ST 520 HW 2.

1.(a) $C.I. = \left(\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$

$= \left(\frac{5}{20} - 1.28 \times \sqrt{\frac{0.25\times 0.75}{20}},\ \frac{5}{20} + 1.28 \times \sqrt{\frac{0.25\times 0.75}{20}}\right).$

$= (0.13,\ 0.37).$

(b) exact $C.I. = (\pi_L, \pi_u)$.

$$\begin{cases} P_{\pi_L}(X \geq 5) = \sum_{j=5}^{20} \binom{20}{j} \pi_L^j (1-\pi_L)^{20-j} \leq \frac{\alpha}{2}. \\ P_{\pi_u}(X \leq 5) = \sum_{j=0}^{5} \binom{20}{j} \pi_u^j (1-\pi_u)^{20-j} \leq \frac{\alpha}{2} \end{cases}$$

$\Rightarrow (\pi_L, \pi_u) = (0.13, 0.41).$

(c)(i) For $X = 0, 1, 2, \cdots, 20$, the exact C.I is:

X	Lower	Upper.
0	0.00	0.11
1	0.01	0.18
2	0.03	0.24
3	0.06	0.30.
⋮	⋮	⋮
6	0.17	0.47
7	0.21	0.52
⋮	⋮	⋮

$P(\pi \in C(X)) = \sum_{k=2}^{6} \binom{20}{k} 0.2^k\, 0.8^{20-k} = 0.84.$

(ii)

X	Lower	Upper
0	⋮	⋮
⋮		
5	0.13	0.41
6	0.17	0.47
7	0.21	0.52
⋮	⋮	⋮
12	0.43	0.75
13	0.48	0.79
14	0.53	0.83

$P(\pi \in C(X)) = \sum_{k=7}^{13} \binom{20}{k} 0.5^k\, 0.5^{20-k} = 0.88$

2. $\pi_0 = 0.25$

For stage I, we need $(1-\pi_0)^{n_0} \le 0.05 \Rightarrow n_0 \ge \frac{\log(0.05)}{\log(0.75)} = 10.41 \approx 11$

For stage II, we need $Z_{\frac{\alpha}{2}} \cdot \left(\frac{\pi_0(1-\pi_0)}{n}\right)^{1/2} \le 0.1 \Rightarrow n \ge 72.03 \approx 73$

The expected sample size is

$$E(N) = n_0 P(X=0|\pi_0) + n(1-P(X=0|\pi_0)) = n-(n-n_0)P(X=0|\pi_0)$$
$$= n-(n-n_0)\,0.75^{n_0}$$

We want to minimize $n-(n-n_0)0.75^{n_0}$ subject to $n_0 \ge 11$, $n \ge 73$, and using R we can get optimal combination $(n_0 = 11, n = 73)$.

Thus, we recruit 11 patients in the first stage, if no patient responds to treatment, we declare it a failure. Otherwise, if at least one patient respond, add 62 patients and count the total number of patients responding to the treatment and then calculate $\hat{\pi}$ and CI for π.

3.

(a) π = true prob. of response, $\underset{\text{responses in stage I}}{X_1} \in \{0,1,2,3\}$, $\underset{\text{response in stage II}}{X_2} \in \{0,1,\cdots,5\}$

$$P(\text{success}) = P(X_1=3) + P(1 \le X_1 \le 2,\ X_1+X_2 > 4)$$
$$= P(X_1=3) + P(X_1=1)P(X_2>3) + P(X_1=2)P(X_2>2)$$
$$= \pi^3 + \binom{3}{1}\pi(1-\pi)^2\left[\binom{5}{4}\pi^4(1-\pi)+\pi^5\right] + \binom{3}{2}\pi^2(1-\pi)\left[\binom{5}{3}\pi^3(1-\pi)^2+\binom{5}{4}\pi^4(1-\pi)+\pi^5\right]$$
$$= \pi^3 + 3\pi(1-\pi)^2\left[5\pi^4(1-\pi)+\pi^5\right] + 3\pi^2(1-\pi)\left[10\pi^3(1-\pi)^2+5\pi^4(1-\pi)+\pi^5\right]$$

(i) $\pi = 0.25$, $P(\text{success}) = 0.037$

(ii) $\pi = 0.50$, $P(\text{success}) = 0.383$

(iii) $\pi = 0.75$, $P(\text{success}) = 0.889$

(b) N = total number of patients, $N = \begin{cases} 3, & \text{if } X_1 = 0 \text{ or } X_1 = 3 \\ 8, & \text{if } X_1 = 1,2 \end{cases}$

$$E(N) = 3(\pi^3+(1-\pi)^3) + 8\left(\binom{3}{1}\pi(1-\pi)^2+\binom{3}{2}\pi^2(1-\pi)\right) = 3(\pi^3+(1-\pi)^3)+24\pi(1-\pi)$$

(i) $\pi=0.25$, $E(N)=5.8125$; (ii) $\pi=0.5$, $E(N)=6.75$; (iii) $\pi=0.75$, $E(N)=5.8125$

4. Using the table, $n_1 = 37$. $r_1 = 8$.

$n = 83$. $r = 22$.

The Simon's two-stage design as follows:

Stage I: Recruit 37 patients and give treatments to them.

If responses less than equal to 8, declare failure.

If responses > 22. declare success.

~~If~~ otherwise, continue to stage II.

Stage II: Add 46 additional patients and give treatments.

If total number of responses is greater than 22, declare success. otherwise, declare failure.

Expected sample size:

$$E(N) = n_1 \cdot P(X_1 \le r_1 \text{ or } X_1 > r) + n\, P(r_1 < X_1 \le r)$$

$$= 37 \times \left\{ \sum_{k=0}^{8} \binom{37}{k} 0.2^k\, 0.8^{37-k} + \sum_{k=23}^{37} \binom{37}{k} 0.2^k\, 0.8^{37-k} \right\}$$

$$+ 83 \times \left\{ \sum_{k=9}^{22} \binom{37}{k} 0.2^k\, 0.8^{37-k} \right\}$$

$$= 51.45.$$