Infinite Products From Primes To Riemann

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Infinite Sum

• At school we learn a lot about infinite sums.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

- "Sum" \leftrightarrow "Series".
- What do we really mean by infinite sum?

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Infinite Sum

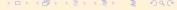
• We say an infinite series converges if

$$\lim_{N\to\infty}\sum_{n=1}^N a_n=S$$

where *S* is finite.

• Tests to check for convergence, eg the ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$$



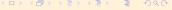
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Infinite Product

At school we don't seem to learn about infinite products.

$$\prod_{n=1}^{\infty} a_n = a_1 \times a_2 \times a_2 \times \dots$$

• What do we really mean by infinite product?



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Intuition

• If elements get larger, does infinite product diverge?

$$2 \times 3 \times 4 \times 5 \times \dots$$

If elements are < 1, does infinite product converge?

$$0.5\times0.55\times0.555\times0.5555\times\dots$$

... even if the elements are getting larger?



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Intuition

What about these infinite products?

$$-2\times -1\times 0\times 1\times 2\times 3\times \dots$$

$$-2 \times -1 \times \emptyset \times 1 \times 2 \times 3 \times \dots$$

- With the 0, product is 0. Without the 0, it diverges.
- Is the zero "powerful" enough to stop divergence on its own?



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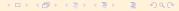
Infinite Product

• We say an infinite product converges if

$$\lim_{N\to\infty}\prod_{n=1}^N a_n=P$$

where *P* is finite and **non-zero**.

• If P = 0, we say the infinite product **diverges to zero**.



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Infinite Product & Logarithms

- Why non-zero P? Why "diverges to zero"?
- Convention is to consider infinite products through a logarithm.

$$\log \prod a_n = \sum \log a_n$$

A product of 0 would give an undefined logarithm.

• Does this converge?

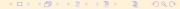
$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n} \right)$$

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• Consider partial product.

$$\prod_{n=1}^{N} \left(1 + \frac{1}{n} \right) = \prod_{n=1}^{N} \left(\frac{n+1}{n} \right)$$
$$= \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \dots \times \frac{N+1}{M}$$
$$= N+1$$

• As $N \to \infty$, product **diverges**.



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• Does this converge?

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)$$

• Note *n* starts at 2.



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• Again, consider partial product.

$$\prod_{n=2}^{N} \left(1 - \frac{1}{n} \right) = \prod_{n=2}^{N} \left(\frac{n-1}{n} \right)$$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{N+1}{N}$$

$$= \frac{1}{N}$$

• As $N \to \infty$, product tends to 0, so product diverges to zero.

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Useful Convergence Test

• If we can write terms as $(1 + a_n)$, where $a_n > 0$

$$\prod_{n=1}^{\infty} (1+a_n) \le \prod_{n=1}^{\infty} \exp(a_n)$$

$$=\exp\sum_{n=1}^{\infty}a_n$$

• Inequality is from $e^x = 1 + x + \dots$

$$\sum_{n=1}^{\infty} a_n \text{ converges } \implies \prod_{n=1}^{\infty} (1+a_n) \text{ converges, for } a_n > 0$$

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• Does this converge?

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2} \right)$$

• Yes - because $\sum_{n=1}^{\infty} 1/n^2$ converges.



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