$\sum \frac{1}{p}$ Grows Like $\log \log x$

From Primes To Riemann

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March 22, 2021

How Does $\sum \frac{1}{p}$ Grow?

- $\sum 1/n$ diverges
- $\sum 1/n$ grows like $\log n$

- $\sum 1/p$ diverges
- How does $\sum 1/p$ grow?

Euler's 1737 Assertion

- *"The sum of the reciprocals of the prime numbers,*

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \dots$$

is infinitely great but is infinitely times less than the sum of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

And the sum of the former is the as the logarithm of the sum of the latter."

- Widely interpreted as meaning, for large x

$$\sum_{p \leq x} \frac{1}{p} \approx \log \log x$$

Euler's 1737 Assertion

Theorema 19.

Summa ſeriei reciprocae numerorum primorum

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \text{etc.}$$

eſt infinite magna; infinities tamen minor, quam ſumma ſeriei harmonicae $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.}$ *Atque illius ſummae eſt huius ſummae quaſi logarithmus.*

- https://scholarlycommons.pacific.edu/euler-works/72/

Pollack's Proof

- Paul Pollack's *"Euler and the Partial Sums of the Prime Harmonic Series"*
  - doesn't require advanced knowledge
  - but does require a bit of bookwork
- `http://pollack.uga.edu/eulerprime.pdf`

Proof Overview

- Find an $S_0 = \sum_{p \leq x} 1/p$
- Find an $S_U \geq S_0$
- Find an $S_L \leq S_0$

- Hopefully S_U and S_L are nice expressions for lower and upper bounds on $\sum_{p \leq x} 1/p$
- Trick is to find S that are easy (enough) to work with

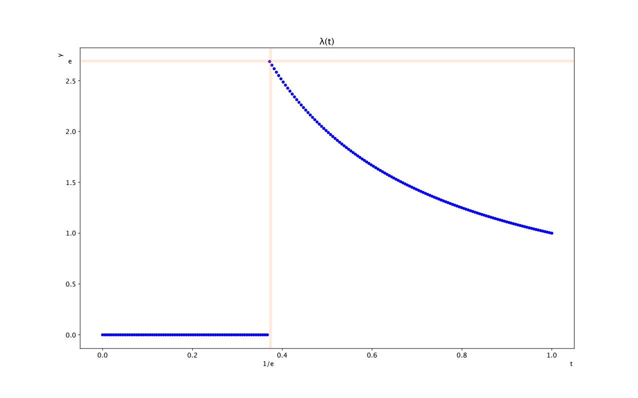
Start With $S_0 = \sum_{p \le x} 1/p$

$$S(\lambda, x) = \sum_{p} p^{-1-\frac{1}{\log x}} \cdot \lambda(p^{-\frac{1}{\log x}})$$

- $S(\lambda, x)$ is a function of a function $\lambda(t)$ defined over a small domain $t \in [0, 1]$
- Looks convoluted but can be simplified easily if we choose $\lambda(t)$ carefully

Start With $S_0 = \sum_{p \le x} 1/p$

$$\lambda_0(t) = \begin{cases} 1/t & \text{if } 1/e \le t \le 1 \\ 0 & \text{if } 0 \le t < 1/e \end{cases}$$



Start With $S_0 = \sum_{p \leq x} 1/p$

- Let's see what $1/e \leq t \leq 1$ means

$$\begin{aligned} 1/e &\leq t \leq 1 \\ 1/e &\leq p^{-\frac{1}{\log x}} \leq 1 \\ -1 &\leq -\frac{1}{\log x} \log p \leq 0 \\ -\log x &\leq -\log p \leq 0 \\ 0 &\leq p \leq x \end{aligned}$$

- This is the range we're interested in for $\sum_{p \leq x} 1/p$.

- Checking $0 \leq t < 1/e$ leads to $x < p < \infty$ which contibutes 0 to $S(\lambda_0, x)$

Start With $S_0 = \sum_{p \leq x} 1/p$

- This definition of $\lambda_0(t)$ reduces $S(\lambda_0, x)$ nicely:

$$\begin{aligned} S(\lambda_0, x) &= \sum_p p^{-1-\frac{1}{\log x}} \cdot \lambda(p^{-\frac{1}{\log x}}) \\ &= \sum_p p^{-1-\frac{1}{\log x}} \cdot p^{+\frac{1}{\log x}} \\ &= \boxed{\sum_p \frac{1}{p}} \end{aligned}$$

for $0 \leq p \leq x$

Linear Forms For λ_U, λ_L

- Let's try a linear form for $\lambda(t) = \alpha + \beta t$

$$
\begin{aligned}
S(\lambda, x) &= \sum_p p^{-1-\frac{1}{\log x}} \cdot \lambda(p^{-\frac{1}{\log x}}) \\
&= \sum_p p^{-1-\frac{1}{\log x}} \cdot (\alpha + \beta \cdot p^{-\frac{1}{\log x}}) \\
&= \alpha \cdot P(1 + \frac{1}{\log x}) + \beta \cdot P(1 + \frac{2}{\log x})
\end{aligned}
$$

- We need something to help us with $P(1+s)$.

First Result $|\log \zeta(s) - P(s)| < \frac{1}{2}$

- Start with Euler product formula

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_n \left(\frac{1}{1 - p^{-s}} \right)$$

- And take logs, legitimate becacse $\zeta(s)$ converges for $s > 1$

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s})$$

First Result $|\log \zeta(s) - P(s)| < \frac{1}{2}$

- Use $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ for $|x| < 1$

$$\begin{aligned}\log \zeta(s) &= \sum_p \sum_k \frac{1}{kp^{ks}} \\ &= P(s) + \sum_p \sum_{k \geq 2} \frac{1}{kp^{ks}}\end{aligned}$$

- We've isolated $P(s) = \sum 1/p^s$, a step in the right direction.
- Since $s > 1$ and $k \geq 2$, we have $1/kp^{ks} \leq 1/2p^k$ so

$$0 < \sum_p \sum_{k \geq 2} \frac{1}{kp^{ks}} \leq \frac{1}{2} \sum_p \sum_{k \geq 2} \frac{1}{p^k}$$

First Result $|\log\zeta(s) - P(s)| < \frac{1}{2}$

- Also, since $(1-x)^{-1} = 1 + x + x^2 + \dots$

$$\begin{aligned}
\sum_p \sum_{k\geq 2} \frac{1}{kp^{ks}} &\leq \frac{1}{2} \sum_p \sum_{k\geq 2} \frac{1}{p^k} \\
&= \frac{1}{2} \sum_p (\frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \dots) \\
&= \frac{1}{2} \sum_p \frac{1}{p} (\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \dots) \\
&= \frac{1}{2} \sum_p \frac{1}{p} (-1 + (1 - \frac{1}{p})^{-1}) \\
&= \frac{1}{2} \sum_p \frac{1}{p(p-1)}
\end{aligned}$$

First Result $|\log \zeta(s) - P(s)| < \frac{1}{2}$

- Since primes p are a subset of counting numbers n

$$\frac{1}{2}\sum_{p}\frac{1}{p(p-1)} < \frac{1}{2}\sum_{n\geq 2}\frac{1}{n(n-1)} = \frac{1}{2}$$

- Last step uses $\sum_{n\geq 2}\frac{1}{n(n-1)} = \sum_{n\geq 2}\frac{1}{n-1} - \frac{1}{n} = 1$.

- Difference between $\log \zeta(s)$ the **prime zeta function** $P(s)$ is bounded.

$$\boxed{0 < \log \zeta(s) - P(s) < \frac{1}{2}}$$

- Not the prime harmonic series $\sum 1/p$ which would require $s = 1$.

Second Result $1 < (s-1) \cdot \zeta(s) < s$

- We have previously used integreal tests to find:

$$\frac{1}{s-1} < \zeta(s) < \frac{1}{s-1} + 1$$

- Re-arranging

$$\boxed{1 < (s-1) \cdot \zeta(s) < s}$$

Third Result $\left|P(s+1) - \log \frac{1}{s}\right| < \frac{1}{2}$

- First result, for $s > 1$

$$0 < \log \zeta(s) - P(s) < \frac{1}{2}$$

- Restricting $0 < s < \frac{1}{2}$, we have to rewrite s to $s+1$

$$-\frac{1}{2} < P(s+1) - \log \zeta(s+1) < 0$$

Third Result $\left|P(s+1) - \log\frac{1}{s}\right| < \frac{1}{2}$

- Second result, for $s > 1$

$$1 < (s-1) \cdot \zeta(s) < s$$

- Restricting $0 < s < \frac{1}{2}$, and rewriting s to $s+1$

$$1 < s \cdot \zeta(s+1) < \frac{3}{2}$$

- Taking logarithms

$$0 < \log\zeta(s+1) - \log\frac{1}{s} < \log\frac{3}{2} < \frac{1}{2}$$

Third Result $\left|P(s+1) - \log\frac{1}{s}\right| < \frac{1}{2}$

- Adding these two inequalities

$$-\frac{1}{2} < P(s+1) - \log\frac{1}{s} < \frac{1}{2}$$

$$\boxed{|P(s+1) - \log\frac{1}{s}| < \frac{1}{2}}$$

for $0 < s < \frac{1}{2}$.

- This can help with $S(\lambda, x) = \alpha \cdot P(1 + \frac{1}{\log x}) + \beta \cdot P(1 + \frac{2}{\log x})$

Linear Forms For λ_U, λ_L

$$S(\lambda, x) = \alpha \cdot P(1 + \frac{1}{\log x}) + \beta \cdot P(1 + \frac{2}{\log x})$$

- Before we apply the third result, we need to ensure $0 < s < \frac{1}{2}$.
- That is, $\frac{2}{\log x} < \frac{1}{2} \implies x > e^4$

$$-\frac{1}{2} < P(1+\frac{1}{\log x}) - \log\log x < \frac{1}{2}$$
$$-\frac{1}{2} < P(1+\frac{2}{\log x}) - \log\log x + \log 2 < \frac{1}{2}$$

- Multiply by α and β, noting they could be negative

$$-\frac{|\alpha|}{2} < \alpha P(1+\frac{1}{\log x}) - \alpha\log\log x < \frac{|\alpha|}{2}$$
$$-\frac{|\beta|}{2} < \beta P(1+\frac{2}{\log x}) - \beta\log\log x + \beta\log 2 < \frac{|\beta|}{2}$$
$$-\frac{|\alpha|}{2} - \frac{|\beta|}{2} < S(\lambda, x) - \alpha\log\log x - \beta\log\log x + \beta\log 2 < \frac{|\alpha|}{2} + \frac{|\beta|}{2}$$

Linear Forms For λ_U, λ_L

- Dealing with that $\beta \log 2$, following is true for both β positive or negative

$$-\frac{|\alpha|}{2}-|\beta|(\frac{1}{2}+\log 2) < S(\lambda, x)-\alpha \log\log x-\beta \log\log x < \frac{|\alpha|}{2}+|\beta|(\frac{1}{2}+\log$$

- Using $\lambda(1) = \alpha + \beta$ to simplify

$$-\frac{|\alpha|}{2} - |\beta|(\frac{1}{2} + \log 2) < S(\lambda, x) - \lambda(1) \log\log x < \frac{|\alpha|}{2} + |\beta|(\frac{1}{2} + \log 2)$$

$$\boxed{|S(\lambda, x) - \lambda(1) \log\log x| < \frac{|\alpha|}{2} + |\beta|(\frac{1}{2} + \log 2)}$$

Two Specific Linear Forms For λ_U, λ_L

- Upper bound

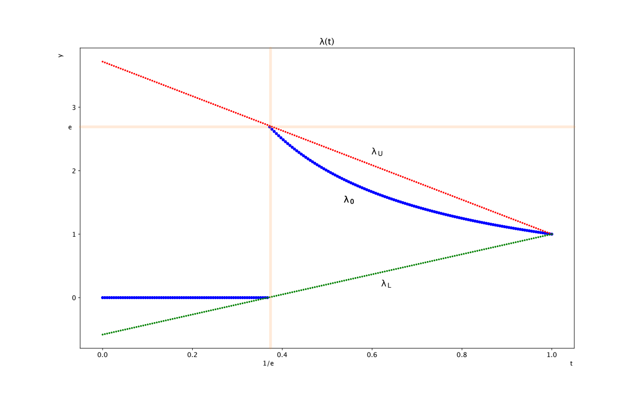
$$\lambda_U(t) = -et + (e+1) \geq \lambda_0(t)$$

- Lower bound

$$\lambda_L(t) = \frac{e}{e-1}t - \frac{1}{e-1} \geq \lambda_0(t)$$

- Note that for both $\alpha + \beta = 1$

Two Specific Linear Forms For λ_U, λ_L



- We can see that $\lambda_L \leq \lambda_0 \leq \lambda_U$ for $t \in [0, 1]$

Upper Bound For $\sum_{p\leq x} 1/p$

- Since $\lambda_U(t) \geq \lambda_0(t)$ we can say $S(\lambda_U, x) \geq S(\lambda_0, x) = \sum_{p\leq x} 1/p$

$$\begin{aligned}
\sum_{p\leq x} \frac{1}{p} &< \log\log x + \frac{|\alpha|}{2} + |\beta|(\frac{1}{2} + \log 2) \\
&= \log\log x + \frac{e+1}{2} + e(\frac{1}{2} + \log 2) \\
\sum_{p\leq x} \frac{1}{p} &< \log\log x + 6
\end{aligned}$$

Lower Bound For $\sum_{p \leq x} 1/p$

- Since $\lambda_L(t) \leq \lambda_0(t)$ we can say $S(\lambda_L, x) \leq S(\lambda_0, x) = \sum_{p \leq x} 1/p$

$$\sum_{p \leq x} \frac{1}{p} > \log \log x - \frac{|\alpha|}{2} - |\beta|(\frac{1}{2} + \log 2)$$
$$= \log \log x + \frac{e+1}{2} + e(\frac{1}{2} + \log 2)$$
$$\sum_{p \leq x} \frac{1}{p} > \log \log x + 3$$

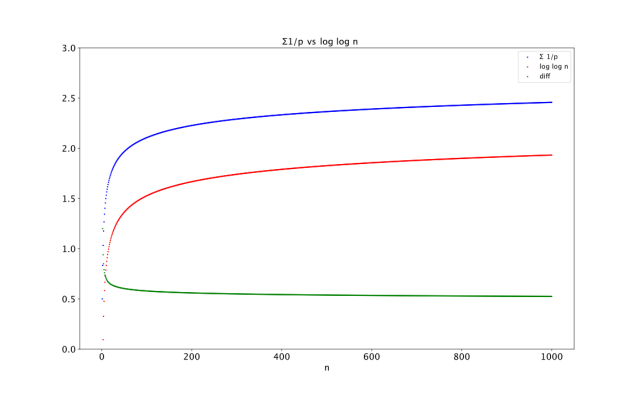
- Putting all this together

$$\log\log x + 3 < \sum_{p\le x} \frac{1}{p} < \log\log x + 6$$

- Finaly we can say

$$\boxed{\sum_{p\le x} \frac{1}{p} \text{ grows like } \log\log x}$$

Visualising $\sum_{p \leq x} 1/p$ and $\log \log x$



- Interesting that the differences appears $\to \frac{1}{2}$

Thoughts

- $\sum_{p \le x} 1/p$ grows very very slowly ... and still diverges!

x	$\log x$	$\log\log x$
10	*2.30*	*0.83*
1,000	*6.91*	*1.93*
1,000,000	*13.82*	*2.63*
1,000,000,000	*20.72*	*3.03*
1,000,000,000,000	*27.63*	*3.32*