Gaps Between Primes

From Primes To Riemann

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Intuition

- We've seen the primes
  - Can't be so thinly spread out that $\sum 1/p$ converges.
  - Occur with an average density of $1/\ln(x)$ around the number x.
- This suggests the gaps between primes must be constrained.
- Let's find out ...

Factorials

- The factorial $n!$ of a whole number n is the product of all the whole numbers between 1 and n.

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

- For example, $4! = 4 \times 3 \times 2 \times 1 = 24$.
- For $n > 2$ factorials are definitely not prime - by definition they are products.

Prime-Free Sequence

- Let's look at the sequence of numbers from $(5! + 2)$ to $(5! + 5)$.

$$
\begin{aligned}
5! + 2 &= (5 \cdot 4 \cdot 3 \cdot 1 + 1) \cdot 2 \\
5! + 3 &= (5 \cdot 4 \cdot 2 \cdot 1 + 1) \cdot 3 \\
5! + 4 &= (5 \cdot 3 \cdot 2 \cdot 1 + 1) \cdot 4 \\
5! + 5 &= (4 \cdot 3 \cdot 2 \cdot 1 + 1) \cdot 5
\end{aligned}
$$

- We can see 2 is a factor of $(5! + 2)$, 3 is a factor of $(5! + 3)$, 4 is a factor of $(5! + 4)$, and 5 is a factor of $(5! + 5)$.

- So the sequence $(5! + 2)$ to $(5! + 5)$ is entirely free of primes.

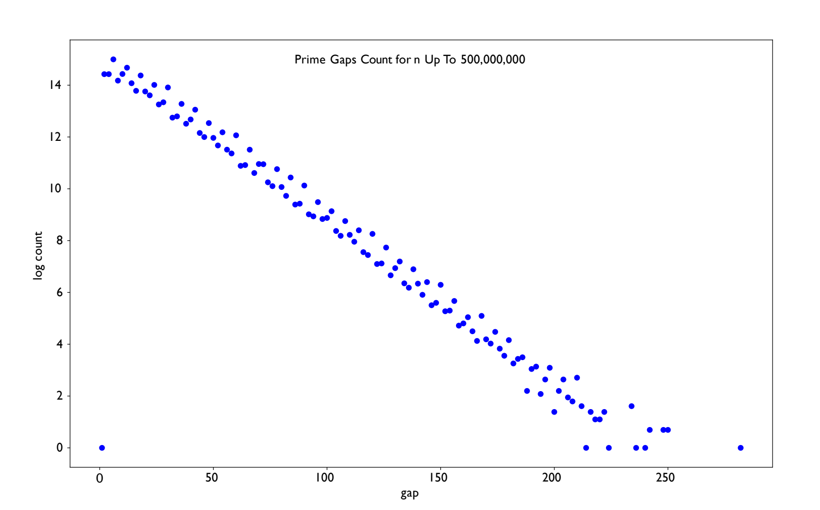
No Upper Limit On Prime Gaps

- Using the same method, we can show any sequence that follows the same pattern $(n! + 2)$ to $(n! + n)$ is prime-free.
- This is a sequence of length $n - 1$.
- Because n can be as large as desired, we've just proved **there is no upper limit on prime gaps**.
- ... Easy!

Thoughts

- The primes continue to surprise us:
  - The primes can't be too sparse.
  - There is no upper limit on the gaps between primes.
- Both facts can be true if large prime gaps are rare.

Distribution Of Prime Gaps



Distribution Of Prime Gaps

- It is surprising to see the distribution so tightly constrained:
  - The overall shape appears to be very linear
  - clear regions above and below the line with no prime gap counts
- Note: prime gap is defined as $g(n) = p_{n+1} - p_n$, so prime gap for 3 is 5, not 1.

Twin Prime Conjecture

- Looking at the distribution, gaps of size 2 seem to occur more often than almost all other sizes.
- These **twin primes** so seem to pop up more often than expected:

$$(3,5),(5,7),(11,13),(17,19)\dots(101,103),(107,109)\dots$$

- The **Twin Prime Conjecture** says there are infinite twin primes.
    - This is yet another simple statement about primes that remains unproven.