

Integral Comparison Tests

From Primes To Riemann

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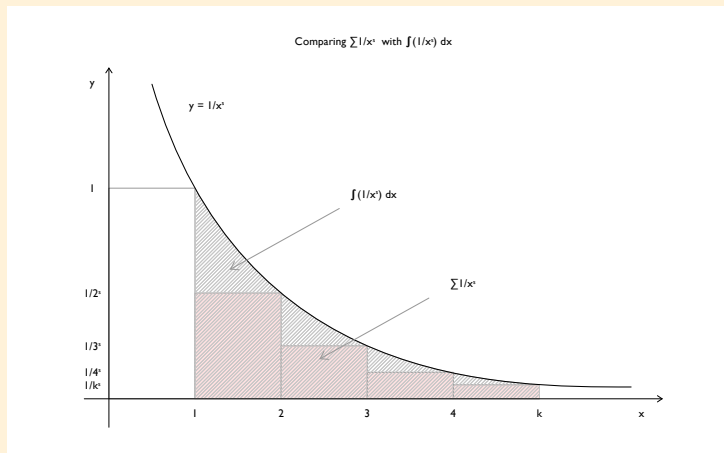
Discrete vs Continuous Functions

- Understanding the behaviour of continuous functions is often easier than discrete functions.

$$\sum \frac{1}{x} \text{ compared with } \int \frac{1}{x} dx$$

- Simple but powerful technique used a lot in number theory.

Convergence Of $\zeta(s) = \sum 1/n^s$



- graph of $y = \frac{1}{x^s}$, with rectangles representing the fractions $\frac{1}{x^s}$

Convergence Of $\zeta(s) = \sum 1/n^s$

- Looking at $1 \leq x \leq 4$, area of three pink rectangles is **less** than the area under curve $\int_1^4 \frac{1}{x^s} dx$.
- By extending range to $1 \leq x \leq k$, we can make a general observation.

$$\sum_2^k \frac{1}{x^s} < \int_1^k \frac{1}{x^s} dx$$

Convergence Of $\zeta(s) = \sum 1/n^s$

- Sum starts at 2 because we're looking at rectangles extending to the left of a given x .
- We can adjust the limit of the sum using $\sum_1^k \frac{1}{x^s} = 1 + \sum_2^k \frac{1}{x^s}$.

$$\sum_1^k \frac{1}{x^s} - 1 < \int_1^k \frac{1}{x^s} dx$$

Convergence Of $\zeta(s) = \sum 1/n^s$

- The integral is easily evaluated.

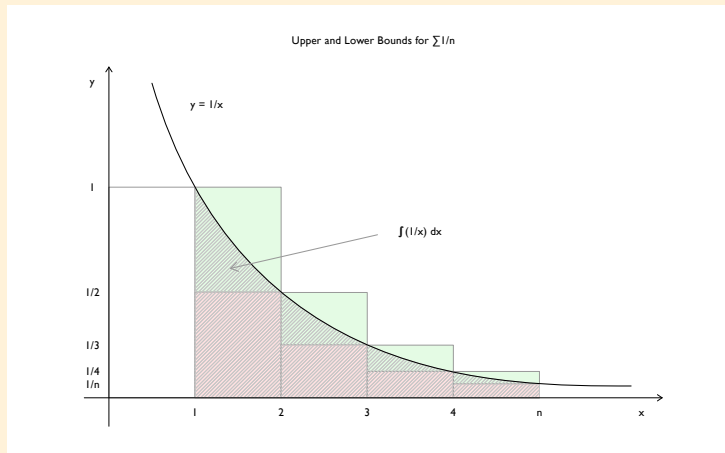
$$\sum_1^k \frac{1}{x^s} - 1 < \frac{k^{1-s} - 1}{1-s}$$

- Looking at the denominator, $s \neq 1$.
- Looking at the numerator, only way k^{1-s} won't diverge as $k \rightarrow \infty$ is if $1-s < 0$. That is, $s > 1$.

Convergence Of $\zeta(s) = \sum 1/n^s$

- **Riemann Zeta** function $\zeta(s) = \sum 1/n^s$ only converges for $s > 1$.

The Growth Of $\sum 1/n$



- graph of $y = \frac{1}{x}$, together with rectangles representing the fractions $\frac{1}{n}$.

Lower Bound For Growth Of $\sum 1/n$

- Looking at $1 \leq x \leq 4$, area of the three taller green rectangles $1 + \frac{1}{2} + \frac{1}{3}$ is **greater** than the area under the curve $\int_1^4 \frac{1}{x} dx$.
- By extending range to $1 \leq x \leq n$, we can make a general observation.

$$\sum_1^n \frac{1}{x} > \int_1^{n+1} \frac{1}{x} dx$$

- $n + 1$ because the width of the last rectangle extends from $x = n$ to $x = n + 1$.

Lower Bound For Growth Of $\sum 1/n$

- We can perform the integral to simplify the expression.

$$\sum_{1}^n \frac{1}{x} > \ln(n+1)$$

- Rather nice lower bound on the growth of the harmonic series.

Upper Bound For Growth Of $\sum 1/n$

- Looking at the range $1 \leq x \leq 4$, area of the three shorter rectangles $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ is **less** than the area under the curve $\int_1^4 \frac{1}{x} dx$.
- Again, by extending the range to n we can make a general observation.

$$\sum_2^n \frac{1}{x} < \int_1^n \frac{1}{x} dx$$

Upper Bound For Growth Of $\sum 1/n$

- Sum starts at 2 because we're looking at rectangles extending to the left of a given x .
- We can adjust the limit of the sum using $\sum_1^n \frac{1}{x} = 1 + \sum_2^n \frac{1}{x}$.

$$\sum_1^n \frac{1}{x} - 1 < \int_1^n \frac{1}{x} dx$$

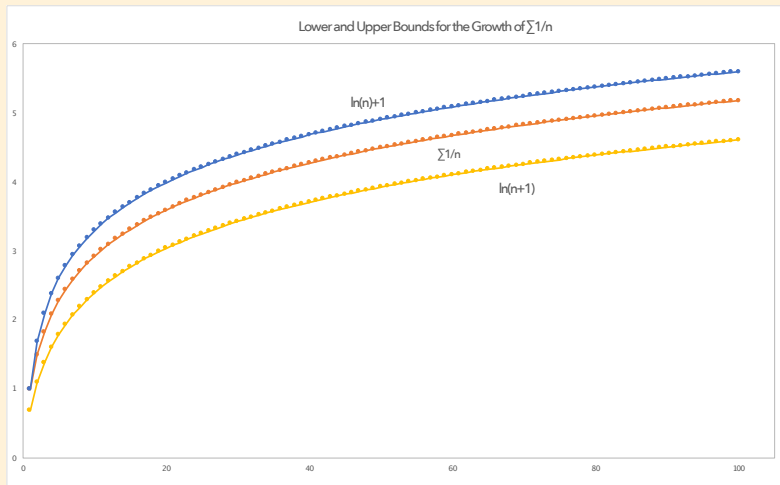
Upper Bound For Growth Of $\sum 1/n$

- We can perform the integral.

$$\sum_{1}^n \frac{1}{x} < \ln(n) + 1$$

- A nice upper bound to the growth of the harmonic series.

Visualisation



- Caution - experimental evidence isn't proof.