Primes Are Rather Elusive

From Primes To Riemann

Tariq Rashid

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Simple Pattern, Simple Formula

- A simple pattern in the primes would mean a simple generating formula.
- Eg the triangle numbers 1,3,6,10,15,... are $\frac{1}{2}n(n+1)$

Polynomials

- Polynomials are both **simple** and rather **flexible**.

$$P(n) = a + bn + cn^2 + dn^3 + \ldots + \alpha n^\beta$$

- **Can a polynomial generate the n^{th} prime?**
  - Are primes simple enough to be modelled by a polynomial?

Proof

$$P(n) = a + bn + cn^2 + dn^3 + \ldots + \alpha n^\beta$$

- **Simple** polynomial:
  - coefficients $a, b, c \ldots \alpha$ are whole numbers.
  - also $b, c, d, \ldots \alpha$ are not all zero $\rightarrow$ to avoid eg $P(n) = 7$

Proof

- Proof by contradition .. again!
- Assume $P(n)$ does generates only primes.
- So when $n = 1$, it generates a prime, which we can call p_1

$$p_1 = P(1) = a + b + c + d + \ldots + \alpha$$

Proof

- Now let's try $n = (1 + p_1)$

$$P(1 + p_1) = a + b(1 + p_1) + c(1 + p_1)^2 + d(1 + p_1)^3 + \dots$$

- Looks scary but .. if we expand, we'll have terms with p_1 and those without.

- Let's collect all those p_1 terms and call them X

$$P(1 + p_1) = (a + b + c + d + e + \dots \alpha) + p_1 \cdot X$$

Proof

- That $(a + b + c + d + e + \ldots \alpha)$ is actually p_1.

$$\begin{aligned} P(1 + p_1) &= p_1 + p_1 \cdot X \\ &= p_1(1 + X) \end{aligned}$$

- This is divisible by p_1 ... and it shouldn't be because $P(n)$ is supposed to generate only primes!

- Contradiction means assumption $P(n)$ only generates primes is wrong.

Stronger Proof Than Intended

- We wannted to prove
  - "$P(n)$ can't generate n^{th} prime".
- We proved
  - "$P(n)$ can't generate only primes"

What about Rational Coefficients?

- This time we'll consider $n = (1 + k \cdot p_1)$

$$
\begin{aligned}
P(1 + k \cdot p_1) &= a + b(1 + k \cdot p_1) + c(1 + k \cdot p_1)^2 + \ldots \\
&= p_1 + k \cdot p_1 \cdot X \\
&= p_1(1 + k \cdot X)
\end{aligned}
$$

- X contains terms that are combinations of rational coefficients $a, b, c \ldots \alpha$ multiplied together.

- We can choose a k which cancels all the denominators of the rational coefficients leaving $k \cdot X$ as an integer.

- .. and the proof continues as before.