$\zeta(s)$ Has One Pole In $\sigma > 0$

From Primes To Riemann

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Previously ...

- Series representation $\zeta(s) = \sum 1/n^s$ converges for $\sigma > 1$, $\implies$ no poles in that domain.

- New series for $\zeta(s)$ based on the (alternating zeta) eta function $\eta(s)$.

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \eta(s)$$

- $\eta(s)$ converges for $\sigma > 0$, so any divergence must come from factor $(1 - 2^{1-s})^{-1}$.

Potential Poles

- The denominator $(1 - 2^{1-s})$ is zero at $s = 1 + 2\pi ia/\ln(2)$ for integer a, so the factor $(1 - 2^{1-s})^{-1}$ diverges at all these points.
- Visualising $\zeta(s)$ suggested it had only **one** pole at $s = 1 + 0i$.
- If true, $\eta(s)$ needs zeros at $s = 1 + 2\pi ia/\ln(2)$ for integers $a \neq 0$ to cancel out the other poles from $(1 - 2^{1-s})^{-1}$.
- To prove this directly isn't easy, but there is a nice indirect path.

Yet Another Series For $\zeta(s)$

- We start with a specially constructed Dirichlet series.

$$X(s) = \frac{1}{1^s} + \frac{1}{2^s} - \frac{2}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} - \frac{2}{6^s} + \dots$$

- The pattern can be exploited to find yet another series for $\zeta(s)$.

$$\zeta(s) - X(s) = \frac{3}{3^s} + \frac{3}{6^s} + \frac{3}{9^s} + \dots$$

$$= \frac{3}{3^s}\zeta(s)$$

$$\zeta(s) = \frac{1}{1 - 3^{1-s}} X(s)$$

Comparing Potential Poles

- Dirichlet series $X(s)$ converges for $\sigma > 0$, so any divergence must come from factor $(1 - 3^{1-s})^{-1}$.
- Denominator $(1 - 3^{1-s})$ is zero when $s = 1 + 2\pi ib/\ln(3)$ for integer b.
- We can equate the two expressions for where poles of $\zeta(s)$ **could** be.

$$1 + \frac{2\pi ia}{\ln(2)}\frac{a}{b} = 1 + \frac{2\pi ib}{\ln(3)}$$

$$\frac{a}{b} = \frac{\ln(2)}{\ln(3)}$$

Comparing Potential Poles

$$\frac{a}{b} = \frac{\ln(2)}{\ln(3)}$$

- No non-zero integers a and b satisfy this because $\ln(2)/\ln(3)$ is irrational.
- $\implies s = 1 + 0i$ as the only pole for $\zeta(s)$ in the domain $\sigma > 0$.

$\zeta(s)$ Has One Pole In $\sigma > 0$