Pairs of Symmetric Zeros

From Primes To Riemann

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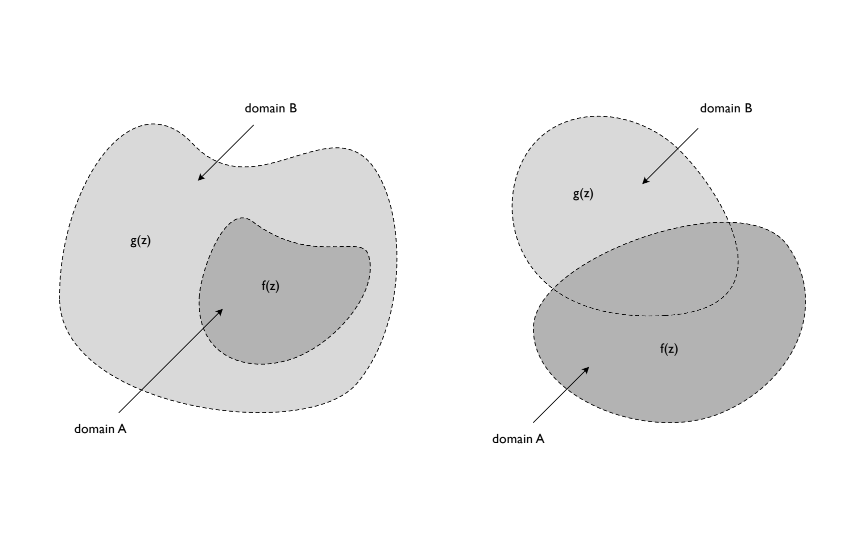
Common Strategy

- A common strategy for wrestling with the Riemann Hypothesis is exploring the properties of its zeros.

Property $\zeta(\bar{s}) = \overline{\zeta(s)}$

- We showed the property $\zeta(\bar{s}) = \overline{\zeta(s)}$ holds for the series $\zeta(s) = \sum 1/n^s$, valid for $\sigma > 1$.
- We can show the new series extending $\zeta(s)$ to $\sigma > 0$ also maintains this property.
- Instead of doing this with slightly laborious algebra, we'll introduce and use the powerful principle of **analytic continuation**.

Intuition For Analytic Continuation



Intuition For Analytic Continuation

- The left picture shows a complex function $f(z)$ defined on a domain A. It also shows a larger domain B which includes domain A.
- The complex function $g(z)$ is defined on domain B. Note $f(z)$ is not valid on the larger domain B.
- If $g(z)$ agrees with $f(z)$ on the domain A, then we can see that $g(z)$ is more general than $f(z)$.
- We call $g(z)$ an **analytic continuation** of $f(z)$.

Intuition For Analytic Continuation

- Let's go a little further. The right picture shows a domain B which overlaps, but doesn't completely cover, domain A.
- If $f(z)$ agrees with $g(z)$ where A and B overlap, we can see how $f(z)$ and $g(z)$ are representations of the same function.
- If we initially know $f(z)$ and then later find $g(z)$, we can say again $g(z)$ is an analytic continuation of $f(z)$.

Intuition For Analytic Continuation

- **Intuition** is that if two complex functions match over a shared domain, they must represent the same function.
- For this intuition to be valid, the functions must be well behaved enough to only allow a single unique extension.
- **Analytic functions** have this good behaviour.
- We'll define analytic functions and explain analytic continuation more precisely next time.

Using Analytic Continuation

- Let's construct a function

$$f(s) = \zeta(s) - \overline{\zeta(\overline{s})}$$

- Wherever $\zeta(s)$ is analytic, so is $f(s)$. This is because $\overline{\zeta(\overline{s})}$ is analytic too.

- We know $f(s) = 0$ along the real line where $\sigma > 1$. Using analytic continuation, $f(s)$ must also be zero in any connected domain where $f(s)$ is analytic.

Using Analytic Continuation

- So $f(s) = 0$ in the complex half-plane $\sigma > 1$.

- ... and also $\sigma > 0$ because we extended $\zeta(s)$ to this larger domain, where it remains analytic except at $s = 1$.

- But $f(s) = 0$ means $\zeta(\bar{s}) = \overline{\zeta(s)}$, which means this property holds in $\sigma > 0$.

- If later we analytically continue $\zeta(s)$ into $\sigma < 0$, this property will continue to hold there too.

Symmetric Zeros

- If $\zeta(s) = 0$ then the property $\zeta(\bar{s}) = \overline{\zeta(s)}$ tells us $\zeta(\bar{s}) = 0$.

$$\zeta(s) = 0 \implies \zeta(\bar{s}) = 0$$

- This means the zeros exist in symmetric pairs $\sigma + it$ and $\sigma - it$.
- Zeros are mirrored above and below the real line, or lie on it.

Thoughts

- For a first attempt, this is quite an enlightening insight into the zeros of the Riemann Zeta function $\zeta(s)$.
- Zeros existing in symmetric pairs $\sigma \pm it$ is compatible with the Riemann Hypothesis
- ... but sadly it doesn't mean they all lie on a single line $\sigma = a$, never mind the holy grail $\sigma = 1/2$.