

Abel's Partial Summation Formula

From Primes To Riemann

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Discrete vs Continuous Functions

- Often easier to understand how a **discrete** function behaves if it can be expressed as a **continuous** function.
- **Abel's partial summation formula** allows us to write a discrete sum as continuous integral.

- Sum over arithmetic function $a(n)$ weighted by smooth function $f(x)$.

$$\sum_{x_1 < n \leq x_2} a(n)f(n)$$

- A useful general object to find an integral form for, because it gives us flexibility in choosing the arithmetic and smooth functions.
- To be precise: $a(n)$ takes only positive integers $n \geq 1$, and $f(x)$ has a continuous derivative over the domain $[x_1, x_2]$. Both $a(n)$ and $f(x)$ can be complex.

Deriving An Integral Form

- The derivation of an integral form for the sum is just lots and lots of simple algebra!
- Because $a(n)$ is only defined over integers $n \geq 1$, we clarify the sum by setting $m_1 = \lfloor x_1 \rfloor$ and $m_2 = \lfloor x_2 \rfloor$.

$$\sum_{x_1 < n \leq x_2} a(n)f(n) = \sum_{n=m_1+1}^{m_2} a(n)f(n)$$

- Remember $\lfloor x \rfloor$ is the largest integer up to, and including, x .

Deriving An Integral Form

- Define $A(x) = \sum_{n \leq x} a(n)$. By definition $a(n) = A(n) - A(n-1)$ so we can replace $a(n)$.

$$\begin{aligned}\sum_{m_1+1}^{m_2} a(n)f(n) &= \sum_{m_1+1}^{m_2} [A(n) - A(n-1)] f(n) \\ &= \sum_{m_1+1}^{m_2} A(n)f(n) - \sum_{m_1}^{m_2-1} A(n)f(n+1)\end{aligned}$$

The two sums have different limits for n , but both cover $[m_1 + 1, m_2 - 1]$.

$$\begin{aligned}\sum_{m_1+1}^{m_2} a(n)f(n) &= \sum_{m_1+1}^{m_2-1} A(n)[f(n) - f(n+1)] \\ &\quad + A(m_2)f(m_2) - A(m_1)f(m_1+1)\end{aligned}$$

Deriving An Integral Form

- Noticing that $\int_n^{n+1} f'(t)dt = f(n+1) - f(n)$ allows us to introduce the integral.

$$\sum_{m_1+1}^{m_2} a(n)f(n) = - \sum_{m_1+1}^{m_2-1} A(n) \int_n^{n+1} f'(t)dt$$
$$+ A(m_2)f(m_2) - A(m_1)f(m_1 + 1)$$

Deriving An Integral Form

- Because $A(t) = A(n)$ over the interval $[n, n+1)$, we can move $A(n)$ it inside the integral as $A(t)$.

$$\begin{aligned}\sum_{m_1+1}^{m_2} a(n)f(n) &= - \sum_{m_1+1}^{m_2-1} \int_n^{n+1} A(t)f'(t)dt \\ &\quad + A(m_2)f(m_2) - A(m_1)f(m_1+1)\end{aligned}$$

- The sum of integrals over consecutive intervals can be simplified to a single integral.

$$\begin{aligned}\sum_{x_1 < n \leq x_2} a(n)f(n) &= - \int_{m_1+1}^{m_2} A(t)f'(t)dt \\ &\quad + A(m_2)f(m_2) - A(m_1)f(m_1+1)\end{aligned}$$

Deriving An Integral Form

- We now need to adjust the integration limits back to x_1 and x_2 , not forgetting the intervals to m_1 and m_2 .
- Writing out the integrals that split the interval $[x_1, x_2]$ is helpful.

$$\int_{x_1}^{x_2} X dt = \int_{m_1+1}^{m_2} X dt + \int_{x_1}^{m_1+1} X dt + \int_{m_2}^{x_2} X dt$$

Deriving An Integral Form

- Using $A(t) = A(x_1)$ over $[x_1, m_1 + 1)$, and $A(t) = A(x_2)$ over $[m_2, x_2]$

$$-\int_{m_1+1}^{m_2} A(t)f'(t)dt = \int_{x_1}^{m_1+1} A(t)f'(t)dt + \int_{m_2}^{x_2} A(t)f'(t)dt$$

$$-\int_{x_1}^{x_2} A(t)f'(t)dt$$

$$= A(x_1)[f(m_1 + 1) - f(x_1)] + A(x_2)[f(x_2) - f(m_2)]$$

$$-\int_{x_1}^{x_2} A(t)f'(t)dt$$

Deriving An Integral Form

- We plug this integral back into our object, then use $A(m_1) = A(x_1)$ and $A(m_2) = A(x_2)$.

$$\sum_{x_1 < n \leq x_2} a(n)f(n) = A(x_2)f(m_2) - A(x_1)f(m_1 + 1)$$

$$+ A(x_1)[f(m_1 + 1) - f(x_1)] + A(x_2)[f(x_2) - f(m_2)]$$

$$- \int_{x_1}^{x_2} A(t)f'(t)dt$$

- Many of these terms cancel out, leaving us with the **Abel Identity**.

$$\boxed{\sum_{x_1 < n \leq x_2} a(n)f(n) = A(x_2)f(x_2) - A(x_1)f(x_1) - \int_{x_1}^{x_2} A(t)f'(t)dt}$$

Example: Growth of $\sum 1/n$

- Abel's identity \rightarrow asymptotic behaviour of discrete functions.
- Let's use it to explore how the harmonic series $\sum_1^N 1/n$ grows.
- We can choose $a(n) = 1$ and $f(x) = 1/x$, which means $A(x) = \lfloor x \rfloor$ and $f'(x) = -1/x^2$.

$$\begin{aligned}\sum_{0 < n \leq N} \frac{1}{n} &= 1 + \sum_{1 < n \leq N} \frac{1}{n} \\ &= 1 + A(N)f(N) - A(1)f(1) - \int_1^N A(t)f'(t)dt \\ &= 1 + \frac{N}{N} - \frac{1}{1} + \int_1^N \frac{t - \{t\}}{t^2} dt\end{aligned}$$

Example: Growth of $\sum 1/n$

- We've written $\lfloor t \rfloor$ as $t - \{t\}$, where $\{t\}$ is the fractional part of t . This lets us split the integral into manageable parts.

$$\sum_{0 < n \leq N} \frac{1}{n} = 1 + \int_1^N \frac{1}{t} dt - \int_1^N \frac{\{t\}}{t^2} dt$$

- Because $\{t\}$ is only ever in the range $[0, 1)$, the last integral is always less than $\int_1^N 1/t^2 dt$, that is, $\mathcal{O}\left(\int_1^N 1/t^2 dt\right)$.

$$\begin{aligned} \sum_{0 < n \leq N} \frac{1}{n} &= 1 + \ln(N) - \mathcal{O}\left(1 - \frac{1}{N}\right) \\ &= \ln(N) + \mathcal{O}(1) \end{aligned}$$

Example: Growth of $\sum 1/n$

- This tells us the harmonic series $\sum_1^N 1/n$ grows like $\ln(N)$.
- It also tells us the difference is bounded by $O(1)$.
- In fact, the difference tends to the Euler–Mascheroni constant $\gamma \approx 0.5772$ which pops up in several areas of number theory and analysis.