$\sum \frac{1}{n}$ Diverges

From Primes To Riemann

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Question

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots \to ?$$

Infinite Series

- Infinite series is a sum of infinite terms.

$$1 + 1 + 1 + 1 + \dots$$

- Easy to see this sum is infinitely large - the series **diverges**.

Infinite Series

- Another example with terms getting smaller.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

- Intuitively we can see this gets closer to 2.
- The sum **tends to 2**. Many simply say the sum is in fact 2.
- This series **converges**.

Harmonic Series

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

- Each term is smaller than the previous one. Is this enough to say it converges?
- In fact it **diverges**.

Oresme's Proof ~1300AD

- Start with harmonic series.

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

- Now group terms in brackets of size $2, 4, 8, 16 \dots$

$$S = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$

Oresme's Proof ~1300AD

- Create new series by replacing each term in a group by its smallerst member.

$$
\begin{aligned}
T &= 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots \\
&= 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \dots
\end{aligned}
$$

- Easy to this series T diverges.
- Because $S > T$ we can say the **harmonic series S diverges.**