

# Into The Complain Domain

## From Primes To Riemann

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# Previously ...

- The Riemann Zeta function encodes information about the primes.
  - infinite primes
  - primes aren't so sparse,  $\sum 1/p$  diverges

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

- We've considered  $s$  as
  - an **integer**,  $s = 1$  harmonic series,  $s = 2$  Basel problem
  - a **real** value, we proved  $\zeta(s)$  diverges for  $s > 1$

- Riemann was the first to consider  $s$  as a **complex number**.
- If we think  $\zeta(s)$  over the complex domain might reveal new insights into the primes, we need to understand how it behaves.
- Exploring where it **converges** is a good start.
- Traditional to write complex  $s$  as  $s = \sigma + it$

$$\sum \frac{1}{n^s} = \sum \frac{1}{n^{\sigma+it}} = \sum \frac{1}{n^{\sigma}} \frac{1}{n^{it}}$$

# Convergence For $\sigma > 1$

- **Triangle inequality** - magnitude of sum  $\leq$  sum of magnitudes.

$$\left| \sum \frac{1}{n^s} \right| \leq \sum \left| \frac{1}{n^\sigma} \frac{1}{n^{it}} \right|$$

- Rewriting  $n^{it}$  as  $e^{it \ln(n)}$  makes clear it has a magnitude of 1.

$$\left| \sum \frac{1}{n^s} \right| \leq \sum \frac{1}{n^\sigma}$$

- We know  $\sum 1/n^\sigma$  converges for real  $\sigma > 1 \implies \sum 1/n^s$  converges when  $\sigma > 1$ .

# Divergence For $\sigma \leq 0$

- Let's look again at the terms in the sum.

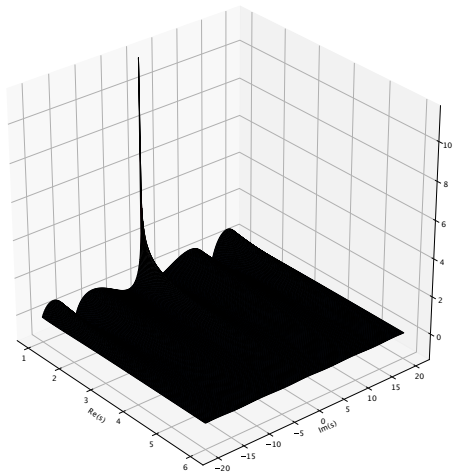
$$\left| \frac{1}{n^\sigma} \right| = \left| \frac{1}{n^\sigma} \frac{1}{n^{it}} \right| = \frac{1}{n^\sigma}$$

- If  $\sigma < 0$ , the magnitude of the terms grows larger than 1.
- If  $\sigma = 0$ , the magnitude of each term is exactly 1.
- For any series to converge, a necessary requirement is that the terms get smaller towards zero  $\implies \sum 1/n^\sigma$  diverges for  $\sigma \leq 0$ .

# Divergence For $\sigma < 1$

- $\zeta(s)$  converges for  $\sigma > 1$ , diverges for  $\sigma \leq 0$ . We're left with a gap  $0 < \sigma \leq 1$ .
- To fill this gap we need to understand more generally when series of the form  $\sum a_n/n^s$ , called **Dirichlet series**, converge or diverge.
- Dirichlet series converge in **half-planes** to the right of an **abscissa of convergence**  $\sigma_c$ . That is, they converge at any  $s = \sigma + it$  where  $\sigma > \sigma_c$ .
- Because  $\zeta(s)$  converges for  $\sigma > 1$ , and we know it diverges at  $s = 1 + 0i$ , then  $\sigma_c = 1 \implies \zeta(s)$  diverges for  $\sigma < 1$ .

# Visualising The Zeta Function For $s > 1$



# Visualising The Zeta Function For $s > 1$

- Spike around  $s = 1 + 0i$  corresponds to divergent harmonic series  $\zeta(1)$ .
- Surface seems to smooth out to the right as  $\sigma$  grows larger. To what value?

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

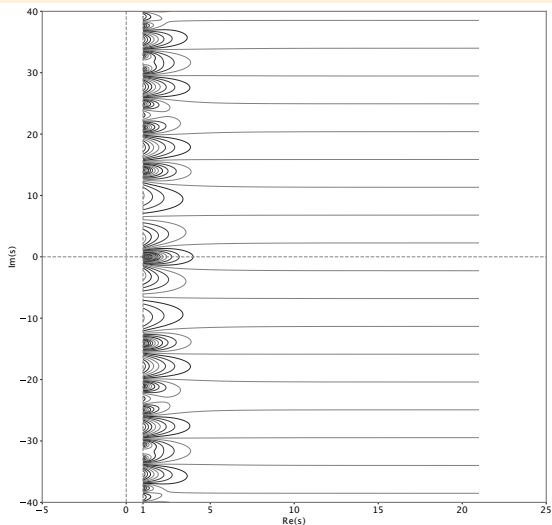
- As  $s \rightarrow \infty$  all the terms  $1/n^s \rightarrow 0$ , except the first term which remains 1.
- To be more precise, the magnitude of each term  $|n^{-s}| = n^{-\sigma}$  tends to zero as  $\sigma \rightarrow \infty$  for all  $n > 1 \implies |\zeta(s)| \rightarrow 1$  as  $\sigma \rightarrow \infty$ .



# Hints The Function Extends Into $\sigma \leq 1$

- Aside from  $s = 1 + 0i$ , the function doesn't seem to diverge along the line  $s = 1 + it$ .
- It looks like the surface has been prematurely cut off, and would continue smoothly into  $\sigma \leq 1$  if allowed.

# Isolines of $|\zeta(s)|$



- The intuition that a function should continue smoothly without abrupt changes corresponds to a powerful property of many functions we come across.
- If we can show  $\zeta(s)$  is one of these well-behaved functions, we are then justified in trying to extend it to the left of  $\sigma = 1$ .