02 - Substitution

Lean: First Steps

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Task

 Let's continue with another simple task. Imagine we have the following formula.

$$y = x + 4$$

• Given x = 3, our task is to prove

$$y = 7$$

• Here, x, y, 3 and 4 are all real numbers.



Maths

• Easy task - an effort to think about the steps involved.

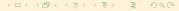
$$y = x + 4$$
 given fact (1)

$$x = 3$$
 given fact (2)

$$y = x + 4$$
 using fact (1)
= (3) + 4 substitution using fact (2)
= 7 using arithmetic

Maths

- We start by listing the two given facts, y = x + 4 and x = 3.
- We want to prove something about y. What is y? The first fact tells us y = x + 4.
- At this point, we have y = x + 4, which is fine, but we do want to resolve that x into a number.
- The second fact tells us x = 3. We can use it to substitute 3 for the x in x + 4. This gives us y = (3) + 4.
- Finally, we can use arithmetic to evaluate (3) + 4 as 7. That gives us what we want, y = 7.



Maths

- It is this kind of structured step-by-step thinking that we'll need to write proofs in Lean.
- It may seem disproportionate for simple tasks, but it is better to develop that thinking with simple tasks.

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```
-- 02 - Simple Proof by Calculation

import Mathlib.Tactic

example {x y : R} (h1 : y = x + 4) (h2 : x = 3) : y = 7 := by

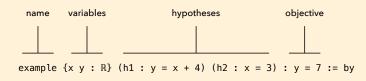
calc

y = x + 4 := by rw [h1]

_ = 3 + 4 := by rw [h2]

_ = 7 := by norm_num
```

- Lines beginning with -- are comments.
- import Mathlib.Tactic loads into Lean information about fundamental results and common methods used in proofs, tactics.
- The next line is the beginning of the proof:
 - Theorem **name**, but example creates anonymous theorem.
 - Variable types {x y : ℝ}.
 - Named hypotheses (h1 : y = x + 4) (h2 : x = 3).
 - After : is the **objective** or **goal** statement we want to prove, y = 7.
 - Finally, := by signals the subsequent code will seek to prove the objective.



- calc tells Lean we intend to do a proof by direct calculation.
- After that is the core of the proof.
 - y = x + 4 is justified using by rw [h1], a tactic for rewriting y using hypothesis h1.
 - Previous expression x + 4, denoted by the shorthand $\underline{}$, is equal to 3 + 4. This is justified by hypothesis h2, which allows us to rewrite x as 3.
 - Finally, y = 7. Simplification of 3 + 4 to 7 is justified with the norm num tactic, which can do numerical arithmetic.

InfoView

- Lean provides feedback through its Infoview, which will appear in a separate pane.
 - Usually to the right of your main code.
- Updated as we edit our code, provides warnings and errors.

InfoView

• With the previous (correct) Lean code, Infoview tells us:

```
All Messages (0)
No messages.
```

• No messages means no warnings or errors. Lean thinks our proof is correct.

InfoView - Deliberate Error

- Change the hypothesis h2, used to rewrite x as 3, to the incorrect hypothesis h1.
- Infoview updates with error message.

```
01_simple.lean:8:24
tactic 'rewrite' failed, did not find instance of the
   pattern in the target expression
y
```

- At line 8 of the code, Lean found the rewrite tactic had failed.
- Changing the hypothesis back to h2 sees all error messages go away.

Easy Exercise

• Write a Lean program to prove y=0 given $y=x^2-9$ and x=-3, where $x,y\in\mathbb{R}$.