Integral Comparison Tests

From Primes To Riemann

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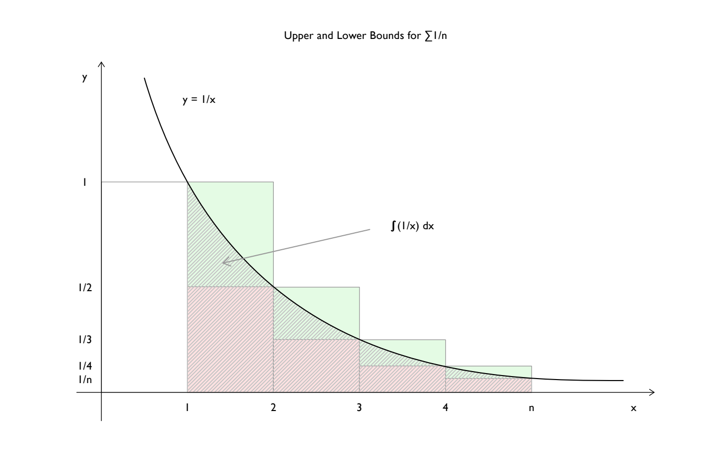
Discrete vs Continuous Functions

- Understanding the behaviour of continuous functions is often easier than discrete functions.

$$\sum \frac{1}{x} \text{ compared with } \int \frac{1}{x} dx$$

- Simple but powerful technique used a lot in number theory.

The Growth Of $\sum 1/n$



- graph of $y = \frac{1}{x}$, together with rectangles representing the fractions $\frac{1}{n}$.

Lower Bound For Growth Of $\sum 1/n$

- Looking at $1 \leq x \leq 4$, area of the three taller green rectangles $1 + \frac{1}{2} + \frac{1}{3}$ is **greater** than the area under the curve $\int_1^4 \frac{1}{x} dx$.

- By extending range to $1 \leq x \leq n$, we can make a general observation.

$$\sum_1^n \frac{1}{x} > \int_1^{n+1} \frac{1}{x} dx$$

- $n + 1$ because the width of the last rectangle extends from $x = n$ to $x = n + 1$.

Lower Bound For Growth Of $\sum 1/n$

- We can perform the integral to simplify the expression.

$$\boxed{\sum_{1}^{n} \frac{1}{x} > \ln(n+1)}$$

- Rather nice lower bound on the growth of the harmonic series.

Upper Bound For Growth Of $\sum 1/n$

- Looking at the range $1 \leq x \leq 4$, area of the three shorter rectangles $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ is **less** than the area under the curve $\int_1^4 \frac{1}{x} dx$.

- Again, by extending the range to n we can make a general observation.

$$\sum_{2}^{n} \frac{1}{x} < \int_{1}^{n} \frac{1}{x} dx$$

Upper Bound For Growth Of $\sum 1/n$

- Sum starts at 2 because we're looking at rectangles extending to the left of a given x.
- We can adjust the limit of the sum using $\sum_1^n \frac{1}{x} = 1 + \sum_2^n \frac{1}{x}$.

$$\sum_1^n \frac{1}{x} - 1 < \int_1^n \frac{1}{x} dx$$

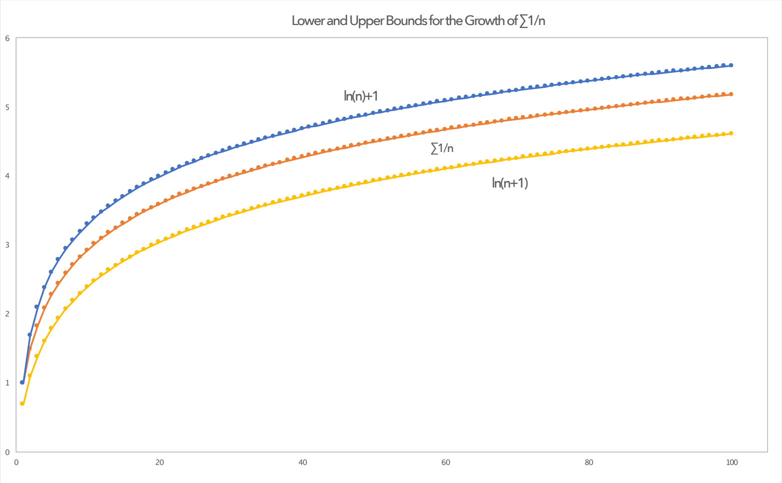
Upper Bound For Growth Of $\sum 1/n$

- We can perform the integral.

$$\boxed{\sum_{1}^{n} \frac{1}{x} < \ln(n) + 1}$$

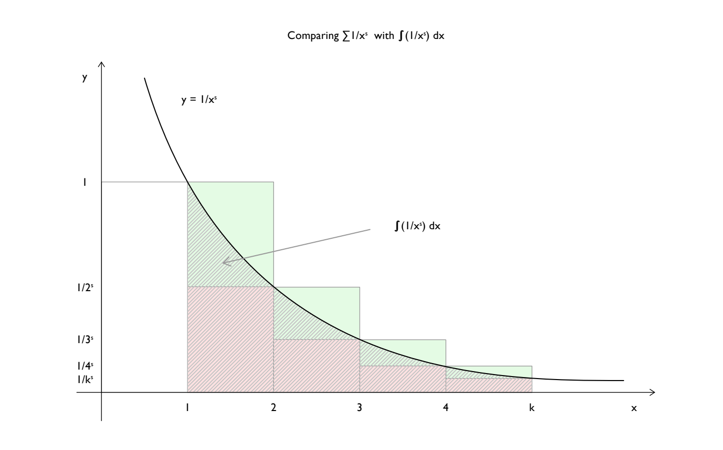
- A nice upper bound to the growth of the harmonic series.

Visualisation



- Caution - experimental evidence isn't proof.

Convergence Of $\zeta(s) = \sum 1/n^s$



- graph of $y = \frac{1}{x^s}$, with rectangles representing the fractions $\frac{1}{x^s}$

Convergence Of $\zeta(s) = \sum 1/n^s$

- Shape assumes $s > 0$, easy to see $\sum 1/n^s$ diverges if $s \leq 0$.
- Looking at $1 \leq x \leq 4$, area of three shorter rectangles is **less** than the area under curve $\int_1^4 \frac{1}{x^s} dx$.
- By extending range to $1 \leq x \leq k$, we can make a general observation.

$$\sum_2^k \frac{1}{x^s} < \int_1^k \frac{1}{x^s} dx$$

Convergence Of $\zeta(s) = \sum 1/n^s$

- Sum starts at 2 because we're looking at rectangles extending to the left of a given x.
- We can adjust the limit of the sum using $\sum_1^k \frac{1}{x^s} = 1 + \sum_2^k \frac{1}{x^s}$.

$$\sum_1^k \frac{1}{x^s} - 1 < \int_1^k \frac{1}{x^s} dx$$

Convergence Of $\zeta(s) = \sum 1/n^s$

- The integral is easily evaluated.

$$\sum_{1}^{k} \frac{1}{x^s} - 1 < \frac{k^{1-s} - 1}{1 - s}$$

- Only way k^{1-s} won't diverge as $k \to \infty$ is if $1 - s < 0$.
- Sum **convergenes** when $s > 1$.
- Possibility the sum might also converge for some $s \leq 1$?

Convergence Of $\zeta(s) = \sum 1/n^s$

- Looking at $1 \leq x \leq 4$, area of three taller rectangles is **more** than the area under curve $\int_1^4 \frac{1}{x^s} dx$.
- By extending range to $1 \leq x \leq k$, we can make a general observation.

$$\sum_1^k \frac{1}{x^s} < \int_1^{k+1} \frac{1}{x^s} dx$$

- Integral upper limit is $k+1$ because we're looking at rectangles extending to the right of a given x.

Convergence Of $\zeta(s) = \sum 1/n^s$

- The integral is easily evaluated.

$$\sum_{1}^{k} \frac{1}{x^s} > \frac{(k+1)^{1-s} - 1}{1-s}$$

- As $k \to \infty$, $(k+1)^{1-s}$ **diverges** when $s \leq 1$.
- Sum $\sum 1/x^s$ also diverges when $s \leq 1$.
- We have now ruled out the possibility the sum might converge for some $s \leq 1$.

- Convergence of Riemann Zeta function

$$\boxed{\zeta(s) = \sum 1/n^s \text{ only converges for } s > 1}$$

- The two inequalities give lower and upper bounds for $\zeta(s)$.

$$\boxed{\frac{1}{s-1} < \zeta(s) < \frac{1}{s-1} + 1}$$