Into The Complex Domain

From Primes To Riemann

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Previously ...

- The Riemann Zeta function encodes information about the primes.
  - infinite primes
  - primes aren't so sparse, $\sum 1/p$ diverges

$$\boxed{\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1}}$$

- We've considered s as
  - an **integer**, $s = 1$ harmonic series, $s = 2$ Basel problem
  - a **real** value, we proved $\zeta(s)$ converges for $s > 1$

Complex s

- Riemann was the first to consider s as a **complex number**.
- If we think $\zeta(s)$ over the complex domain might reveal new insights into the primes, we need to understand how it behaves.
- Exploring where it **converges** is a good start.
- Traditional to write complex s as $s = \sigma + it$

$$\sum \frac{1}{n^s} = \sum \frac{1}{n^{\sigma + it}} = \sum \frac{1}{n^\sigma} \frac{1}{n^{it}}$$

Convergence For $\sigma > 1$

- Let's look at the series with each term replaced by its magnitude.

$$\sum \left| \frac{1}{n^s} \right| = \sum \left| \frac{1}{n^\sigma} \frac{1}{n^{it}} \right|$$

- Rewriting n^{it} as $e^{it \ln(n)}$ makes clear it has a magnitude of 1.

$$\sum \left| \frac{1}{n^s} \right| = \sum \frac{1}{n^\sigma}$$

- We know $\sum 1/n^\sigma$ converges for real $\sigma > 1 \implies \sum 1/n^s$ converges **absolutely** for for $\sigma > 1$.
- **Absolute convergence implies convergence** $\implies \sum 1/n^s$ converges for $\sigma > 1$.

Divergence For $\sigma \leq 0$

- Let's look again at the terms in the sum.

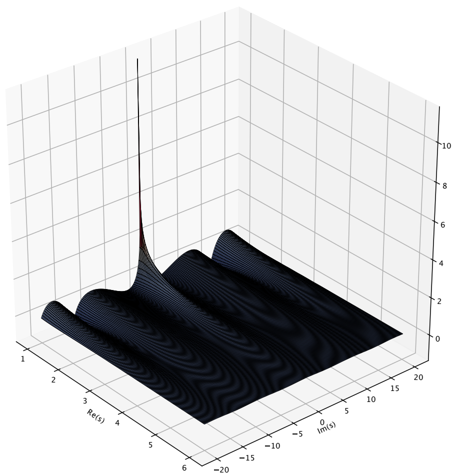
$$\left|\frac{1}{n^s}\right| = \left|\frac{1}{n^\sigma}\frac{1}{n^{it}}\right| = \frac{1}{n^\sigma}$$

- If $\sigma < 0$, the magnitude of the terms grows larger than 1.
- If $\sigma = 0$, the magnitude of each term is exactly 1.
- For any series to converge, a necessary requirement is that the terms get smaller towards zero $\implies \sum 1/n^s$ diverges for $\sigma \leq 0$.

Divergence For $\sigma < 1$

- $\zeta(s)$ converges for $\sigma > 1$, diverges for $\sigma \leq 0$. We're left with a gap $0 < \sigma \leq 1$.
- To fill this gap we need to understand more generally when series of the form $\sum a_n/n^s$, called **Dirichlet series**, converge or diverge.
- Dirichlet series converge in **half-planes** to the right of an **abscissa of convergence** σ_c. That is, they converge at any $s = \sigma + it$ where $\sigma > \sigma_c$.
- Because $\zeta(s)$ converges for $\sigma > 1$, and we know it diverges at $s = 1 + 0i$, then $\sigma_c = 1 \implies \zeta(s)$ diverges for $\sigma < 1$.

Visualising The Zeta Function For $s > 1$



Visualising The Zeta Function For $s > 1$

- Spike around $s = 1 + 0i$ corresponds to divergent harmonic series $\zeta(1)$.
- Surface seems to smooth out to the right as σ grows larger. To what value?

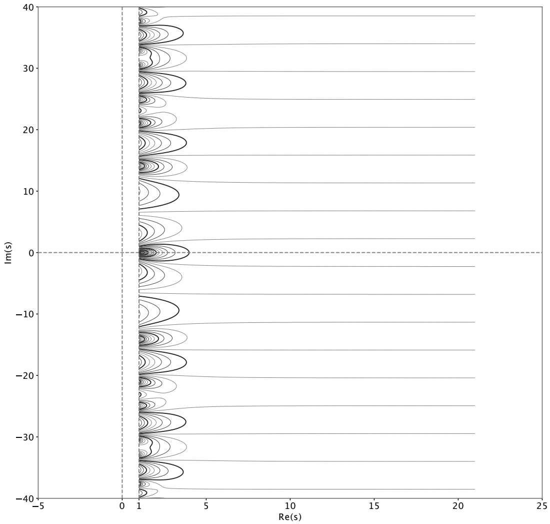
$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

- As $s \to \infty$ all the terms $1/n^s \to 0$, except the first term which remains 1.
- To be more precise, the magnitude of each term $|n^{-s}| = n^{-\sigma}$ tends to zero as $\sigma \to \infty$ for all $n > 1 \implies |\zeta(s)| \to 1$ as $\sigma \to \infty$.

Hints The Function Extends Into $\sigma \leq 1$

- Aside from $s = 1 + 0i$, the function doesn't seem to diverge along the line $s = 1 + it$.
- It looks like the surface has been prematurely cut off, and would continue smoothly into $\sigma \leq 1$ if allowed.

Isolines of $|\zeta(s)|$



Improved Model

- The intuition that a function should continue smoothly without abrupt changes corresponds to a powerful property of many functions we come across.