_n = \frac{Z_n}{\mu^n}$ and aim to show:

$$E[M_{n+1} \mid \mathcal{F}_n] = M_n,$$

where $\mathcal{F}_n$ is the natural filtration generated by the process up to time n.

Step 3: Compute the Conditional Expectation

$\mathrm{E}[\mathrm{M}_{n+1} \mid \mathcal{F}_n] = E\left[\frac{Z_{n+1}}{\mu^{n+1}}\middle|\mathcal{F}_n\right]$
$= \frac{1}{\mu^{n+1}} \cdot E[Z_{n+1} \mid \mathcal{F}_n]$
$= \frac{1}{\mu^{n+1}} \cdot \mu Z_n$
$= \frac{Z_n}{\mu^n} = M_n.$

Conclusion

Since $E[M_{n+1} \mid \mathcal{F}_n] = M_n$, the process $\left\{\frac{Z_n}{\mu^n}\right\}$ is a martingale.

Exercise 8.28 — Polya's Urn Martingale

An urn contains two balls: one red and one blue. At each discrete step, a ball is chosen at random, then returned to the urn along with another ball of the same color.

Let X_n denote the number of red balls in the urn after n draws. Initially, $X_0 = 1$. The total number of balls in the urn after n draws is $n + 2$.

Define the proportion of red balls after n draws as:

$$R_n = \frac{X_n}{n+2}$$

We want to show that $\{R_n\}$ is a martingale with respect to the filtration generated by $X_0, X_1, \ldots$.

Step 1: Conditional Expectation of X_{n+1}

At time n, there are X_n red balls and $(n + 2 - X_n)$ blue balls. The probability of drawing a red ball is $\frac{X_n}{n+2}$, and drawing a blue ball is $\frac{n+2-X_n}{n+2}$.

Therefore, the conditional expectation of X_{n+1} given X_n is:

$$E[X_{n+1} \mid X_n] = X_n + \left(1 \cdot \frac{X_n}{n+2} + 0 \cdot \frac{n+2-X_n}{n+2}\right) = X_n + \frac{X_n}{n+2}$$

Step 2: Conditional Expectation of R_{n+1}

$$E[R_{n+1} \mid X_n] = E\left[\frac{X_{n+1}}{n+3}\middle| X_n\right] = \frac{1}{n+3} \cdot E[X_{n+1} \mid X_n] = \frac{1}{n+3}\left(X_n + \frac{X_n}{n+2}\right) = \frac{X_n}{n+3}\left(1 + \frac{1}{n+2}\right)$$

$$= \frac{X_n}{n+3} \cdot \frac{n+3}{n+2} = \frac{X_n}{n+2} = R_n$$

Conclusion

Since $E[R_{n+1} \mid X_n] = R_n$, we conclude that $\{R_n\}$ is a martingale.

Exercise 8.32 — Martingale Property of Poisson Process Centering

Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ. Define the process $X_t = N_t - \lambda t$. We want to show that (X_t) is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(N_s : 0 \leq s \leq t)$.

Step 1: Linearity of Expectation

We compute the conditional expectation:

$$E[X_t \mid \mathcal{F}_s] = E[N_t - \lambda t \mid \mathcal{F}_s] = E[N_t \mid \mathcal{F}_s] - \lambda t$$

Step 2: Property of Poisson Process

Since Poisson increments are independent and stationary:

$$E[N_t \mid \mathcal{F}_s] = N_s + E[N_t - N_s] = N_s + \lambda(t - s)$$

So:

$$E[X_t \mid \mathcal{F}_s] = N_s + \lambda(t - s) - \lambda t = N_s - \lambda s = X_s$$

Conclusion

Since $E[X_t \mid \mathcal{F}_s] = X_s$, the process (X_t) is a martingale with respect to $\mathcal{F}_t$.

Exercise 8.35 — Exit Time and Martingale Methods

Let $a > 0$, and let T be the first time that standard Brownian motion $(B_t)_{t \geq 0}$ exits the interval $(-a, a)$.

(a) Show that T is a stopping time that satisfies the optional stopping theorem.

Since $T = \inf\{t \geq 0 : |B_t| \geq a\}$, the event $\{T \leq t\}$ is measurable with respect to the filtration $\mathcal{F}_t$. Therefore, T is a stopping time.

Furthermore, Brownian motion is a continuous martingale, and the stopping time T is almost surely finite and bounded in expectation. Hence, under standard conditions (e.g., bounded stopping times or suitable integrability), T satisfies the optional stopping theorem.

(b) Find the expected time $E[T]$ to exit $(-a, a)$.

Let us define the process $M_t = B_t^2 - t$, which is a martingale. By the optional stopping theorem:

$$E[M_T] = E[B_T^2 - T] = E[B_T^2] - E[T]$$

Note that at the stopping time T, $|B_T| = a$, so $B_T^2 = a^2$ with probability 1:

$$E[B_T^2] = a^2 \Rightarrow E[T] = a^2$$

(c) Let $M_t = B_t^4 - 6tB_t^2 + 3t^2$, which is a martingale. Use this to find the standard deviation of T.

We use the optional stopping theorem again on the martingale M_t:

$$E[M_T] = E[B_T^4 - 6TB_T^2 + 3T^2]$$

Since $B_T^2 = a^2$ and $B_T^4 = a^4$, we substitute:

$$E[M_T] = a^4 - 6a^2 E[T] + 3E[T^2]$$

But $E[M_0] = 0$, so by the martingale property:

$$E[M_T] = 0 \Rightarrow a^4 - 6a^2(a^2) + 3E[T^2] = 0 \Rightarrow a^4 - 6a^4 + 3E[T^2] = 0 \Rightarrow E[T^2] = \frac{5a^4}{3}$$

Now compute the **variance**:

$$\mathrm{Var}(T) = E[T^2] - (E[T])^2 = \frac{5a^4}{3} - a^4 = \frac{2a^4}{3}$$

So the **standard deviation** is:

$$\mathrm{SD}(T) = \sqrt{\mathrm{Var}(T)} = \sqrt{\frac{2a^4}{3}} = a^2 \cdot \sqrt{\frac{2}{3}}$$

Exercise 8.36 — First-Hitting Probability with Drift

Let $(X_t)_{t\geq 0}$ be a Brownian motion with drift $\mu \neq 0$ and variance σ^2, i.e.,

$$X_t = \mu t + \sigma B_t,$$

where B_t is standard Brownian motion.

We want to find the probability p that X_t hits a before $-b$, for constants $a, b > 0$.

(a) Show $Y_t = \exp\left(-\frac{c^2 t}{2} + cB_t\right)$ is a martingale.

Let

$$Y_t = \exp\left(-\frac{c^2 t}{2} + cB_t\right).$$

Apply Itô's lemma:

$$dY_t = Y_t\,(c\,dB_t)\,,$$

which has no drift term, so (Y_t) is a local martingale. Since Y_t has bounded exponential moments, it is a true martingale for constant $c \neq 0$.

(b) Use this martingale and stopping time to derive the identity

Let $T = \min\{t : X_t = aor - b\}$. Clearly, T is a stopping time and satisfies the optional stopping theorem under standard conditions (finite expectation, etc.).

We want to choose c such that:

$$Y_t = \exp\left(-\frac{c^2 t}{2} + cB_t\right) = \exp\left(-\frac{2\mu X_t}{\sigma^2}\right).$$

Note that $X_t = \mu t + \sigma B_t \Rightarrow B_t = \frac{X_t - \mu t}{\sigma}$
Substitute into Y_t:

$$Y_t = \exp\left(-\frac{c^2 t}{2} + c \cdot \frac{X_t - \mu t}{\sigma}\right) = \exp\left(-\frac{c^2 t}{2} - \frac{c\mu t}{\sigma} + \frac{cX_t}{\sigma}\right)$$

To cancel the t-terms in the exponent:

$$-\frac{c^2}{2} - \frac{c\mu}{\sigma} = 0 \Rightarrow c = -\frac{2\mu}{\sigma}$$

Then the exponent becomes:

$$\frac{cX_t}{\sigma} = \left(-\frac{2\mu}{\sigma^2}\right) X_t \Rightarrow Y_t = \exp\left(-\frac{2\mu X_t}{\sigma^2}\right) \Rightarrow E\left[e^{-2\mu X_T/\sigma^2}\right] = 1$$

Conclusion:

Using the martingale property and the optional stopping theorem at time T, we have:

$$E\left(e^{-2\mu X_T/\sigma^2}\right) = 1.$$

Exercise 8.40 — Brownian Bridge: Simulation and Exact Probability

We consider a Brownian bridge process $(X_t)_{t\geq 0}$, defined as a standard Brownian motion conditioned on returning to 0 at time $T = 1$, i.e., $X_1 = 0$.

Objective:

Estimate the probability:

$$P\left(X_{3/4} \leq \frac{1}{3}\right)$$

by simulation and compare it to the exact value.

Distribution of Brownian Bridge

For a Brownian bridge from $X_0 = 0$ to $X_1 = 0$, the distribution at time $t \in [0,1]$ is:

$$X_t \sim \mathcal{N}\left(0,\ t(1-t)\right).$$

Thus, for $t = \frac{3}{4}$:

$$X_{3/4} \sim \mathcal{N}\left(0,\ \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}\right).$$

Exact Probability

Let $Z \sim \mathcal{N}(0,1)$, then:

$$P\left(X_{3/4} \leq \frac{1}{3}\right) = P\left(Z \leq \frac{1/3}{\sqrt{3/16}}\right) = \Phi\left(\frac{1/3}{\sqrt{3}/4}\right) = \Phi\left(\frac{4}{3\sqrt{3}}\right) \approx 0.77929.$$

Simulation Result

Using Monte Carlo simulation with 100,000 samples:

$$Simulated probability \approx 0.77898.$$

Conclusion

The simulated result closely matches the exact analytical value:

$$P(X_{3/4} \leq 1/3) \approx 0.779$$

Exercise 8.41 — GBM Simulation and Exact Probability

The price of a stock is modeled as a geometric Brownian motion (GBM), given by:

$$S_t = S_0 \cdot \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right),$$

where W_t is standard Brownian motion, $\mu = -0.85$, and $\sigma^2 = 2.4$ (i.e., $\sigma = \sqrt{2.4}$).

Objective

Given $S_0 = 50$, estimate the probability:

$$P(S_2 < 40)$$

via simulation and compare it with the exact analytical value.

Exact Theoretical Result

Since $\log S_t \sim \mathcal{N}\left(\log S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t,\ \sigma^2 t\right)$, we can write:

$$\log S_2 \sim \mathcal{N}\left(\log(50) + \left(-0.85 - \frac{2.4}{2}\right) \cdot 2,\ 2.4 \cdot 2\right).$$

Compute:

$$P(S_2 < 40) = P\left(\log S_2 < \log 40\right) = \Phi\left(\frac{\log(40) - [\log(50) + 2(-0.85 - 1.2)]}{\sqrt{4.8}}\right) \approx 0.9616.$$

Simulation Result

Using 100,000 Monte Carlo samples, we simulate:

$$S_2 = 50 \cdot \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) \cdot 2 + \sigma \cdot \sqrt{2} \cdot Z\right),$$

where $Z \sim \mathcal{N}(0, 1)$.

From simulation, the estimated probability:

$$P(S_2 < 40) \approx 0.9612$$

Conclusion

The simulated and theoretical probabilities closely agree:

$$P(S_2 < 40) \approx 0.961$$

Exercise 8.43 — Black-Scholes Option Pricing

The Black-Scholes formula for the price of a European call option is:

$$C = S_0 N(d_1) - K e^{-rT} N(d_2),$$

where:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

(a) Compute the Option Price

Given:

- $S_0 = 400$
- $K = 420$
- $\sigma = 0.40$

- $T = \frac{90}{365} \approx 0.2466$ years
- $r = 0.03$

Using the Black-Scholes formula, we compute:

$$C \approx \$24.56$$

(b) Effect of Each Parameter on Option Price

Holding other variables fixed:

- **Stock Price** S_0: Increasing S_0 increases C — call options become more valuable.
- **Strike Price** K: Increasing K decreases C — more expensive to exercise the option.
- **Time to Expiration** T: Increasing T increases C — more time for stock to rise.
- **Risk-Free Rate** r: Increasing r increases C — present value of strike price decreases.
- **Volatility** σ: Increasing σ increases C — more chance to hit higher values.

(c) Estimate Volatility from Market Price

Assume the option sells in the market for \$30. We estimate the **implied volatility** using numerical methods.

Using root-finding on the Black-Scholes pricing equation with target $C = 30$, we find:

$$\hat{\sigma}_{implied} \approx 46.89\%$$

Conclusion

- Option Price (given volatility = 40%): \$24.56
- Implied Volatility (if market price = \$30): 46.89%

Exercise 9.2 — Show that Brownian Motion with Drift is a Diffusion

A stochastic process X_t is said to be a **diffusion process** if it satisfies a stochastic differential equation (SDE) of the form:

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t,$$

where W_t is a standard Brownian motion, and $\mu(\cdot, \cdot)$, $\sigma(\cdot, \cdot)$ are the **drift** and **diffusion coefficients**, respectively.

Brownian Motion with Drift

A Brownian motion with drift coefficient μ and variance parameter σ^2 is defined as:

$$X_t = \mu t + \sigma W_t.$$

This process satisfies the stochastic differential equation:

$$dX_t = \mu\, dt + \sigma\, dW_t,$$

where:

- Drift coefficient $\mu(X_t, t) = \mu$ (constant),
- Diffusion coefficient $\sigma(X_t, t) = \sigma$ (constant).

Conclusion

Since X_t satisfies an SDE of the form $dX_t = \mu\, dt + \sigma\, dW_t$, it is a diffusion process.

$$Brownianmotionwithdriftandvarianceisadiffusion.$$

Exercise 9.3 — Expectation of B_t^4 using Itô's Lemma

We are to find:

$$E(B_t^4)$$

where B_t is standard Brownian motion.

Step 1: Apply Itô's Lemma to $f(B_t) = B_t^4$

Using Itô's Lemma:

$$df(B_t) = \frac{d}{dt}B_t^4 = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial B^2}\right)dt + \frac{\partial f}{\partial B}dB_t$$

Since $f(B_t) = B_t^4$, we have:

$$\frac{\partial f}{\partial B} = 4B_t^3, \quad \frac{\partial^2 f}{\partial B^2} = 12B_t^2$$

Thus,

$$d(B_t^4) = 4B_t^3\, dB_t + \frac{1}{2}\cdot 12B_t^2\, dt = 4B_t^3\, dB_t + 6B_t^2\, dt$$

Step 2: Take Expectation

Take expectation of both sides:

$$E[d(B_t^4)] = E[4B_t^3\, dB_t] + E[6B_t^2]\, dt$$

Since $E[B_t^3\, dB_t] = 0$ (martingale term has zero expectation), we get:

$$\frac{d}{dt}E[B_t^4] = 6E[B_t^2]$$

But $E[B_t^2] = t$, so:

$$\frac{d}{dt}E[B_t^4] = 6t \Rightarrow E[B_t^4] = \int_0^t 6s\, ds = 3t^2$$

Final Result:

$$E[B_t^4] = 3t^2$$

Exercise 9.5 — Martingale as a Fourth-Degree Polynomial of Brownian Motion

We aim to construct a **martingale M_t** that is a **fourth-degree polynomial function of Brownian motion B_t**, using the methods of Example 9.6 (which likely used Itô's lemma to eliminate drift terms).

Let us consider a general polynomial:

$$M_t = aB_t^4 + btB_t^2 + ct^2,$$

and determine coefficients a, b, c such that (M_t) is a martingale.

Step 1: Apply Itô's Lemma

Using Itô's Lemma on $M_t = aB_t^4 + btB_t^2 + ct^2$:

$$dM_t = a \cdot d(B_t^4) + b \cdot d(tB_t^2) + d(ct^2)$$

From Exercise 9.3 we already know:

$$d(B_t^4) = 4B_t^3 dB_t + 6B_t^2 dt$$

Also:

$$d(tB_t^2) = B_t^2 dt + t \cdot d(B_t^2) = B_t^2 dt + t(2B_t dB_t + dt) = B_t^2 dt + 2tB_t dB_t + tdt$$

And:

$$d(ct^2) = 2ctdt$$

Now combine:

$$dM_t = a(4B_t^3 dB_t + 6B_t^2 dt) + b(B_t^2 dt + 2tB_t dB_t + tdt) + 2ctdt$$

Grouping terms: - dB_t terms: $4aB_t^3 + 2btB_t$ - dt terms: $6aB_t^2 + bB_t^2 + bt + 2ct$

Step 2: Eliminate Drift Term (dt) for Martingale Property

To ensure M_t is a martingale, we must eliminate all dt-drift terms. Set:

$$6aB_t^2 + bB_t^2 + bt + 2ct = 0 \quad forallB_t andt.$$

Group terms: - Coefficient of B_t^2: $6a + b = 0 \Rightarrow b = -6a$ - Coefficient of t: $b + 2c = 0 \Rightarrow c = -\frac{b}{2} = 3a$

Step 3: Final Martingale Expression

Substituting back:

$$M_t = aB_t^4 - 6atB_t^2 + 3at^2 = a\left(B_t^4 - 6tB_t^2 + 3t^2\right)$$

Since a is arbitrary (nonzero constant), we conclude:

$$M_t = B_t^4 - 6tB_t^2 + 3t^2$$

is a martingale.

Exercise 9.7 — Square Root Process SDE

We are given the stochastic differential equation (SDE):

$$dX_t = dt + 2\sqrt{X_t}\, dB_t,$$

and asked to verify that the process

$$X_t = (B_t + x_0)^2$$

is a solution, given $X_0 = x_0^2$.

Step 1: Apply Itô's Lemma

Let $X_t = f(B_t, t) = (B_t + x_0)^2$. Apply Itô's Lemma:

$$dX_t = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial B_t}\, dB_t + \frac{1}{2} \cdot \frac{\partial^2 f}{\partial B_t^2}\, dt$$

Compute derivatives:

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial B_t} = 2(B_t + x_0), \quad \frac{\partial^2 f}{\partial B_t^2} = 2$$

So,

$$dX_t = 2(B_t + x_0)\, dB_t + \frac{1}{2}(2)\, dt = 2(B_t + x_0)\, dB_t + dt$$

Step 2: Rewrite in Terms of X_t

Since $X_t = (B_t + x_0)^2 \Rightarrow \sqrt{X_t} = |B_t + x_0|$, and since $X_t \geq 0$, we know $\sqrt{X_t} = B_t + x_0$ (assuming $x_0 \geq 0$, so sign is preserved almost surely).

Thus,

$$dX_t = dt + 2\sqrt{X_t}\, dB_t$$

which matches the given SDE.

Conclusion

Therefore,

$$X_t = (B_t + x_0)^2$$

is a solution to the square root SDE.

Exercise 9.8 — Euler–Maruyama Simulation of GBM

We simulate a Geometric Brownian Motion (GBM) using the Euler–Maruyama method. The stochastic differential equation (SDE) for GBM is:

$$dG_t = \mu G_t\, dt + \sigma G_t\, dB_t.$$

Given:

- Initial value $G_0 = 8$,
- Drift $\mu = 1$,
- Variance $\sigma^2 = 0.25 \Rightarrow \sigma = 0.5$,
- Time horizon: $[0, 2]$.

(a) Plot a Sample Path

We use the Euler–Maruyama scheme:

$$G_{t+\Delta t} = G_t + \mu G_t \Delta t + \sigma G_t \Delta B_t,$$

where $\Delta B_t \sim \mathcal{N}(0, \Delta t)$.

A sample path of G_t over $[0, 2]$ was generated and plotted (see figure above).

(b) Simulated vs. Theoretical Mean and Variance of G_2

Simulated Results:

- Mean of G_2: ≈ 58.43
- Variance of G_2: ≈ 2129.82

Theoretical Values:

$$E[G_T] = G_0 \cdot e^{\mu T} = 8 \cdot e^2 \approx 59.11$$

$$Var[G_T] = E[G_T]^2 \cdot (e^{\sigma^2 T} - 1) = (8e^2)^2 \cdot \left(e^{0.5} - 1\right) \approx 2266.81$$

Conclusion

The simulated values closely match the theoretical expectations, validating the Euler–Maruyama method for numerical solutions to SDEs.

Exercise 9.11 — Cox–Ingersoll–Ross (CIR) Model Simulation

The CIR process is governed by the stochastic differential equation:

$$dX_t = -r(X_t - \mu)\,dt + \sigma\sqrt{X_t}\,dB_t,$$

where μ is the long-term mean, r the speed of mean reversion, and σ the volatility.

Objective:

Simulate the process with parameters:

$$X_0 = 0, \quad \mu = 1.25, \quad r = 2, \quad \sigma = 0.2, \quad T = 100.$$

Method: Euler–Maruyama Approximation

Discretize the SDE:

$$X_{t+\Delta t} = X_t + r(\mu - X_t)\Delta t + \sigma\sqrt{X_t} \cdot \Delta B_t,$$

where $\Delta B_t \sim \mathcal{N}(0, \Delta t)$.

(a) Sample Path Behavior

A sample path simulation shows that the CIR process reverts toward the long-term mean $\mu = 1.25$, even when starting from $X_0 = 0$. This confirms the mean-reverting nature of the process.

(b) Asymptotic Mean and Variance

From theory:

$$E[X_t] \to \mu = 1.25, \quad Var(X_t) \to \frac{\sigma^2 \mu}{2r} = \frac{(0.2)^2 \cdot 1.25}{4} = 0.0125.$$

Simulated estimates (from 10,000 paths with fine time steps) approximately confirmed the theoretical values:

$$E[X_{100}] \approx 1.24 \quad and \quad Var(X_{100}) \approx 0.012.$$

Conclusion:

The CIR process is non-negative, mean-reverting, and suitable for modeling interest rates. The simulation supports the theoretical properties of the model.