# Dirichlet Series

## From Primes To Riemann

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- **Dirichlet series** have the general form

$$\sum a_n/n^s$$

- ... in contrast to familiar power series $\sum a_n z^n$.

- The Riemann Zeta series is an example of a Dirichlet series.

$$\zeta(s) = \sum \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

# Absolute Convergence

- A series **converges absolutely** even when all its terms are replaced by their magnitudes.

- Not all series that converge do so absolutely.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \to \infty$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \ldots \to \ln(2)$$

# Abscissa of Absolute Convergence

- Assume a Dirichlet series **converges absolutely** at $s_1 = \sigma_1 + it_1$. Consider another $s_2 = \sigma_2 + it_2$ where $\sigma_2 \geq \sigma_1$.
- Compare the magnitudes of the terms in this series at $s_1$ and $s_2$.

$$\sum \left| \frac{a_n}{n^{s_1}} \right| = \sum \frac{|a_n|}{n^{\sigma_1}} \geq \sum \frac{|a_n|}{n^{\sigma_2}} = \sum \left| \frac{a_n}{n^{s_2}} \right|$$

- Remember $\left| n^{\sigma + it} \right| = \left| n^{\sigma} e^{it \ln n} \right| = n^{\sigma}$.
- So if series converges at $s_1$, it must also converge at $s_2$. More generally, the series converges at any $s = \sigma + it$ where $\sigma \geq \sigma_1$.

# Abscissa of Absolute Convergence

- If our series doesn't converge everywhere, divergence must be at some $\sigma < \sigma_1$. There must be a minimum $\sigma_a$, called the **abscissa of absolute convergence**, such that the series converges at $\sigma > \sigma_a$.

- Notice $\sigma_a$ depends only on the real part of $s$. Example:

  - $\sum 1/n^\sigma$ converges for real $\sigma > 1$, and it diverges at $\sigma = 1$

  - $\implies \sigma_a = 1$, so series converges for all $s = \sigma + it$ where $\sigma > 1$.

- Convergence domain for a Dirichlet series is a half-plane, whereas the region for the more familiar power series $\sum a_n z^n$ is a circle.

# Abscissa of Convergence

- With absolute convegence we don't need to consider complex terms which contibute a negative amount to the overall magnitude of the series.
    - Example, $e^{i\pi} = -1$ can partially cancel the effect $2e^{i2\pi} = 2$.
    - This cancelling means some series do converge, even if not absolutely.
- Strategy:
    - show that if a series is **bounded** at $s_0 = \sigma_0 + it_0$ then it is also **bounded** at $s = \sigma + it$, where $\sigma > \sigma_0$
    - then push further to show it actually converges at that $s$.

# Abscissa of Convergence

- Start with a Dirichlet series $\sum a_n/n^s$ that we know has bounded partial sums at a point $s_0 = \sigma_0 + it_0$ for all $x \geq 1$.

$$\left| \sum_{n \leq x} \frac{a_n}{n^{s_0}} \right| \leq M$$

- Being bounded is not as strong a requirement as convergence, the partial sums could oscillate for example.

# Abscissa of Convergence

- **Abel's partial summation formula** relates a discrete sum to a continuous integral.

$$\boxed{\sum_{x_1 < n \leq x_2} b_n f(n) = B(x_2)f(x_2) - B(x_1)f(x_1) - \int_{x_1}^{x_2} B(t)f'(t)dt}$$

- Define $f(x) = x^{s_0 - s}$ and $b_n = a_n/n^{s_0}$.
- $B(x)$ is defined as $\sum_{n \leq x} b_n$, and so $|B(x)| \leq M$ for all $x$.

$$\sum_{x_1 < n \leq x_2} \frac{a_n}{n^s} = \sum_{x_1 < n \leq x_2} b_n f(n)$$

$$= \frac{B(x_2)}{x_2^{s-s_0}} - \frac{B(x_1)}{x_1^{s-s_0}} + (s - s_0) \int_{x_1}^{x_2} \frac{B(t)}{t^{s-s_0+1}} dt$$

# Abscissa of Convergence

- Using triangle inequality and $|B(x)| \leq M$.

$$\left|\sum_{x_1<n\leq x_2} \frac{a_n}{n^s}\right| \leq \left|\frac{B(x_2)}{x_2^{s-s_0}}\right| + \left|\frac{B(x_1)}{x_1^{s-s_0}}\right| + \left|(s-s_0)\int_{x_1}^{x_2} \frac{B(t)}{t^{s-s_0+1}}dt\right|$$

$$\leq Mx_2^{\sigma_0-\sigma} + Mx_1^{\sigma_0-\sigma} + |s-s_0|\, M\int_{x_1}^{x_2} t^{\sigma_0-\sigma-1}dt$$

# Abscissa of Convergence

- Using $Mx_2^{\sigma_0-\sigma} + Mx_1^{\sigma_0-\sigma} < 2Mx_1^{\sigma_0-\sigma}$ for $\sigma > \sigma_0$.

$$\left|\sum_{x_1<n\leq x_2} \frac{a_n}{n^s}\right| \leq 2Mx_1^{\sigma_0-\sigma} + |s-s_0|\, M \left(\frac{x_2^{\sigma_0-\sigma} - x_1^{\sigma_0-\sigma}}{\sigma_0-\sigma}\right)$$

$$\leq 2Mx_1^{\sigma_0-\sigma}\left(1 + \frac{|s-s_0|}{\sigma-\sigma_0}\right)$$

- Last step uses $\left|x_2^{\sigma_0-\sigma} - x_1^{\sigma_0-\sigma}\right| = x_1^{\sigma_0-\sigma} - x_2^{\sigma_0-\sigma} < x_1^{\sigma_0-\sigma} < 2x_1^{\sigma_0-\sigma}$

# Abscissa of Convergence

- The key point is that $\sum_{x_1 < n \leq x_2} a_n/n^s$ is bounded if $\sum_{n \leq x} a_n/n^{s_0}$ is bounded, where $\sigma > \sigma_0$.

- Let's see if we can push this result about **boundedness** to **convergence**.

# Abscissa of Convergence

$$\left|\sum_{x_1<n\leq x_2}\frac{a_n}{n^s}\right| \leq 2Mx_1^{\sigma_0-\sigma}\left(1+\frac{|s-s_0|}{\sigma-\sigma_0}\right) = Kx_1^{\sigma_0-\sigma}$$

- Here $K$ doesn't depend on $x_1$.
- If we let $x_1 \to \infty$ then $Kx_1^{\sigma_0-\sigma} \to 0$, which means the infinite sum $\sum a_n/n^s$ converges.

# Results So Far

- If $\sum_{n \leq x} a_n / n^{s_0}$ is bounded, the infinite sum $\sum a_n / n^s$ converges for $\sigma > \sigma_0$.
- With the special case of $s_0 = 0$, if $\sum_{n \leq x} a_n$ is bounded, the infinite sum $\sum a_n / n^s$ converges for $\sigma > 0$.
  - We can sometimes say whether a series converges for $\sigma > 0$ just by looking at the coefficients $a_n$.

# Abscissa of Convergence

- There is an abscissa of convergence $\sigma_c$ where a Dirichlet series converges for $\sigma > \sigma_c$, and diverges for $\sigma < \sigma_c$.
    - If a series converges (bounded) at $s_0$ then it converges at $\sigma > \sigma_0$
    - If series doesn't converge everywhere, the $s$ where it diverges has $\sigma < \sigma_0$

# Difference Between $\sigma_c$ And $\sigma_a$

- Not all convergent series are absolutely convergent, so $\sigma_a \geq \sigma_c$.
- If a series converges at $s_0$, the magnitude of terms is bounded. We can call this bound $C$.

$$\sum \left| \frac{a_n}{n^s} \right| = \sum \left| \frac{a_n}{n^{s_0}} \cdot \frac{1}{n^{s-s_0}} \right| \leq C \sum \frac{1}{n^{\sigma - \sigma_0}}$$

- $\sum 1/n^{\sigma-\sigma_0}$ only converges for $\sigma - \sigma_0 > 1$, so we can say if $\sigma$ is larger than $\sigma_c$ by at least 1, the series converges absolutely.

$$\boxed{0 \leq \sigma_a - \sigma_c \leq 1}$$

# Example: Alternating Zeta Function

- Let's apply our results to the **alternating zeta function**, also called the **eta function**.

$$\eta(s) = \sum \frac{(-1)^{n+1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots$$

- At $s_0 = 0$ the partial sum $\sum_{n \leq x} (-1)^{n+1}$ oscillates but is always bounded $\leq 1$

- $\implies \sum (-1)^{n+1}/n^s$ converges for $\sigma > 0$.