$\sum \frac{1}{n^2}$ Diverges

From Primes To Riemann

Tariq Rashid

January 27, 2021

The Basel Problem

- Difficult problem first posed around 1650.

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \ldots \to ?$$

- Several modern proofs exist but we'll follow Euler's audacious 1734 proof which made him famous, and inspired Riemann's later work on prime numbers.

Taylor Series For sin(x)

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

- valid for all x.

Polynomial Factors And Roots

- The polynomial

$$f(x) = (1 - \frac{x}{a})(1 + \frac{x}{a})$$

- has factors $(1 - \frac{x}{a})$ and $(1 + \frac{x}{a})$, and zeros at $+a$ and $-a$.
- We can shorten it to $f(x) = (1 - \frac{x^2}{a^2})$.

Euler's Idea

- Euler's novel idea was to write $\sin(x)$ as a product of similar linear factors ...

Euler's Factorisation Of $\sin(x)$

- The zeros of $\sin(x)$ are at $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$$\sin(x) = A \cdot x \cdot \left(1 - \frac{x^2}{\pi^2}\right) \cdot \left(1 - \frac{x^2}{(2\pi)^2}\right) \cdot \left(1 - \frac{x^2}{(3\pi)^2}\right) \cdot \ldots$$

- A is 1 because we know $\frac{\sin(x)}{x} \to 1$ as $x \to 0$.
  - Alternatively, taking the first derivative of both sides gives $A = 1$ when $x = 0$.
- The second factor is x and not x^2 because the zero of $\sin(x)$ at $x = 0$ has multiplicity 1.

Euler's New Formula For $\sin(x)$

- Euler then expanded out his new formula for $\sin(x)$

$$\sin(x) = x \cdot \left[1 - \frac{x^2}{\pi^2} \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right) + X \right]$$

- Inside the square brackets, the terms with powers of x higher than 2 are contained in X.

Comparing The Two Series

- Euler compared the Taylor series and his new series for $sin(x)$.
- Picked out the x^3 terms from both, we get

$$\frac{x^3}{3!} = \frac{x^3}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right)$$

- Simple rearranging

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Basel Problem Solved!

- Aged 28, Euler had solved the long standing Basel problem!
- He not only proved the infinite series of squared reciprocals **converged**, but gave it an exact value.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

Rigour

- Euler was adventurous in expressing $\sin(x)$ as an infinite product of simple linear factors.
- 100 years later Weierstass developed and proved a factorisation theorem that confirmed Euler's leap was legitimate.