# Swapping $\lim \sum$ For $\sum \lim$

## From Primes To Riemann

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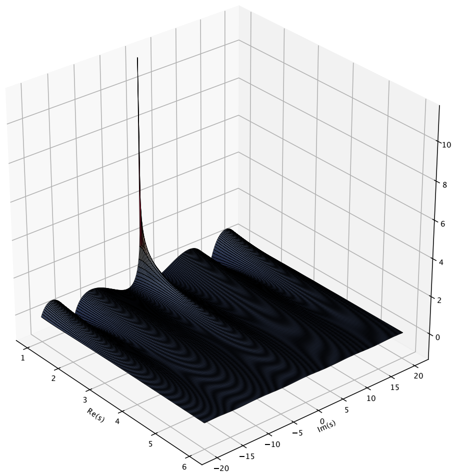
# Previously ...

- $|\zeta(s)|$ looks like it $\to 1$ as $\sigma \to +\infty$



# $\zeta(s)$ as $\sigma \to +\infty$

$$\lim_{\sigma\to\infty} \sum_n \frac{1}{n^s} = \lim_{\sigma\to\infty} \left( \frac{1}{1^s} + \frac{1}{2^s} + \dots \right)$$

- Tempting to say $|n^{-s}| = n^{-\sigma} \to 0$ as $\sigma \to \infty$ for all $n$ except $n = 1$, then conclude $\zeta(s) \to 1$ as $\sigma \to \infty$.

- In effect, this takes the limit inside the sum.

$$\sum_n \lim_{\sigma\to\infty} \frac{1}{n^s} = \lim_{\sigma\to\infty} \left( \frac{1}{1^s} \right) + \lim_{\sigma\to\infty} \left( \frac{1}{2^s} \right) + \dots$$

- However, the limit of an infinite sum is not always the sum of the limits.
- Tannery's Theorem tells us when we can swap sum and limit operators.

# Hints The Function Extends Into $\sigma \leq 1$

- The theorem has three requirements

  1. An infinite sum $S_j = \sum_k f_k(j)$ that converges
  2. The limit $\lim_{j\to\infty} f_k(j) = f_k$ exists
  3. An $M_k \geq |f_k(j)|$ independent of $j$, where $\sum_k M_k$ converges

- If the requirements are met, we can take the limit inside the sum.

$$\lim_{j\to\infty} \sum_k f_k(j) = \sum_k \lim_{j\to\infty} f_k(j)$$

# Application To $\zeta(s)$

1. We start with the convergent infinite sum. Here $f_k(j)$ is $f_n(s) = 1/n^s$.

$$\zeta(s) = \sum_n \frac{1}{n^s} \text{ converges for } \sigma > 1$$

2. We confirm $f_n(s)$ exists when $\sigma \to \infty$.

$$\lim_{\sigma \to \infty} \frac{1}{n^s} = f_n = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

3. We also find an $M_n \geq |f_n(s)|$ independent of $\sigma$.

$$\left| \frac{1}{n^s} \right| = \frac{1}{n^\sigma} \leq M_n = \frac{1}{n^\alpha}$$

Here $1 < \alpha \leq \sigma$. The sum $\sum_n M_n$ converges because $\alpha > 1$.

# Application To $\zeta(s)$

- The criteria have been met, so we can legitimately move the limit inside the sum.

$$\lim_{\sigma \to \infty} \sum_n \frac{1}{n^s} = \sum_n \lim_{\sigma \to \infty} \frac{1}{n^s} = 1 + 0 + 0 + \dots$$

- So $\zeta(s) \to 1$, as $\sigma \to +\infty$.

# Proof

$$\lim_{j\to\infty} \sum_k f_k(j) = \sum_k \lim_{j\to\infty} f_k(j)$$

- Let's first show the RHS sum of the limit actually exists.
- By definition, $|f_k(j)| \leq M_k$, and $\sum_k M_k$ converges, so $\sum_k |f_k(j)|$ also converges.
- Therefore $\sum_k f_k(j)$ converges absolutely, including as $j \to \infty$.
- That is, the sum of limits $\sum_k \lim_{j\to\infty} f_k(j)$ converges.

# Proof

- Now let's show the LHS limit of the sum is the RHS sum of the limits.
- The following easy inequality will be useful.

$$|f_k(j) - f_k| \leq |f_k(j)| + |f_k| \leq M_k + M_k = 2M_k$$

- Since $\sum_k M_k$ converges there must be an $N$ so that $\sum_{k=N} M_k < \epsilon$, where $\epsilon$ is as small as we require.

$$\left|\sum_{k=N} f_k(j)\right| \leq \sum_{k=N} |f_k(j)| \leq \sum_{k=N} M_k < \epsilon$$

- The following is the case when $j \to \infty$.

$$\left|\sum_{k=N} f_k\right| \leq \sum_{k=N} |f_k| \leq \sum_{k=N} M_k < \epsilon$$

# Proof

- Consider the absolute difference between $\sum_k f_k(j)$ and $\sum_k f_k$.
- Looks complicated, but it is simply splitting the sums over $[0, \infty]$ into sums over $[0, N-1]$ and $[N, \infty]$.

$$\left|\sum_k f_k(j) - \sum_k f_k\right| = \left|\sum_k^{N-1} f_k(j) + \sum_{k=N} f_k(j) - \sum_k^{N-1} f_k - \sum_{k=N} f_k\right|$$

$$\leq \left|\sum_{k=N} f_k(j)\right| + \left|\sum_{k=N} f_k\right| + \left|\sum_k^{N-1} f_k(j) - \sum_k^{N-1} f_k\right|$$

$$< 2\epsilon + \left|\sum_k^{N-1} (f_k(j) - f_k)\right|$$

# Proof

- As $j \to \infty$, the finite sum $\sum_k^{N-1} (f_k(j) - f_k) \to 0$, which leaves a simpler inequality.

$$\lim_{j\to\infty} \left| \sum_k f_k(j) - \sum_k f_k \right| < 2\epsilon$$

- Because $\epsilon$ can be as small as we require, we have $\lim_{j\to\infty} \sum_k f_k(j) = \sum_k f_k$, which proves the theorem.

$$\lim_{j\to\infty} \sum_k f_k(j) = \sum_k f_k = \sum_k \lim_{j\to\infty} f_k(j)$$