

# Dirichlet Series

## From Primes To Riemann

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June 20, 2021

- **Dirichlet series** have the general form

$$\sum a_n/n^s$$

- ... in contrast to familiar power series  $\sum a_n z^n$ .
- The Riemann Zeta series is an example of a Dirichlet series.

$$\zeta(s) = \sum \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

# Absolute Convergence

- A series **converges absolutely** even when all its terms are replaced by their magnitudes.
- Not all series that converge do so absolutely.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \rightarrow \infty$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots \rightarrow \ln(2)$$

# Abscissa of Absolute Convergence

- Assume a Dirichlet series **converges absolutely** at  $s_1 = \sigma_1 + it_1$ . Consider another  $s_2 = \sigma_2 + it_2$  where  $\sigma_2 \geq \sigma_1$ .
- Compare the magnitudes of the terms in this series at  $s_1$  and  $s_2$ .

$$\sum \left| \frac{a_n}{n^{s_1}} \right| = \sum \frac{|a_n|}{n^{\sigma_1}} \geq \sum \frac{|a_n|}{n^{\sigma_2}} = \sum \left| \frac{a_n}{n^{s_2}} \right|$$

- Remember  $|n^{\sigma+it}| = |n^{\sigma} e^{it \ln n}| = n^{\sigma}$ .
- So if series converges at  $s_1$ , it must also converge at  $s_2$ . More generally, the series converges at any  $s = \sigma + it$  where  $\sigma \geq \sigma_1$ .

# Abscissa of Absolute Convergence

- If our series doesn't converge everywhere, divergence must be at some  $\sigma < \sigma_1$ . There must be a minimum  $\sigma_a$ , called the **abscissa of absolute convergence**, such that the series converges at  $\sigma > \sigma_a$ .
- Notice  $\sigma_a$  depends only on the real part of  $s$ . Example:
  - $\sum 1/n^\sigma$  converges for real  $\sigma > 1$ , and it diverges at  $\sigma = 1$
  - $\implies \sigma_a = 1$ , so series converges for all  $s = \sigma + it$  where  $\sigma > 1$ .
- Convergence domain for a Dirichlet series is a half-plane, whereas the region for the more familiar power series  $\sum a_n z^n$  is a circle.

# Abscissa of Convergence

- With absolute convergence we don't need to consider complex terms which contribute a negative amount to the overall magnitude of the series.
  - Example,  $e^{i\pi} = -1$  can partially cancel the effect  $2e^{i2\pi} = 2$ .
  - This cancelling means some series do converge, even if not absolutely.
- Strategy:
  - show that if a series is **bounded** at  $s_0 = \sigma_0 + it_0$  then it is also **bounded** at  $s = \sigma + it$ , where  $\sigma > \sigma_0$
  - then push further to show it actually converges at that  $s$ .

# Abscissa of Convergence

- Start with a Dirichlet series  $\sum a_n/n^s$  that we know has bounded partial sums at a point  $s_0 = \sigma_0 + it_0$  for all  $x \geq 1$ .

$$\left| \sum_{n \leq x} \frac{a_n}{n^{s_0}} \right| \leq M$$

- Being bounded is not as strong a requirement as convergence, the partial sums could oscillate for example.

# Abcissa of Convergence

- **Abel's partial summation formula** relates a discrete sum to a continuous integral.

$$\sum_{x_1 < n \leq x_2} b_n f(n) = B(x_2)f(x_2) - B(x_1)f(x_1) - \int_{x_1}^{x_2} B(t)f'(t)dt$$

- Define  $f(x) = x^{s_0-s}$  and  $b_n = a_n/n^{s_0}$ .
- $B(x)$  is defined as  $\sum_{n \leq x} b_n$ , and so  $|B(x)| \leq M$  for all  $x$ .

$$\begin{aligned} \sum_{x_1 < n \leq x_2} \frac{a_n}{n^s} &= \sum_{x_1 < n \leq x_2} b_n f(n) \\ &= \frac{B(x_2)}{x_2^{s-s_0}} - \frac{B(x_1)}{x_1^{s-s_0}} + (s-s_0) \int_{x_1}^{x_2} \frac{B(t)}{t^{s-s_0+1}} dt \end{aligned}$$



# Abscissa of Convergence

- Using triangle inequality and  $|B(x)| \leq M$ .

$$\begin{aligned} \left| \sum_{x_1 < n \leq x_2} \frac{a_n}{n^s} \right| &\leq \left| \frac{B(x_2)}{x_2^{s-s_0}} \right| + \left| \frac{B(x_1)}{x_1^{s-s_0}} \right| + \left| (s-s_0) \int_{x_1}^{x_2} \frac{B(t)}{t^{s-s_0+1}} dt \right| \\ &\leq M x_2^{\sigma_0-\sigma} + M x_1^{\sigma_0-\sigma} + |s-s_0| M \int_{x_1}^{x_2} t^{\sigma_0-\sigma-1} dt \end{aligned}$$

# Abscissa of Convergence

- Using  $Mx_2^{\sigma_0-\sigma} + Mx_1^{\sigma_0-\sigma} < 2Mx_1^{\sigma_0-\sigma}$  for  $\sigma > \sigma_0$ .

$$\left| \sum_{x_1 < n \leq x_2} \frac{a_n}{n^s} \right| \leq 2Mx_1^{\sigma_0-\sigma} + |s - s_0| M \left| \frac{x_2^{\sigma_0-\sigma} - x_1^{\sigma_0-\sigma}}{\sigma_0 - \sigma} \right|$$
$$\leq 2Mx_1^{\sigma_0-\sigma} \left( 1 + \frac{|s - s_0|}{\sigma - \sigma_0} \right)$$

- Last step uses  $|x_2^{\sigma_0-\sigma} - x_1^{\sigma_0-\sigma}| = |x_1^{\sigma_0-\sigma} - x_2^{\sigma_0-\sigma}| < x_1^{\sigma_0-\sigma} < 2x_1^{\sigma_0-\sigma}$

# Abscissa of Convergence

- The key point is that  $\sum_{x_1 < n \leq x_2} a_n/n^s$  is bounded if  $\sum_{n \leq x} a_n/n^{s_0}$  is bounded, where  $\sigma > \sigma_0$ .
- Let's see if we can push this result about **boundedness** to **convergence**.

# Abscissa of Convergence

$$\left| \sum_{x_1 < n \leq x_2} \frac{a_n}{n^s} \right| \leq 2Mx_1^{\sigma_0 - \sigma} \left( 1 + \frac{|s - s_0|}{\sigma - \sigma_0} \right) = Kx_1^{\sigma_0 - \sigma}$$

- Here  $K$  doesn't depend on  $x_1$ .
- If we let  $x_1 \rightarrow \infty$  then  $Kx_1^{\sigma_0 - \sigma} \rightarrow 0$  for  $\sigma > \sigma_0$ , which means the infinite sum  $\sum a_n/n^s$  converges.

- If  $\sum_{n \leq x} a_n/n^{s_0}$  is bounded, the infinite sum  $\sum a_n/n^s$  converges for  $\sigma > \sigma_0$ .
- With the special case of  $s_0 = 0$ , if  $\sum_{n \leq x} a_n$  is bounded, the infinite sum  $\sum a_n/n^s$  converges for  $\sigma > 0$ .
  - The special case  $s_0 = 0$  allows us to say a series converges for  $\sigma > 0$  just by looking at the coefficients  $a_n$ .

# Abscissa of Convergence

- There is an abscissa of convergence  $\sigma_c$  where a Dirichlet series converges for  $\sigma > \sigma_c$ , and diverges for  $\sigma < \sigma_c$ .
  - If a series converges (bounded) at  $s_0$  then it converges at  $\sigma > \sigma_0$
  - If series doesn't converge everywhere, the  $s$  where it diverges has  $\sigma < \sigma_0$
- Note  $\sigma_a \geq \sigma_c$ , in fact

$$0 \leq \sigma_a - \sigma_c \leq 1$$

## Example: Alternating Zeta Function

- Let's apply our results to the **alternating zeta function**, also called the **eta function**.

$$\eta(s) = \sum \frac{(-1)^{n+1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots$$

- At  $s_0 = 0$  the partial sum  $\sum_{n \leq x} (-1)^{n+1}$  oscillates but is always bounded  $\leq 1$
- $\implies \sum (-1)^{n+1}/n^s$  converges for  $\sigma > 0$ .