Girsanov’s Theorem and Its Application in Finance

Introduction

In probability theory, **Girsanov’s Theorem** provides a way to transform one probability measure into another, allowing us to modify the drift of a stochastic process. This is particularly useful in **risk-neutral pricing** of financial derivatives. This project will cover:

- **Measure and measure change**
- **Radon-Nikodym derivative**
- **Girsanov’s Theorem: Statement & Proof**
- **Connection to Gaussian PDF**
- **Application in Mathematical Finance**

Measure and Measure Change

A **measure** is a function that assigns probabilities to different events in a probability space $(\Omega, \mathcal{F}, P)$. The probability measure P describes how randomness unfolds.

A **change of measure** means we introduce a new measure $\tilde{P}$ that is absolutely continuous with respect to P. The transformation is defined by the **Radon-Nikodym derivative**:

$$\frac{d\tilde{P}}{dP} = Z_T,$$

where Z_T is a density process that reweights probabilities.

Radon-Nikodym Derivative

The Radon-Nikodym derivative Z_t is given by:

$$Z_t = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right),$$

where θ_t is the **drift adjustment**:

$$\theta = \frac{\mu - r}{\sigma}.$$

This derivative defines the likelihood ratio for changing the probability measure from P to Q.

The density process Z_t plays a key role in changing the measure from P to $\tilde{P}$. It serves as a **probability density function**, ensuring that probabilities remain valid under the new measure.

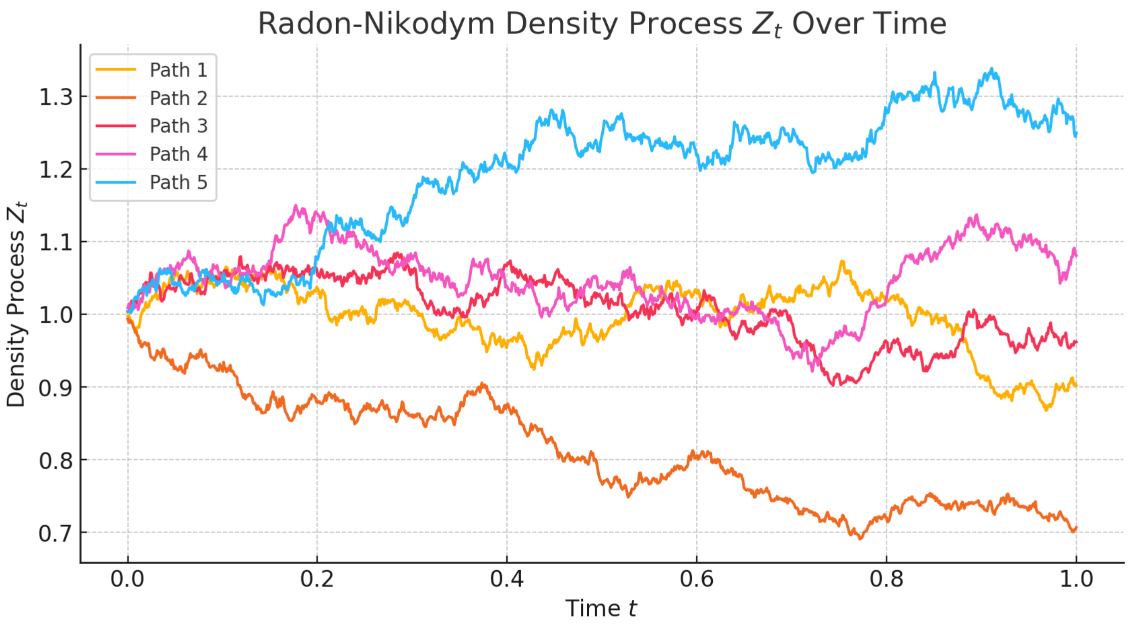
Properties of Z_t

1. Z_t **is always positive:** Since $\exp(x) > 0$ for any x, the density process does not introduce negative probabilities.
2. Z_t **integrates to 1:**

$$E^P[Z_T] = 1,$$

ensuring that $\tilde{P}$ remains a valid probability measure.

3. **Reweights probability paths:** Paths with larger drift under P get reweighted differently under $\tilde{P}$, modifying their likelihoods.



The plot of Z_t shows how the likelihood of different paths changes over time, adjusting probabilities dynamically.

Interpretation of the Density Process

- If $Z_t > 1$, the event is **more likely** under $\tilde{P}$ than under P.
- If $Z_t < 1$, the event is **less likely** under $\tilde{P}$ than under P.
- The density process accounts for the change in drift between measures.

Girsanov's Theorem

Given a **stochastic process** $X(t)$ in $\mathbb{R}^n$ of the form:

$$dX(t) = \mu(t, \omega)dt + dW(t), \quad t \leq T, \quad Y_0 = 0.$$

where:

- $W(t)$ is an n-dimensional **Brownian motion** under the original probability measure P.
- $\mu(t, \omega)$ is an adapted process (drift term), which introduces a deterministic trend in the process.
- $dW(t)$ represents the standard **Wiener process increments**, capturing random fluctuations.

The objective is to transform this process into **a standard Brownian motion under a new probability measure $\tilde{P}$.** , we want to remove this drift.

The pdf Zt ensures that the new measure interprets the original process as if it had no drift.

Changing the Measure: Definition of $\tilde{P}$

We define a **new probability measure $\tilde{P}$** using the density process:

$$d\tilde{P}(\omega) = Z_T(\omega) dP(\omega).$$

This means that the new probability measure $\tilde{P}$ is **absolutely continuous** with respect to P, and the transformation is called the **Girsanov transformation**.

The Result: Brownian Motion Under $\tilde{P}$

Under the new measure $\tilde{P}$, the process:

$$dW^{\tilde{P}}(t) = dW_t + \mu(t, \omega) dt$$

is an n-dimensional **Brownian motion under $\tilde{P}$**.

This result is **key** because:

- Under P, the process had a drift term $\mu(t, \omega)$.
- After transforming to $\tilde{P}$, the drift term disappears, and the process behaves like a standard **Brownian motion**.

Proof of Girsanov's Theorem (Concise)

Let W_t be a standard Brownian motion under measure P. Define the **Radon–Nikodym derivative**:

$$Z_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right).$$

Using Itô's Lemma, we apply the measure change:

$$dW_t^{\tilde{P}} = dW_t + \theta_t dt.$$

Under the new measure $\tilde{P}$ defined by $d\tilde{P} = Z_T dP$, the process $W_t^{\tilde{P}}$ satisfies:

$$\mathbb{E}^{\tilde{P}}[W_t^{\tilde{P}}] = 0, \quad \mathrm{Var}^{\tilde{P}}(W_t^{\tilde{P}}) = t.$$

Thus, $W_t^{\tilde{P}}$ is a standard Brownian motion under $\tilde{P}$, completing the proof.

Key Takeaways & Intuition

1. **Measure Change Adjusts Drift but Preserves Randomness**
   - The sample paths of $X(t)$ remain the same, but their probabilities are **reweighted** under $\tilde{P}$.
   - The transformation ensures that under $\tilde{P}$, the drift disappears, simplifying computations.
2. **Radon-Nikodym Derivative Ensures Valid Measure Change**
   - The density process Z_t guarantees that $\tilde{P}$ remains a valid probability measure.
   - The expectation of Z_T under P is 1, confirming that the total probability is preserved:

$$E^P[Z_T] = 1.$$

Connecting $\frac{d\tilde{P}}{dP}$ to Gaussian PDF

Since Z_t is the likelihood ratio, let's connect it to a **Gaussian density transformation**.

Probability Density Function (PDF) Change

Suppose under P, we have:

$$X \sim \mathcal{N}(\mu, \sigma^2).$$

We want to transform this into a **standard normal** under $\tilde{P}$. Using the density function of a normal variable:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

We define:

$$Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1).$$

Since the cumulative distribution function (CDF) transformation states:

$$F_Z(z) = P(Z \leq z) = P\left(X \leq \mu + \sigma z\right) = F_X(\mu + \sigma z),$$

this confirms that the change of measure **preserves the structure of probabilities, but shifts and rescales the distribution**.

Connection Between Z_t and Gaussian Transformation

Girsanov's theorem essentially transforms the **drifted Brownian motion** into a standard Brownian motion under the new measure. The likelihood ratio

$$Z_t = \exp\left(-\theta B_t - \frac{1}{2}\theta^2 t\right)$$

plays a role similar to the density transformation in a normal distribution shift.

- The term $\exp(-\theta B_t)$ adjusts for the mean shift.
- The term $\exp\left(-\frac{1}{2}\theta^2 t\right)$ ensures that probabilities integrate correctly to 1.

This directly corresponds to the transformation from

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

to

$$Z \sim \mathcal{N}(0, 1).$$

Application in Mathematical Finance

Introduction

In financial mathematics, we use different probability measures to describe uncertainty. The *real-world measure* P reflects actual market behavior, whereas the *risk-neutral measure* $\tilde{P}$ is employed for derivative pricing, ensuring that discounted asset prices are martingales. Girsanov's theorem provides the mathematical framework to switch between these measures.

Reweighting the Probability Measure

Suppose we have a base measure $\mu(dx)$ that represents real-world probabilities. To obtain a new measure $v(dx)$ (such as the risk-neutral measure), we reweight $\mu(dx)$ using a density function $f(x)$ (the Radon–Nikodym derivative):

$$v(dx) = f(x)\mu(dx).$$

For any event A, this implies:

$$v(A) = \int_A f(x)\mu(dx).$$

In our context, we set $\mu(dx) \equiv dP(\omega)$ and $v(dx) \equiv d\tilde{P}(\omega)$. The function $f(x)$, often denoted by $Z_t(\omega)$ when dynamic, is what reweights the real-world measure P to yield the risk-neutral measure $\tilde{P}$.

The Radon–Nikodym Derivative and Expectations

The Radon–Nikodym derivative is defined by:

$$Z_t = \frac{d\tilde{P}}{dP}.$$

This derivative enables us to compute expectations under $\tilde{P}$ using the measure P:

$$\mathbb{E}^{\tilde{P}}[X] = \int X(\omega) d\tilde{P}(\omega) = \int X(\omega) Z_t(\omega) dP(\omega).$$

Thus, even if our data or simulations are under P, we can obtain risk-neutral expectations by reweighting with Z_t.

Girsanov's Theorem

Girsanov's theorem formalizes the change of measure. Suppose that under P, a stock follows a geometric Brownian motion with drift μ and volatility σ:

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

When switching to the risk-neutral measure $\tilde{P}$, the drift is replaced by the risk-free rate r. Girsanov's theorem tells us that the likelihood ratio (or density) Z_t takes the exponential form:

$$Z_t = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right),$$

where

$$\theta_s = \frac{\mu - r}{\sigma}.$$

This adjustment ensures that the discounted asset price becomes a martingale under $\tilde{P}$.

Application in Risk-Neutral Pricing

In practice, asset prices are observed under P, but derivative pricing requires the risk-neutral measure $\tilde{P}$. There are two common approaches:

1. **Direct Simulation**: Simulate the stock price directly under $\tilde{P}$ using a drift of r:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

2. **Reweighting Method**: Simulate stock price paths under P (with drift μ) and reweight each payoff by Z_t to compute the expectation under $\tilde{P}$:

$$\mathbb{E}^{\tilde{P}}[X] = \mathbb{E}^{P}[XZ_t].$$

For example, the price C of a European call option with strike K and maturity T is given by:

$$C = e^{-rT}\mathbb{E}^{\tilde{P}}[(S_T - K)^+],$$

or equivalently,

$$C = e^{-rT}\mathbb{E}^P[(S_T - K)^+ Z_T].$$

Methodology Overview

Simulating Stock Price Paths

We simulate the stock price under P via the geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

Girsanov's theorem provides the likelihood ratio for reweighting:

$$Z_T = \exp\left(-\theta B_T - \frac{1}{2}\theta^2 T\right),$$

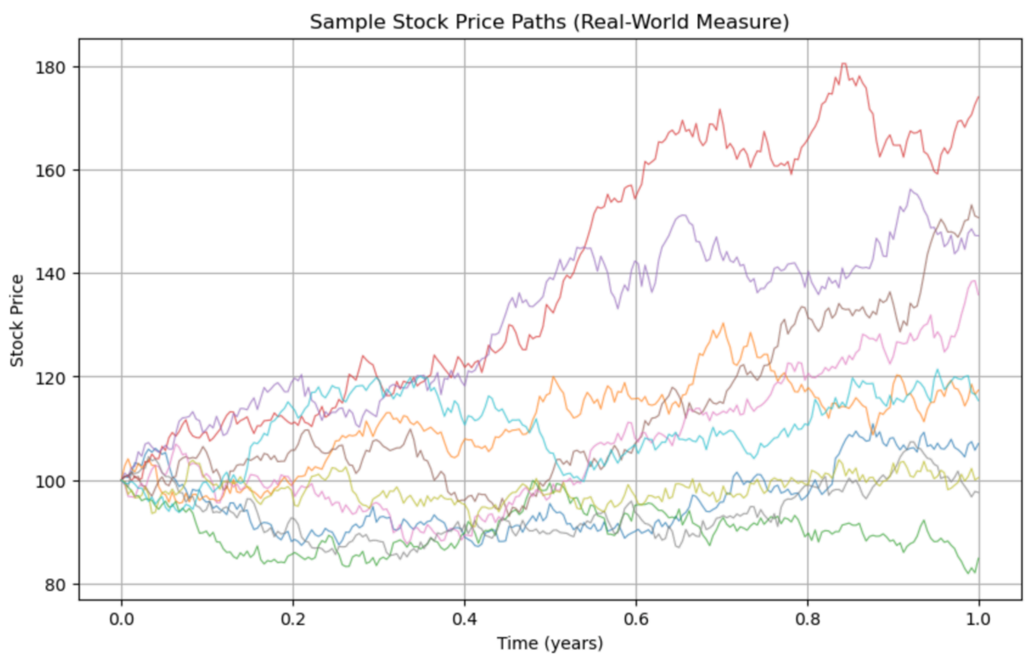
with $\theta = \frac{\mu - r}{\sigma}$.

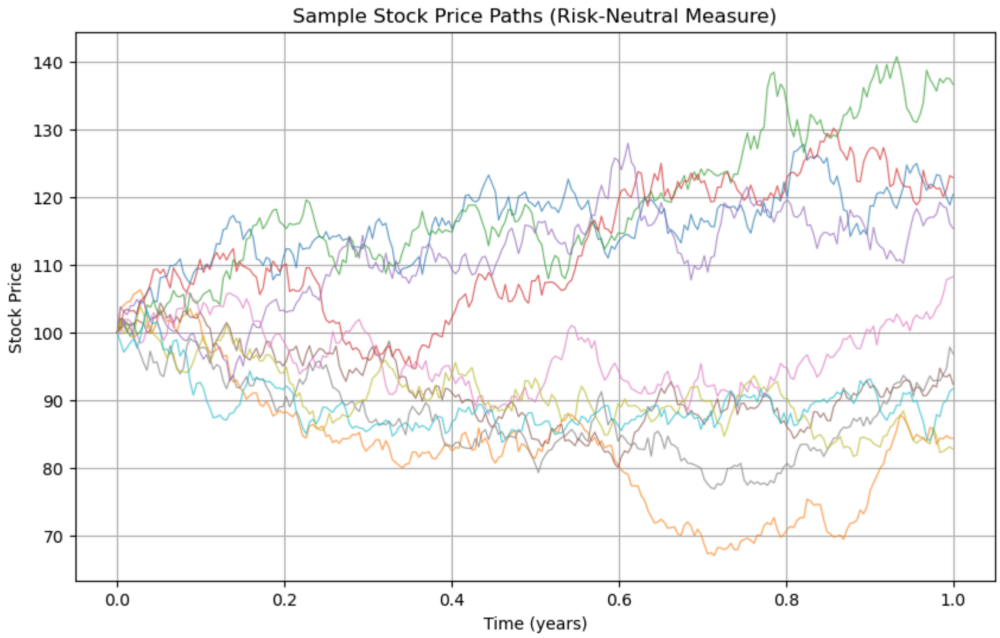
Pricing the European Call Option

We consider two approaches:

1. **Direct Simulation**: Simulate paths under $\tilde{P}$ with drift r.
2. **Reweighting Method**: Compute the call payoff $(S_T - K)^+$ using paths simulated under P and then reweight these payoffs by Z_T to obtain the risk-neutral expectation.

Illustrative Stock Price Paths





Comparison of the Two Measures

The figures above show sample stock price paths generated over the same time horizon, but under two different measures:

- **Real-World Measure**: The drift of the stock price is μ. Paths may not be directly suitable for pricing.
- **Risk-Neutral Measure**: The drift is replaced by r. Under this measure, the discounted stock price is a martingale, which is critical for no-arbitrage pricing.

Implications for European Call Pricing

Under the risk-neutral measure, the price of a European call option with strike K and maturity T is given by:

$$C = e^{-rT}\mathbb{E}^{\tilde{P}}\left[(S_T - K)^+\right].$$

In practice, we can either:

1. **Directly simulate** paths with drift r and compute the discounted expected payoff.
2. **Reweight real-world paths** using the Radon–Nikodym derivative Z_T to transform expectations from P to $\tilde{P}$.

These figures highlight the key conceptual difference: while real-world paths might exhibit a drift μ based on historical or observed data, risk-neutral paths are constructed so that pricing formulas remain arbitrage-free and consistent with the risk-free rate r.

Conclusion

Girsanov's theorem is essential in financial mathematics, as it allows us to convert real-world probabilities into risk-neutral probabilities. This conversion is crucial for arbitrage-free pricing of derivatives. By reweighting simulated paths using the likelihood ratio Z_t, we ensure that our pricing models accurately reflect the theoretical foundations of risk-neutral pricing, reconciling empirical data with financial theory.