Infinite Products From Primes To Riemann

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August 4, 2021

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Infinite Sum

• At school we learn a lot about infinite sums.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

- "Sum" \leftrightarrow "Series".
- What do we really mean by infinite sum?

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Infinite Sum

 We say an infinite series converges if the limit of partial sums tends to a finite value.

$$\lim_{N\to\infty}\sum_{n=1}^N a_n=S$$

Tests exist to check for convergence, eg the ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$$



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Infinite Product

At school we don't seem to learn about infinite products.

$$\prod_{n=1}^{\infty} a_n = a_1 \times a_2 \times a_2 \times \dots$$

• What do we really mean by infinite product?



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Initial Observations

 Example 1 - Easy to see the infinite product diverges. Each factor increases the size of the product.

$$2 \times 3 \times 4 \times 5 \times \dots$$

 Example 2 - Fundamental idea that multiplying by zero causes a product to be zero.

$$2 \times 0 \times 4 \times 5 \times \dots$$



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Initial Observations

• Example 3 - Each factor reduces the size of the product.

$$\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \dots$$

- Infinite number of such factors, the product \rightarrow 0.
- We have found two different ways an infinite product can be zero.

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Definition

 An infinite product is defined, like infinite series, as the limit of a sequence.

$$\prod_{n=1}^{\infty} a_n = \lim_{N \to \infty} \prod_{n=1}^{N} a_n$$



• Does this converge?

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n} \right)$$

Consider partial product.

$$\prod_{n=1}^{N} \left(1 + \frac{1}{n} \right) = \prod_{n=1}^{N} \left(\frac{n+1}{n} \right)$$

$$= \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \dots \times \frac{N+1}{M}$$

$$= N+1$$

• As $N \to \infty$, product **diverges**.



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• Does this converge?

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)$$

• Note *n* starts at 2.



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• Again, consider partial product.

$$\prod_{n=2}^{N} \left(1 - \frac{1}{n} \right) = \prod_{n=2}^{N} \left(\frac{n-1}{n} \right)$$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{N+1}{N}$$

$$= \frac{1}{N}$$

• As $N \to \infty$, product tends to 0, so product diverges to zero.

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Convergence

- For an infinite series $\sum a_n$ to converge, the terms $a_n \to 0$
 - Intuition: if each term $a_n > \epsilon$, then $\sum a_n > \sum \epsilon = \infty$
 - (see Cauchy criterion for more rigour)
- ullet For an infinite product $\prod a_n$ to converge, the terms $a_n o 1$
 - Intuition: if each term $a_n > 1$, the product gets ever larger.
 - If each term $a_n < 1$, the product gets ever smaller towards zero.

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Removing Zero Factors

- A single factor 0 collapses entire product to zero.
- If an infinite product has a **finite number of zero-valued factors**, they can be removed and remaining product studied.
- Example:

$$\prod_{n=1} (1 - \frac{1}{n^2}) = 0$$

Removing first factor leaves an interesting infinite product:

$$\prod_{n=2} (1 - \frac{1}{n^2}) = \frac{1}{2}$$

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• Useful to write factors as $(1 + a_n)$

$$P=\prod(1+a_n)$$

Turn product into sum by takings logarithm

$$ln(P) = ln \prod (1+a_n) = \sum ln(1+a_n)$$

• Using $1 + x \le e^x$

$$ln(P) \leq \sum a_n$$

• If the sum is **bounded** \implies the product is bounded. If $a_n > 0$ then boundedness is **convergence** (no oscillation).

• If we expand out product $\prod (1+a_n)$ we see another inequality.

$$1+\sum a_n \leq \prod (1+a_n)=P$$

- The expansion creates the terms $1 + \sum a_n$ and many more
- ullet This tells us that if the product converges \Longrightarrow so does the sum.

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• The two results together give us

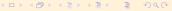
$$\sum a_n$$
 converges $\Leftrightarrow \prod (1+a_n)$ converges, for $a_n>0$

- This allows us to say:
 - $\prod (1+1/n)$ diverges because $\sum 1/n$ diverges
 - $\prod (1+1/n^2)$ converges because $\sum 1/n^2$ converges

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Divergence To Zero

- The logarithmic view of infinite products has an interesting side effect.
- ullet If the partial products o 0 then the logarithm o $-\infty$
- This is why we say the product diverges to zero.



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- First convergence critera applies to **real** values $a_n > 0$.
- Would be good to have criteria for **complex** a_n .
- To do that we'll need an intermediate result about absolute values $|a_n|$

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• Start by assuming $\sum |a_n|$ converges to a finite S

$$S = \sum |a_n| < \infty$$

Consider partial product

$$p_N = \prod^N (1 + |a_n|)$$

• Using $1 + x \le e^x$

$$p_N \le e^{\sum^N |a_n|} \le e^S < \infty$$

• p_N are monotonically increasing, but always $\leq e^S \implies p_N$ converges

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• Need to show opposite direction too. Assume product converges.

$$P = \prod (1 + |a_n|)$$

- We know $|a_n| \to 0$, so $|a_n| < 2$ for n at least some finite value M
- We can use $e^{x/2} \le 1 + x$ for $0 \le x \le 2$

$$e^{|a_n|/2} \leq 1 + |a_n|$$
 for $n \geq M$



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- Set Q to be the infinite product but starting at n = M.
- Q converges because it is P but with a finite number of factors removed.

$$Q=\prod_M(1+|a_n|)$$

• Using $|a_n| < 2$

$$e^{rac{1}{2}\sum_{M}|a_{n}|}\leq Q<\infty ext{ for } n\geq M$$

• We can see that $\sum_{M} |a_n| \le 2 \ln(Q) < \infty$, so $\sum |a_n|$ converges.

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• We have a new constraint

$$\sum |a_n|$$
 converges $\Leftrightarrow \prod (1+|a_n|)$

• We can use this to show $\prod_2 (1 - 1/n)$ diverges.

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- We're interested in $\prod (1+a_n)$ for complex a_n , not just $\prod (1+|a_n|)$.
- The key:

$$\sum |a_n|$$
 converges $\implies \sum a_n$ converges

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Let's start with two partial products

$$ho_N = \prod^N (1+a_n)$$

$$q_N = \prod^N (1 + |a_n|)$$

- We assert $a_n \neq -1$ to ensure no zero-valued factors.
- Should be intuitively clear that

$$|p_N-1|\leq q_N-1$$

• For $N > M \ge 1$, we can compare $|p_N - p_M|$ with $|q_N - q_M|$

$$egin{aligned} |p_N - p_M| &= |p_M| \cdot |rac{p_N}{p_M} - 1| \ &= |p_M| \cdot |\prod_{M + 1}^N (1 + a_n) - 1| \ &\leq |q_M| \cdot |\prod_{M + 1}^N (1 + |a_n|) - 1| \ &= |q_M| \cdot |rac{q_N}{q_M} - 1| \ &= |q_M - q_M| \end{aligned}$$

• If $|q_N - q_M| < \epsilon$, then $|p_N - p_M| < \epsilon$. Cauchy criterion for convergence.

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• Finally we have

$$\sum |a_n|$$
 converges $\implies \prod (1+a_n)$ converges, for $a_n
eq -1$

 This is one way, we can't say the sum comverges if the product converges.

Summary

• Real a_n

$$\sum a_n$$
 converges $\Leftrightarrow \prod (1+a_n)$ converges, for $a_n>0$

• Complex *a_n*

$$\sum |a_n|$$
 converges $\Leftrightarrow \prod (1+|a_n|)$

Complex a_n

$$\sum |a_n|$$
 converges $\implies \prod (1+a_n)$ converges, for $a_n
eq -1$

Riemann Zeta Function

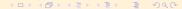
• For $\sigma > 1$

$$\zeta(s) = \sum \frac{1}{n^s} = \prod (1 - \frac{1}{\rho^s})^{-1}$$

- No factor $(1-1/p^s)^{-1}$ is zero.
 - That would require p^s to be zero.
 - This isn't possible, and is easy to see by writing

$$|p^s| = \left|e^{s\ln(p)}\right| = e^{\sigma\ln(p)} > 0$$

Also need to check product doesn't diverge to zero.



Riemann Zeta Function

$$\sum |a_n|$$
 converges $\implies \prod (1+a_n)$ converges, for $a_n
eq -1$

• Let's see if $\sum |-1/p^s|$ converges

$$\sum \left| -\frac{1}{p^s} \right| = \sum \frac{1}{p^{\sigma}} \le \sum \frac{1}{n^{\sigma}}$$

- So $1/\zeta(s)=\prod (1-\frac{1}{\rho^s})$ converges to a non-zero value. And so $\zeta(s)=\prod (1-\frac{1}{\rho^s})^{-1}$ converges to a non-zero value.
- \Longrightarrow Riemann Zeta function has no zeros $\sigma > 1$