1 $\Sigma 1/n^2$ Converges

The sum of the reciprocals of the square numbers was a particularly difficult challenge, first posed around 1650, and later named the Basel problem.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Although there are more modern proofs, we will follow Euler's original proof from 1734 because his methods were pretty audacious, and later influenced Reimann's work on the prime number theorem.

Taylor Series For sin(x)

We start with the familiar Taylor series for $sin(x)$.

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$$

Euler's New Series For sin(x)

The polynomial $f(x) = (1 - \frac{x}{a})(1 + \frac{x}{a})$ has factors $(1 - \frac{x}{a})$ and $(1 + \frac{x}{a})$, and zeros at $+a$ and $-a$. We can shorten it to $f(x) = (1 - \frac{x^2}{a^2})$.

Euler's novel idea was to write $sin(x)$ as a product of similar linear factors, which would lead him to a different series.

The zeros of $sin(x)$ are at $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ so the product of factors looks like the following.

$$sin(x) = A \cdot x \cdot \left(1 - \frac{x^2}{\pi^2}\right) \cdot \left(1 - \frac{x^2}{(2\pi)^2}\right) \cdot \left(1 - \frac{x^2}{(3\pi)^2}\right) \cdot \ldots$$

The constant A is 1 because we know $\frac{sin(x)}{x} \to 1$ as $x \to 0$. Alternatively, taking the first derivative of both sides gives $A = 1$ when $x = 0$.

The second factor is x and not x^2 because the zero of $sin(x)$ at $x = 0$ has multiplicity 1.

Euler then expanded out the series.

$$sin(x) = x \cdot \left[1 + \frac{x^2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) + X\right]$$

Inside the square brackets, the terms with powers of x higher than 2 are contained in X.

Comparing The Two Series

The terms in x must be the same in Euler's new series and the Taylor series because they both represent $sin(x)$. Let's pick out the x^3 terms from both series.

$$\frac{x^3}{3!} = \frac{x^3}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

We can easily rearrange this to give us the desired infinite sum.

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Euler, aged 28, had solved the long standing Basel problem, not only proving the infinite series of squared reciprocals converged, but giving it an exact value.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Rigour

Euler's original proof was adventurous in expressing $sin(x)$ as an infinite product of simple linear factors. It made intuitive sense, but at the time was not rigourously justified.

It was almost 100 years later when Weierstass developed and proved a factorisation theorem that confirmed Euler's leap was legitimate.