ST540 HW2

Q2

(a) We first validate that $f(x) = \frac{1}{b-a} \geq 0, \forall x \in [a,b]$. It is obvious since $b > a$. Then we need to show that $\int_{[a,b]} f(x)dx = 1$:

$$\int_{[a,b]} f(x)dx = \int_{[a,b]} \frac{1}{b-a}dx = \frac{1}{b-a}x|_a^b = 1$$

(b)

$$E(X) = \int_{[a,b]} xf(x)dx = \int_{[a,b]} \frac{x}{b-a}dx = \frac{1}{b-a}\frac{1}{2}x^2|_a^b = \frac{a+b}{2}$$

$$E(X^2) = \int_{[a,b]} x^2 f(x)dx = \int_{[a,b]} \frac{x^2}{b-a}dx = \frac{1}{b-a}\frac{1}{3}x^3|_a^b = \frac{b^2+ab+a^2}{3}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{(b-a)^2}{12}$$

Q3

$Gamma(\alpha, \beta)$ is the easiest one that satisfies all of the constraints. It has positive support and the following properties:

Suppose $X \sim Gamma(\alpha, \beta)$, then

$$E(X) = \alpha\beta$$

$$Var(X) = \alpha\beta^2$$

Let $E(X) = 5$ and $Var(X) = 3$ and we have $\alpha = 25/3$ and $\beta = 3/5$.

Some other distributions with positive support also works like lognormal distribution but can be more complicated.

Q4

(a)

$$p(X_1 = i) = \sum_{j=0}^{1} p(X_1 = i, X_2 = j), \forall i = 0, 1, 2$$

Thus, the marginal distribution of X_1 is:

X_1	**0**	**1**	**2**

p	0.3	0.35	0.35

(b)

$$p(X_2 = j) = \sum_{i=0}^{2} p(X_1 = i, X_2 = j), \forall j = 0, 1$$

Thus, the marginal distribution of X_2 is:

X_2	**0**	**1**
p	0.45	0.55

(c)

$$p(X_1 = i | X_2 = j) = \frac{p(X_1 = i, X_2 = j)}{p(X_2 = j)}$$

Thus, the conditional distribution of $X_1 | X_2$ is:

$X_1 \setminus X_2$	**0**	**1**
0	1/3	3/11
1	1/3	4/11
2	1/3	4/11

(d)

$$p(X_2 = j | X_2 = i) = \frac{p(X_1 = i, X_2 = j)}{p(X_1 = i)}$$

Thus, the conditional distribution of $X_1 | X_2$ is:

$X_2 \setminus X_1$	**0**	**1**	**2**
0	1/2	3/7	3/7
1	1/2	4/7	4/7

(d) X_1 and X_2 are not independent.

$p(X_1 = 0, X_2 = 0) = 0.15 \neq p(X_1 = 0)p(X_2 = 0) = 0.3 \times 0.45$

Q4

Method 1: Exact posterior

We first work out the kernel of the posterior of n.

$$\begin{aligned}
P(n=k|Y) &\propto P(Y|n=k)P(n=k) \\
&= \binom{k}{Y}\theta^Y(1-\theta)^{k-Y}\cdot\frac{5^k e^{-5}}{k!} \\
&\propto \frac{(1-\theta)^{k-Y}5^k}{(k-Y)!} \\
&\propto \frac{(5-5\theta)^{k-Y}}{(k-Y)!}
\end{aligned}$$

where k is an **integer** and $k \geq Y$.

Recall that the kernel of $X \sim Poisson(\lambda)$ is:

$$P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!} \propto \frac{\lambda^k}{k!}$$

where $k \geq 0$.

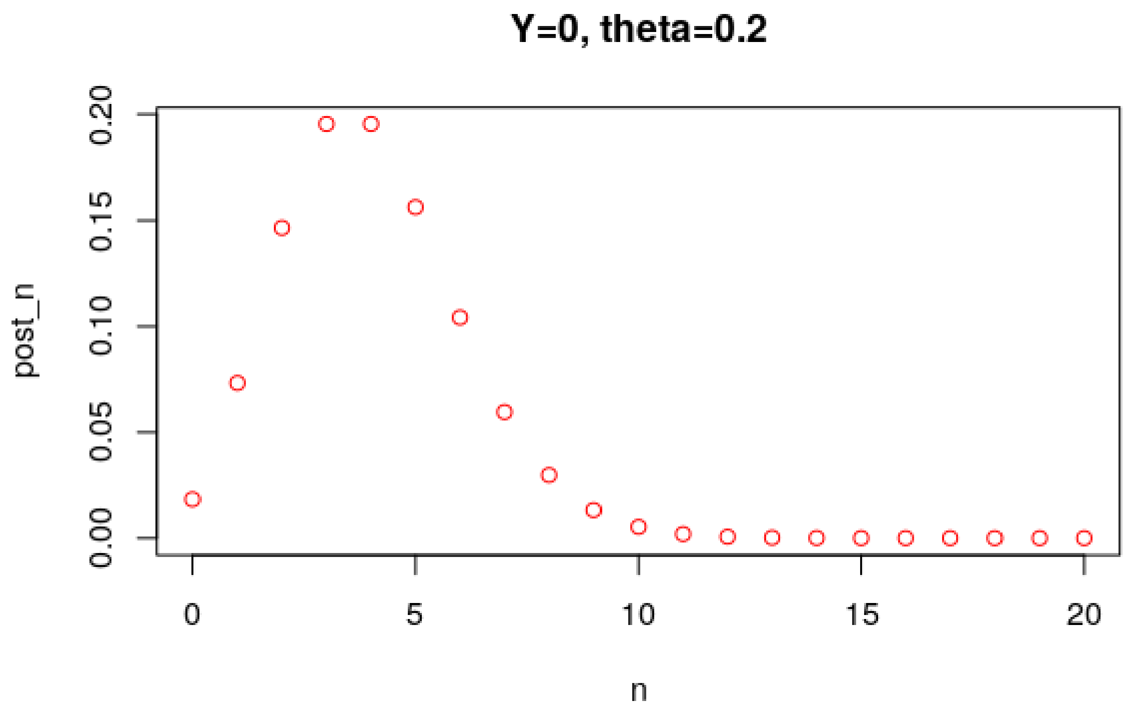
Comparing the kernels of $P(n=k|Y)$ and $Poisson(\lambda)$, we conclude that $P(n=k|Y)$ follows a translated Poisson distribution with coefficient $\lambda = 5-5\theta$. Thus, we have:

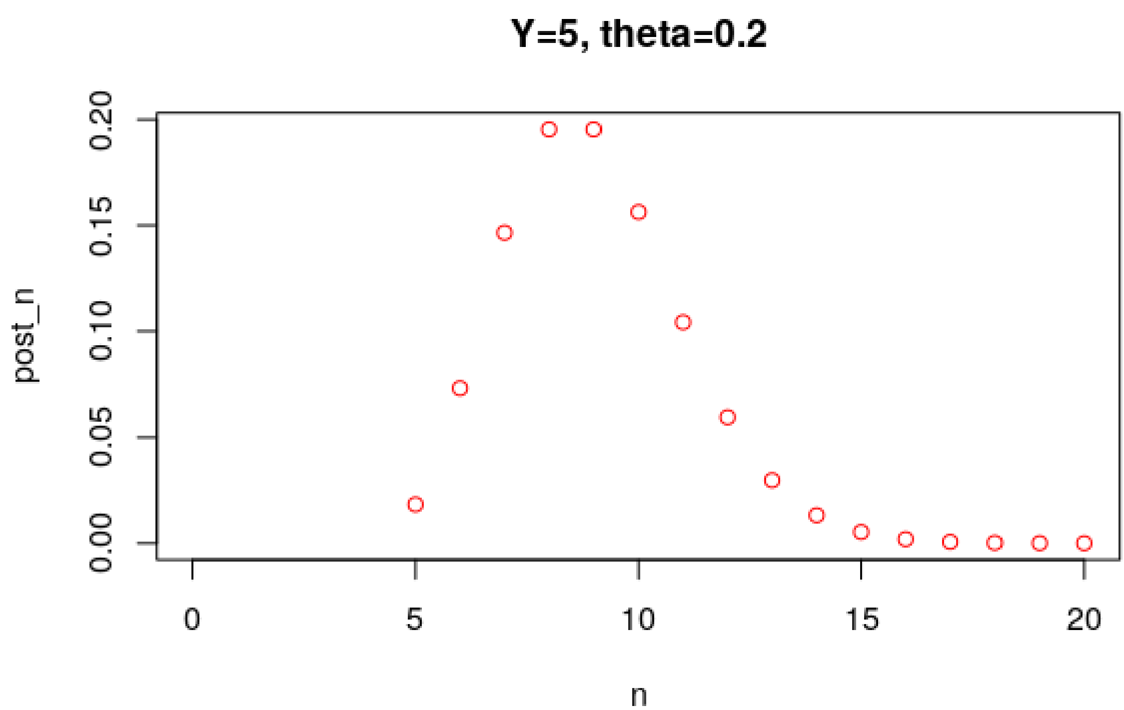
$$P(n=k|Y) = P(X=k-Y)$$

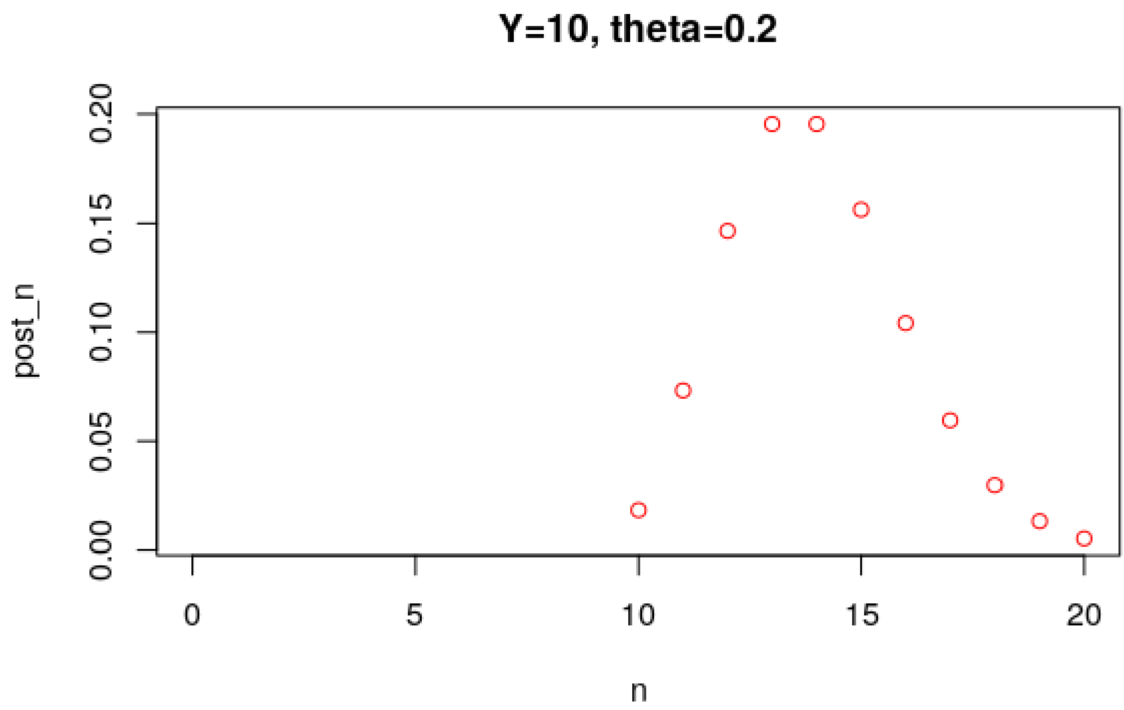
where $X \sim Poisson(5-5\theta)$.

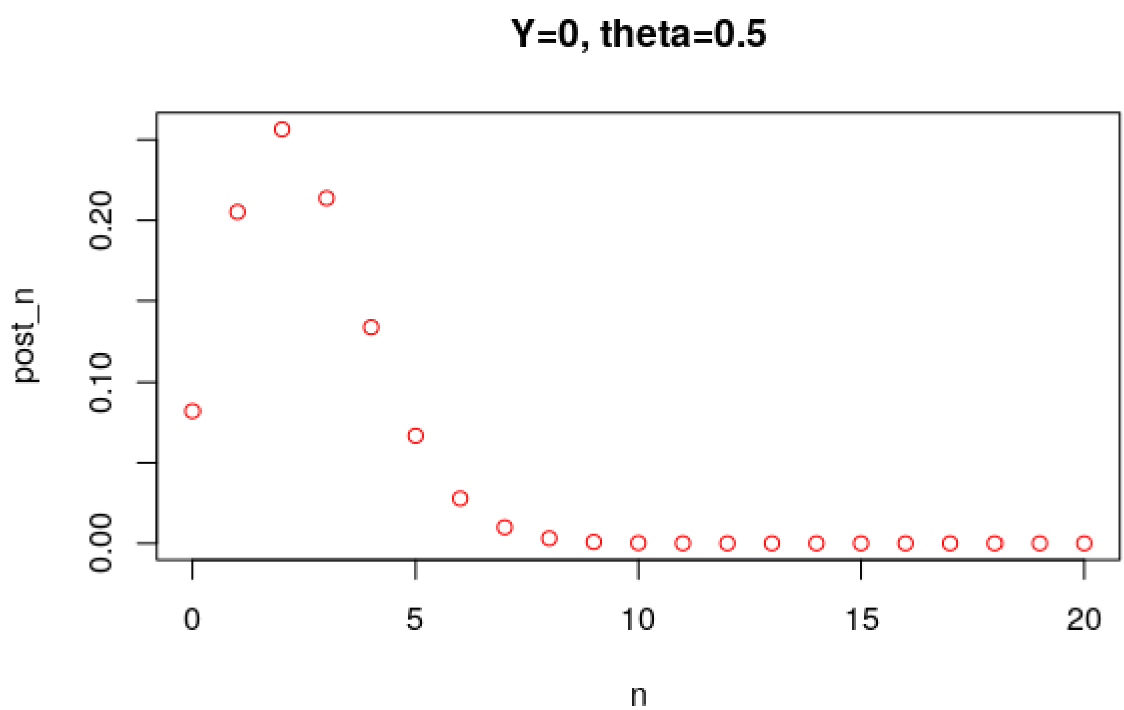
The following code is to plot the posterior of $P(n=k|Y)$ when $Y=0$ and $\theta=0.2$. To plot with other values of n and $theta$, just change the first two lines of code.

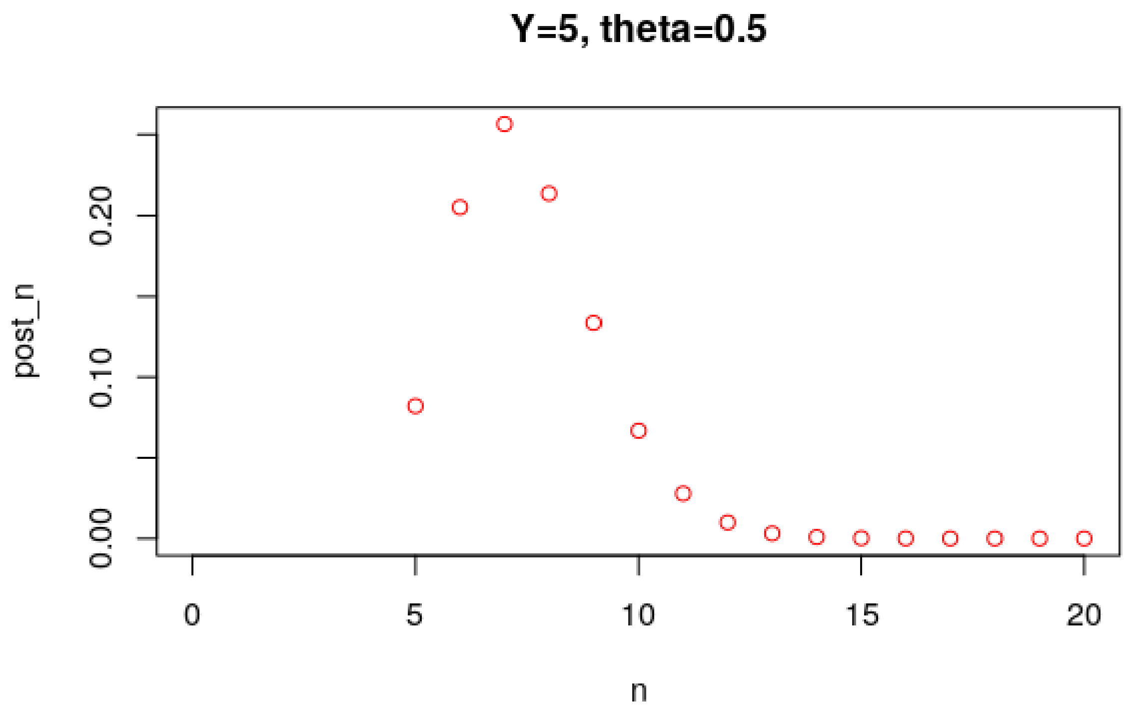
```
Y = 0
theta = 0.2
# n is an integer and n >= Y
# we will calculate P(n|Y) at n = Y, Y+1, ..., 20
n = seq(Y, 20, 1)
post_n = dpois(n-Y, 5-5*theta)
# plot
plot(n, post_n, col = 'red', xlim = range(0:20), main = 'Y=0, theta=0.2')
```

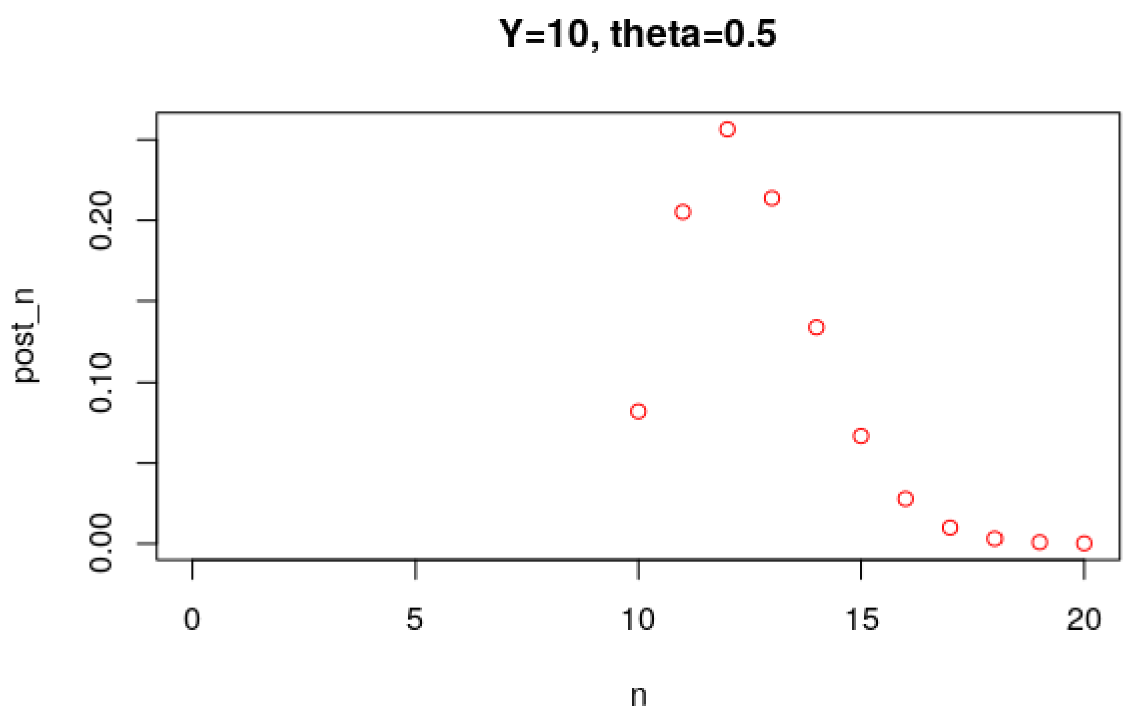

Y=0, theta=0.2
post_n
0.00
0.05
0.10
0.15
0.20
0
5
10
15
20
n


Y=5, theta=0.2
post_n
0.00
0.05
0.10
0.15
0.20
0
5
10
15
20
n


Y=10, theta=0.2
post_n
0.00
0.05
0.10
0.15
0.20
0
5
10
15
20
n


Y=0, theta=0.5
post_n
0.00
0.10
0.20
0
5
10
15
20
n





comment on θ and Y

1. Given θ, the posterior just moves to the right as Y increases with the shape stays the same the distribution to the right.
2. Given Y, the posterior gets more concentrated around the mode as θ increases. distribution.

Method 2: Approximate posterior

Instead of working on the kernel of the posterior, we can also approximate the posterior:

$$P(n=k|Y) = \frac{P(Y|n=k)P(n=k)}{\sum_{k_0=Y}^{\infty} P(Y|n=k_0)P(n=k_0)}$$

Note that $n \sim Poisson(5)$ so that $P(n=k_0)$ becomes very small if k_0 is large. So we have:

$$\sum_{k_0=Y}^{\infty} P(Y|n=k_0)P(n=k_0) \approx \sum_{k_0=Y}^{K} P(Y|n=k_0)P(n=k_0)$$

where K is a large integer (e.g. 100).

```
Y = 0
theta = 0.2

# approximate the denominator. Take K = 100
n_grid = seq(Y, 100, 1)
like  = dbinom(Y, n_grid, theta)
prior = dpois(n_grid, 5)
denominator  = sum(like*prior)

# posterior of n at n=Y,Y+1,...,100
post = like*prior/denominator
plot(n_grid, post, col = 'red', xlim = range(0:20), main = 'Y=0, theta=0.2'
```