1 $\sum 1/n$ Diverges

Infinite Series

Have a look at the following infinite series.

$$1 + 1 + 1 + 1 + \dots$$

We can easily see this sum is infinitely large. The series **diverges**.

The following shows a different infinite series. Each term is half the size of the previous one.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

We can intuitively see this series gets ever closer to 2. Many would simply say the sum is in fact 2. The series **converges**.

Harmonic Series

Now let's look at this infinite series, called the **harmonic series**.

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

Each term is smaller than the previous one, and so contributes an ever smaller amount to the sum. Perhaps suprisingly, the harmonic series doesn't converge. The sum is infinitely large.

The following, rather fun, proof is based on Oresme's which dates back to the early 1300s.

We start by grouping the terms in the series as follows.

$$S = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$

The brackets will have $2, 4, 8, 16 \dots$ terms. Replacing each term in a group by its smallest member gives us the following new series.

$$\begin{aligned} T &= 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots \\ &= 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \dots \end{aligned}$$

We can see straight away this series diverges.

Because we replaced terms in S by smaller ones to make T, we can say $S > T$.

And because T diverges, so must the harmonic series S.