

Gaps Between Primes

From Primes To Riemann

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- We've seen the primes
 - Can't be so thinly spread out that $\sum 1/p$ converges.
 - Occur with an average density of $1/\ln(x)$ around the number x .
- This suggests the gaps between primes must be constrained.
- Let's find out ...

Factorials

- The factorial $n!$ of a whole number n is the product of all the whole numbers between 1 and n .

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$$

- For example, $4! = 4 \times 3 \times 2 \times 1 = 24$.
- For $n > 2$ factorials are definitely not prime - by definition they are products.

Prime-Free Sequence

- Let's look at the sequence of numbers from $(5! + 2)$ to $(5! + 5)$.

$$5! + 2 = (5 \cdot 4 \cdot 3 \cdot 1 + 1) \cdot 2$$

$$5! + 3 = (5 \cdot 4 \cdot 2 \cdot 1 + 1) \cdot 3$$

$$5! + 4 = (5 \cdot 3 \cdot 2 \cdot 1 + 1) \cdot 4$$

$$5! + 5 = (4 \cdot 3 \cdot 2 \cdot 1 + 1) \cdot 5$$

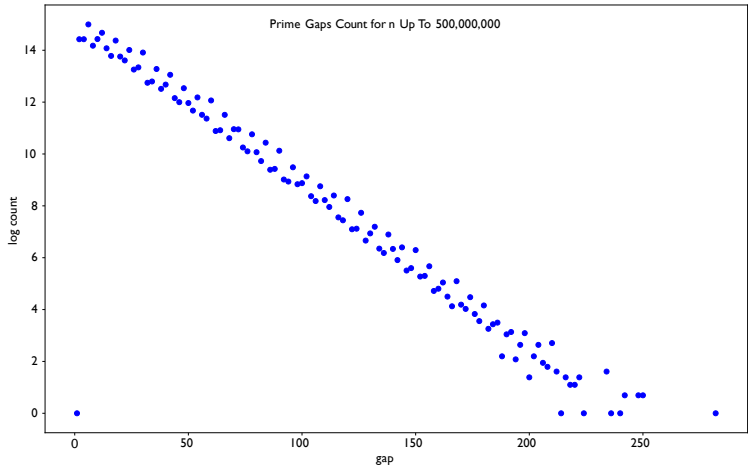
- We can see 2 is a factor of $(5! + 2)$, 3 is a factor of $(5! + 3)$, 4 is a factor of $(5! + 4)$, and 5 is a factor of $(5! + 5)$.
- So the sequence $(5! + 2)$ to $(5! + 5)$ is entirely free of primes.

No Upper Limit On Prime Gaps

- Using the same method, we can show any sequence that follows the same pattern $(n! + 2)$ to $(n! + n)$ is prime-free.
- This is a sequence of length $n - 1$.
- Because n can be as large as desired, we've just proved **there is no upper limit on prime gaps**.
- ... Easy!

- The primes continue to surprise us:
 - The primes can't be too sparse.
 - There is no upper limit on the gaps between primes.
- Both facts can be true if large prime gaps are rare.

Distribution Of Prime Gaps



Distribution Of Prime Gaps

- It is surprising to see the distribution so tightly constrained:
 - The overall shape appears to be very linear
 - clear regions above and below the line with no prime gap counts
- Note: prime gap is defined as $g(p_n) = p_{n+1} - p_n$, so prime gap for 3 is 2, not 1.

Twin Prime Conjecture

- Looking at the distribution, gaps of size 2 seem to occur more often than almost all other sizes.
- These **twin primes** do seem to pop up more often than expected:

$(3, 5), (5, 7), (11, 13), (17, 19) \dots (101, 103), (107, 109) \dots$

- The **Twin Prime Conjecture** says there are infinite twin primes.
 - This is yet another simple statement about primes that remains unproven.