Infinite Products

From Primes To Riemann

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August 2, 2021

Infinite Sum

- At school we learn a lot about **infinite sums**.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

- "Sum" $\leftrightarrow$ "Series".

- What do we really mean by infinite sum?

Infinite Sum

- We say an **infinite series** converges if the limit of partial sums tends to a **finite** value.

$$\lim_{N \to \infty} \sum_{n=1}^{N} a_n = S$$

- Tests exist to check for convergence, eg the **ratio test**.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

Infinite Product

- At school we don't seem to learn about **infinite products**.

$$\prod_{n=1}^{\infty} a_n = a_1 \times a_2 \times a_2 \times \ldots$$

- What do we really mean by infinite product?

Initial Observations

- Example 1 - Easy to see the infinite product diverges. Each factor increases the size of the product.

$$2 \times 3 \times 4 \times 5 \times \dots$$

- Example 2 - Fundamental idea that multiplying by zero causes a product to be zero.

$$2 \times 0 \times 4 \times 5 \times \dots$$

Initial Observations

- Example 3 - Each factor reduces the size of the product.

$$\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \dots$$

- Infinite number of such factors, the product $\to 0$.

- We have found **two different ways** an infinite product can be zero.

Definition

- An infinite product is defined, like infinite series, as the limit of a sequence.

$$\prod_{n=1}^{\infty} a_n = \lim_{N\to\infty} \prod_{n=1}^{N} a_n$$

Example 1

- Does this converge?

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$$

Example 1

- Consider partial product.

$$\prod_{n=1}^{N}\left(1+\frac{1}{n}\right)=\prod_{n=1}^{N}\left(\frac{n+1}{n}\right)$$

$$=\frac{\cancel{2}}{1}\times\frac{\cancel{3}}{\cancel{2}}\times\frac{4}{\cancel{3}}\times\ldots\times\frac{N+1}{\cancel{N}}$$

$$=N+1$$

- As $N\to\infty$, product **diverges**.

Example 2

- Does this converge?

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$$

- Note n starts at 2.

Example 2

- Again, consider partial product.

$$\prod_{n=2}^{N}\left(1-\frac{1}{n}\right)=\prod_{n=2}^{N}\left(\frac{n-1}{n}\right)$$

$$=\frac{1}{\cancel{2}}\times\frac{\cancel{2}}{\cancel{3}}\times\frac{\cancel{3}}{\cancel{4}}\times\ldots\times\frac{\cancel{N-1}}{N}$$

$$=\frac{1}{N}$$

- As $N \to \infty$, product tends to 0, so product **diverges to zero**.

Convergence

- For an infinite series $\sum a_n$ to converge, the terms $a_n \to 0$
  - Intuition: if each term $a_n > \epsilon$, then $\sum a_n > \sum \epsilon = \infty$
  - (see Cauchy criterion for more rigour)
- For an infinite product $\prod a_n$ to converge, the terms $a_n \to 1$
  - Intuition: if each term $a_n > 1$, the product gets ever larger.
  - If each term $a_n < 1$, the product gets ever smaller towards zero.

Removing Zero Factors

- A single factor 0 collapses entire product to zero.
- If an infinite product has a **finite number of zero-valued factors**, they can be removed and remaining product studied.
- Example:

$$\prod_{n=1}(1 - \frac{1}{n^2}) = 0$$

- Removing first factor leaves an interesting infinite product:

$$\prod_{n=2}(1 - \frac{1}{n^2}) = \frac{1}{2}$$

Convergence Criteria 1

- Useful to write factors as $(1 + a_n)$

$$P = \prod (1 + a_n)$$

- Turn product into sum by takings logarithm

$$\ln(P) = \ln \prod (1 + a_n) = \sum \ln(1 + a_n)$$

- Using $1 + x \leq e^x$

$$\ln(P) \leq \sum a_n$$

- If the sum is **bounded** $\implies$ the product is bounded. If $a_n > 0$ then boundedness is **convergence** (no oscillation).

Convergence Criteria 1

- If we expand out product $\prod(1 + a_n)$ we see another inequality.

$$1 + \sum a_n \leq \prod(1 + a_n) = P$$

- The expansion creates the terms $1 + \sum a_n$ and many more
- This tells us that if the product converges $\implies$ so does the sum.

Convergence Criteria 1

- The two results together give us

$$\boxed{\sum a_n \text{ converges } \Leftrightarrow \prod (1 + a_n) \text{ converges, for } a_n > 0}$$

- This allows us to say:
  - $\prod(1 + 1/n)$ diverges because $\sum 1/n$ diverges
  - $\prod(1 + 1/n^2)$ converges because $\sum 1/n^2$ converges

Divergence To Zero

- The logarithmic view of infinite products has an interesting side effect.
- If the partial products $\to 0$ then the logarithm $\to -\infty$
- This is why we say the product **diverges to zero**.

Convergence Criteria 2

- First convergence critera applies to **real** values $a_n > 0$.
- Would be good to have criteria for **complex** a_n.
- To do that we'll need an intermediate result about absolute values $|a_n|$

Convergence Criteria 2

- Start by assuming $\sum |a_n|$ converges to a finite S

$$S = \sum |a_n| < \infty$$

- Consider partial product

$$p_N = \prod^{N} (1 + |a_n|)$$

- Using $1 + x \leq e^x$

$$p_N \leq e^{\sum^N |a_n|} \leq e^S < \infty$$

- p_N are monotonically increasing, but always $\leq e^S \implies p_N$ converges

Convergence Criteria 2

- Need to show opposite direction too. Assume product converges.

$$P = \prod(1 + |a_n|)$$

- We know $|a_n| \to 0$, so $|a_n| < 2$ for n at least some finite value M
- We can use $e^{x/2} \leq 1 + x$ for $0 \leq x \leq 2$

$$e^{|a_n|/2} \leq 1 + |a_n| \text{ for } n \geq M$$

Convergence Criteria 2

- Set Q to be the infinite product but starting at $n = M$.

- Q converges because it is P but with a finite number of factors removed.

$$Q = \prod_{M} (1 + |a_n|)$$

- Using $|a_n| < 2$

$$e^{\frac{1}{2}\sum_M^N |a_n|} \leq \prod_{M}^{N} (1 + |a_n|) \leq Q < \infty \text{ for } n \geq M$$

- We can see that $\sum_M^N |a_n| \leq 2\ln(Q) < \infty$, so $\sum |a_n|$ converges.

Convergence Criteria 2

- We have a new constraint

$$\boxed{\sum |a_n| \text{ converges } \Leftrightarrow \prod (1 + |a_n|)}$$

- We can use this to show $\prod_2 (1 - 1/n)$ diverges.

Convergence Criteria 3

- We're interested in $\prod(1 + a_n)$ for complex a_n, not just $\prod(1 + |a_n|)$.
- The key:

$$\boxed{\sum |a_n| \text{ converges } \implies \sum a_n \text{ converges}}$$

Convergence Criteria 3

- Let's start with two partial products

$$p_N = \prod^{N} (1 + a_n)$$

$$q_N = \prod^{N} (1 + |a_n|)$$

- We assert $a_n \neq -1$ to ensure no zero-valued factors.

- Should be intuitively clear that

$$|p_N - 1| \leq q_N - 1$$

Convergence Criteria 3

- For $N > M \geq 1$, we can compare $|p_N - p_M|$ with $|q_N - q_M|$

$$
\begin{aligned}
|p_N - p_M| &= |p_M| \cdot |\frac{p_N}{p_M} - 1| \\
&= |p_M| \cdot |\prod_{M+1}^{N}(1 + a_n) - 1| \\
&\leq |q_M| \cdot |\prod_{M+1}^{N}(1 + |a_n|) - 1| \\
&= |q_M| \cdot |\frac{q_N}{q_M} - 1| \\
&= |q_N - q_M|
\end{aligned}
$$

- If $|q_N - q_M| < \epsilon$, then $|p_N - p_M| < \epsilon$. Cauchy criterion for convergence.

Convergence Criteria 3

- Finally we have

$$\boxed{\sum |a_n| \text{ converges } \implies \prod (1+a_n) \text{ converges, for } a_n \neq -1}$$

- This is one way, we can't say the sum comverges if the product converges.

Summary

- Real a_n

$$\boxed{\sum a_n \text{ converges } \Leftrightarrow \prod(1+a_n) \text{ converges, for } a_n > 0}$$

- Complex a_n

$$\boxed{\sum |a_n| \text{ converges } \Leftrightarrow \prod(1+|a_n|)}$$

- Complex a_n

$$\boxed{\sum |a_n| \text{ converges } \implies \prod(1+a_n) \text{ converges, for } a_n \neq -1}$$

Riemann Zeta Function

- For $\sigma > 1$

$$\zeta(s) = \sum \frac{1}{n^s} = \prod (1 - \frac{1}{p^s})^{-1}$$

- No factor $(1 - 1/p^s)^{-1}$ is zero.
  - That would require p^s to be zero.
  - This isn't possible, and is easy to see by writing

$$|p^s| = \left| e^{s \ln(p)} \right| = e^{\sigma \ln(p)} > 0$$

- Also need to check product doesn't **diverge to zero**.

Riemann Zeta Function

$$\sum |a_n| \text{ converges } \implies \prod(1 + a_n) \text{ converges, for } a_n \neq -1$$

- Let's see if $\sum | - 1/p^s|$ converges

$$\sum \left| -\frac{1}{p^s} \right| = \sum \frac{1}{p^\sigma} \leq \sum \frac{1}{n^\sigma}$$

- So $1/\zeta(s) = \prod(1 - \frac{1}{p^s})$ converges to a non-zero value. And so $\zeta(s) = \prod(1 - \frac{1}{p^s})^{-1}$ converges to a non-zero value.

- $\implies$ Riemann Zeta function has no zeros $\sigma > 1$