Infinite Products

From Primes To Riemann

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Infinite Product

- At school we don't seem to learn about **infinite products**.

$$\prod_{n=1}^{\infty} a_n = a_1 \times a_2 \times a_2 \times \ldots$$

- What do we really mean by infinite product?

Initial Observations

- Example 1 - Easy to see the infinite product diverges. Each factor increases the size of the product.

$$2 \times 3 \times 4 \times 5 \times \ldots$$

- Example 2 - Fundamental idea that multiplying by zero causes a product to be zero.

$$2 \times 0 \times 4 \times 5 \times \ldots$$

Initial Observations

- Example 3 - Each factor reduces the size of the product.

$$\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \dots$$

- Infinite number of such factors, the product $\rightarrow 0$.

- We have found **two different ways** an infinite product can be zero.

Definition

- Similar to **infinite series**, we say an **infinite product** converges if the limit of the partial products is a **finite** value.

$$\lim_{N \to \infty} \prod_{n=1}^{N} a_n = P$$

- We'll see why it is conventional to insist the finite value is **non-zero**.

Example 1

- Does this converge?

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)$$

- Each factor $(1 + 1/n)$ is larger than one, so we expect the product to keep growing.

Example 1

- Consider partial product.

$$\prod_{n=1}^{N}\left(1+\frac{1}{n}\right)=\prod_{n=1}^{N}\left(\frac{n+1}{n}\right)$$

$$=\frac{\cancel{2}}{1}\times\frac{\cancel{3}}{\cancel{2}}\times\frac{4}{\cancel{3}}\times\ldots\times\frac{N+1}{\cancel{N}}$$

$$=N+1$$

- As $N\to\infty$, product **diverges**.

Example 2

- Does this converge?

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$$

- Note n starts at 2 to ensure the first factor is not zero.

- Each factor $(1-1/n)$ is smaller than one, so we expect the product to keep shrinking.

Example 2

$$\prod_{n=2}^{N}\left(1-\frac{1}{n}\right)=\prod_{n=2}^{N}\left(\frac{n-1}{n}\right)$$

$$=\frac{1}{\cancel{2}}\times\frac{\cancel{2}}{\cancel{3}}\times\frac{\cancel{3}}{\cancel{4}}\times\ldots\times\frac{\cancel{N+1}}{N}$$

$$=\frac{1}{N}$$

- As $N\to\infty$, product tends to 0, we say product **diverges to zero**.

Convergence And a_n

- For an infinite series $\sum a_n$ to converge, the terms $a_n \to 0$
- For an infinite product $\prod a_n$ to converge, the terms $a_n \to 1$
  - if each term $a_n > 1$, the product gets ever larger.
  - if each term $a_n < 1$, the product gets ever smaller towards zero.
  - negative a_n cause partial product to oscillate.
  - .. meaning convergence only happens if $a_n \to 1$.

Removing Zero Factors

- A single factor 0 collapses entire product to zero.

- If an infinite product has a **finite number of zero-valued factors**, they can be removed to leave a different potentially interesting product.

- Example:

$$\prod_{n=1} \left(1 - \frac{1}{n^2}\right) = 0$$

- Removing first factor leaves an interesting infinite product:

$$\prod_{n=2} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$$

Convergence Criterion 1

- Useful to write factors as $(1+a_n)$

$$P = \prod (1+a_n)$$

- Turn product into sum by takings logarithm

$$\ln(P) = \ln \prod (1+a_n) = \sum \ln(1+a_n)$$

- Using $1+x \leq e^x$

$$\ln(P) \leq \sum a_n$$

- If the sum is **bounded** $\implies$ the product is bounded. If $a_n > 0$ then boundedness is **convergence** (no oscillation).

Convergence Criterion 1

- If we expand out product $\prod(1 + a_n)$ we see another inequality.

$$1 + \sum a_n \leq \prod(1 + a_n) = P$$

- The expansion creates the terms $1 + \sum a_n$ and many more
- This tells us that if the product converges $\implies$ so does the sum.

Convergence Criterion 1

- The two results together give us

$$\boxed{\sum a_n \text{ converges } \Leftrightarrow \prod (1 + a_n) \text{ converges, for } a_n > 0}$$

- This allows us to say:
  - $\prod(1 + 1/n)$ diverges because $\sum 1/n$ diverges
  - $\prod(1 + 1/n^2)$ converges because $\sum 1/n^2$ converges

Convergence Criterion 2

- Using $1 - x \leq e^{-x}$ leads to a criterion for products of the form $\prod(1 - a_n)$.

$$\boxed{\sum a_n \text{ converges } \Leftrightarrow \prod (1 - a_n) \text{ converges, for } 0 < a_n < 1}$$

- So $\prod_2^\infty (1 - 1/n)$ diverges, because $\sum 1/n$ diverges.

Divergence To Zero

- The logarithmic view of infinite products has an interesting side effect.
- If the partial products $\to 0$ then the logarithm $\to -\infty$
- This is why we say the product **diverges to zero**.

Convergence Criterion 3

- Previous convergence critera apply for **real** values $a_n > 0$.
- Would be good to have criteria for **complex** a_n.
- To do that we'll need an intermediate result about $|a_n|$
- For complex a_n, we have $|a_n| > 0$ for all $a_n \neq 0$.

$$\sum |a_n| \text{ converges } \Leftrightarrow \prod (1 + |a_n|) \text{ converges}$$

Convergence Criterion 3

- We're interested in $\prod(1 + a_n)$ for complex a_n, not just $\prod(1 + |a_n|)$.

$$p_N = \prod^{N}(1 + a_n)$$

$$q_N = \prod^{N}(1 + |a_n|)$$

- We assert $a_n \neq -1$ to ensure no zero-valued factors.

Convergence Criterion 3

- For $N > M \geq 1$, we can compare $|p_N - p_M|$ with $|q_N - q_M|$

$$
\begin{aligned}
|p_N - p_M| &= |p_M| \cdot \left| \frac{p_N}{p_M} - 1 \right| \\
&= |p_M| \cdot \left| \prod_{M+1}^{N} (1 + a_n) - 1 \right| \\
&\leq |q_M| \cdot \left| \prod_{M+1}^{N} (1 + |a_n|) - 1 \right| \\
&= |q_M| \cdot \left| \frac{q_N}{q_M} - 1 \right|
\end{aligned}
$$

$$|p_N - p_M| \leq |q_N - q_M|$$

- If $|q_N - q_M| < \epsilon$, then $|p_N - p_M| < \epsilon$. Cauchy criterion for convergence. If q_N converges, so does p_N.

Convergence Criteria 3

- So $\sum |a_n|$ converges $\implies \prod(1+|a_n|)$ converges $\implies \prod(1+a_n)$ converges.

$$\boxed{\sum |a_n| \text{ converges } \implies \prod(1+a_n) \text{ converges, for } a_n \neq -1}$$

- This is one way, we can't say the sum comverges if the product converges.

Summary

- Real a_n

$$\boxed{\sum a_n \text{ converges } \Leftrightarrow \prod (1+a_n) \text{ converges, for } a_n > 0}$$

- Real a_n

$$\boxed{\sum a_n \text{ converges } \Leftrightarrow \prod (1-a_n) \text{ converges, for } 0 < a_n < 1}$$

- Complex a_n

$$\boxed{\sum |a_n| \text{ converges } \implies \prod (1+a_n) \text{ converges, for } a_n \neq -1}$$

Why Convergence Is Non-Zero

- Convergence according to these criteria $\implies$ products converge to a **non-zero** value.
- Why?

Why Convergence Is Non-Zero

- We've know $a_n \to 0$, so $|a_n| < 1/2$ except for a finite number of terms.
- Useful inequality $1 + x \leq e^x$ gives us

$$1 \leq \prod (1 + |a_n|) < e^{\sum |a_n|}$$

- $\sum |a_n|$ converges $\implies \prod (1 + |a_n|)$ converges and is non-zero.

Why Convergence Is Non-Zero

- Another inequality $1 - x \geq e^{-2x}$ for $0 \leq x \leq 1/2$

$$0 < e^{-2\sum |a_n|} \leq \prod (1 - |a_n|) \leq 1$$

- Uses $e^y > 0$ for all real y.
- $\sum |a_n|$ converges $\implies$ $\prod (1 - |a_n|)$ converges and is non-zero.

Why Convergence Is Non-Zero

- Another inequality

$$1 - |a_n| \leq |1 \pm a_n| \leq 1 + |a_n|$$

$$\prod(1 - |a_n|) \leq \prod(1 \pm a_n) \leq \prod(1 + |a_n|)$$

- $\sum |a_n|$ converges $\implies \prod(1 \pm a_n)$ is non-zero
  - because its value is between two known non-zero values.

Riemann Zeta Function

- For $\sigma > 1$

$$\zeta(s) = \sum \frac{1}{n^s} = \prod (1 - \frac{1}{p^s})^{-1}$$

- No factor $(1 - 1/p^s)^{-1}$ is zero.
  - That would require p^s to be zero.
  - This isn't possible, and is easy to see by writing

$$|p^s| = \left| e^{s \ln(p)} \right| = e^{\sigma \ln(p)} > 0$$

- Also need to check product doesn't **diverge to zero**.

Riemann Zeta Function

- Consider $1/\zeta(s) = \prod(1 - 1/p^s)$. Using third convergence criterion, check $\sum |1/p^s|$ converges.

$$\sum \left| \frac{1}{p^s} \right| = \sum \frac{1}{p^\sigma} \leq \sum \frac{1}{n^\sigma}$$

- Because $\sum 1/n^\sigma$ converges for $\sigma > 1$, so does $\sum |1/p^s|$.
- This means $1/\zeta(s)$ converges to a non-zero value, and therefore so does $\zeta(s)$.
- $\implies$ Riemann Zeta function has no zeros $\sigma > 1$