A New Riemann Zeta Series

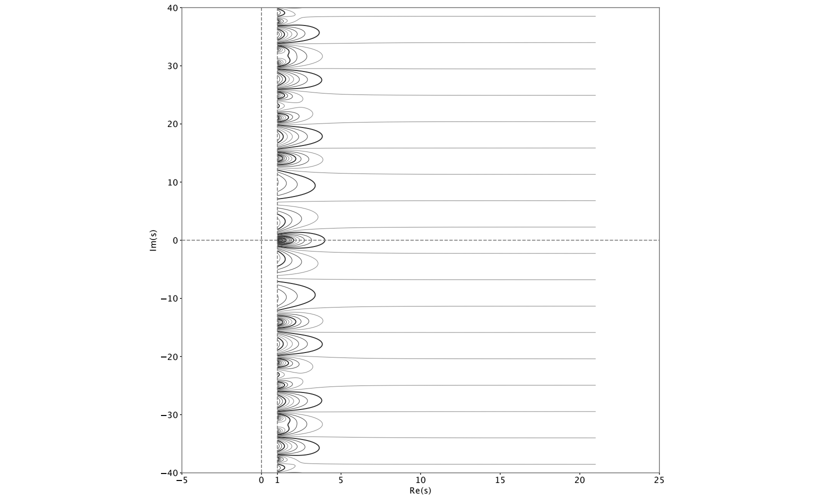
From Primes To Riemann

Tariq Rashid

August 27, 2021

Previously ...

- visual hint the Riemann Zeta function was artificially cut off at $\sigma = 1$



Series & Functions

- Taylor series expansion of $f(x) = (1-x)^{-1}$ developed around $x = 0$.

$$S_0 = 1 + x + x^2 + x^3 + \dots$$

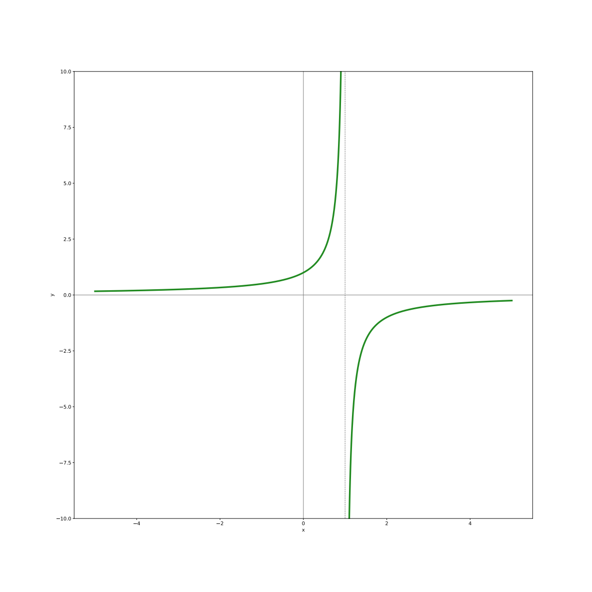
- S_0 is only valid for $|x| < 1$
- $f(x)$ is defined for all x except $x = 1$.
- Discrepancy?

Series & Functions

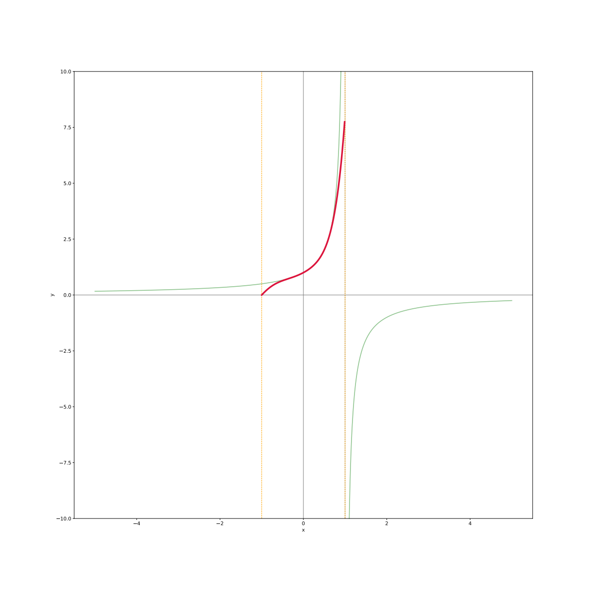
- That series S_0 is **just one** representation of the function, valid for **some** of that function's domain, specifically $|x| < 1$.
- We can find a different representation of $f(x)$ valid outside $|x| < 1$.
- For example, S_3 is a Taylor series developed around $x = 3$, and is valid for $1 < x < 5$.

$$S_3 = -\frac{1}{2} + \frac{1}{4}(x-3) - \frac{1}{8}(x-3)^2 + \frac{1}{16}(x-3)^3 - \dots$$

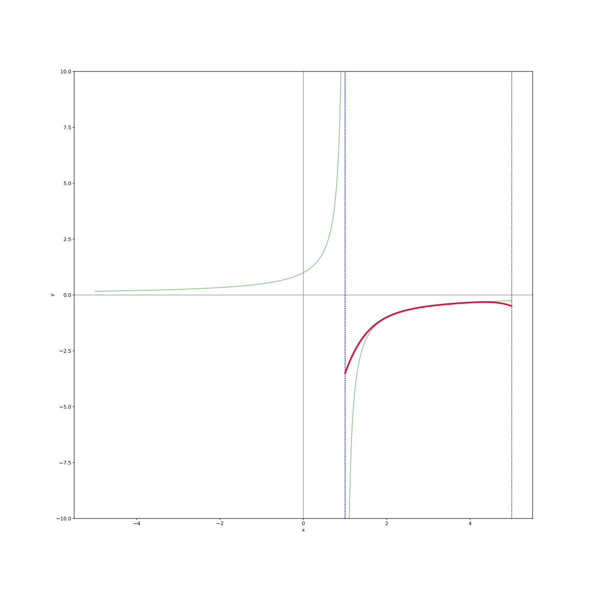
$f(x) = (1 - x)^{-1}$



S_0 Valid For $-1 < x < 1$



S_3 Valid For $1 < x < 5$



Series & Functions

- So S_0 and S_3 **both** represent $f(x) = (1 - x)^{-1}$ but over **different** parts of its domain.

- Distinction between a function, and series which represent it in different parts of its domain.

Question

- Perhaps the series $\sum 1/n^s$ only gives us a partial view of a much richer function that encodes information about the primes.
- Could that function be represented by a different series over a different domain?

A New Series

- An alternating version of the zeta function is called the **eta** function $\eta(s)$.

$$\eta(s) = \sum \frac{(-1)^{n+1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \dots$$

- This is a Dirichlet series which converges for $\sigma > 0$.
- If we could express $\zeta(s)$ in terms of $\eta(s)$, we would have a new series for the Riemann Zeta function that extends to the left of $\sigma = 1$, as far as $\sigma > 0$.

A New Series

- Let's write out the familiar series for $\zeta(s)$.

$$\zeta(s) = \sum \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$$

- Looking at the difference $\zeta(s) - \eta(s)$, we can see a pattern to exploit.

$$\begin{aligned}
\zeta(s) - \eta(s) &= \frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + \dots \\
&= \frac{2}{2^s}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots\right) \\
&= 2^{1-s}\zeta(s)
\end{aligned}$$

A New Series

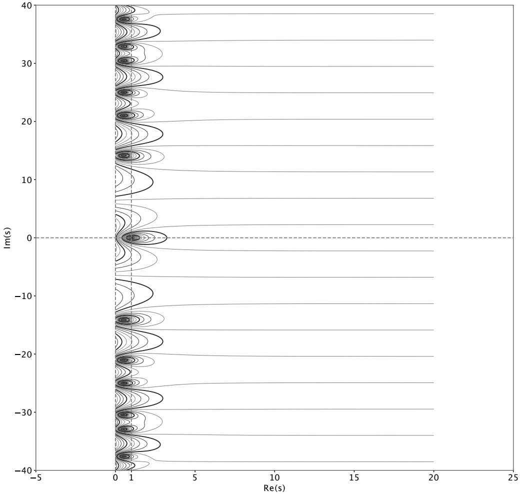
- Isolating $\zeta(s)$ gives us a new series that is valid in the larger domain $\sigma > 0$

$$\boxed{\zeta(s) = \frac{1}{1-2^{1-s}} \sum \frac{(-1)^{n+1}}{n^s}}$$

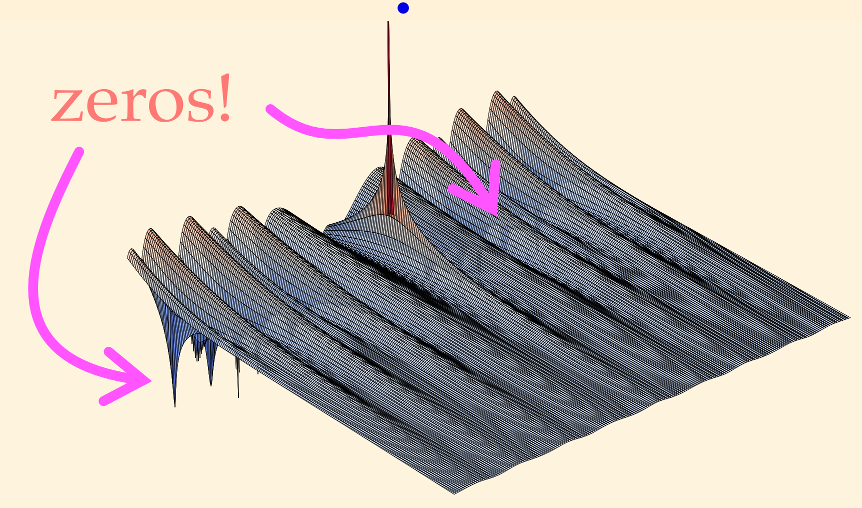
- The denominator $(1 - 2^{1-s})$ is zero at $s = 1 + 0i$, and provides $\zeta(s)$ with its divergence at that point.

Visualising The New Series

- Surface does continue smoothly to the left of $\sigma = 1$ ... and looks like it should continue $\sigma < 0$.



Visualising The New Series



Zeros

- All the zeros seem to be on the line $s = 1/2 + it$.
- What does it mean that the Riemann Zeta function has zeros in the complex domain?
- What is the significance of them appearing to be along the line $s = 1/2 + it$?