Towards The Riemman Zeta Hypothesis

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Contents

1. Introduction

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Approach

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Part I.

Exploring Prime Numbers

2. What Are Prime Numbers?

Let's start by looking at the most ordinary numbers we know, the **counting numbers**.

$$1, 2, 3, 4, 5, 6, 7, 8, \ldots$$

We became familiar with these numbers when we were just toddlers, counting apples in a bowl, for example.

Multiplication

We soon learned to add and multiply these numbers. Many of us learned our times tables by heart. Almost without thinking we could recite multiplications like $2 \times 4 = 8$, and $5 \times 5 = 25$.

When we multiply 3 by 4, the answer is 12. This 12 is called a **product**, and the 3 and 4 are called **factors**.

If we pick any two numbers a and b and multiply them, the result is another number, which we can call c.

$$a \times b = c$$

Because a and b are whole numbers, so is c.

An Innocent Question

Those factors a and b can be any counting number we feel like choosing. Does this freedom apply to c as well?

Surely some combination of a and b can give us any number c that we desire. Let's try a couple of examples.

- If we want c to be 12, we could choose $a = 3$ and $b = 4$. We could have chosen $a = 2$ and $b = 6$, and that would work too.

- If we want c to be 100, we could choose $a = 2$ and $b = 50$. Another combination that works is $a = 10$ and $b = 10$.

What if we want c to be 7?

If we try for a short while, we'll find there doesn't seem to be a combination of factors a and b that gives 7 as a product. In fact, if we try all the numbers in the range $2 \ldots 6$ we'll see for ourselves there really is no combination that gives $a \times b = 7$.

What if we want c to be 11? Again we'll find no combination of whole number factors gives 11 as a product.

So the answer to our innocent-looking question is no, c can't be any whole number.

Numbers like 7 and 11 that don't have whole number factors, are called **prime numbers**. Here are the first few.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, \ldots$$

In short, if we multiply two counting numbers, the answer is never a prime number.

What About 1?

You might have spotted that when we were trying to find factors of 7 we didn't consider combinations like $a = 1$ and $b = 7$. That's because we exclude 1 as a legitimate factor. Why? Because every number has 1 as a factor, and that's not particularly interesting.

If we didn't exclude 1, there would be no prime numbers because every number c would have factors $a = 1$ and $b = c$.

Even worse, a number could have lots of factors as 1, which is also rather unhelpful. The number 12 could have an infinite number of factors.

$$12 = 4 \times 3 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times \ldots$$

Negative Numbers?

Prime numbers were known about and discussed in ancient times, well before the idea of a negative number was accepted.

Over the hundreds of years since then, new ideas and insights were developed about prime numbers, and they were built on the original assumption that prime numbers could only be **positive** whole numbers.

Today almost all exploration of prime numbers continues under the same constraint that products, factors and primes are positive whole numbers greater than 1. This constraint really doesn't limit the mysteries and surprises that prime numbers hold.

Apparent Randomness

Looking back at the list of prime numbers, there doesn't seem to be a pattern to them. Apart from never being even numbers, with the exception of 2, they seem to be fairly randomly located along the number line.

For hundreds of years, mathematicians puzzled over the primes, attacking them with all sorts of exotic tools, trying to crack them open to reveal any elusive rules that govern their location. That endeavour continues to this day.

3. How Many Primes Are There?

At first thought it might seem obvious that there is an unending supply of prime numbers.

If we think a little longer, a bit of doubt might intrude on our certainty. A small number like 6 has factors 2 and 3. Every multiple of 2 is not a prime number, every multiple of 3 is not a prime number, every multiple of 4 is not a prime number, and so on. All these multiples are reducing the probability that a large number is prime.

We might be tempted to think that eventually prime numbers just fizzle out. Instead of relying on intuition, let's decide the matter with rigorous mathematical proof.

Proof There Are Infinitely Many Primes

A proof is not an intuition, nor is it a set of convincing examples. A proof is a watertight logical argument that leads to a conclusion we can't argue with.

The proof that there is no limit to the number of primes is ancient and rather elegant, due to Euclid around 300 BC, and a nice one to have as our first example.

Let's start by assuming the number of primes is not endless but finite.

If there are n primes, we can list them.

$$p_1, p_2, p_3, p_4 \dots p_n$$

We can create a new number x by multiplying all these primes together.

$$x = p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot \ldots \cdot p_n$$

This x is clearly not a prime number. It's full of factors like p_1 , p_3 and p_n.

Let's make another number y in the same way, but this time we'll also add 1.

$$y = p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot \ldots \cdot p_n + 1$$

Now y could be a prime number, or it could not be a prime number. These are the only two options for any positive whole number.

If y is prime then we have a problem because we've just found a new prime number which isn't part of the original finite set $p_1, p_2 \dots p_n$. How do we know it's not part of the original set? Well y is bigger than any of the primes in the list because we created it by multiplying them all together, and adding 1 for good measure.

So perhaps y is not a prime. In this case, it must have factors. And the factors must be one or more of the known primes $p_1, p_2 \dots p_n$. That means y can be divided by one of those primes p_i exactly, leaving no remainder. Let's write this out.

$$\frac{y}{p_i} = \frac{p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot \ldots \cdot p_n}{p_i} + \frac{1}{p_i}$$

The first part divides neatly without a remainder because p_i is one of the primes $p_1, p_2 \ldots p_n$. The second part doesn't divide neatly at all. That means y can't be divided by any of known primes. Which again suggests it is a new prime, not in the original list.

Both of these options point to the original list of primes being incomplete.

And that's the proof. No finite list of primes can be a complete list of primes. So there are infinitely many primes.

A Common Misunderstanding

It is easy to think that $p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot \ldots \cdot p_n + 1$ is a way of generating prime numbers. This is not correct. The proof only asks what the consequences are if $p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot \ldots \cdot p_n + 1$ is prime, under the assumption that we have a limited list of primes $p_1, p_2, p_3, p_4 \ldots p_n$.

We can prove that $p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot \ldots \cdot p_n + 1$ is not always prime by finding just one counter-example. If we use prime numbers 2, 3, 5, 7, 11 and 13, we can see that $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031$ which is not prime because $30031 = 59 \cdot 509$.

4. Primes Are The Building Blocks Of Numbers

We saw earlier that positive whole numbers have factors if they're not a prime number. Let's explore this a little further.

Breaking A Number Into Its Factors

Let's think about the number 12 and its factors. We can think of two combinations straight away.

$$12 = 2 \times 6$$
$$12 = 3 \times 4$$

Looking again at those factors we can see that 6 itself can be broken down into smaller factors 3 and 2. That 4 can also be broken down into factors 2 and 2.

$$12 = 2 \times (3 \times 2)$$
$$12 = 3 \times (2 \times 2)$$

We can't break these smaller factors down any further, which means they're prime numbers. Both combinations now look very similar. If we put those factors in order of size, we can see they are in fact exactly the same.

$$12 = 2 \times 2 \times 3$$
$$12 = 2 \times 2 \times 3$$

Perhaps every number can be broken down into a list of prime factors that is unique to that number, much like DNA is unique to people. Let's prove it.

Fundamental Theorem Of Arithmetic

We'll split this proof into two steps.

- First we'll show that any positive whole number can be broken down into a list of factors that are all prime.
- Second we'll show this list of primes is unique to that number.

Let's imagine a number N and write it out as a product of factors.

$$N = f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_n$$

We can look at each of these factors f_i in turn. If a factor is not prime, we can break it down into smaller factors. For example, the factor f_1 might be broken down as $f_1 = g_1 \cdot g_2$. If a factor is prime, $f_2 = p_1$ for example, we leave it because we can't break it into smaller factors.

$$N = (g_1 \cdot g_2) \cdot p_1 \cdot (g_3 \cdot g_4 \cdot g_5) \cdot \ldots \cdot (g_x \cdot g_y)$$

If we keep repeating this process, all the factors will eventually be prime. How can we be so sure? Well, if any number in the list isn't prime, we can apply the process again, breaking that number down into smaller factors. The only thing that stops us applying the process again is when all the factors are eventually prime.

Figure 4.1 shows an example of this iterative process applied to the number 720.

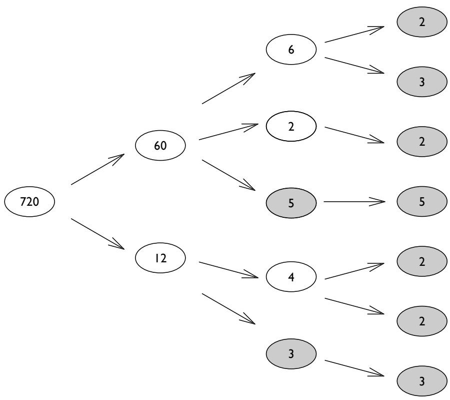


Figure 4.1.: Breaking 720 into factors until only primes remain.

We can now write N as a product of these primes.

$$N = p_2 \cdot p_3 \cdot p_1 \cdot p_5 \cdot p_4 \cdot p_6 \cdot p_7 \cdot \ldots \cdot p_n$$

These primes won't necessarily be in order of size. They may also repeat, for example p_1 might be the same as p_7. It doesn't matter. We've shown that any positive whole number can be written as a product of primes.

Let's now show that this list of primes is unique to that number N. For the moment, imagine this isn't true and a number N can be written as a product of two different lists of primes.

$$
\begin{aligned}
N &= p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_a \\
N &= q_1 \cdot q_2 \cdot q_3 \cdot \ldots \cdot q_a \cdot q_b \cdot q_c \cdot q_d
\end{aligned}
$$

These primes are not necessarily in order of size, and some might be repeated, so p_2 could be the same as p_4. Again, we won't let that bother us. To keep our argument general, we'll assume that the number of primes in the second list, d, is larger than the number of primes in the first list, a.

Now, we can see that p_1 is a factor of N. That means it must also be a factor of the second list. That means p_1 is one of the factors q_i. Because we we didn't assume any order in these primes, let's say it is q_1. That means we can divide both lists by $p_1 = q_1$.

$$\cancel{p_1} \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_a = \cancel{q_1} \cdot q_2 \cdot q_3 \cdot \ldots \cdot q_a \cdot q_b \cdot q_c \cdot q_d$$

We can apply the same logic again. The first list has a factor p_2 which means it must also be a factor of the second list. We can say that $p_2 = q_2$, and divide both lists by this factor.

$$\cancel{p_1} \cdot \cancel{p_2} \cdot p_3 \cdot \ldots \cdot p_a = \cancel{q_1} \cdot \cancel{q_2} \cdot q_3 \cdot \ldots \cdot q_a \cdot q_b \cdot q_c \cdot q_d$$

We can keep doing this until all the factors in the first list have been matched up with factors in the second list. It doesn't matter if a prime repeats, for example if p_1 is the same as p_3, the factors will still be matched correctly, in this case $p_1 = q_1$ and $p_3 = q_3$.

$$\cancel{p_1} \cdot \cancel{p_2} \cdot \cancel{p_3} \cdot \ldots \cdot \cancel{p_a} = \cancel{q_1} \cdot \cancel{q_2} \cdot \cancel{q_3} \cdot \ldots \cdot \cancel{q_a} \cdot q_b \cdot q_c \cdot q_d$$

Let's simplify the algebra.

$$1 = q_b \cdot q_c \cdot q_d$$

What we've just shown is that if a number N can be written as two separate lists of prime factors, their factors can be paired up as being equal, and if any are left over, they must equal 1. That is, the two lists are identical.

We've shown that any whole number N greater than 1 can be decomposed into a list of prime factors, and this list of primes is unique to that number. This is rather profound, and is called the Fundamental Theorem of Arithmetic.

5. Primes Are Rather Elusive

If we listed all the counting numbers $1, 2, 3, 4, 5, \ldots$, excluded 1, and then crossed out all the multiples of $2, 3, 4, \ldots$ we'd be left with the primes. This sieving process emphasises that primes are defined more by what they are not, than by what they are.

If there was a simple pattern in the primes, we'd be able to encode it into a simple formula for generating them. For example, the triangle numbers 1,3,6,10,15,... can be generated by the simple expression $\frac{1}{2}n(n+1)$. The prime numbers, however, have resisted attempts by mathematicians over hundreds of years to find precise and simple patterns in them.

One of the first questions anyone enthusiastic about prime numbers asks is whether a polynomial can generate the n^{th} prime. Polynomials are both simple and rather flexible, and it would be quite pleasing if one could generate primes.

Let's prove that prime numbers are so elusive that no simple polynomial in n can generate the n^{th} prime.

No Simple Polynomial Generates Only Primes

A **polynomial** in n has the following general form, simple yet flexible.

$$P(n) = a + bn + cn^2 + dn^3 + \ldots + \alpha n^{\beta}$$

By simple polynomial we mean the coefficients $a, b, c \dots \alpha$ are whole numbers. Let's also say that $b, c, d, \dots \alpha$ are not all zero. This way we exclude trivial polynomials like $P(n) = 7$ that only generate a single value no matter what n is.

Let's start our proof by assuming there is indeed a $P(n)$ that generates only primes, given a counting number n. When $n = 1$, it generates a prime, which we can call p_1.

$$p_1 = P(1) = a + b + c + d + \dots + \alpha$$

Now let's try $n = (1 + p_1)$.

$$P(1 + p_1) = a + b(1 + p_1) + c(1 + p_1)^2 + d(1 + p_1)^3 + \dots$$

That looks complicated, but all we need to notice is that if we expand out all the terms, we'll have two kinds, those with p_1 as a factor, and those without. We can collect together all those terms with factor p_1 and call them $p_1 \cdot X$.

$$P(1 + p_1) = (a + b + c + d + e + \dots \alpha) + p_1 \cdot X$$

We then notice that $(a + b + c + d + e + \dots \alpha)$ is actually p_1.

$$\begin{aligned} P(1 + p_1) &= p_1 + p_1 \cdot X \\ &= p_1(1 + X) \end{aligned}$$

Since X is a whole number, this is divisible by p_1. It shouldn't be because $P(1 + p_1)$ is supposed to be a prime. This contradiction means the starting assumption that there is a simple polynomial $P(n)$ that generates only primes is wrong.

We've actually proved a stronger statement than we intended. We intended to prove that there is no simple polynomial $P(n)$ that generates the n^{th} prime. We ended up proving that no simple polynomial $P(n)$ can generate only primes.

Polynomials With Rational Coefficients

Insisting on integer coefficients for polynomials might seem overly restrictive. Let's broaden our definition to allow **rational** coefficients of the form $\frac{s}{t}$ where s and t are integers.

We again assume $P(n)$ does indeed generate only primes, and so $p_1 = P(1)$ is prime. This time we'll consider $n = (1 + k \cdot p_1)$.

$$\begin{aligned} P(1 + k \cdot p_1) &= a + b(1 + k \cdot p_1) + c(1 + k \cdot p_1)^2 + \dots \\ &= p_1 + k \cdot p_1 \cdot X \\ &= p_1(1 + k \cdot X) \end{aligned}$$

Here X contains terms that are combinations of the rational coefficients $a, b, c \dots \alpha$ multiplied together. We can choose a k which cancels all the denominators of the rational coefficients leaving $k \cdot X$ as an integer. The lowest common multiple of all the denominators is one way to do this.

Our proof by contradiction then continues as before because we've found an example of $P(n)$ that is not prime.

The primes really are rather elusive if even polynomials with rational coefficients can't generate only primes.

6. Primes Aren't That Spread Out

We've seen there is no limit to the supply of primes. A good question to ask next is how frequently they occur.

One way to explore this is by looking at the sum of their inverses, or **reciprocals**.

Infinite Sum Of Reciprocals

The counting numbers $1, 2, 3, 4, \ldots$ are spaced 1 apart. The sum of their inverses is called the **harmonic series**.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

This series is known to diverge, that is, the sum is infinitely large. Appendix A has an easy short proof.

The square numbers $1, 4, 9, 16, \ldots$ are spaced further apart than the counting numbers.

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$$

The sum of their inverses converges. Appendix B walks through Euler's historic and rather adventurous proof showing it converges to $\frac{\pi^2}{6}$.

We can interpret this to mean the squares n^2 are so spread out that the terms in the series become small quickly enough to avoid sum becoming infinitely large.

It is natural to then ask the same question about the primes. Are they so spread out that the infinite sum of their inverses converges too?

Square-Free Numbers

We can write any counting number as a unique product of a square and **square-free** factor.

To see why, let's remember that any integer is a unique product of primes. We can split these primes into two groups, one of which has any primes raised to an even power, which together can be written as a square. This leaves the other group of primes not raised to any power.

$$
\begin{aligned}
12 &= 2 \times 3 \times 3 = (2) \cdot (3)^2 \\
30 &= 2 \times 3 \times 5 = (2 \cdot 3 \cdot 5) \\
72 &= 2 \times 2 \times 2 \times 3 \times 3 = (2) \cdot (2 \cdot 3)^2
\end{aligned}
$$

The number 12 has a square-free factor of 2, and a square factor of 3. The number 30 is entirely square-free as it has no square factor. The last example shows how a prime factor 2, repeated 3 times, is grouped into square and square-free factors.

Infinite Sum Of Prime Reciprocals

We can now write the following inequality for any integer n, where k are only the square-free integers less than n.

$$\left(\sum_{k<n} 1/k\right)\left(\sum_{j<n} \frac{1}{j^2}\right) \geq \sum_{m<n} \frac{1}{m}$$

The inequality is true because multiplying out the two series would give us not just the terms $1/m$, but also many more. The two sides are only equal when $n = 2$.

As $n \to \infty$ we can see the right hand side is the harmonic series which we now know diverges. We also know the second sum converges to $\pi^2/6$. That means the sum over square-free integers $\sum 1/k$ must diverge, a result we'll use very soon.

Let's set up a proof by contradition, and assume the sum of prime reciprocals $\sum 1/p$ converges to a finite value β.

Now, a partial sum of a series with only positive terms is less than the full sum.

$$\exp(\beta) > \exp\left(\sum_{p<n} \frac{1}{p}\right) = \prod_{p<n} \exp(\frac{1}{p})$$

That last equality simply uses $e^{x+y} = e^x e^y$.

We can trancate the Taylor series expansion of e^x to say that $e^x > 1+x$, and apply it to $\exp(1/p)$.

$$\prod_{p<n} \exp(\frac{1}{p}) > \prod_{p<n} (1 + \frac{1}{p})$$

Multiplying out that product would give a series with terms $1/k$ where k is square-free. This is because there is no way to combine the same prime twice in any k. However, multiplying out the product gives us more terms than are in $\sum_{p<n} \frac{1}{k}$ for $n > 2$.

$$\prod_{p<n} \exp(\frac{1}{p}) > \prod_{p<n} (1 + \frac{1}{p}) \geq \sum_{k<n} \frac{1}{k}$$

Putting this all together gives us:

$$\exp(\beta) > \exp\left(\sum_{p<n} \frac{1}{p}\right) = \prod_{p<n} \exp(\frac{1}{p}) > \prod_{p<n} (1 + \frac{1}{p}) \geq \sum_{k<n} \frac{1}{k}$$

This suggests that as $n \to \infty$ the finite $\exp(\beta)$ is greather than $\sum 1/k$, which we showed was divergent. This is clearly a contradiction, so our assumption that $\sum 1/p$ converges was wrong.

That $\sum 1/p$ diverges is a little surprising because our intuition was that primes thin out rather rapidly.

Legendre's Conjecture

The fact that $\sum 1/n^2$ converges suggests the primes are not as sparse as the squares. This leads us to an interesting proposal attributed to Legendre, but actually first published by Desboves in 1855, that there is at least one prime number between two consecutive squares.

$$n^2 < p < (n+1)^2$$

This remains a deep mystery of mathematics. Nobody has been able to prove or disprove it.

7. Distribution Of Primes

Given primes are so resistant to encoding into a simple generating formula, let's take a detour and try a different approach, **experimental mathematics**.

Number of Primes Up To A Number

We showed that primes don't run out as we explore larger and larger numbers. We also showed they don't thin out as quickly as the squares. So how quickly do they thin out?

One way to explore this is to keep a count of the number of primes as we progress along the whole numbers.

The expression $\pi(n)$ has become an abbreviation for 'the number of primes up to, and including, n'. For example, $\pi(5) = 3$ because there are 3 primes up to, and including, 5. The next number 6 is not prime, so $\pi(6)$ remains 3. The use of the symbol π can be confusing at first.

Figure 7.1 shows $\pi(n)$ for n up to 100. A fairly smooth curve seems to be emerging. This is slightly unexpected because the primes appear to be randomly placed amongst the numbers. The curve suggests the primes are governed by some kind of constraint. It wouldn't be too adventurous to say the curve looks logarithmic, like $\ln(n)$.

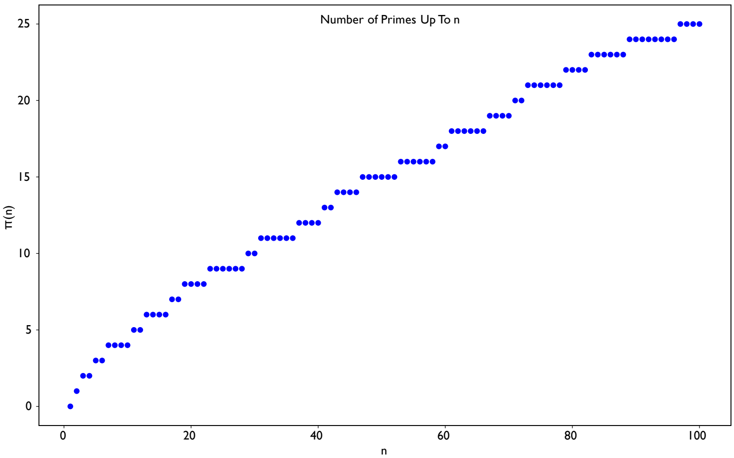


Figure 7.1.: $\pi(n)$ for n from 1 to 100.

Rather Good Approximations for $\pi(n)$

Gauss, one the most prolific mathematicians in history, was the first to find an expression that approximates $\pi(n)$ fairly well. He was aged about 15 at the time.

$$\pi(n) \approx \frac{n}{\ln(n)}$$

The expression is surprisingly simple. It is worth pondering on what hidden pattern in the primes is captured by the natural logarithm $\ln(n)$.

Just a year later, Gauss developed a different expression that approximates $\pi(n)$ even more closely.

$$\pi(n) \approx \int_0^n \frac{1}{\ln(x)} dx$$

At first glance, this logarithmic integral function, shortened to $\mathrm{li}(n)$, appears to be a continuous form of the first approximation.

Figure 7.2 shows a comparison of these approximations with the actual $\pi(n)$ for n all the way up to 10,000. It's clear the logarithmic integral function is much closer to the actual prime counts.

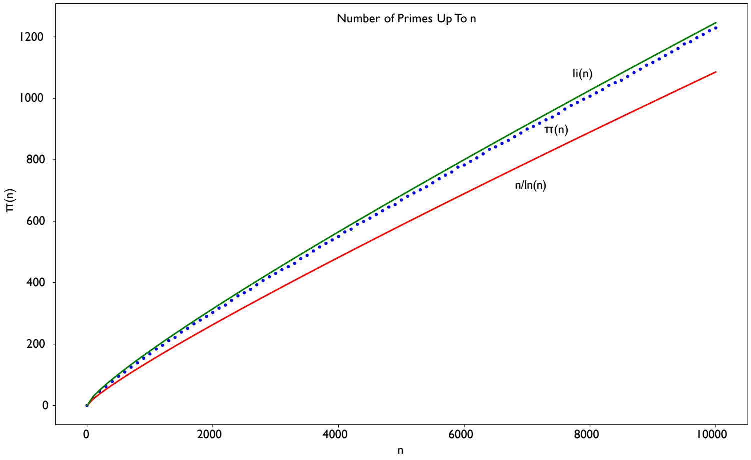


Figure 7.2.: Comparing $\mathrm{li}(n)$ and $n/\ln(n)$ with $\pi(n)$.

Proportional Error

Looking again at the previous chart, the prime counting approximation $n/\ln(n)$ appears to be diverging away from the true prime count $\pi(n)$ as n gets larger. That is, the error appears to be getting ever larger.

If we looked at the numbers, we'd also see li(n) diverging away from $\pi(n)$ too. Does this mean the approximations become useless as n gets larger?

Figure 7.3 paints a different picture. It shows the error as a proportion of $\pi(n)$. We can see this proportional error becomes smaller as n grows to 10,000. It's also clear that li(n) has a distinctly smaller proportional error than $n/\ln(n)$.

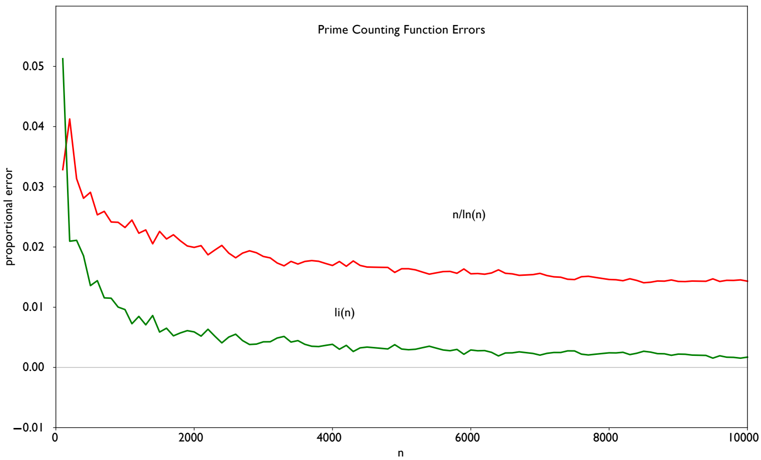


Figure 7.3.: Proportional errors for li(n) and $n/\ln(n)$.

There are 1229 primes amongst the first 10,000 whole numbers. The logarithmic integral gives us $\text{li}(10,000) = 1246$. The error is just 17, and as a proportion of 1229, an impressively small 0.0138.

If we extended n to even larger values, we'd find the proportional error would fall further towards zero. Perhaps these approximations are correct in the limit $n \to \infty$?

Prime Density

Let's look again at those approximations and see if we can interpret their form. The following compares Gauss' first approximation with a general expression for calculating the mass of a volume of stuff with a given average density.

$$\text{mass} = \text{density} \times \text{volume}$$

$$\pi(n) \approx \frac{1}{\ln(n)} \times n$$

The comparison suggests that $1/\ln(n)$ is the average density of primes. If true, this would be a remarkable insight into the primes.

We can apply a similar analogy to Gauss' second approximation too. This time we compare it with another general expression for calculating mass where the density is not assumed to be constant throughout its volume.

$$\text{mass} = \int (\text{density}) dv$$

$$\pi(n) \approx \int_0^n \frac{1}{\ln(x)} dx$$

Again, $1/\ln(x)$ emerges as a more locally accurate density of primes around a number x.

It was this density of primes around a number that the young Gauss first noticed as he studied the number of primes in successive ranges of whole numbers, 1-1000, 1001-2000, 2001-3000, and so on.

Imperfect History

The question of who first developed an approximation for $\pi(n)$ is not perfectly clear. Gauss didn't always publish his work, leaving us to reconstruct history from notes and letters.

In his 1797 book on number theory, Legendre first published a form $n/(A\ln(n) + B)$, which he updated in his 1808 second edition to $n/(\ln(n) - 1.08366)$.

However, in 1849 Gauss wrote a letter to astronomer, and former student, Encke telling him that he had, in '1792 or 1793', developed the logarithmic integral approximation, which he wrote as $\int \frac{dn}{\mathrm{logn}}$. His collected works also reveal that in 1791 he had written about the simpler approximation, $\frac{a}{la}$ as he wrote it.

Appendix C presents reproductions of the relevant parts of these historical works.

8. The Prime Number Theorem

We've just seen experimental evidence that $n/\ln(n)$ approximates $\pi(n)$ fairly well. Although the error itself grows as $n \to \infty$, the proportional error gets ever smaller.

Let's write that out.

$$\lim_{n\to\infty} \frac{\pi(n) - n/\ln(n)}{\pi(n)} = 0$$

Rearranging this gives us the following.

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\ln(n)} = 1$$

This says the ratio of $\pi(n)$ and the approximation $n/\ln(n)$ tends to 1 as $n \to \infty$. And this is precisely what the **prime number theorem** says.

$$\pi(n) \sim n/\ln(n)$$

The symbol $\sim$ says that both sides are **asymptotically equivalent**. For example, $f(n) \sim g(n)$ means $f(n)/g(n) = 1$ as $n \to \infty$.

Asymptotic Equivalence

Some examples of asymptotic equivalence will help clarify its meaning.

If $f(x) = x^2 + x$ and $g(x) = x^2$, then $f \sim g$. Both f and g have the same dominant term x^2.

$$\lim_{x\to\infty} \frac{x^2 + x}{x^2} = \lim_{x\to\infty} 1 + \frac{1}{x} = 1$$

Swapping f and g doesn't break asymptotic equivalence, $g \sim f$. This is clear from its definition as a ratio.

However, if $f(x) = x^3$ and $g(x) = x^2$, then f and g are not asymptotically equivalent because the ratio f/g tends to x, not 1.

If we know that $f \sim g$ and $g \sim h$, then we can also say $f \sim h$. This property, called **transitivity**, is familiar from normal equality.

What About li(n)?

Gauss' second approximation $\text{li}(n)$ appeared to be a better approximation for $\pi(n)$. You'll find the prime number theorem is sometimes expressed using the logarithmic integral.

$$\pi(n) \sim \text{li}(n)$$

Surely the prime number theorem must be about one of the approximations, not both? The only solution is for both approximations to be asymptotically equivalent. Let's see that this is indeed the case.

Let's set $f(n) = \frac{n}{\ln(n)}$ and $g(n) = \int_0^n \frac{1}{\ln(x)} dx$.

To show $f \sim g$ we need to find the limit of $f(n)/g(n)$ as $n \to \infty$ and confirm it is 1. Sadly, both $f(n)$ and $g(n)$ become infinitely large as $n \to \infty$, which is a little unhelpful.

When this happens, we usually try l'Hopital's rule as an alternative way to find the limit.

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f'(n)}{g'(n)}$$

It's fairly easy to work out $f'(n) = \frac{\ln(n)-1}{\ln^2(n)}$, and $g'(n) = \frac{1}{\ln(n)}$ pops out of the definition of $\mathrm{li}(n)$.

$$\begin{aligned}
\lim_{n \to \infty} \frac{f'(n)}{g'(n)} &= \lim_{n \to \infty} \frac{(\ln(n) - 1)\ln(n)}{\ln^2(n)} \\
&= \lim_{n \to \infty} 1 - \frac{1}{\ln(n)} \\
&= 1
\end{aligned}$$

So the prime number theorem can refer to either of the two approximations, $n/ln(n)$ and $\mathrm{li}(n)$, because they are asymptotically equivalent.

What Does The Prime Number Theorem Really Say?

The prime number theorem says that $\pi(n)$ grows in a way that is asymptotically equivalent to functions like $n/\ln(n)$ and $\mathrm{li}(n)$.

It doesn't say that these are the only or best functions for approximating $\pi(n)$, which leaves open the intriguing possibility of other functions that are even better than $\mathrm{li}(n)$.

Bertrand's Postulate

The prime number theorem, even if it looks imprecise, can provide easy insights into questions about the primes.

In 1845 Bertrand proposed that there is at least one prime between a counting number and its double, $n < p < 2n$. A proof would take a few pages to walk through.

We can use the prime number theorem to asymptotically compare the number of primes up to $2n$, with the number of primes up to n.

$$\frac{\pi(2n)}{\pi(n)} \sim \frac{2n}{\ln(2n)} \cdot \frac{\ln(n)}{n} \sim 2$$

With very little work, this tells us that between n and $2n$, there are approximately $n/\ln(n)$ primes, an approximation that becomes truer for larger n.

This is actually a stronger statement than Bertrand's postulate which merely suggests there is at least one prime.

9. Euler's Golden Bridge

Euler was the first to find a connection between the world of primes and the world of ordinary counting numbers. Many insights about the primes have been revealed by travelling over this 'golden bridge'.

Let's recreate Euler's discovery for ourselves.

Reimann Zeta Function

We know the harmonic series $\sum 1/n$ diverges. We also know the series $\sum 1/n^2$ converges.

It's natural to ask for which values of s the more general series, known as the **Riemann Zeta** function $\zeta(s)$, converges.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots$$

Series like $\sum 1/n^3$ and $\sum 1/n^4$ converge because each term is smaller than the corresponding one in $\sum 1/n^2$. Less obvious is when $s < 2$.

Appendix D presents a short proof that $\zeta(s)$ converges for $s > 1$.

The proof compares the discrete sum $\zeta(s)$ with a related continuous integral that is easier to analyse. This simple technique is used a lot in number theory, and worth becoming familiar with.

Sieving The Zeta Function

Let's write out the zeta function again, noting that $1^s = 1$.

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \ldots$$

We can divide this series by 2^s.

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{12^s} \ldots$$

These denominators are multiples of 2^s. By subtracting these terms from $\zeta(s)$, we sieve out terms with these multiples of 2^s.

$$(1 - \frac{1}{2^s}) \cdot \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \ldots$$

This dividing and subtracting of infinite series is only valid because they are absolutely convergent for $s > 1$.

Let's now divide this series by 3^s.

$$\frac{1}{3^s} \cdot (1 - \frac{1}{2^s}) \cdot \zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \ldots$$

These denominators are all multiples of 3^s, but not all multiples of 3^s are here. Some like 6^s and 12^s were removed in the previous step. Subtracting this series from the previous series leaves terms with denominators that are not multiples of 2 or 3.

$$(1 - \frac{1}{3^s}) \cdot (1 - \frac{1}{2^s}) \cdot \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \ldots$$

We can't remove terms with multiples of 4^s because they were sieved out when we removed multiples of 2^s. The next useful step is to remove multiples of 5^s. Doing this leaves terms with denominators that are not multiples of 2, 3 or 5.

$$(1-\frac{1}{5^s})\cdot(1-\frac{1}{3^s})\cdot(1-\frac{1}{2^s})\cdot\zeta(s) = 1+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\ldots$$

Repeating this several times, we'll see that only multiples of successive primes are available to be removed. We'll also see that after each removal, the very first term 1 always survives.

If we kept going, we'd end up with an all the primes on the left, and only 1 on the right. This is because every n in $1/n^s$ is either a prime, or a multiple of a prime, which we know from the fundamental theroem of arithmetic.

$$\ldots\cdot(1-\frac{1}{11^s})\cdot(1-\frac{1}{7^s})(1-\frac{1}{5^s})\cdot(1-\frac{1}{3^s})\cdot(1-\frac{1}{2^s})\cdot\zeta(s) = 1$$

We can rearrange this to isolate $\zeta(s)$.

$$\zeta(s) = \prod_p (1-\frac{1}{p^s})^{-1}$$

The symbol $\prod$ means product, just like $\sum$ means sum.

Euler's Product Formula

We've arrived at **Euler's product formula**.

$$\boxed{\sum_n \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1}}$$

The product of $(1 - \frac{1}{p^s})^{-1}$ over all primes p is the sum of $\frac{1}{n^s}$ over all positive integers n, as long as we remember to keep $s > 1$.

To say this result is amazing would not be an exaggeration. It reveals a deep connection between the primes and the ordinary counting numbers, a connection that doesn't appear too complicated at first sight.

10. Walking Euler's Bridge

The simpliciy of Euler's product formula almost demands we explore its implications.

$$\sum_n \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1}$$

Another Proof Of Infinite Primes

We know the harmonic series $\zeta(1) = \sum 1/n$ diverges. We can use Euler's formula to write this sum as a product over primes.

$$\sum \frac{1}{n} = \prod_p (1 - \frac{1}{p})^{-1}$$

Each of the factors $(1 - \frac{1}{p})^{-1}$ is always finite, and never zero. For the product to diverge, there must be an infinite number of these factors. This means there must be an infinite number of primes.

Euler's product formula gave us the first new proof of infinite primes since Euclid's from around 300 BC.

You might be concerned this proof is a circular argument, that the derivation of the Euler product formula itself assumed an infinity of primes. Looking back, the derivation sieves out all integers greater than 1 because they are either a prime, or a multiple of a prime. This is the fundamental theorem of arithmetic, which makes no claims about the number of primes.

π Is Irrational

We also know that $\zeta(2) = \sum 1/n^2$ converges to $\pi^2/6$. Again, we can write the sum as a product over primes.

$$\frac{\pi^2}{6} = \prod_p (1 - \frac{1}{p^2})^{-1}$$

$$= \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{16}{15} \cdot \frac{25}{24} \cdot \frac{36}{35} \cdot \frac{49}{48} \cdot \ldots$$

This is an infinite product that can't be simplified as a fraction of two whole numbers. So π is irrational, a nice easy result.

Growth Of Prime Reciprocals

Euler was gifted with a deep intuition that led him to groundbreaking results, even if his routes didn't always meet modern standards of rigour. Here we'll follow his famous intuition to find the rate at which the inverse primes diverge.

Turning a product into a sum, by taking the natural logarithm, is often the start of a fruitful journey.

$$\ln\left(\sum_n \frac{1}{n}\right) = \ln\left(\prod_p \frac{1}{(1-\frac{1}{p})}\right)$$

Let's focus on the right hand side.

$$\begin{aligned}\ln\left(\prod_p \frac{1}{(1-\frac{1}{p})}\right) &= \sum_p \ln \frac{1}{(1-\frac{1}{p})} \\ &= -\sum_p \ln(1-\frac{1}{p})\end{aligned}$$

We can use the well known series $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$ to expand $\ln(1-\frac{1}{p})$.

$$\begin{aligned}-\sum_p \ln(1-\frac{1}{p}) &= \sum_p(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4}\ldots) \\ &= \sum_p \frac{1}{p} + C\end{aligned}$$

The expansion gives us the sum over the reciprocals of primes, which we now know is divergent. The rest of the expansion C converges.

How do we know C converges? Well, we know the sum of the reciprocals of squares $\sum \frac{1}{n^2}$, and indeed higher powers $\sum \frac{1}{n^x}$, converges. Any series $\sum \frac{1}{p^x}$, summming only over primes p and not all integers n, also converges because it is a subseries of $\sum \frac{1}{n^x}$.

Let's now focus on the left hand side, and look more closely at a finite form of the harmonic series $\sum_1^n \frac{1}{x}$.

$$
\begin{aligned}
\ln\left(\sum_{1}^{n}\frac{1}{x}\right) &> \ln\int_{1}^{n+1}\frac{1}{x}dx \\
&= \ln\left(\ln(n+1)\right)
\end{aligned}
$$

Appendix D explains how the inequality emerges by comparing the area under the curve $y = \frac{1}{x}$ with discrete sums of $\frac{1}{n}$.

Examining these two results, Euler made a rather brave leap to say the sum of prime reciprocals less than n grows like $\ln(\ln(n))$.

$$
\sum_{p<n}\frac{1}{p} \sim \ln\left(\ln(n)\right)
$$

His conclusion was correct, but his argument wasn't quite watertight. We can't naively take the logarithm of infinity. Even so, this is a deep insight about primes prompted by Euler's product formula.

Density of Primes

We can try Eulerian brave leaps ourselves. The following expressions are all equivelent, with the last one derived using $u = \ln(x)$ and $du = \frac{1}{x}dx$.

$$
\ln\left(\ln(n)\right) = \int_{1}^{\ln(n)}\frac{1}{u}du = \int_{e}^{n}\frac{1}{x}\frac{1}{\ln(x)}dx
$$

Now let's write the sum of prime reciprocals as an integral.

$$
\sum_{p<n}\frac{1}{p} = \int_{1}^{n}\frac{1}{x}P(x)dx
$$

Here $P(x)$ is a selector for primes, and is 1 if x is prime, 0 if not. We can make a small leap to say that in the neighbourhood of x, $P(x)$ is like a prime density.

Let's compare the two integrals.

$$\sum_{p<n} \frac{1}{p} \sim \int_e^n \frac{1}{x} \frac{1}{\ln(x)} dx$$
$$\sum_{p<n} \frac{1}{p} = \int_1^n \frac{1}{x} P(x) dx$$

This suggests that $1/\ln(x)$ is the density of primes around x. We saw experimental evidence for this earlier, and now we've drawn the same conclusion, albeit not rigorously, from Euler's golden bridge.

11. Prime Reciprocals Grow Like LogLog

We're going to end Part 1 by being ambitious and proving that sum of prime reciprocals grows like $\ln(\ln(x))$. This is ambitious because we going to go much further than simpler proofs which only prove a lower bound to the sum.

Part II.

Prime Number Theorem

Part III.

Appendices

A. $\sum 1/n$ Diverges

Infinite Series

Have a look at the following infinite series.

$$1 + 1 + 1 + 1 + \ldots$$

We can easily see this sum is infinitely large. The series **diverges**.

The following shows a different infinite series. Each term is half the size of the previous one.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

We can intuitively see this series gets ever closer to 2. Many would simply say the sum is in fact 2. The series **converges**.

Harmonic Series

Now let's look at this infinite series, called the **harmonic series**.

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots$$

Each term is smaller than the previous one, and so contributes an ever smaller amount to the sum. Perhaps surprisingly, the harmonic series doesn't converge. The sum is infinitely large.

The following, rather fun, proof is based on Oresme's which dates back to the early 1300s.

We start by grouping the terms in the series as follows.

$$S = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$

The brackets will have $2, 4, 8, 16\dots$ terms inside them. Replacing each term in a group by its smallest member gives us the following new series.

$$\begin{aligned} T &= 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots \\ &= 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \dots \end{aligned}$$

We can see straight away this series diverges.

Because we replaced terms in S by smaller ones to make T, we can say $S > T$.

And because T diverges, so must the harmonic series S.

B. $\sum 1/n^2$ Converges

The sum of the reciprocals of the square numbers was a particularly difficult challenge, first posed around 1650, and later named the Basel problem.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Although there are more modern proofs, we will follow Euler's original proof from 1734 because his methods were pretty audacious, and later influenced Riemann's work on the prime number theorem.

Taylor Series For sin(x)

We start with the familiar Taylor series for $\sin(x)$.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$$

Euler's New Series For sin(x)

The polynomial $f(x) = (1 - \frac{x}{a})(1 + \frac{x}{a})$ has factors $(1 - \frac{x}{a})$ and $(1 + \frac{x}{a})$, and zeros at $+a$ and $-a$. We can shorten it to $f(x) = (1 - \frac{x^2}{a^2})$.

Euler's novel idea was to write $\sin(x)$ as a product of similar linear factors, which would lead him to a different series.

The zeros of $\sin(x)$ are at $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ so the product of factors looks like the following.

$$\sin(x) = A \cdot x \cdot \left(1 - \frac{x^2}{\pi^2}\right) \cdot \left(1 - \frac{x^2}{(2\pi)^2}\right) \cdot \left(1 - \frac{x^2}{(3\pi)^2}\right) \cdot \ldots$$

The constant A is 1 because we know $\frac{\sin(x)}{x} \to 1$ as $x \to 0$. Alternatively, taking the first derivative of both sides gives $A = 1$ when $x = 0$.

The second factor is x and not x^2 because the zero of $\sin(x)$ at $x = 0$ has multiplicity 1.

Euler then multiplied out his new formula.

$$\sin(x) = x \cdot \left[1 - \frac{x^2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) + X\right]$$

Inside the square brackets, the terms with powers of x higher than 2 are contained in X.

Comparing The Two Series

The terms in Euler's new series and the Taylor series must be equivalent because they both represent $\sin(x)$. Let's pick out the x^3 terms from both series.

$$\frac{x^3}{3!} = \frac{x^3}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

We can easily rearrange this to give us the desired infinite sum.

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Euler, aged 28, had solved the long standing Basel problem, not only proving the infinite series of squared reciprocals converged, but giving it an exact value.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Rigour

Euler's original proof was adventurous in expressing $\sin(x)$ as an infinite product of simple linear factors. It made intuitive sense, but at the time was not rigorously justified.

It was almost 100 years later when Weierstrass developed and proved a factorisation theorem that confirmed Euler's leap was legitimate.

C. Historical References For $\pi(n)$

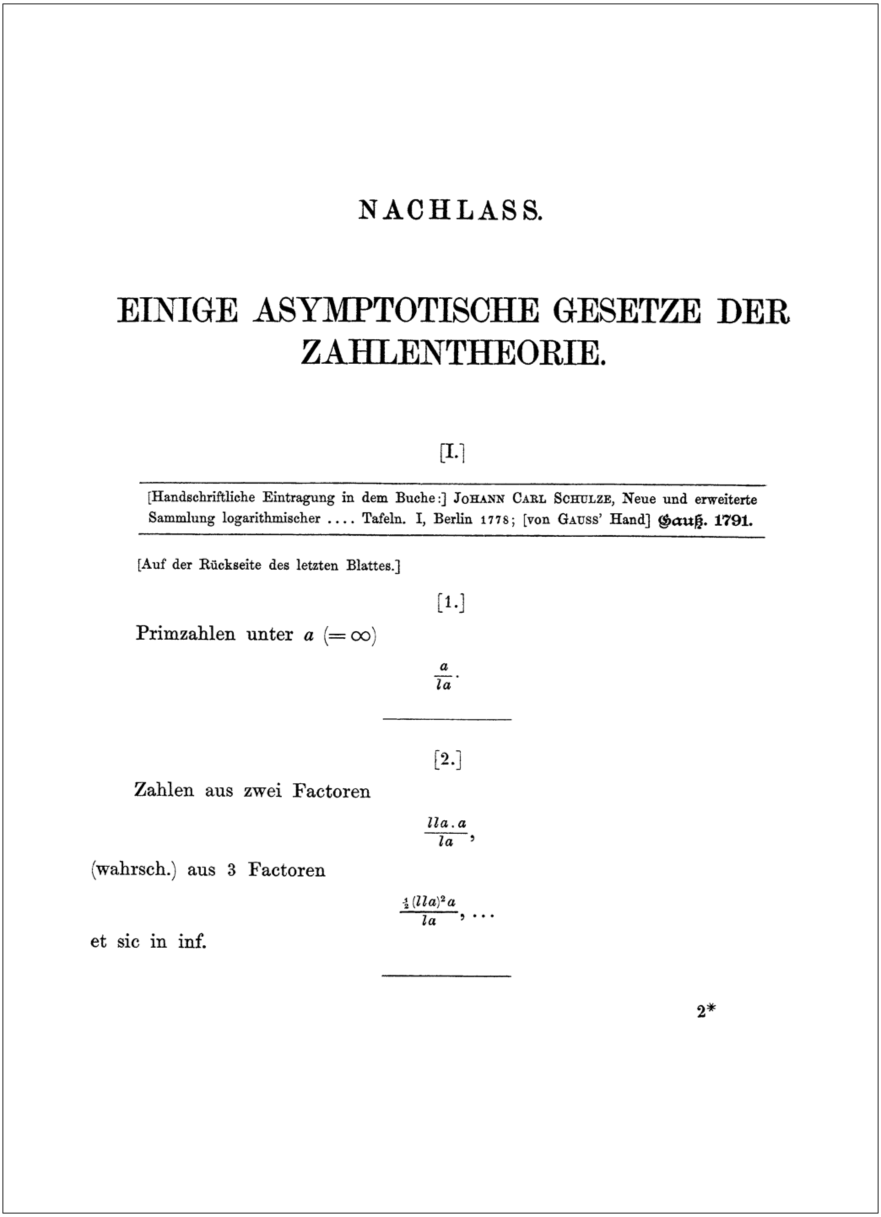
Gauss, 1791

Gauss' 1791 'Some Asymptotic Laws Of Number Theory' can be found in volume 10 of his collected works. In it he presents his approximation for $\pi(n)$.

$$\frac{a}{la}$$

Today, this would be written as $n/\ln(n)$.

Source: `http://resolver.sub.uni-goettingen.de/purl?PPN236018647`



NACHLASS.

EINIGE ASYMPTOTISCHE GESETZE DER ZAHLENTHEORIE.

[I.]

[Handschriftliche Eintragung in dem Buche:] JOHANN CARL SCHULZE, Neue und erweiterte Sammlung logarithmischer .... Tafeln. I, Berlin 1778; [von GAUSS' Hand] Gauß. 1791.

[Auf der Rückseite des letzten Blattes.]

[1.]

Primzahlen unter a $(=\infty)$

$$\frac{a}{la}.$$

[2.]

Zahlen aus zwei Factoren

$$\frac{lla.a}{la},$$

(wahrsch.) aus 3 Factoren

$$\frac{\frac{1}{2}(lla)^2 a}{la}, \ldots$$

et sic in inf.

2*

Figure C.1.: Gauss' 1971 Some Asymptotic Laws Of Number Theory.

Legendre, 1797

Legendre in his first edition of 'Essai Sur La Theorie Des Nombres' presented his approximation.

$$\frac{a}{A\log(a)+B}$$

The logarithm is the natural $\ln(a)$. In his 1808 second edition he quantifies the constants.

$$\frac{x}{\log(x)-1.08366}$$

Source: `https://gallica.bnf.fr/ark:/12148/btv1b8626880r/f55.image` qu'à 1000000 la proportion sera encore moindre et ainsi de suite. En effet, la probabilité qu'un nombre pris au hasard sera premier, est d'autant moindre que ce nombre est plus grand; car plus le nombre est grand, plus il y a de divisions à essayer pour s'assurer si le nombre est premier ou s'il ne l'est pas.

XXX. Nous remarquerons encore, que si on considère les seize suites dont les termes généraux sont : $60x+1$, $60x-1$, $60x+7$, $60x-7$, $60x+11$, $60x-11$, &c. (art. XV), et qu'on cherche, par exemple, combien il y a de nombres premiers dans un million des premiers termes de chaque suite, on trouveroit sensiblement le même nombre pour chacune; d'où il suit que tous les nombres premiers (sauf 2, 3 et 5) sont répartis également entre ces différentes suites, et que chacune peut être censée contenir la seizième partie de la totalité des nombres premiers.

---

de a pris dans les tables ordinaires; cette formule très-simple peut être regardée comme suffisamment approchée, au moins lorsque a n'excède pas 1000000. Ainsi si on demande combien il y a de nombres premiers depuis 1 jusqu'à 400000, on trouvera que ce nombre est $\frac{400000}{2\times 5,602}$ ou 35700 à-peu-près.

Au reste, il est vraisemblable que la formule rigoureuse qui donne la valeur de b lorsque a est très-grand, est de la forme $b=\frac{a}{A\ \log.\ a+B}$, A et B étant des coefficiens constans, et log. a désignant un logarithme hyperbolique. La détermination exacte de ces coefficiens seroit un problême curieux et digne d'exercer la sagacité des Analystes.

C 2

Figure C.2.: Legendre's 1797 Essai Sur La Theorie Des Nombres.

Gauss, 1849

Gauss wrote a letter to astronomer Encke dated Decemer 24th 1849, in which he first presents an integral form of a prime counting function. He states this is based on work he started in 1792 or 1793.

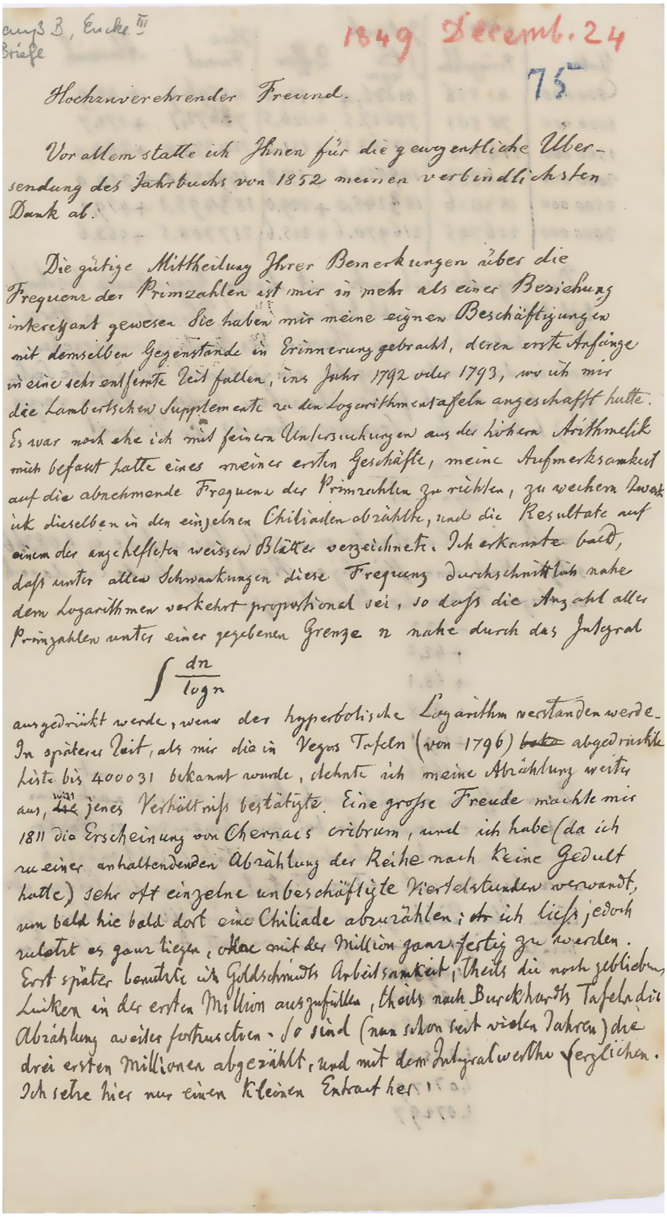
Gauss uses the following expression.

$$\int \frac{dn}{\mathrm{logn}}$$

Today this would be written as the logarithmic integral function.

$$\int_0^n \frac{1}{ln(x)} dx$$

Source: `https://gauss.adw-goe.de/handle/gauss/199`



aus B, Encke III
Briefe

1849 Decemb. 24

75

Hochzuverehrender Freund.

Vor allem statte ich Ihnen für die gewogentliche Übersendung des Jahrbuchs von 1852 meinen verbindlichsten Dank ab.

Die gütige Mittheilung Ihrer Bemerkungen über die Frequenz der Primzahlen ist mir in mehr als einer Beziehung interessant gewesen. Sie haben mir meine eignen Beschäftigungen mit demselben Gegenstande in Erinnerung gebracht, deren erste Anfänge in eine sehr entfernte Zeit fallen, ins Jahr 1792 oder 1793, wo ich mir die Lambertschen Supplemente zu den Logarithmentafeln angeschafft hatte. Es war noch ehe ich mit feinern Untersuchungen aus der höhern Arithmetik mich befasst hatte eines meiner ersten Geschäfte, meine Aufmerksamkeit auf die abnehmende Frequenz der Primzahlen zu richten, zu welchem Zweck ich dieselben in den einzelnen Chiliaden abzählte, und die Resultate auf einem der angehefteten weissen Blätter verzeichnete. Ich erkannte bald, dass unter allen Schwankungen diese Frequenz durchschnittlich nahe dem Logarithmen verkehrt proportional sei, so dass die Anzahl aller Primzahlen unter einer gegebenen Grenze n nahe durch das Integral

$$\int \frac{dn}{\log n}$$

ausgedrückt werde, wenn der hyperbolische Logarithm verstanden werde. In späterer Zeit, als mir die in Vegas Tafeln (von 1796) ~~bek~~ abgedruckte Liste bis 400031 bekannt wurde, dehnte ich meine Abzählung weiter aus, ~~die~~ was jenes Verhältniss bestätigte. Eine grosse Freude machte mir 1811 die Erscheinung von Chernacs cribrum, und ich habe (da ich zu einer anhaltenden Abzählung der Reihe nach keine Geduld hatte) sehr oft einzelne unbeschäftigte Viertelstunden verwandt, um bald hie bald dort eine Chiliade abzuzählen; doch liess ich es zuletzt ganz liegen, ohne mit der Million ganz fertig zu werden. Erst später benutzte ich Goldschmidts Arbeitsamkeit, theils die noch gebliebenen Lücken in der ersten Million auszufüllen, theils nach Burckhardts Tafeln die Abzählung weiter fortzusetzen. So sind (nun schon seit vielen Jahren) die drei ersten Millionen abgezählt, und mit dem Integralwerthe verglichen. Ich setze hier nur einen kleinen Extract her:

Figure C.3.: First page of Gauss' 1849 letter to Encke.

D. Integral Comparison

Understanding the behaviour of continuous functions is often easier than discrete functions. We can gain insights into discrete sums like $\sum \frac{1}{x}$ by exploring the related continuous integral $\int \frac{1}{x} dx$.

Lower & Upper Bounds For The Growth Of $\sum 1/n$

Figure D.1 shows a graph of $y = \frac{1}{x}$, together with rectangles representing the fractions $\frac{1}{n}$.

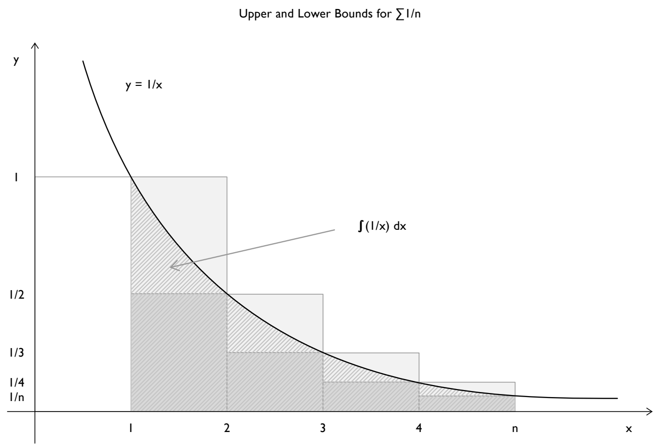


Figure D.1.: Comparing discrete $1/n$ with continuous $1/x$.

If we consider the range $1 \leq x \leq 4$ we can see the area of the three taller rectangles $1 + \frac{1}{2} + \frac{1}{3}$ is greater than the area under the curve $\int_1^4 \frac{1}{x} dx$. By extending the range to n, we can make a general observation.

$$\sum_1^n \frac{1}{x} > \int_1^{n+1} \frac{1}{x} dx$$

The integral has an upper limit of $n + 1$ because the width of the last rectangle extends from $x = n$ to $x = n+1$. We can perform the integral to simplify the expression.

$$\boxed{\sum_1^n \frac{1}{x} > \ln(n + 1)}$$

This is a rather nice lower bound on the growth of the harmonic series.

Let's now look at the shorter rectangles. In the range $1 \leq x \leq 4$ we can see the area of the three shorter rectangles $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ is less than the area under the curve $\int_1^4 \frac{1}{x} dx$. Again, by extending the range to n we can make a general observation.

$$\sum_2^n \frac{1}{x} < \int_1^n \frac{1}{x} dx$$

The harmonic sum starts at 2 because this time we're looking at rectangles extending to the left of a given x. We can adjust the limit of the sum using $\sum_1^n \frac{1}{x} = 1 + \sum_2^n \frac{1}{x}$.

$$\sum_1^n \frac{1}{x} - 1 < \int_1^n \frac{1}{x} dx$$

Again, we can perform the integral.

$$\boxed{\sum_1^n \frac{1}{x} < \ln(n) + 1}$$

This is a nice upper bound to the growth of the harmonic series.

Convergence Of $\zeta(s) = \sum 1/n^s$

Figure D.2 shows a graph of $y = \frac{1}{x^s}$, together with rectangles representing the fractions $\frac{1}{x^s}$.

The shape of the graph assumes $s > 0$. If s was ≤ 0 then it is easy to see $\sum 1/n^s$ would diverge because each term would be ≥ 1.

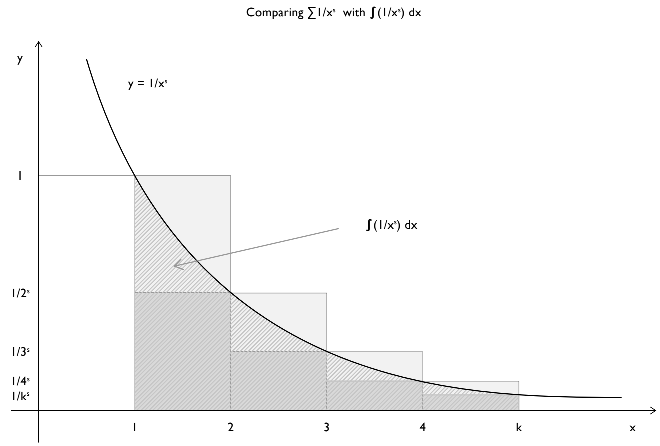


Figure D.2.: Comparing discrete $\sum 1/x^s$ with continuous $\int 1/x^s dx$.

If we consider the range $1 \leq x \leq 4$ we can see the area of the three shorter rectangles $\frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s}$ is less than the area under the curve $\int_1^4 \frac{1}{x^s} dx$. By extending the range to k, we can make a general observation.

$$\sum_2^k \frac{1}{x^s} < \int_1^k \frac{1}{x^s} dx$$

The sum starts at 2 because we're looking at rectangles extending to the left of a given x. We can adjust the limit of the sum using $\sum_1^k \frac{1}{x^s} = 1 + \sum_2^k \frac{1}{x^s}$.

$$\sum_{1}^{k} \frac{1}{x^s} - 1 < \int_{1}^{k} \frac{1}{x^s} dx$$

The integral is easily evaluated.

$$\sum_{1}^{k} \frac{1}{x^s} < \frac{k^{1-s} - 1}{1 - s} + 1$$

As $k \to \infty$, the right hand side only **converges** when $s > 1$. Because it is less than the right hand side, the sum $\sum 1/x^s$ also converges when $s > 1$. We haven't yet ruled out the possibility the sum might also converge for some $s \leq 1$.

If we now consider the three taller rectangles $1 + \frac{1}{2^s} + \frac{1}{3^s}$ in the range $1 \leq x \leq 4$, we can see their area is greater than the area under the curve $\int_1^4 \frac{1}{x^s} dx$. By extending the range to k, we can make a general observation.

$$\sum_{1}^{k} \frac{1}{x^s} > \int_{1}^{k+1} \frac{1}{x^s} dx$$

The integral has an upper limit of $k + 1$ because we're looking at rectangles extending to the right of a given x. We can perform the integral to simplify the expression.

$$\sum_{1}^{k} \frac{1}{x^s} > \frac{(k+1)^{1-s} - 1}{1 - s}$$

As $k \to \infty$, the right hand side **diverges** when $s \leq 1$. Because it is greater than the right hand side, the sum $\sum 1/x^s$ also diverges when

$s \le 1$. We have now ruled out the possibility the sum might converge for some $s \le 1$.

$$\boxed{\zeta(s) = \sum 1/n^s \text{ only converges for } s > 1}$$

We can go further. The two inequalities together provide a lower and upper bound for the zeta function.

$$\boxed{\frac{1}{s-1} < \zeta(s) < \frac{1}{s-1} + 1}$$