17 - Using Our Own Lemma Lean: First Steps

Tariq Rashid

November 25, 2024

Using Our Own Lemma

- In the last chapter we created our own lemma.
- Here we'll make use of that lemma to prove a disequality.

Task

• For any natural number n, show that

$$n^2 \neq 7$$



Maths

- Proof strategy split the natural number sequence into two
 - all the smaller numbers result in $n^2 < 7$
 - all the larger numbers in $n^2 > 7$
- If every possible choice for n leads to either $n^2 < 7$ or $n^2 > 7$, then $n^2 \neq 7$.

Maths

$$n^2 \neq 7$$

$$n \leq m \vee m + 1 \leq n$$

our lemma, for
$$m, n \in \mathbb{N}$$
 (1)

$$n \le 2 \lor 3 \le n$$

lemma (1) with
$$m=2$$
 (2)

case $n \le 2$

using (2)

$$n^2 \le 4$$
< 7

$$n^2 \neq 7$$

$$\mathsf{lemma} \ a < b \implies a \neq b$$

case $n \ge 3$

 $n^2 \ge 9$

> 7

 $n^2 \neq 7$

using (2)

 $\mathsf{lemma}\ a > b \implies a \neq b$

conclusion of both cases < = > = <20 <

(3)

(4)

5/12

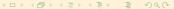
 $n \neq 7$

17 - Using Our Own Lemma

November 25, 2024

Maths

- Our lemma (1) gives a disjunction of two cases, $n \le 2$ and $n \ge 3$.
- The first case $n \le 2$
 - means $n^2 \le 4$, which also means $n^2 < 7$
 - Using small lemma $a < b \implies a \neq b$, we conclude $n^2 \neq 7$
- The second case $n \ge 3$
 - means $n^2 \ge 9$, which also means $n^2 > 7$.
 - Using small lemma $a > b \implies a \neq b$, we conclude $n^2 \neq 7$.
- Both cases lead to $n^2 \neq 7$, completing the proof.



Code 1/2

```
-- 17 - Using Our Own Lemma

import Mathlib.Tactic

---

lemma Nat.le_or_succ_le (a b : N): a \le b \le b + 1 \le a := by

rw [Nat.add_one_le_iff]

exact le_or_lt a b

---
```

Code 2/2

Code

- The code is separated into two sections, separated by comment dashes.
- The first section is a copy of the lemma we developed,
 Nat.le_or_succ_le.
- The second section is the Lean proof that $n^2 \neq 7$.

Code

- Our lemma Nat.le_or_succ_le with parameters n and 2, gives us n \leq 2 \vee 2 + 1 \leq n.
- We capture this as hypothesis h. We use obtain to split h into two cases, $n \le 2$ and $2 + 1 \le n$.
- The first case starts by changing the goal from n² ≠ 7 to n² < 7 by applying Mathlib lemma ne_of_lt. Its definition confirms that if we can prove n² < 7 then we've proven n² ≠ 7:

```
lemma ne_of_lt (h : a < b) : a \neq b :=
```

• We use a calc section to show $n^2 \le 2^2$ from the current case $n \le 2$, then $2^2 < 7$, all of which resolves the goal $n^2 < 7$.

Code

• The second case is similar. It starts by changing the goal from $n^2 \neq 7$ to $n^2 > 7$ by applying Mathlib lemma ne_of_gt . Its definition confirms that if we can prove $n^2 > 7$ then we've proven $n^2 \neq 7$:

```
lemma ne_of_gt (h : b < a) : a \neq b :=
```

• Again, we use a calc section to show $n^2 \ge 3^2$ from the current case $2 + 1 \le n$, then $3^2 > 7$, all of which resolves the goal $n^2 > 7$.

Easy Exercise

• Write a Lean proof to show that $n^3 \neq 10$, for any natural number n.