Infinite Products

From Primes To Riemann

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Infinite Sum

- At school we learn a lot about **infinite sums**.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

- "Sum" $\leftrightarrow$ "Series".
- What do we really mean by infinite sum?

Infinite Sum

- We say an infinite series converges if

$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n = S$$

where S is finite.

- Tests to check for convergence, eg the **ratio test**.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

Infinite Product

- At school we don't seem to learn about **infinite products**.

$$\prod_{n=1}^{\infty} a_n = a_1 \times a_2 \times a_2 \times \ldots$$

- What do we really mean by infinite product?

Intuition

- If elements get larger, does infinite product diverge?

$$2 \times 3 \times 4 \times 5 \times \ldots$$

- If elements are < 1, does infinite product converge?

$$0.5 \times 0.55 \times 0.555 \times 0.5555 \times \ldots$$

... even if the elements are getting larger?

Intuition

- What about these infinite products?

$$-2 \times -1 \times 0 \times 1 \times 2 \times 3 \times \ldots$$

$$-2 \times -1 \times \cancel{0} \times 1 \times 2 \times 3 \times \ldots$$

- With the 0, product is 0. Without the 0, it diverges.
- Is the zero "powerful" enough to stop divergence on its own?

Infinite Product

- We say an infinite product converges if

$$\lim_{N \to \infty} \prod_{n=1}^{N} a_n = P$$

where P is finite and **non-zero**.

- If $P = 0$, we say the infinite product **diverges to zero**.

Infinite Product & Logarithms

- Why non-zero P? Why "diverges to zero"?
- Convention is to consider infinite products through a logarithm.

$$\log \prod a_n = \sum \log a_n$$

- A product of 0 would give an undefined logarithm.

Example 1

- Does this converge?

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)$$

Example 1

- Consider partial product.

$$\prod_{n=1}^{N}\left(1+\frac{1}{n}\right)=\prod_{n=1}^{N}\left(\frac{n+1}{n}\right)$$

$$=\frac{\cancel{2}}{1}\times\frac{\cancel{3}}{\cancel{2}}\times\frac{4}{\cancel{3}}\times\ldots\times\frac{N+1}{\cancel{N}}$$

$$=N+1$$

- As $N\to\infty$, product **diverges**.

Example 2

- Does this converge?

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n}\right)$$

- Note n starts at 2.

Example 2

- Again, consider partial product.

$$\prod_{n=2}^{N}\left(1-\frac{1}{n}\right)=\prod_{n=2}^{N}\left(\frac{n-1}{n}\right)$$

$$=\frac{1}{\cancel{2}}\times\frac{\cancel{2}}{\cancel{3}}\times\frac{\cancel{3}}{\cancel{4}}\times\ldots\times\frac{\cancel{N+1}}{N}$$

$$=\frac{1}{N}$$

- As $N\to\infty$, product tends to 0, so product **diverges to zero**.

Useful Convergence Test

- If we can write terms as $(1 + a_n)$, where $a_n > 0$

$$\prod_{n=1}^{\infty}(1 + a_n) \leq \prod_{n=1}^{\infty} \exp(a_n)$$

$$= \exp \sum_{n=1}^{\infty} a_n$$

- Inequality is from $e^x = 1 + x + \dots$

$$\boxed{\sum_{n=1}^{\infty} a_n \text{ converges } \implies \prod_{n=1}^{\infty}(1 + a_n) \text{ converges, for } a_n > 0}$$

Example 3

- Does this converge?

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$$

- Yes - because $\sum_{n=1}^{\infty} 1/n^2$ converges.