

# Niven's Proof $\sum 1/p$ Diverges

## From Primes To Riemann

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# Niven's 1971 Proof

- Another proof that  $\sum 1/p$  diverges based on Ivan Niven's 1971.
- Short, fun, but moves a little faster.

## A PROOF OF THE DIVERGENCE OF $\sum 1/p$

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First we prove that  $\sum' 1/k$  diverges, where  $\sum'$  denotes the sum over the squarefree positive integers. Each positive integer is uniquely expressible as a product of a squarefree positive integer and a square, so for any positive integer  $n$ ,

$$\left( \sum'_{k < n} 1/k \right) \left( \sum_{j^2 < n} 1/j^2 \right) \geq \sum_{m < n} 1/m.$$

Here the second sum is bounded but the third sum is unbounded as  $n$  increases, so the first sum must be unbounded. Next suppose that  $\sum 1/p$  converges to  $\beta$ , the sum taken over all primes  $p$ . By dropping all terms beyond  $x$  in the series expansion of  $e^x$  or  $\exp(x)$ , we see that  $\exp(x) > 1 + x$  for  $x > 0$ . Hence for each positive integer  $n$

$$\exp(\beta) > \exp\left(\sum_{p < n} 1/p\right) = \prod_{p < n} \exp(1/p) > \prod_{p < n} (1 + 1/p) \geq \sum'_{k < n} 1/k.$$

But this contradicts the unboundedness of the last sum, so  $\sum 1/p$  diverges.

- <https://www.tandfonline.com/doi/abs/10.1080/00029890.1971.11992740>.

# Square-Free Numbers

- We can write any counting number  $m$  as a unique product of a square  $j^2$  and **square-free** factor  $k$ .

$$m = k \cdot j^2$$

# Square-Free Numbers

- Any integer is a unique product of primes.
- We can split these primes into two groups:
  - one group with primes raised to an even power, which together can be written as a square.
  - other group with primes not raised to any power.
- Examples
  - $360 = (2 \cdot 5) \cdot (2 \cdot 3)^2$  square-free factor of 10, square factor of 36.
  - $30 = (2 \cdot 3 \cdot 5)$  is entirel square-free.

# Sum of Square-Free Integers Diverges

- In the following inequality  $k$  are square-free integers less than  $n$ .

$$\left( \sum_{k < n} \frac{1}{k} \right) \left( \sum_{j < n} \frac{1}{j^2} \right) \geq \sum_{m < n} \frac{1}{m}$$

- Inequality because multiplying out the two series gives us terms  $1/m$  and many more. Equality when  $n = 2$ .
- As  $n \rightarrow \infty$ 
  - $\sum 1/m$  diverges, and  $\sum 1/j^2$  converges to  $\pi^2/6$
  - $\rightarrow \sum 1/k$  must diverge !

# Sum Of Prime Reciprocals

- Assume  $\sum 1/p$  converges to a finite  $\beta$ .
- Partial sum is less than the full sum  $\sum_{p < n} 1/p < \beta$
- so we can write

$$\exp(\beta) > \exp\left(\sum_{p < n} \frac{1}{p}\right) = \prod_{p < n} \exp\left(\frac{1}{p}\right)$$

# Sum Of Prime Reciprocals

- Taylor series for  $e^x$  lets us say  $e^x > 1 + x$

$$\prod_{p < n} \exp\left(\frac{1}{p}\right) > \prod_{p < n} \left(1 + \frac{1}{p}\right)$$

- Multiplying out product gives terms  $1/k$  where  $k$  is square-free, because each prime contributes to any  $k$  at most once.
- Also gives more terms than are in  $\sum_{p < n} 1/p$  for  $n > 3$ .

$$\prod_{p < n} \left(1 + \frac{1}{p}\right) \geq \sum_{k < n} \frac{1}{k}$$



# Sum of Prime Reciprocals

- Putting everything together ...

$$\exp(\beta) > \exp\left(\sum_{p < n} \frac{1}{p}\right) = \prod_{p < n} \exp\left(\frac{1}{p}\right) > \prod_{p < n} \left(1 + \frac{1}{p}\right) \geq \sum_{k < n} \frac{1}{k}$$

- Suggests that as  $n \rightarrow \infty$ , the **finite**  $\exp(\beta)$  is greater than **divergent**  $\sum 1/k$  ... contradiction.

$$\boxed{\sum \frac{1}{p} \rightarrow \infty}$$