1 Bounds For $\sum \frac{1}{x}$

Understanding the behaviuor of the sum $\sum_1^n \frac{1}{x}$ would be easier if it was a continuous function of x. Instead, we can try find continuous functions for upper and lower bounds for the sum instead.

Figure 1.1 shows a graph of $y = \frac{1}{x}$, together with rectangles representing the fractions $\frac{1}{n}$.

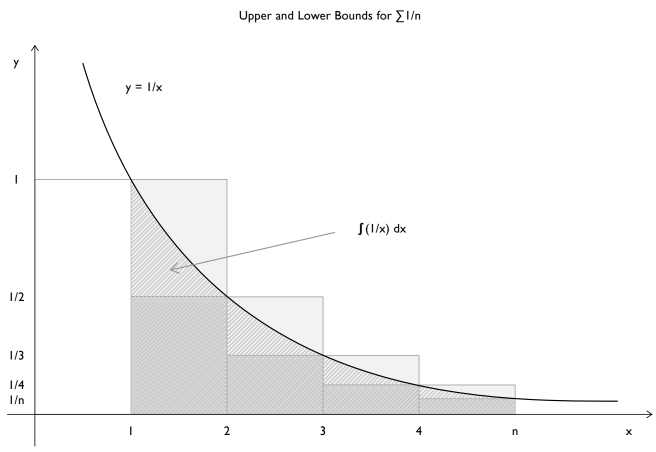

Figure 1.1: Comparing discrete $1/n$ with continuous $1/x$.

Lower Bound

If we consider the range $1 \leq x \leq 4$ we can see the area of the three taller rectangles $1 + \frac{1}{2} + \frac{1}{3}$ is greater than the area under the curve $\int_1^4 \frac{1}{x} dx$. By extending the range to n we can make a general observation.

$$\sum_1^n \frac{1}{x} > \int_1^{n+1} \frac{1}{x} dx$$

The integral has an upper limit of $n+1$ because the width of the last rectangle extends from $x = n$ to $x = n+1$. We can perform the integral to simplify the expression.

$$\boxed{\sum_1^n \frac{1}{x} > \ln(n+1)}$$

This is a rather nice lower bound on the growth of the harmonic series.

Upper Bound

Let's now look at the shorter rectanges. In the range $1 \leq x \leq 4$ we can see the area of the three shorter rectangles $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ is less than the area under the curve $\int_1^4 \frac{1}{x} dx$. Again, by extending the range to n we can make a general observation.

$$\sum_2^n \frac{1}{x} < \int_1^n \frac{1}{x} dx$$

The harmonic sum starts at 2 because this time we're looking at rectanges extending to the left of a given x. We can easily fix the limits of the sum using $\sum_1^n \frac{1}{n} = 1 + \sum_2^n \frac{1}{n}$.

$$\sum_{1}^{n} \frac{1}{x} - 1 < \int_{1}^{n} \frac{1}{x} dx$$

Again, we can perform the integral.

$$\boxed{\sum_{1}^{n} \frac{1}{x} < \ln(n) + 1}$$

This is another rather nice upper bound to the growth of the harmonic series.

Comparison Tests

This method of comparing a discrete series with a continuous function to obtain nice functions for lower or upper bounds is fairly powerful, and very common in the pratice of number theory.