Prime Number Theorem

From Primes To Riemann

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Experimental Maths

- We just saw experimental evidence that

$$\pi(n) \approx \frac{n}{\ln(n)}$$

Ever Smaller Proportional Error

- Specifically we saw the **proportional error** gets smaller as $n \to \infty$.

$$\lim_{n \to \infty} \frac{\pi(n) - n/\ln(n)}{\pi(n)} = 0$$

- Rearranging ...

$$\lim_{n \to \infty} \frac{\pi(n)}{n/\ln(n)} = 1$$

- The ratio of $\pi(n)$ and $n/\ln(n)$ tends to 1 as $n \to \infty$.

- This is what the **prime number theorem** says.

Prime Number Theorem

$$\pi(n) \sim n/\ln(n)$$

- The symbol $\sim$ says that both sides are **asymptotically equivalent**.
- For example, $f(n) \sim g(n)$ means $f(n)/g(n) = 1$ as $n \to \infty$.

Asymptotic Equivalence Example 1

- If $f(x) = x^2 + x$ and $g(x) = x^2$, then $f \sim g$.

$$\lim_{x\to\infty} \frac{x^2 + x}{x^2} = \lim_{x\to\infty} 1 + \frac{1}{x} = 1$$

- Notice how f and g have the same dominant term x^2.
- Swapping f and g doesn't break asymptotic equivalence, $g \sim f$.
  - This is clear from its definition as a ratio.

Asymptotic Equivalence Example 2

- However, if $f(x) = x^3$ and $g(x) = x^2$, then f and g are not asymptotically equivalent.

$$\lim_{x \to \infty} \frac{x^3}{x^2} = x \neq 1$$

- Notice how f and g have different dominant terms.

Asymptotic Equivalence

- If we know that $f \sim g$ and $g \sim h$, then we can also say $f \sim h$.
- This property, called **transitivity**, is familiar from normal equality.

What About li(n)?

- PNT is sometimes written with Gauss' better approximation li(n) .

$$\pi(n) \sim \mathrm{li}(n)$$

- Can both li(n) and $n/\ln(n)$ both be $\sim \pi(n)$?
  - for this to work $\mathrm{li}(n) \sim n/\ln(n)$

Is $\mathrm{li}(n) \sim n/\ln(n)$?

$$f(n) = \frac{n}{\ln(n)}$$

$$g(n) = \int_0^n \frac{1}{\ln(x)} dx$$

- To show $f \sim g$ we need to confirm $\lim_{n\to\infty} f(n)/g(n)$ is 1.
- Sadly, both $f(n)$ and $g(n)$ become infinitely large as $n \to \infty$, which is a little unhelpful.

Is $\mathrm{li}(n) \sim n/\ln(n)$?

- We can try **l'Hopital's rule**.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

- $f'(n) = \frac{\ln(n)-1}{\ln^2(n)}$ and $g'(n) = \frac{1}{\ln(n)}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{f'(n)}{g'(n)} &= \lim_{n\to\infty} \frac{(\ln(n)-1)\ln(n)}{\ln^2(n)} \\
&= \lim_{n\to\infty} 1 - \frac{1}{\ln(n)} = 1 \\
&= 1
\end{aligned}$$

Is $\mathrm{li}(n) \sim n/\ln(n)$?

- $n/ln(n)$ and $\mathrm{li}(n)$ are asymptotically equivalent.

$$li(n) \sim n/\ln(n)$$

- So the PNT can refer to either $n/ln(n)$ or $\mathrm{li}(n)$.

$$\pi(n) \sim li(n) \sim n/\ln(n)$$

What Does The PNT Really Say?

- The PNT says that $\pi(n)$ grows in a way that is asymptotically equivalent to functions like $n/\ln(n)$ and $\mathrm{li}(n)$.
- It **doesn't** say that these are the **only** or **best** functions for approximating $\pi(n)$.
  - ... which leaves open the intriguing possibility of other functions that are even better than $\mathrm{li}(n)$.

Bertrand's Postulate

- PNT can provide easy insights into questions about the primes.
- 1845 Bertrand proposed that there is at least one prime $n < p < 2n$.
- Using PNT, we can compare the $\pi(2x)$ with $\pi(x)$.

$$\frac{\pi(2x)}{\pi(x)} \sim \frac{2x}{\ln(2x)} \cdot \frac{\ln(x)}{x} \sim 2$$

- Between n and $2n$, there are approximately $n/\ln(n)$ primes - approximation becomes truer for larger n.
- This is actually a stronger statement than Bertrand's postulate which merely suggests there is at least one prime.