Abel's Partial Summation Formula

From Primes To Riemann

Tariq Rashid

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Discrete vs Continuous Functions

- Often easier to understand how a **discrete** function behaves if it can be expressed as a **continuous** function.

- **Abel's partial summation formula** allows us to write a discrete sum as continuous integral.

Useful Object

- Sum over arithmetic function $a(n)$ weighted by smooth function $f(x)$.

$$\sum_{x_1 < n \leq x_2} a(n) f(n)$$

- A useful general object to find an integral form for, because it gives us flexibility in choosing the arithmetic and smooth functions.

- To be precise: $a(n)$ takes only positive integers $n \geq 1$, and $f(x)$ has a continuous derivative over the domain $[x_1, x_2]$. Both $a(n)$ and $f(x)$ can be complex.

Deriving An Integral Form

- The derivation of an integral form for the sum is just lots and lots of simple algebra!
- Because $a(n)$ is only defined over integers $n \geq 1$, we clarify the sum by setting $m_1 = \lfloor x_1 \rfloor$ and $m_2 = \lfloor x_2 \rfloor$.

$$\sum_{x_1 < n \leq x_2} a(n)f(n) = \sum_{n=m_1+1}^{m_2} a(n)f(n)$$

- Remember $\lfloor x \rfloor$ is the largest integer up to, and including, x.

Deriving An Integral Form

- Define $A(x) = \sum_{n \leq x} a(n)$. By definition $a(n) = A(n) - A(n-1)$ so we can replace $a(n)$.

$$\sum_{m_1+1}^{m_2} a(n)f(n) = \sum_{m_1+1}^{m_2} [A(n) - A(n-1)]\, f(n)$$

$$= \sum_{m_1+1}^{m_2} A(n)f(n) - \sum_{m_1}^{m_2-1} A(n)f(n+1)$$

The two sums have different limits for n, but both cover $[m_1+1, m_2-1]$.

$$\sum_{m_1+1}^{m_2} a(n)f(n) = \sum_{m_1+1}^{m_2-1} A(n)\, [f(n) - f(n+1)]$$

$$+ A(m_2)f(m_2) - A(m_1)f(m_1+1)$$

Deriving An Integral Form

- Noticing that $\int_n^{n+1} f'(t)dt = f(n+1) - f(n)$ allows us to introduce the integral.

$$\sum_{m_1+1}^{m_2} a(n)f(n) = -\sum_{m_1+1}^{m_2-1} A(n) \int_n^{n+1} f'(t)dt$$

$$+ A(m_2)f(m_2) - A(m_1)f(m_1+1)$$

Deriving An Integral Form

- Because $A(t) = A(n)$ over the interval $[n, n+1)$, we can move $A(n)$ it inside the integral as $A(t)$.

$$\sum_{m_1+1}^{m_2} a(n)f(n) = -\sum_{m_1+1}^{m_2-1} \int_n^{n+1} A(t)f'(t)dt$$

$$+ A(m_2)f(m_2) - A(m_1)f(m_1+1)$$

- The sum of integrals over consecutive intervals can be simplified to a single integral.

$$\sum_{x_1 < n \leq x_2} a(n)f(n) = -\int_{m_1+1}^{m_2} A(t)f'(t)dt$$

$$+ A(m_2)f(m_2) - A(m_1)f(m_1+1)$$

Deriving An Integral Form

- We now need to adjust the integration limits back to x_1 and x_2, not forgetting the intervals to m_1 and m_2.

- Writing out the integrals that split the interval $[x_1, x_2]$ is helpful.

$$\int_{x_1}^{x_2} X dt = \int_{m_1+1}^{m_2} X dt + \int_{x_1}^{m_1+1} X dt + \int_{m_2}^{x_2} X dt$$

Deriving An Integral Form

- Using $A(t) = A(x_1)$ over $[x_1, m_1 + 1)$, and $A(t) = A(x_2)$ over $[m_2, x_2]$

$$
\begin{aligned}
-\int_{m_1+1}^{m_2} A(t)f'(t)dt &= \int_{x_1}^{m_1+1} A(t)f'(t)dt + \int_{m_2}^{x_2} A(t)f'(t)dt \\
&- \int_{x_1}^{x_2} A(t)f'(t)dt \\
&= A(x_1)\left[f(m_1+1) - f(x_1)\right] + A(x_2)\left[f(x_2) - f(m_2)\right] \\
&- \int_{x_1}^{x_2} A(t)f'(t)dt
\end{aligned}
$$

Deriving An Integral Form

- We plug this integral back into our object, then use $A(m_1) = A(x_1)$ and $A(m_2) = A(x_2)$.

$$\sum_{x_1 < n \leq x_2} a(n)f(n) = A(x_2)f(m_2) - A(x_1)f(m_1 + 1)$$

$$+ A(x_1)\left[f(m_1 + 1) - f(x_1)\right] + A(x_2)\left[f(x_2) - f(m_2)\right]$$

$$- \int_{x_1}^{x_2} A(t)f'(t)dt$$

- Many of these terms cancel out, leaving us with the **Abel Identity**.

$$\boxed{\sum_{x_1 < n \leq x_2} a(n)f(n) = A(x_2)f(x_2) - A(x_1)f(x_1) - \int_{x_1}^{x_2} A(t)f'(t)dt}$$

Simpler Abel Identify

- In many cases n starts at 1, and so the formula reduces further.

$$\boxed{\sum_{1 \le n \le x_2} a(n)f(n) = A(x_2)f(x_2) - \int_1^{x_2} A(t)f'(t)dt}$$

- Lower limit of the integral is 1 because $A(t) = 0$ in the range $[0, 1)$.

Example: Growth of $\sum 1/n$

- Abel's identity $\rightarrow$ asymptotic behaviour of discrete functions.
- Let's use it to explore how the harmonic series $\sum_1^N 1/n$ grows.
- We can choose $a(n) = 1$ and $f(x) = 1/x$, which means $A(x) = \lfloor x \rfloor$ and $f'(x) = -1/x^2$.

$$\sum_{n \leq N} \frac{1}{n} = A(N)f(N) - \int_1^N A(t)f't)$$

$$= \frac{N - \{N\}}{N} + \int_1^N \frac{t - \{t\}}{t^2} dt$$

Example: Growth of $\sum 1/n$

- We've used $\lfloor x \rfloor = x - \{x\}$, where $\{x\}$ is the fractional part of x.

$$\sum_{0<n\leq N} \frac{1}{n} = 1 + \mathcal{O}\left(\frac{1}{N}\right) + \int_1^N \frac{1}{t} dt - \int_1^N \frac{\{t\}}{t^2} dt$$

- Because $\{t\}$ is only ever in the range $[0, 1)$, the last integral is always less than $\int_1^N 1/t^2 dt$, that is, $\mathcal{O}\left([-1/t]_1^N\right)$.

$$\sum_{0<n\leq N} \frac{1}{n} = 1 + \mathcal{O}\left(\frac{1}{N}\right) + \ln(N) + \mathcal{O}\left(1 - \frac{1}{N}\right)$$

$$= \ln(N) + \mathcal{O}(1)$$

Example: Growth of $\sum 1/n$

- This tells us the harmonic series $\sum_1^N 1/n$ grows like $\ln(N)$.
- It also tells us the difference is bounded by $O(1)$.
- In fact, the difference tends to the Euler–Mascheroni constant $\gamma \approx 0.5772$ which pops up in several areas of number theory and analysis.