

Probabilistic Primes

From Primes To Riemann

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May 3, 2021

Probabilistic Models Of Primes

- Probabilistic models of primes can match numerical evidence fairly well.
- Predictions can become worthy conjectures.
- We'll build a simple probabilistic model to predict the distribution of prime gaps.

- Prime Number Theorem: density of primes around x is approximately

$$\frac{1}{\ln(x)}$$

- Not a big leap to interpret this density as a probability.

$$\Pr(n \text{ prime}) = \frac{1}{\ln(n)}$$

- A prime gap at n of size $g(n) = 4$ is a sequence

prime, not prime, not prime, not prime, prime

- The probability of this sequence is

$$\frac{1}{\ln(n)} \cdot \left(1 - \frac{1}{\ln(n+1)}\right) \cdot \left(1 - \frac{1}{\ln(n+2)}\right) \cdot \left(1 - \frac{1}{\ln(n+3)}\right) \cdot \frac{1}{\ln(n+4)}$$

- For $n \gg a$ we can approximate $\ln(n + a) \approx \ln(n)$

$$\left(1 - \frac{1}{\ln(n)}\right)^3 \cdot \left(\frac{1}{\ln(n)}\right)^2$$

- Generalise to prime gaps of size k having a sequence {prime, $k-1$ not prime, prime}

$$\left(1 - \frac{1}{\ln(n)}\right)^{k-1} \cdot \left(\frac{1}{\ln(n)}\right)^2$$

- **Expected** number of gaps of size k amongst first N numbers is approximately

$$\sum_n^N \left(1 - \frac{1}{\ln(n)}\right)^{k-1} \cdot \left(\frac{1}{\ln(n)}\right)^2$$

- For most n between 1 and N , $\ln(n) \approx \ln(N)$, so probability simplifies

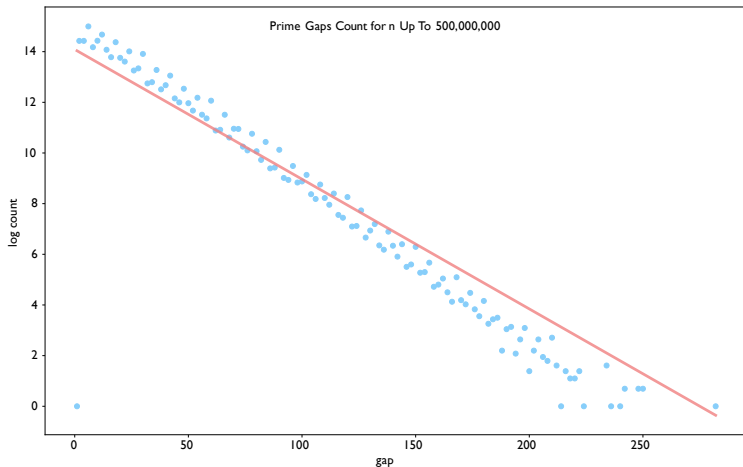
$$N \cdot \left(1 - \frac{1}{\ln(N)}\right)^{k-1} \cdot \left(\frac{1}{\ln(N)}\right)^2$$

- Taking logs gives us a linear function of gap size k of the form $Ak + B$

$$(k - 1) \cdot \ln \left(1 - \frac{1}{\ln(N)} \right) + 2 \ln \left(\frac{1}{\ln(N)} \right) + \ln(N)$$

- The line will have a negative gradient A because it is the logarithm of $\left(1 - \frac{1}{\ln(N)} \right) < 1$.

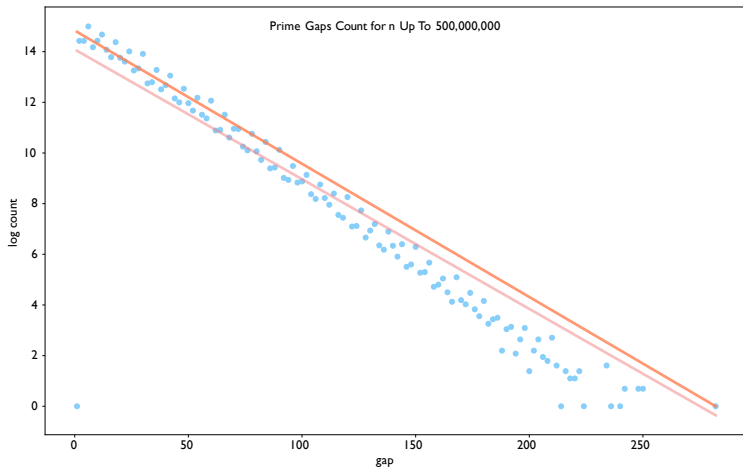
Prime Gap Counts



- Simple model ignores fact even numbers are never prime.
- Idea
 - if we want to assert $\Pr(\text{even}) = 0$
 - but also preserve PNT density of primes is $1/\ln(x)$ in the neighbourhood of x
 - we can double $\Pr(\text{even})$ to $2/\ln(x)$
- Same method, but applied to $N/2$ numbers, gives expected gaps

$$\frac{N}{2} \cdot \left(1 - \frac{2}{\ln(N)}\right)^{k-1} \cdot \left(\frac{2}{\ln(N)}\right)^2$$

Improved Model



Prime Gaps And Estimates Animated

- Youtube: <https://www.youtube.com/watch?v=85hrakisiP0>

What Have We Learned?

- A coin-tossing experiment would give us a linear log count of Head or Tail sequences.
 - suggests occurrence of primes approximately mimics independent “random events”
- The order-of-magnitude suggests the PNT based $1/\ln(x)$ probability is about right.

Compare to Coin Tosses



- <https://www.desmos.com/calculator/zyyiczghuy>