20 - Contradictory Cases

Lean: First Steps

Tariq Rashid

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Contradictory Cases

- When we explored proof by cases, each case led to the proof objective.
- In general, not all cases permitted by a hypothesis lead to the proof objective.
- Some cases may lead to a contradiction, ruling them out as possible cases.

Task

- If P is a proposition, show that

$$\neg(\neg P) \implies P$$

- For this task we'll pretend we don't yet know two negations cancel out.
- Proving $\neg(\neg P) \implies P$ will justify our intuition that two negations do indeed cancel out.

Law of the Excluded Middle

- A proposition is either **true** or **false**, and there is no other possibility.
- This is called the **Law of the Excluded Middle**.
- So there are only two possibilities for our proposition P, true or false.

- Let's consider each case.
  - P is **true**. This is the proof objective, so there is nothing more to do.
  - P is **false**. That is, $\neg P$ is true. This contradicts the given hypothesis $\neg(\neg P)$ is true. A contradiction arises from an invalid assumption, so it can't be the case that P is false.
- We've shown P true is possible, but P false is not.

- A couple of points worth explaining:
- Two statements of the form Q and $\neg Q$ are contradictory. This is how, in the second case, we justify $\neg P$ and $\neg(\neg P)$ are contradictory.
- It may look like a **circular argument** that the first case assumes P is true to prove P is true. It isn't circular because we consider **all the possibilities** for P, which here means considering P is false.

Maths

$\neg(\neg P) \implies P$	proof objective	
$\neg(\neg P)$	hypothesis	(1)
P	proof goal	(2)
$P \vee \neg P$	law of excluded middle	(3)
case P	using fact (3)	
P	proof goal	
case $\neg P$	using fact (3)	
$\neg P$	contradicts hypothesis (1)	
$\neg(\neg P) \implies P$	only consistent case	$\square$

- We've not proved an implication before so let's clarify how it's done.
- To prove an implication $A \implies B$, we assume A as a hypothesis and then prove B.
- So to prove $\neg(\neg P) \implies P$, we take $\neg(\neg P)$ as a hypothesis (1) and set P as a proof goal (2).
- The rest is a proof by cases, and proceeds just as we discussed above.

Code

```
-- 20 - Contradictory Cases

import Mathlib.Tactic

example {P : Prop} : ¬(¬P) → P := by
  intro g
  by_cases h : P
  · exact h
  · contradiction
```

Code

- The proof header declares `P` as a **proposition** using `{P : Prop}`.
- The proof starts with `intro g` which converts the proof objective, the implication `¬(¬P) → P`, into a hypothesis `g : ¬¬P` and a new goal `P`.
- `by_cases h : P` creates the two cases for `P` using the Law of the Excluded Middle. Lean will replace the current goal with two new goals:
  - one with `h : P` as a hypothesis
  - the other with `h : ¬P` as a hypothesis.

Code

- The rest of the proof handles each case:
  - The first case is `h : P`. This exactly matches the goal `P`, so we can use `exact h` to resolve this case.
  - The second case is `h : ¬P`. This directly contradicts the hypothesis `g : ¬¬P` created at the start of the proof, so we can use `contradiction` to resolve this case. That means `¬P` is ruled out as a possibility.

Infoview

- Placing the cursor before `intro g` shows the original proof goal.

```
⊢ ¬¬P → P
```

- Moving the cursor to the start of the next line shows `¬¬P` established as a new hypothesis `g`, and `P` set as the new goal.

```
g : ¬¬P
⊢ P
```

Infoview

- Placing the curser after `by_cases h : P` shows the two new goals, one for each case.

```
case pos
P : Prop
g : ¬¬PP
h : P
⊢ P

case neg
P : Prop
g : ¬¬P
h : ¬P
⊢ P
```

Easy Exercise

- If P is a proposition, show that

$$P \implies \neg(\neg P)$$

- To keep the proof as short as the example above, try applying the law of the excluded middle, not to P, but to $(\neg P)$.