1 Euler's Golden Bridge

Euler was the first to find a connection between the world of primes and the world of ordinary counting numbers. This 'golden bridge' has become a path through which many new insights about the primes have been revealed.

Let's recreate Euler's discovery and experience some of his genius ourselves.

A Simple Series

Let's start with a simple and familiar series, valid for $|x| < 1$.

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \ldots$$

Every power of x is in the series, something we'll be making use of.

In our enthusiasm to involve the primes, we might be tempted to say that x is a prime p. This won't work because $|x|$ needs to be < 1, and the primes are all larger than 1.

One idea is to set x to $\frac{1}{p}$ which is always < 1.

$$\frac{1}{(1-\frac{1}{p})} = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \ldots$$

Again, every power of that prime p can be found in this series.

Multiplying Two Series

Now imagine taking that expression for one prime p_1, and multiplying it with the expression for a different prime p_2.

$$\frac{1}{(1-\frac{1}{p_1})} \cdot \frac{1}{(1-\frac{1}{p_2})} = \left(1 + \frac{1}{p_1} + \frac{1}{p_1^2} + \dots\right) \cdot \left(1 + \frac{1}{p_2} + \frac{1}{p_2^2} + \dots\right)$$

If we multiplied out the product on the right, we would get every combination of powers for p_1 and p_2 . For example, somewhere in that series will be denominators $p_1^7 \cdot p_2^3$, as well as p_1^8 and p_2^{99}.

Also, each combination will appear only once. We wouldn't get p_1^2 appearing twice, for example.

All The Primes

Let's now extend the product, from two primes p_1 and p_2, to all primes p_i. The symbol $\prod$ means product, just like $\sum$ means sum.

$$\prod_{p_i} \frac{1}{(1-\frac{1}{p_i})} = \prod_{p_i} \left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \dots\right)$$

If we multiplied out the product, again, we would get every combination of powers for every combination of primes p_i. For example, somewhere in that series will be denominators $p_1^2 \cdot p_2^{10} \cdot p_5^2$, as well as p_{23}^{43} and p_{76}.

And again, each combination will appear only once.

Unique Prime Factors

We saw earlier the fundamental theorem of arithmetic tell us that every positive integer n can be expressed as a unique product of prime factors.

The series we just multipled out contains every combination of powers for every combination of all primes. That is, it contains all the unique prime factorisations of every positive integer.

For example, the number 15750 has prime factors $2 \cdot 3^2 \cdot 5^3 \cdot 7$, and that series will have a term $(2 \cdot 3^2 \cdot 5^3 \cdot 7)^{-1}$ in it.

Euler's Product Formula

If that series contains the unique prime factorisation of every whole number n, and only contains it once, then we can finally make the magic leap that leads to **Euler's product formula**.

$$\boxed{\prod_p \frac{1}{(1-\frac{1}{p})} = \sum_n \frac{1}{n}}$$

The product of $\frac{1}{(1-\frac{1}{p})}$ over all primes p, is the sum of $\frac{1}{n}$ over all positive integers.

To say this result is amazing would not be an exaggeration. It reveals a previously well-hidden connection between the primes and the ordinary counting numbers.

Even better, that connection is beautifully simple. And that simplicity looks ripe for revealing more insights into the primes.