# Infinite Products From Primes To Riemann

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August 2, 2021

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#### Infinite Sum

• At school we learn a lot about infinite sums.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

- "Sum"  $\leftrightarrow$  "Series".
- What do we really mean by infinite sum?

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#### Infinite Sum

 We say an infinite series converges if the limit of partial sums tends to a finite value.

$$\lim_{N\to\infty}\sum_{n=1}^N a_n=S$$

Tests exist to check for convergence, eg the ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$$



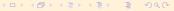
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#### Infinite Product

At school we don't seem to learn about infinite products.

$$\prod_{n=1}^{\infty} a_n = a_1 \times a_2 \times a_2 \times \dots$$

What do we really mean by infinite product?



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#### Initial Observations

 Example 1 - Easy to see the infinite product diverges. Each factor increases the size of the product.

$$2 \times 3 \times 4 \times 5 \times \dots$$

 Example 2 - Fundamental idea that multiplying by zero causes a product to be zero.

$$2 \times 0 \times 4 \times 5 \times \dots$$



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#### Initial Observations

• Example 3 - Each factor reduces the size of the product.

$$\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \dots$$

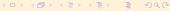
- Infinite number of such factors, the product  $\rightarrow$  0.
- We have found two different ways an infinite product can be zero.

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#### Definition

 An infinite product is defined, like infinite series, as the limit of a sequence.

$$\prod_{n=1}^{\infty} a_n = \lim_{N \to \infty} \prod_{n=1}^{N} a_n$$



• Does this converge?

$$\prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)$$

• Consider partial product.

$$\prod_{n=1}^{N} \left( 1 + \frac{1}{n} \right) = \prod_{n=1}^{N} \left( \frac{n+1}{n} \right)$$

$$= \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \dots \times \frac{N+1}{M}$$

$$= N+1$$

• As  $N \to \infty$ , product **diverges**.



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• Does this converge?

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)$$

• Note *n* starts at 2.



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• Again, consider partial product.

$$\prod_{n=2}^{N} \left( 1 - \frac{1}{n} \right) = \prod_{n=2}^{N} \left( \frac{n-1}{n} \right)$$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{N+1}{N}$$

$$= \frac{1}{N}$$

• As  $N \to \infty$ , product tends to 0, so product diverges to zero.

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#### Convergence

- For an infinite series  $\sum a_n$  to converge, the terms  $a_n \to 0$ 
  - Intuition: if each term  $a_n > \epsilon$ , then  $\sum a_n > \sum \epsilon = \infty$
  - (see Cauchy criterion for more rigour)
- ullet For an infinite product  $\prod a_n$  to converge, the terms  $a_n o 1$ 
  - Intuition: if each term  $a_n > 1$ , the product gets ever larger.
  - If each term  $a_n < 1$ , the product gets ever smaller towards zero.

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#### Removing Zero Factors

- A single factor 0 collapses entire product to zero.
- If an infinite product has a **finite number of zero-valued factors**, they can be removed and remaining product studied.
- Example:

$$\prod_{n=1} (1 - \frac{1}{n^2}) = 0$$

Removing first factor leaves an interesting infinite product:

$$\prod_{n=2} (1 - \frac{1}{n^2}) = \frac{1}{2}$$

• Useful to write factors as  $(1 + a_n)$ 

$$P = \prod (1 + a_n)$$

Turn product into sum by takings logarithm

$$ln(P) = ln \prod (1+a_n) = \sum ln(1+a_n)$$

• Using  $1 + x \le e^x$ 

$$ln(P) \leq \sum a_n$$

• If the sum is **bounded**  $\implies$  the product is bounded. If  $a_n > 0$  then boundedness is **convergence** (no oscillation).

• If we expand out product  $\prod (1+a_n)$  we see another inequality.

$$1+\sum a_n \leq \prod (1+a_n) = P$$

- The expansion creates the terms  $1 + \sum a_n$  and many more
- ullet This tells us that if the product converges  $\Longrightarrow$  so does the sum.

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• The two results together give us

$$\sum a_n$$
 converges  $\Leftrightarrow \prod (1+a_n)$  converges, for  $a_n>0$ 

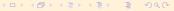
- This allows us to say:
  - $\prod (1+1/n)$  diverges because  $\sum 1/n$  diverges
  - $\prod (1+1/n^2)$  converges because  $\sum 1/n^2$  converges

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#### Divergence To Zero

- The logarithmic view of infinite products has an interesting side effect.
- ullet If the partial products o 0 then the logarithm o  $-\infty$
- This is why we say the product diverges to zero.



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- First convergence critera applies to **real** values  $a_n > 0$ .
- Would be good to have criteria for **complex**  $a_n$ .
- To do that we'll need an intermediate result about absolute values  $|a_n|$

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• Start by assuming  $\sum |a_n|$  converges to a finite S

$$S = \sum |a_n| < \infty$$

Consider partial product

$$p_N = \prod^N (1 + |a_n|)$$

• Using  $1 + x \le e^x$ 

$$p_N \le e^{\sum^N |a_n|} \le e^S < \infty$$

•  $p_N$  are monotonically increasing, but always  $\leq e^S \implies p_N$  converges

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• Need to show opposite direction too. Assume product converges.

$$P = \prod (1 + |a_n|)$$

- We know  $|a_n| \to 0$ , so  $|a_n| < 2$  for n at least some finite value M
- We can use  $e^{x/2} \le 1 + x$  for  $0 \le x \le 2$

$$e^{|a_n|/2} \le 1 + |a_n|$$
 for  $n \ge M$ 



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- Set Q to be the infinite product but starting at n = M.
- Q converges because it is P but with a finite number of factors removed.

$$Q=\prod_{M}(1+|a_n|)$$

• Using  $|a_n| < 2$ 

$$e^{rac{1}{2}\sum_{M}^{N}|a_n|}\leq \prod_{M}^{N}(1+|a_n|)\leq Q<\infty ext{ for } n\geq M$$

• We can see that  $\sum_{M}^{N} |a_n| \leq 2 \ln(Q) < \infty$ , so  $\sum |a_n|$  converges.

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• We have a new constraint

$$\sum |a_n|$$
 converges  $\Leftrightarrow \prod (1+|a_n|)$ 

• We can use this to show  $\prod_2 (1 - 1/n)$  diverges.

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- We're interested in  $\prod (1+a_n)$  for complex  $a_n$ , not just  $\prod (1+|a_n|)$ .
- The key:

$$\sum |a_n|$$
 converges  $\implies \sum a_n$  converges

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Let's start with two partial products

$$p_N = \prod^N (1+a_n)$$

$$q_N = \prod^N (1 + |a_n|)$$

- We assert  $a_n \neq -1$  to ensure no zero-valued factors.
- Should be intuitively clear that

$$|p_N-1|\leq q_N-1$$

• For  $N > M \ge 1$ , we can compare  $|p_N - p_M|$  with  $|q_N - q_M|$ 

$$egin{aligned} |p_N - p_M| &= |p_M| \cdot |rac{p_N}{p_M} - 1| \ &= |p_M| \cdot |\prod_{M + 1}^N (1 + a_n) - 1| \ &\leq |q_M| \cdot |\prod_{M + 1}^N (1 + |a_n|) - 1| \ &= |q_M| \cdot |rac{q_N}{q_M} - 1| \ &= |q_M - q_M| \end{aligned}$$

• If  $|q_N - q_M| < \epsilon$ , then  $|p_N - p_M| < \epsilon$ . Cauchy criterion for convergence.

• Finally we have

$$\sum |a_n|$$
 converges  $\implies \prod (1+a_n)$  converges, for  $a_n 
eq -1$ 

 This is one way, we can't say the sum comverges if the product converges.

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## Summary

• Real  $a_n$ 

$$\sum a_n$$
 converges  $\Leftrightarrow \prod (1+a_n)$  converges, for  $a_n>0$ 

• Complex a<sub>n</sub>

$$\sum |a_n|$$
 converges  $\Leftrightarrow \prod (1+|a_n|)$ 

Complex a<sub>n</sub>

$$\sum |a_n|$$
 converges  $\implies \prod (1+a_n)$  converges, for  $a_n 
eq -1$ 

#### Riemann Zeta Function

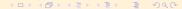
• For  $\sigma > 1$ 

$$\zeta(s) = \sum \frac{1}{n^s} = \prod (1 - \frac{1}{p^s})^{-1}$$

- No factor  $(1-1/p^s)^{-1}$  is zero.
  - That would require  $p^s$  to be zero.
  - This isn't possible, and is easy to see by writing

$$|p^s| = \left|e^{s\ln(p)}\right| = e^{\sigma\ln(p)} > 0$$

Also need to check product doesn't diverge to zero.



#### Riemann Zeta Function

$$\sum |a_n|$$
 converges  $\implies \prod (1+a_n)$  converges, for  $a_n 
eq -1$ 

• Let's see if  $\sum |-1/p^s|$  converges

$$\sum \left| -\frac{1}{p^s} \right| = \sum \frac{1}{p^{\sigma}} \le \sum \frac{1}{n^{\sigma}}$$

- So  $1/\zeta(s) = \prod (1-\frac{1}{p^s})$  converges to a non-zero value. And so  $\zeta(s) = \prod (1 - \frac{1}{p^s})^{-1}$  converges to a non-zero value.
- $\Longrightarrow$  Riemann Zeta function has no zeros  $\sigma > 1$

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