$\zeta(s)$ Is Almost Symmetric $\sigma > 0$

From Primes To Riemann

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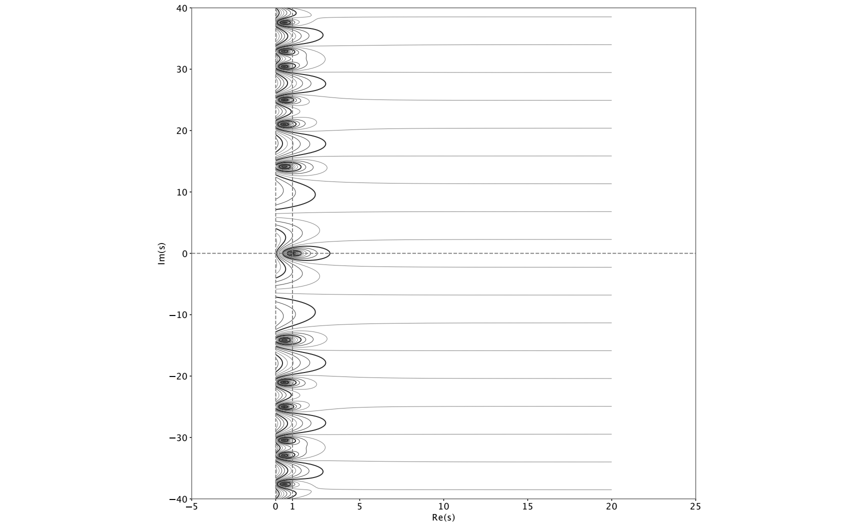
$\zeta(s)$ Looks Symmetric About The Real Axis



Caution

- Plotting the magnitude $|\zeta(s)|$ hides information about **phase**.

$\zeta(s) = \sum 1/n^s$ for $\sigma > 1$

- The **complex conjugate** $\bar{s}$ is a reflection of s in the real axis, for example $\overline{3+2i} = 3 - 2i$

$$n^{-\bar{s}} = e^{-\bar{s}\ln(n)} = \overline{e^{-s ln(n)}} = \overline{n^{-s}}$$

- $\implies \sum n^{-\bar{s}}$ is the complex conjugate of $\sum \overline{n^{-s}}$

$$\zeta(\bar{s}) = \overline{\zeta(s)}$$

- $|\zeta(s)|$ is mirrored in the real axis, the phase is inverted, for $\sigma > 1$

$\zeta(s) = (1 - s^{2-1})^{-1}\eta(s)$ for $\sigma > 0$

- Similar logic tells us $\eta(\bar{s}) = \overline{\eta(s)}$.
- Using $1/\bar{z} = \overline{1/z}$

$$(1 - 2^{1-\bar{s}})^{-1} = \overline{(1 - 2^{1-s})^{-1}}$$

- Putting the two together

$$\begin{aligned}
\zeta(\bar{s}) &= (1 - 2^{1-\bar{s}})^{-1} \cdot \eta(\bar{s}) \\
&= \overline{(1 - 2^{1-s})^{-1}} \cdot \overline{\eta(s)} \\
&= \overline{\zeta(s)}
\end{aligned}$$

- $|\zeta(s)|$ is mirrored in the real axis, the phase is inverted, for $\sigma > 0$

$\zeta(s)$ Has One Pole In $\sigma > 0$