$\sum \frac{1}{p}$ Grows Like $\log \log x$

From Primes To Riemann

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How Does $\sum \frac{1}{p}$ Grow?

- $\sum 1/n$ diverges
- $\sum 1/n$ grows like $\log n$

- $\sum 1/p$ diverges
- How does $\sum 1/p$ grow?

Euler's 1737 Assertion

- *"The sum of the reciprocals of the prime numbers,*

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \dots$$

is infinitely great but is infinitely times less than the sum of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

And the sum of the former is the as the logarithm of the sum of the latter."

- Widely interpreted as meaning, for large x

$$\sum_{p \leq x} \frac{1}{p} \approx \log \log x$$

Euler's 1737 Assertion

Theorema 19.

Summa ſeriei reciprocae numerorum primorum

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

eſt infinite magna; infinities tamen minor, quam ſumma ſeriei harmonicae $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ *etc. Atque illius ſummae eſt huius ſummae quaſi logarithmus.*

- https://scholarlycommons.pacific.edu/euler-works/72/

Pollack's Proof

- Paul Pollack's *"Euler and the Partial Sums of the Prime Harmonic Series"*
    - doesn't require advanced knowledge
    - but does require a bit of bookwork
- `http://pollack.uga.edu/eulerprime.pdf`

Proof Overview

- Find an $S_0 = \sum_{p \le x} 1/p$
- Find an $S_U \ge S_0$
- Find an $S_L \le S_0$

- Hopefully S_U and S_L are nice expressions for lower and upper bounds on $\sum_{p \le x} 1/p$
- Trick is to find S that are easy (enough) to work with

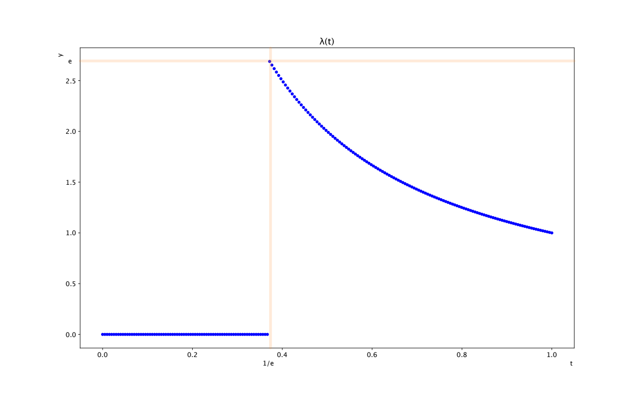
Start With $S_0 = \sum_{p \le x} 1/p$

$$S(\lambda, x) = \sum_{p} p^{-1-\frac{1}{\log x}} \cdot \lambda(p^{-\frac{1}{\log x}})$$

- $S(\lambda, x)$ is a function of a function $\lambda(t)$ defined over a small domain $t \in [0, 1]$
- Looks convoluted but can be simplified easily if we choose $\lambda(t)$ carefully

Start With $S_0 = \sum_{p \le x} 1/p$

$$\lambda_0(t) = \begin{cases} 1/t & \text{if } 1/e \le t \le 1 \\ 0 & \text{if } 0 \le t < 1/e \end{cases}$$



Start With $S_0 = \sum_{p \leq x} 1/p$

- Let's see what $1/e \leq t \leq 1$ means

$$
\begin{aligned}
1/e &\leq t \leq 1 \\
1/e &\leq p^{-\frac{1}{\log x}} \leq 1 \\
-1 &\leq -\frac{1}{\log x} \log p \leq 0 \\
-\log x &\leq -\log p \leq 0 \\
0 &\leq p \leq x
\end{aligned}
$$

- This is the range we're interested in for $\sum_{p \leq x} 1/p$.
- Checking $0 \leq t < 1/e$ leads to $x < p < \infty$ which contibutes 0 to $S(\lambda_0, x)$

Start With $S_0 = \sum_{p \le x} 1/p$

- This definition of $\lambda_0(t)$ reduces $S(\lambda_0, x)$ nicely:

$$
\begin{aligned}
S(\lambda_0, x) &= \sum_p p^{-1-\frac{1}{\log x}} \cdot \lambda(p^{-\frac{1}{\log x}}) \\
&= \sum_p p^{-1-\frac{1}{\log x}} \cdot p^{+\frac{1}{\log x}} \\
&= \boxed{\sum_p \frac{1}{p}}
\end{aligned}
$$

for $0 \le p \le x$

Linear Forms For λ_U, λ_L

- Let's try a linear form for $\lambda(t) = \alpha + \beta t$

$$
\begin{aligned}
S(\lambda, x) &= \sum_p p^{-1-\frac{1}{\log x}} \cdot \lambda(p^{-\frac{1}{\log x}}) \\
&= \sum_p p^{-1-\frac{1}{\log x}} \cdot (\alpha + \beta \cdot p^{-\frac{1}{\log x}}) \\
&= \alpha \cdot P(1 + \frac{1}{\log x}) + \beta \cdot P(1 + \frac{2}{\log x})
\end{aligned}
$$

- We need something to help us with $P(1+s)$.