Primes Aren't That Spread Out

From Primes To Riemann

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January 23, 2021

How Common Are Primes?

- We've seen there is no limit to the supply of primes.
- A good next question - how often do they occur?
- One way is to compare the sum of their inverses with other sequences.

Sum of Inverse Counting Numbers

- The counting numbers $1, 2, 3, 4, \ldots$ are **spaced 1 apart**.
- The sum of their inverses is called the **harmonic series**.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

- This series **diverges** - the sum grows infinitely large.
- If the numbers were spaced further apart, perhaps the infinite series might **converge**?

Sum of Inverse Squares

- The square numbers $1, 4, 9, 16, \ldots$ are **spaced further apart**.
- The sum of their inverses is known as the **Basel problem**.

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$$

- This series **converges** to $\frac{\pi^2}{6}$.
- Interpretation $\rightarrow$ the squares n^2 are so spread out the terms become small quickly enough to avoid divergence.

Sum of Inverse Primes

$$\sum \frac{1}{n} \to \infty$$

$$\sum \frac{1}{n^2} \to \frac{\pi^2}{6}$$

$$\sum \frac{1}{p} \to ?$$

- Are primes as spread out as n^2 ?

Sum of Inverse Primes

- Proof by contradiction .. again!
- Assume the infinite series of prime **reciprocals** does indeed converge.

$$S = \sum_{n=1} \frac{1}{p_n}$$

Sum of Inverse Primes

- Because S is finite, and the terms get smaller, there must be a k such that

$$\sum_{n=k+1} \frac{1}{p_n} < 1$$

- Let's call it x

$$x = \sum_{n=k+1} \frac{1}{p_n} < 1$$

Sum of Inverse Primes

- Let's build an infinite geometric series based on this x.

$$G = x + x^2 + x^3 + x^4 + \dots$$

- G converges because the ratio between terms x is less than 1.

Sum of Inverse Primes

$$G = x + x^2 + x^3 + x^4 + \dots$$

- Any term in G will be of the form $\frac{1}{N}$ where N has prime factors p_{k+1} or larger.
- This is because x was intentionally constructed with primes p_{k+1} or larger.

Sum of Inverse Primes

- Let's create a second series F but this time use primes p_k and smaller.

$$F = \sum_{j=1} \frac{1}{1 + j \cdot (p_1 \cdot p_2 \cdot p_3 \ldots p_k)}$$

- Looking more closely at $1 + j \cdot (p_1 \cdot p_2 \cdot p_2 \ldots p_k)$
  - no prime factors in range p_1 to p_k
  - so any prime factors must be p_{k+1} or larger
- That means F is a **subseries** of G.

Sum of Inverse Primes

- Does F diverge?
    - **Limit comparision** test with harmonic series which does diverge.
    - If ratio of terms is finite, they both converge or both diverge.

Sum of Inverse Primes

- Limit comparison test

$$\lim_{j \to \infty} \frac{1 + j \cdot (p_1 \cdot p_2 \cdot p_3 \ldots p_k)}{j} = p_1 \cdot p_2 \cdot p_3 \ldots p_k$$

- Ratio is finite $\rightarrow$ since harmonic series diverges, so does F.

Sum of Inverse Primes

- Since F diverges, and is a subseries of G, then G must also diverge.
- But, we constructed G to converge !
- Contradiction $\rightarrow$ initial assumption that infinite sum of inverse primes was wrong.

Sum of Inverse Primes Diverges

$$\sum 1/p_n \to \infty$$

- Perhaps suprising because we thought primes thinned out rapidly.
- They seem not to thin out as fast as squares n^2.

Legendre's Conjecture

- That $\sum 1/n^2$ converges suggests primes not as sparse as squares.
- Interesting idea attributed to Legendre that there is at least one prime between two consecutive squares.

$$n^2 < p < (n+1)^2$$

- Still unproven today!