Infinite Products From Primes To Riemann

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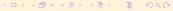
1/27

Infinite Product

At school we don't seem to learn about infinite products.

$$\prod_{n=1}^{\infty} a_n = a_1 \times a_2 \times a_2 \times \dots$$

• What do we really mean by infinite product?



Initial Observations

 Example 1 - Easy to see the infinite product diverges. Each factor increases the size of the product.

$$2 \times 3 \times 4 \times 5 \times \dots$$

 Example 2 - Fundamental idea that multiplying by zero causes a product to be zero.

$$2 \times 0 \times 4 \times 5 \times \dots$$



3 / 27

Initial Observations

• Example 3 - Each factor reduces the size of the product.

$$\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \dots$$

- Infinite number of such factors, the product \rightarrow 0.
- We have found two different ways an infinite product can be zero.

Definition

• Similar to **infinite series**, we say an **infinite product** converges if the limit of the partial products is a **finite** value.

$$\lim_{N\to\infty}\prod_{n=1}^N a_n=P$$

• We'll see why it is conventional to insist the finite value is **non-zero**.

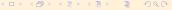


5 / 27

• Does this converge?

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n} \right)$$

• Each factor (1+1/n) is larger than one, so we expect the product to keep growing.



• Consider partial product.

$$\prod_{n=1}^{N} \left(1 + \frac{1}{n} \right) = \prod_{n=1}^{N} \left(\frac{n+1}{n} \right)$$
$$= \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \dots \times \frac{N+1}{M}$$
$$= N+1$$

• As $N \to \infty$, product **diverges**.

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7 / 27

• Does this converge?

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)$$

- Note *n* starts at 2 to ensure the first factor is not zero.
- Each factor (1-1/n) is smaller than one, so we expect the product to keep shrinking.

$$\prod_{n=2}^{N} \left(1 - \frac{1}{n} \right) = \prod_{n=2}^{N} \left(\frac{n-1}{n} \right)$$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{N+1}{N}$$

$$= \frac{1}{N}$$

• As $N \to \infty$, product tends to 0, we say product **diverges to zero**.

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9 / 27

Convergence And a_n

- For an infinite series $\sum a_n$ to converge, the terms $a_n \to 0$
- For an infinite product $\prod a_n$ to converge, the terms $a_n \to 1$
 - if each term $a_n > 1$, the product gets ever larger.
 - if each term $a_n < 1$, the product gets ever smaller towards zero.
 - negative a_n cause partial product to oscillate.
 - .. meaning convergence only happens if $a_n \to 1$.

Removing Zero Factors

- A single factor 0 collapses entire product to zero.
- If an infinite product has a finite number of zero-valued factors, they can be removed to leave a different potentially interesting product.
- Example:

$$\prod_{n=1} \left(1 - \frac{1}{n^2} \right) = 0$$

Removing first factor leaves an interesting infinite product:

$$\prod_{n=2} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$$

• Useful to write factors as $(1 + a_n)$

$$P=\prod(1+a_n)$$

Turn product into sum by takings logarithm

$$ln(P) = ln \prod (1+a_n) = \sum ln(1+a_n)$$

• Using $1 + x \le e^x$

$$ln(P) \leq \sum a_n$$

• If the sum is **bounded** \implies the product is bounded. If $a_n > 0$ then boundedness is **convergence** (no oscillation).

Tariq Rashid Infinite Products August 18, 2021 12 / 27

• If we expand out product $\prod (1+a_n)$ we see another inequality.

$$1+\sum a_n \leq \prod (1+a_n) = P$$

- The expansion creates the terms $1 + \sum a_n$ and many more
- ullet This tells us that if the product converges \Longrightarrow so does the sum.

Tariq Rashid Infinite Products August 18, 2021 13 / 27

• The two results together give us

$$\sum a_n$$
 converges $\Leftrightarrow \prod (1+a_n)$ converges, for $a_n>0$

- This allows us to say:
 - $\prod (1+1/n)$ diverges because $\sum 1/n$ diverges
 - $\prod (1+1/n^2)$ converges because $\sum 1/n^2$ converges

Tariq Rashid Infinite Products A

14 / 27

• Using $1 - x \le e^{-x}$ leads to a criterion for products of the form $\prod (1 - a_n)$.

$$\sum a_n$$
 converges $\Leftrightarrow \prod (1-a_n)$ converges, for $0 < a_n < 1$

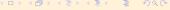
• So $\prod_{2}^{\infty} (1 - 1/n)$ diverges, because $\sum 1/n$ diverges.



Tariq Rashid Infinite Products August 18, 2021 15 / 27

Divergence To Zero

- The logarithmic view of infinite products has an interesting side effect.
- ullet If the partial products o 0 then the logarithm o $-\infty$
- This is why we say the product **diverges to zero**.



Tariq Rashid Infinite Products August 18, 2021 16 / 27

- Previous convergence critera apply for real values $a_n > 0$.
- Would be good to have criteria for **complex** a_n .
- To do that we'll need an intermediate result about $|a_n|$
- For complex a_n , we have $|a_n| > 0$ for all $a_n \neq 0$.

$$\sum |a_n|$$
 converges $\Leftrightarrow \prod (1+|a_n|)$ converges

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• We're interested in $\prod (1+a_n)$ for complex a_n , not just $\prod (1+|a_n|)$.

$$p_N = \prod^N (1+a_n)$$

$$q_N = \prod^N (1 + |a_n|)$$

• We assert $a_n \neq -1$ to ensure no zero-valued factors.

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18 / 27

• For $N > M \ge 1$, we can compare $|p_N - p_M|$ with $|q_N - q_M|$

$$egin{aligned} |p_N-p_M| &= |p_M| \cdot \left| rac{p_N}{p_M} - 1
ight| \ &= |p_M| \cdot \left| \prod_{M=1}^N (1+a_n) - 1
ight| \ &\leq |q_M| \cdot \left| \prod_{M=1}^N (1+|a_n|) - 1
ight| \ &= |q_M| \cdot \left| rac{q_N}{q_M} - 1
ight| \end{aligned}$$

$$|p_N - p_M| \le |q_N - q_M|$$

• If $|q_N - q_M| < \epsilon$, then $|p_N - p_M| < \epsilon$. Cauchy criterion for convergence. If q_N converges, so does p_N .

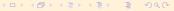
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19 / 27

• So $\sum |a_n|$ converges $\implies \prod (1+|a_n|)$ converges $\implies \prod (1+a_n)$ converges.

$$\sum |a_n|$$
 converges $\implies \prod (1+a_n)$ converges, for $a_n \neq -1$

 This is one way, we can't say the sum comverges if the product converges.



Tariq Rashid Infinite Products August 18, 2021 20 / 27

Summary

• Real a_n

$$\sum a_n$$
 converges $\Leftrightarrow \prod (1+a_n)$ converges, for $a_n>0$

• Real a_n

$$\sum a_n$$
 converges $\Leftrightarrow \prod (1-a_n)$ converges, for $0 < a_n < 1$

Complex a_n

$$\sum |a_n|$$
 converges $\implies \prod (1+a_n)$ converges, for $a_n \neq -1$

Tariq Rashid Infinite Products August 18, 2021 21 / 27

- Why?

Tariq Rashid Infinite Products August 18, 2021 22 / 27

- We've know $a_n \to 0$, so $|a_n| < 1/2$ except for a finite number of terms.
- Useful inequality $1 + x \le e^x$ gives us

$$1 \leq \prod (1+|a_n|) < e^{\sum |a_n|}$$

• $\sum |a_n|$ converges $\implies \prod (1+|a_n|)$ converges and is non-zero.



Tariq Rashid Infinite Products August 18, 2021 23 / 27

• Another inequality $1 - x \ge e^{-2x}$ for $0 \le x \le 1/2$

$$0 < e^{-2\sum |a_n|} \le \prod (1-|a_n|) \le 1$$

- Uses $e^y > 0$ for all real y.
- $\sum |a_n|$ converges $\implies \prod (1-|a_n|)$ converges and is non-zero.

Tariq Rashid Infinite Products August 18, 2021 24 / 27

Another inequality

$$1 - |a_n| \le |1 \pm a_n| \le 1 + |a_n|$$

$$\prod (1-|a_n|) \leq \left| \prod (1\pm a_n)
ight| \leq \prod (1+|a_n|)$$

- $\sum |a_n|$ converges $\implies \prod (1 \pm a_n)$ is non-zero
 - because its absoslute value is between two known non-zero values.

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Riemann Zeta Function

• For $\sigma > 1$

$$\zeta(s) = \sum \frac{1}{n^s} = \prod (1 - \frac{1}{p^s})^{-1}$$

- No factor $(1-1/p^s)^{-1}$ is zero.
 - That would require p^s to be zero.
 - This isn't possible, and is easy to see by writing

$$|p^s| = \left|e^{s\ln(p)}\right| = e^{\sigma\ln(p)} > 0$$

Also need to check product doesn't diverge to zero.



26 / 27

Riemann Zeta Function

• Consider $1/\zeta(s) = \prod (1 - 1/p^s)$. Using third convergence criterion, check $\sum |1/p^s|$ converges.

$$\sum \left| \frac{1}{p^s} \right| = \sum \frac{1}{p^{\sigma}} \le \sum \frac{1}{n^{\sigma}}$$

- Because $\sum 1/n^{\sigma}$ converges for $\sigma > 1$, so does $\sum |1/p^{s}|$.
- This means $1/\zeta(s)$ converges to a non-zero value, and therefore so does $\zeta(s)$.
- \Longrightarrow Riemann Zeta function has no zeros $\sigma > 1$



Tariq Rashid Infinite Products August 18, 2021 27 / 2