# Infinite Products From Primes To Riemann

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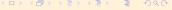
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#### Infinite Product

At school we don't seem to learn about infinite products.

$$\prod_{n=1}^{\infty} a_n = a_1 \times a_2 \times a_2 \times \dots$$

What do we really mean by infinite product?



#### Initial Observations

 Example 1 - Easy to see the infinite product diverges. Each factor increases the size of the product.

$$2 \times 3 \times 4 \times 5 \times \dots$$

 Example 2 - Fundamental idea that multiplying by zero causes a product to be zero.

$$2 \times 0 \times 4 \times 5 \times \dots$$



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#### Initial Observations

• Example 3 - Each factor reduces the size of the product.

$$\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \dots$$

- Infinite number of such factors, the product  $\rightarrow$  0.
- We have found two different ways an infinite product can be zero.

#### Definition

• Similar to **infinite series**, we say an **infinite product** converges if the limit of the partial products is a **finite** value.

$$\lim_{N\to\infty}\prod_{n=1}^N a_n=P$$

• We'll see why it is conventional to insist the finite value is **non-zero**.

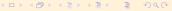


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• Does this converge?

$$\prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)$$

• Each factor (1+1/n) is larger than one, so we expect the product to keep growing.



Consider partial product.

$$\prod_{n=1}^{N} \left( 1 + \frac{1}{n} \right) = \prod_{n=1}^{N} \left( \frac{n+1}{n} \right)$$
$$= \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \dots \times \frac{N+1}{M}$$
$$= N+1$$

• As  $N \to \infty$ , product **diverges**.



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• Does this converge?

$$\prod_{n=2}^{\infty} \left( 1 - \frac{1}{n} \right)$$

- Note *n* starts at 2 to ensure the first factor is not zero.
- Each factor (1-1/n) is smaller than one, so we expect the product to keep shrinking.

$$\prod_{n=2}^{N} \left( 1 - \frac{1}{n} \right) = \prod_{n=2}^{N} \left( \frac{n-1}{n} \right)$$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{N+1}{N}$$

$$= \frac{1}{N}$$

• As  $N \to \infty$ , product tends to 0, we say product **diverges to zero**.

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# Convergence And $a_n$

- For an infinite series  $\sum a_n$  to converge, the terms  $a_n \to 0$
- For an infinite product  $\prod a_n$  to converge, the terms  $a_n \to 1$ 
  - if each term  $a_n > 1$ , the product gets ever larger.
  - if each term  $a_n < 1$ , the product gets ever smaller towards zero.
  - negative a<sub>n</sub> cause partial product to oscillate.
  - .. meaning convergence only happens if  $a_n \to 1$ .

#### Removing Zero Factors

- A single factor 0 collapses entire product to zero.
- If an infinite product has a finite number of zero-valued factors, they can be removed to leave a different potentially interesting product.
- Example:

$$\prod_{n=1} \left( 1 - \frac{1}{n^2} \right) = 0$$

Removing first factor leaves an interesting infinite product:

$$\prod_{n=2} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$$

• Useful to write factors as  $(1 + a_n)$ 

$$P=\prod(1+a_n)$$

Turn product into sum by takings logarithm

$$ln(P) = ln \prod (1+a_n) = \sum ln(1+a_n)$$

• Using  $1 + x \le e^x$ 

$$ln(P) \leq \sum a_n$$

• If the sum is **bounded**  $\implies$  the product is bounded. If  $a_n > 0$  then boundedness is **convergence** (no oscillation).

• If we expand out product  $\prod (1+a_n)$  we see another inequality.

$$1+\sum a_n \leq \prod (1+a_n)=P$$

- The expansion creates the terms  $1 + \sum a_n$  and many more
- ullet This tells us that if the product converges  $\Longrightarrow$  so does the sum.

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• The two results together give us

$$\sum a_n$$
 converges  $\Leftrightarrow \prod (1+a_n)$  converges, for  $a_n>0$ 

- This allows us to say:
  - $\prod (1+1/n)$  diverges because  $\sum 1/n$  diverges
  - $\prod (1+1/n^2)$  converges because  $\sum 1/n^2$  converges

• Using  $1 - x \le e^{-x}$  leads to a criterion for products of the form  $\prod (1 - a_n)$ .

$$\sum a_n$$
 converges  $\Leftrightarrow \prod (1-a_n)$  converges, for  $0 < a_n < 1$ 

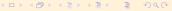
• So  $\prod_{2}^{\infty} (1 - 1/n)$  diverges, because  $\sum 1/n$  diverges.



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#### Divergence To Zero

- The logarithmic view of infinite products has an interesting side effect.
- If the partial products ightarrow 0 then the logarithm  $ightarrow -\infty$
- This is why we say the product diverges to zero.



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- Previous convergence critera apply for real values  $a_n > 0$ .
- Would be good to have criteria for **complex** a<sub>n</sub>.
- To do that we'll need an intermediate result about  $|a_n|$
- For complex  $a_n$ , we have  $|a_n| > 0$  for all  $a_n \neq 0$ .

$$\sum |a_n|$$
 converges  $\Leftrightarrow \prod (1+|a_n|)$  converges

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• We're interested in  $\prod (1+a_n)$  for complex  $a_n$ , not just  $\prod (1+|a_n|)$ .

$$p_N = \prod^N (1+a_n)$$

$$q_N = \prod^N (1 + |a_n|)$$

• We assert  $a_n \neq -1$  to ensure no zero-valued factors.

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• For  $N > M \ge 1$ , we can compare  $|p_N - p_M|$  with  $|q_N - q_M|$ 

$$egin{aligned} |p_N-p_M| &= |p_M| \cdot \left| rac{p_N}{p_M} - 1 
ight| \ &= |p_M| \cdot \left| \prod_{M=1}^N (1+a_n) - 1 
ight| \ &\leq |q_M| \cdot \left| \prod_{M=1}^N (1+|a_n|) - 1 
ight| \ &= |q_M| \cdot \left| rac{q_N}{q_M} - 1 
ight| \end{aligned}$$

$$|p_N - p_M| \le |q_N - q_M|$$

• If  $|q_N - q_M| < \epsilon$ , then  $|p_N - p_M| < \epsilon$ . Cauchy criterion for convergence. If  $q_N$  converges, so does  $p_N$ .

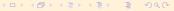
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• So  $\sum |a_n|$  converges  $\implies \prod (1+|a_n|)$  converges  $\implies \prod (1+a_n)$  converges.

$$\sum |a_n|$$
 converges  $\implies \prod (1+a_n)$  converges, for  $a_n \neq -1$ 

 This is one way, we can't say the sum comverges if the product converges.



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#### Summary

• Real  $a_n$ 

$$\sum a_n$$
 converges  $\Leftrightarrow \prod (1+a_n)$  converges, for  $a_n>0$ 

• Real a<sub>n</sub>

$$\sum a_n$$
 converges  $\Leftrightarrow \prod (1-a_n)$  converges, for  $0 < a_n < 1$ 

Complex a<sub>n</sub>

$$\sum |a_n|$$
 converges  $\implies \prod (1+a_n)$  converges, for  $a_n 
eq -1$ 

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- Why?

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- We've know  $a_n \to 0$ , so  $|a_n| < 1/2$  except for a finite number of terms.
- Useful inequality  $1 + x \le e^x$  gives us

$$1 \leq \prod (1+|a_n|) < e^{\sum |a_n|}$$

•  $\sum |a_n|$  converges  $\implies \prod (1+|a_n|)$  converges and is non-zero.

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• Another inequality  $1 - x \ge e^{-2x}$  for  $0 \le x \le 1/2$ 

$$0 < e^{-2\sum |a_n|} \le \prod (1-|a_n|) \le 1$$

- Uses  $e^y > 0$  for all real y.
- $\sum |a_n|$  converges  $\implies \prod (1-|a_n|)$  converges and is non-zero.

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Another inequality

$$1-|a_n| \leq |1 \pm a_n| \leq 1+|a_n|$$

$$\prod (1-|a_n|) \leq \prod (1\pm a_n) \leq \prod (1+|a_n|)$$

- $\sum |a_n|$  converges  $\implies \prod (1 \pm a_n)$  is non-zero
  - because its value is between two known non-zero values.

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#### Riemann Zeta Function

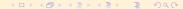
• For  $\sigma > 1$ 

$$\zeta(s) = \sum \frac{1}{n^s} = \prod (1 - \frac{1}{p^s})^{-1}$$

- No factor  $(1-1/p^s)^{-1}$  is zero.
  - That would require p<sup>s</sup> to be zero.
  - This isn't possible, and is easy to see by writing

$$|p^s| = \left|e^{s\ln(p)}\right| = e^{\sigma\ln(p)} > 0$$

Also need to check product doesn't diverge to zero.



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#### Riemann Zeta Function

• Consider  $1/\zeta(s) = \prod (1 - 1/p^s)$ . Using third convergence criterion, check  $\sum |1/p^s|$  converges.

$$\sum \left| \frac{1}{p^s} \right| = \sum \frac{1}{p^{\sigma}} \le \sum \frac{1}{n^{\sigma}}$$

- Because  $\sum 1/n^{\sigma}$  converges for  $\sigma > 1$ , so does  $\sum |1/p^{s}|$ .
- This means  $1/\zeta(s)$  converges to a non-zero value, and therefore so does  $\zeta(s)$ .
- $\Longrightarrow$  Riemann Zeta function has no zeros  $\sigma > 1$

