## 15 - Zero Product

Lean: First Steps

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#### Bidirectional Lemmas

- Some lemmas are bidirectional implications  $\iff$  .
- To apply them, we decide which of the two directions match our task.

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## Task

Given the zero product

$$(p-1)\cdot(q-2)=0$$

show that

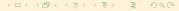
$$p=1$$
 or  $q=2$ 

• where *p* and *q* are rational numbers.



- It is common knowledge that the product of two numbers is zero, if and only if, one or the other is zero.
- We'll think of this as a lemma,

$$a \cdot b = 0 \iff (a = 0) \lor (b = 0)$$



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- $P \iff Q$  is **equivalent** to both of the following being true:
  - $P \implies Q$
  - $Q \implies P$
- We say "P is true if and only if Q is true"
  - "P is true **iff** Q is true."



So our lemma is equivalent to the following two statements:

• 
$$a \cdot b = 0 \implies (a = 0) \lor (b = 0)$$

• 
$$(a=0) \lor (b=0) \implies a \cdot b = 0$$

• The antecedent of the first matches our hypothesis  $(p-1)\cdot (q-2)=0$ , so applying it gives us a disjunction.

$$(p-1=0) \lor (q-2=0)$$

• We could conclude p = 1 or q = 2 by inspection, but a lean proof needs more detail.



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$$(p-1)\cdot(q-2)=0$$

$$a \cdot b = 0 \iff (a = 0) \lor (b = 0)$$

$$(p-1=0) \lor (q-2=0)$$

case 
$$p-1=0$$

$$p=1$$

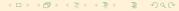
using (3)

case 
$$q - 2 = 0$$
  
 $q = 2$ 

$$(p = 1) \lor (q = 2)$$

conclusion

- Key step is applying the lemma (2) to the hypothesis (1) to derive a new intermediate result (3).
- This intermediate result is a disjunction  $(p-1=0) \lor (q-2=0)$ , so the proof proceeds as a proof by cases.



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-- 15 - Using Lemmas: Multiplied Factors Equal Zero
import Mathlib.Tactic
example \{p \ q : \mathbb{Q}\}\ (h : (p - 1) * (q - 2) = 0) : p = 1 \lor q = 2 :=
     by
  apply mul_eq_zero.mp at h
  obtain hp | hq := h
  · left
    linarith
  · right
    linarith
```

- Mathlib lemma mul\_eq\_zero says  $a \cdot b = 0 \iff (a = 0) \lor (b = 0)$ .
- Let's check its definition:

theorem mul\_eq\_zero : 
$$a * b = 0 \iff a = 0 \lor b = 0 :=$$

- The first line of the proof applies mul\_eq\_zero to hypothesis h, but the lemma name has a .mp appended.
  - Adding .mp to a bidirectional lemma selects forward direction.
  - Adding .mpr to a bidirectional lemma selects reverse direction.
- mp is "modus ponens", terminology from logic for the process of reaching a conclusion using an implication.
- Infoview will confirm the hypothesis has been **replaced** with the equivalent disjunction  $p 1 = 0 \lor q 2 = 0$ .

- The proof then proceeds as a proof by cases.
  - obtain splits hypotheses h into two cases, p 1 = 0 and q 2 = 0.
  - First case uses left to prove p=1. The **linarith** tactic can perform "add a number to both sides" to justify the goal p=1 from the current hypotheses, one of which is hp:p-1=0.
  - The second case is almost the same, but we use right to prove
     q = 2.
- Both cases, p 1 = 0 and q 2 = 0, confirm the same disjunctive proof goal, so the proof is complete.



# Replacing A Hypothesis?

- In the maths proof we derived an intermediate result from the hypothesis.
- But in the Lean code the hypothesis was replaced.
- That's ok, it was replaced by an equivalent statement.

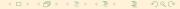
#### Infoview

 Placing the cursor before apply mul\_eq\_zero.mp at h shows the original hypothesis.

$$h : (p - 1) * (q - 2) = 0$$

 Moving the cursor to the beginning of the next line shows the hypothesis has been replaced.

$$h : p - 1 = 0 \lor q - 2 = 0$$



## Easy Exercise

- Write a Lean program to prove  $(p-1) \cdot (q-2) \neq 0$  given  $p-1 \neq 0$  and  $q-2 \neq 0$ , where p and q are rational numbers.
- Use the lemma mul\_ne\_zero\_iff.
  - Look up the definition to understand what it says, then decide in which direction to apply it.