Niven's Proof $\sum 1/p$ Diverges

From Primes To Riemann

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February 28, 2021

Niven's 1971 Proof

- Another proof that $\sum 1/p$ diverges based on Ivan Niven's 1971.
- Short, fun, but moves a little faster.

Niven's 1971 Proof

A PROOF OF THE DIVERGENCE OF $\Sigma\, 1/p$

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First we prove that $\Sigma'\, 1/k$ diverges, where Σ' denotes the sum over the squarefree positive integers. Each positive integer is uniquely expressible as a product of a squarefree positive integer and a square, so for any positive integer n,

$$\left(\sum_{k<n}{}' \, 1/k\right)\left(\sum_{j<n} 1/j^2\right) \geqq \sum_{m<n} 1/m.$$

Here the second sum is bounded but the third sum is unbounded as n increases, so the first sum must be unbounded. Next suppose that $\Sigma\, 1/p$ converges to β, the sum taken over all primes p. By dropping all terms beyond x in the series expansion of e^x or $\exp(x)$, we see that $\exp(x) > 1 + x$ for $x > 0$. Hence for each positive integer n

$$\exp(\beta) > \exp\left(\sum_{p<n} 1/p\right) = \prod_{p<n} \exp(1/p) > \prod_{p<n} (1 + 1/p) \geqq \sum_{k<n}{}' \, 1/k.$$

But this contradicts the unboundedness of the last sum, so $\Sigma\, 1/p$ diverges.

- https://www.tandfonline.com/doi/abs/10.1080/00029890.1971.11992740.

Square-Free Numbers

- We can write any counting number m as a unique product of a square j^2 and **square-free** factor k.

$$m = k \cdot j^2$$

Square-Free Numbers

- Any integer is a unique product of primes.
- We can split these primes into two groups:
  - one group with primes raised to an even power, which together can be written as a square.
  - other group with primes not raised to any power.
- Examples
  - $360 = (2 \cdot 5) \cdot (2 \cdot 3)^2$ square-free factor of 10, square factor of 36.
  - $30 = (2 \cdot 3 \cdot 5)$ is entirely square-free.

Sum of Square-Free Integers Diverges

- In the following inequality k are square-free integers less than n.

$$\left(\sum_{k<n} \frac{1}{k}\right)\left(\sum_{j<n} \frac{1}{j^2}\right) \geq \sum_{m<n} \frac{1}{m}$$

- Inequality because multiplying out the two series gives us terms $1/m$ and many more. Equality when $n = 2$.

- As $n \to \infty$
  - $\sum 1/m$ diverges, and $\sum 1/j^2$ converges to $\pi^2/6$
  - $\to \sum 1/k$ must diverge !

- Assume $\sum 1/p$ converges to a finite β.
- Partial sum is less than the full sum $\sum_{p<n} 1/p < \beta$
- so we can write

$$\exp(\beta) > \exp\left(\sum_{p<n} \frac{1}{p}\right) = \prod_{p<n} \exp(\frac{1}{p})$$

Sum Of Prime Reciprocals

- Taylor series for e^x lets us say $e^x > 1 + x$

$$\prod_{p<n} \exp(\frac{1}{p}) > \prod_{p<n} (1 + \frac{1}{p})$$

- Multiplying out product gives terms $1/k$ where k is square-free, because each prime contributes to any k at most once.

- Also gives more terms than are in $\sum_{p<n} 1/k$ for $n > 3$.

$$\prod_{p<n} (1 + \frac{1}{p}) \geq \sum_{k<n} \frac{1}{k}$$

Sum of Prime Reciprocals

- Putting everything together ...

$$\exp(\beta) > \exp\left(\sum_{p<n} \frac{1}{p}\right) = \prod_{p<n} \exp(\frac{1}{p}) > \prod_{p<n}(1 + \frac{1}{p}) \geq \sum_{k<n} \frac{1}{k}$$

- Suggests that as $n \to \infty$, the **finite** $\exp(\beta)$ is greater than **divergent** $\sum 1/k$ ... contradiction.

$$\boxed{\sum \frac{1}{p} \to \infty}$$