Probabilistic Primes

From Primes To Riemann

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Probabilistic Models Of Primes

- Probabilistic models of primes can match numerical evidence fairly well.
- Predictions can become worthy conjectures.
- We'll build a simple probabilistic model to predict the distribution of prime gaps.

Probability Of A Prime

- Prime Number Theorem: density of primes around x is approximately

$$\frac{1}{\ln(x)}$$

- Not a big leap to interpret this density as a probability.

$$\Pr(n \text{ prime}) = \frac{1}{\ln(n)}$$

Prime Gaps

- A prime gap at n of size $g(n) = 4$ is a sequence

prime, not prime, not prime, not prime, prime

- The probability of this sequence is

$$\frac{1}{\ln(n)} \cdot \left(1 - \frac{1}{\ln(n+1)}\right) \cdot \left(1 - \frac{1}{\ln(n+2)}\right) \cdot \left(1 - \frac{1}{\ln(n+3)}\right) \cdot \frac{1}{\ln(n+4)}$$

Prime Gaps

- For $n \gg a$ we can approximate $\ln(n+a) \approx \ln(n)$

$$\Pr\left(g(n) = 4\right) = \left(1 - \frac{1}{\ln(n)}\right)^3 \cdot \left(\frac{1}{\ln(n)}\right)^2$$

- Generalise to prime gaps of size k having a sequence {prime, k-1 not prime, prime}

$$\Pr\left(g(n) = k\right) = \left(1 - \frac{1}{\ln(n)}\right)^{k-1} \cdot \left(\frac{1}{\ln(n)}\right)^2$$

Prime Gap Counts

- **Expected** number of gaps of size k amongst first N numbers is $N \cdot \Pr\left(g(n) = k\right)$

$$N \cdot \left(1 - \frac{1}{\ln(N)}\right)^{k-1} \cdot \left(\frac{1}{\ln(N)}\right)^{2}$$

- This uses approximation that for most n between 1 and N, $\ln(n) \approx \ln(N)$.

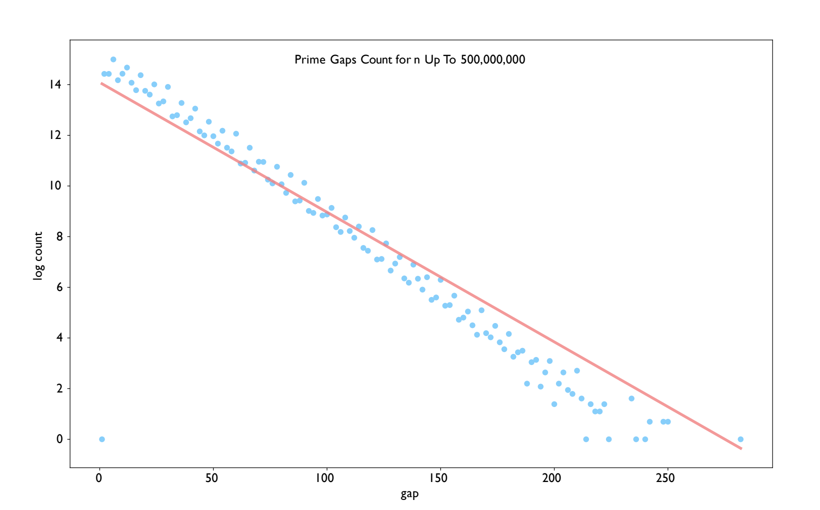
Prime Gap Counts

- Taking logs gives us a linear function of gap size k of the form $Ak + B$

$$(k-1) \cdot \ln\left(1 - \frac{1}{\ln(N)}\right) + 2\ln\left(\frac{1}{\ln(N)}\right) + \ln(N)$$

- The line will have a negative gradient A because it is the logarithm of $\left(1 - \frac{1}{\ln(N)}\right) < 1$.

Prime Gap Counts

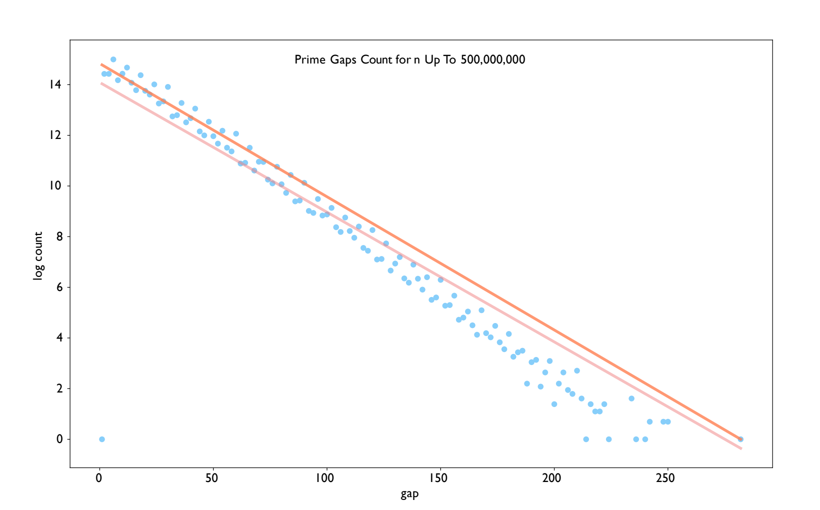


Improved Model

- Simple model ignores fact even numbers are never prime.
- Idea
  - if we want to assert $\Pr(\text{even}) = 0$
  - but also preserve PNT density of primes is $1/\ln(x)$ in the neighbourhood of x
  - we can double $\Pr(\text{even})$ to $2/\ln(x)$
- Same method, but applied to $N/2$ numbers, gives expected gaps

$$\frac{N}{2} \cdot \left(1 - \frac{2}{\ln(N)}\right)^{k-1} \cdot \left(\frac{2}{\ln(N)}\right)^{2}$$

Improved Model



- Youtube: https://www.youtube.com/watch?v=85hrakisiP0

What Have We Learned?

- A coin-tossing experiment would give us a linear log count of Head or Tail sequences.
  - suggests occurence of primes looks "random"
- The order-of-magnitude suggests the PNT based $1/\ln(x)$ probabilty is about right.