Euler's Golden Bridge

From Primes To Riemann

Tariq Rashid

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Golden Bridge

- Euler first found a "simple" connection between primes and integers.

$$p \leftrightarrow n$$

Riemann Zeta Function

- We know $\sum 1/n$ diverges.
- We know $\sum 1/n^2$ converges.
- It's natural to ask for which values of s the more general series converges.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

- This is the **Riemann Zeta** function $\zeta(s)$.

Riemann Zeta Function

- Series like $\sum 1/n^3$ and $\sum 1/n^4$ converge
  - each term is smaller than the corresponding one in $\sum 1/n^2$.
- Less obvious is when $s < 2$.
- In a separate tutorial we'll show a short proof that

$$\zeta(s) \text{ converges for } s > 1$$

Sieving The Zeta Function

- Let's write out the zeta function again.

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \dots$$

- Divide this series by 2^s.

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \frac{1}{12^s} \dots$$

- Denominators are multiples of 2^s.

Sieving The Zeta Function

- Subtracting them from $\zeta(s)$ sieves out multiples of 2^s.

$$(1-\frac{1}{2^s})\cdot\zeta(s) = 1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+\dots$$

- This dividing and subtracting of infinite series is only valid because they are absolutely convergent for $s>1$.

Sieving The Zeta Function

- Now divide resulting series by 3^s.

$$\frac{1}{3^s} \cdot (1 - \frac{1}{2^s}) \cdot \zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \dots$$

- Denominators are all multiples of 3^s, but not all multiples of 3^s are here - eg 6^s removed in previous step.
- Subtracting from previous series leaves terms with denominators not multiples of 2 or 3.

$$(1 - \frac{1}{3^s}) \cdot (1 - \frac{1}{2^s}) \cdot \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \dots$$

Sieving The Zeta Function

- We can't remove terms with multiples of 4^s because they were sieved out when we removed multiples of 2^s.
- The next useful step is to remove multiples of 5^s.
- Repeating this process leaves terms with denominators that are not multiples of 2, 3 or 5.

$$(1 - \frac{1}{5^s}) \cdot (1 - \frac{1}{3^s}) \cdot (1 - \frac{1}{2^s}) \cdot \zeta(s) = 1 + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \dots$$

Sieving The Zeta Function

- Again, we can't remove multiples of 6^s because they were already removed.
- Repeating several times, we see
  - we can only remove multiples of **successive primes**.
  - after each removal, the very first term 1 always survives.

Sieving The Zeta Function

- If we kept going $\rightarrow$ infinite product on the left, and only 1 on the right.

$$\ldots \cdot (1 - \frac{1}{11^s}) \cdot (1 - \frac{1}{7^s})(1 - \frac{1}{5^s}) \cdot (1 - \frac{1}{3^s}) \cdot (1 - \frac{1}{2^s}) \cdot \zeta(s) = 1$$

- Rearrange to isolate $\zeta(s)$.

$$\zeta(s) = \prod_p (1 - \frac{1}{p^s})^{-1}$$

- $\prod$ means product

Euler's Product Formula

- We've arrived at **Euler's product formula**.

$$\boxed{\sum_n \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1}}$$

- To say this result is amazing would not be an exaggeration.