16 - Writing Our Own Lemma

Lean: First Steps

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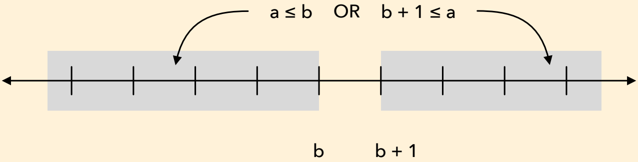
Writing Our Own Lemmas

- Mathlib has many lemmas and theorems for us to use.
- We can also write our own.
- We'll create a small convenient lemma about the natural numbers.

Task

- If we pick a natural number b, then any other natural number a must be less than or equal to b, or greater than or equal to $b + 1$.



- Let's create a lemma for natural numbers a and b

$$\boxed{a \leq b \ \vee \ b + 1 \leq a}$$

- Example: setting $b = 7$ lemma says $a \leq 7$ or $8 \leq a$.

Maths

- Aim to use lemmas that exist in Mathlib to ease the transition to Lean.

$$a \leq b \ \vee \ b + 1 \leq a \qquad \text{proof objective, } a, b \in \mathbb{N}$$

$$a \leq b \ \vee \ b < a \qquad \text{known lemma, } a, b \in \mathbb{N} \tag{1}$$

$$m + 1 \leq n \iff m < n \qquad \text{known lemma, } m, n \in \mathbb{N} \tag{2}$$

$$a \leq b \ \vee \ b + 1 \leq a \qquad \text{apply lemma (2) to (1)} \qquad \square$$

- Lemma (1) is already very close to our proof objective.
- Lemma (2) lets us rewrite $b < a$ as $b + 1 \leq a$, which gives us the proof objective.
- Both lemmas (1) and (2) exist in Mathlib.

Code

```
-- 16 - Writing And Using Our Own Lemma

import Mathlib.Tactic

lemma Nat.le_or_succ_le (a b: ℕ): a ≤ b ∨ b + 1 ≤ a := by
  rw [Nat.le_or_succ_le]
  exact le_or_lt a b

example {c : ℕ} :  c ≤ 2 ∨ 3 ≤ c  := by
  exact Nat.le_or_succ_le c 2
```

Code

- `lemma Nat.le_or_succ_le` tells Lean we want to create a new lemma named `Nat.le_or_succ_le`.
  - `Nat.` prefix is conventional for lemmas about natural numbers.
  - `le_or_succ_le` tries to follow the naming convention to describe what the lemma is about.
- `a` and `b` declared as natural numbers, and states the lemma's proposal `a ≤ b ∨ b + 1 ≤ a`.
- The round brackets `(a b : ℕ)` require anyone using the lemma to always provide `a` and `b` as parameters.

Code

- The first line of the proof rewrites the goal using Mathlib lemma `Nat.add_one_le_iff`. Let's see its definition:

```
theorem add_one_le_iff : n + 1 ≤ m ↔ n < m :=
```

- The **rw** tactic is like "find and replace", so it looks at the goal `a ≤ b ∨ b + 1 ≤ a` and finds `b + 1 ≤ a` matches the LHS of the `Nat.add_one_le_iff` lemma. The matched `b + 1 ≤ a` is rewritten with the RHS of the lemma `b < a`.

- We'll see in the Infoview, the goal is now `a ≤ b ∨ b < a`.

Code

- The final line of the proof applies Mathlib lemma `le_or_lt`. Let's look at its definition:

```
lemma le_or_lt (a b : α) : a ≤ b ∨ b < a :=
```

- This matches the current goal exactly, so we can use `exact` to resolve the goal and complete the proof.

- Notice the definition of the Mathlib lemma `le_or_lt (a b : α)` uses round brackets requiring us to provide `a` and `b` when we use it.

Infoview

- Placing the cursor before `rw [Nat.add_one_le_iff]` shows the original proof goal.

```
⊢ a ≤ b ∨ b + 1 ≤ a
```

- Moving the cursor to the start of the next line shows the rewritten goal.

```
⊢ a ≤ b ∨ b < a
```

- This proof goal now matches exactly the Mathlib lemma `le_or_lt a b`, and can be resolved with **exact**.

Minimal Example

- When writing your own lemmas, it is considerate to provide a **minimal example** showing how to use them.
- The provided example illustrates how to prove $c \leq 2 \vee 3 \leq c$ using our lemma.
- With `a` set to `c`, and `b` set to `2`, the lemma becomes `c ≤ 2 ∨ 3 ≤ c`.
  - This matches the proof goal exactly, allowing us to use **exact** with our lemma to complete the proof.

Forwards & Backwards

- Maths and Lean proofs go in opposite directions:
  - maths proof starts with a lemma $a \leq b \vee b < a$ and arrives at goal $a \leq b \vee b + 1 \leq a$.
  - Lean proof starts with the goal $a \leq b \vee b + 1 \leq a$ and arrives at equivalent known lemma $a \leq b \vee b < a$.
- That's ok.
  - Lean tactics bias a little towards operating on current goal.
  - So we see more "backwards" proofs in Lean than pen-n-paper.

“Forwards” Lean Proof

- For interest only, this Lean proof follows the “forward” direction of the maths proof.

```
lemma Nat.le_or_succ_le (a b : ℕ): a ≤ b ∨ b + 1 ≤ a := by
  have h : a ≤ b ∨ b < a := le_or_lt a b
  rw [← Nat.succ_le] at h
  exact h
```

Easy Exercise

- Write a lemma for **integers** a and b that says

$$a \leq b \ \vee \ b + 1 \leq a$$

- You can copy the provided lemma for natural numbers and modify it to work with integers.

- Create a minimal example illustrating how your lemma can prove the following for any integer c.

$$c \leq -5 \ \vee \ -4 \leq c$$