CS 4780/5780 Homework 9

Due: Tuesday 05/08/18 11:55pm on Gradescope

Problem 1: Gaussian Process Regression

Consider you have the following dataset:

No.	$\mathcal{X}$	$\mathcal{Y}$
1	-1	5
2	1	5
3	2	8

and you are going to use a Gaussian Process

$$f(x) \sim GP(m, k)$$

to model the underlying relationship between $\mathcal{X}$ and $\mathcal{Y}$. After discussing with the professors, you decide to use a zero mean function and a quadratic kernel for your GP, namely,

$$m(x) = 0$$
$$k(x, x') = (x \cdot x' + 1)^2$$

(a) What is the mean and covariance of your GP prior? Please use the order provided to compute the mean and the covariance of the prior.

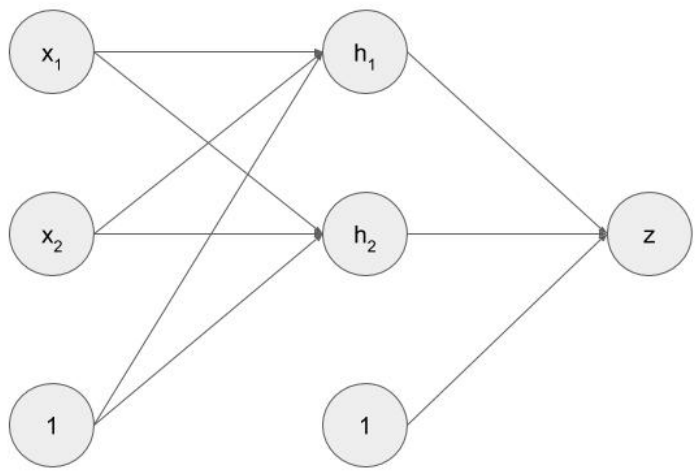
(b) Now, you are given the following test points:

No.	$\mathcal{X}$	$\mathcal{Y}$
1	-2	8
2	0	4

Assume a noise free setup. What is the mean and covariance of your GP posterior? Does the mean of your GP posterior match with your the target values of your test point?

Problem 2: RELU-network

In this question, you are going to explore a 2-layer fully connected network with RELU activation function.



Suppose you have the architecture above, namely,

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

$$z = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} f(h_1) \\ f(h_2) \\ 1 \end{bmatrix}$$

$$t = \sigma(z)$$

where $f(h_i) = max(0, h_i)$, $\sigma(z) = \frac{1}{1+e^{-z}}$ and t is the output of the network.

(a) Suppose $\begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}$. Draw the decision boundary of the network, namely, $\sigma(z) = 0.5$. Please indicate the positive and negative side of the boundary.

(b) Assume the weights in (a), what is the prediction for $[x_1, x_2]^T = [1, 1]^T$?

(c) Usually, the weights $W = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{11} & w_{12} & w_{13} \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ have to be learned using Stochastic Gradient Descent. For neural network binary classification, a common loss function is the cross entropy loss

$$l(y, t) = -(y \log(t) + (1 - y) \log(1 - t))$$

where y is the true label for a sample and t is the output of the neural network. In order to use SGD, we have to derive the gradient with respect to W and V. Show that for a single training example,

$x = [x_1, x_2]$

$$
\begin{aligned}
\frac{\partial l}{\partial v_i} &= (t-y)f(h_i) \text{ for } i \neq 3 \\
\frac{\partial l}{\partial v_3} &= (t-y) \\
\frac{\partial l}{\partial w_{ij}} &= (t-y)v_i\mathbb{I}(h_i > 0)x_j \text{ for } j \neq 3 \\
\frac{\partial l}{\partial w_{i3}} &= (t-y)v_i\mathbb{I}(h_i > 0)
\end{aligned}
$$

where $\mathbb{I}(\cdot)$ is the indicator function.