

Math Problem Set 1

Open Source Macroeconomics Laboratory Boot Camp

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1. Exercises from the book:

3.6 Since $A \in \mathcal{F}$, which is the power set of Ω , then $A \subset \Omega = \cup_{i \in I} B_i$. Therefore, $A = \cup_{i \in I} A \cap B_i$. By the definition of a probability space, P is countably additive on \mathcal{F} so $P(\cup_{i \in I} A \cap B_i) = P(A) = \sum_{i \in I} P(A \cap B_i)$.

3.8 First, we will prove that if E_1 and E_2 are independent events, then E_1^c and E_2^c are also independent events. Note that:

$$\begin{aligned} P(E_1^c \cap E_2^c) &= P((E_1 \cup E_2)^c) \\ &= 1 - (P(E_1) + P(E_2) - P(E_1 \cap E_2)) \\ &= 1 - P(E_1) - P(E_2) + P(E_1) \times P(E_2) \\ &= (1 - P(E_1)) \times (1 - P(E_2)) \\ &= P(E_1^c) \times P(E_2^c) \end{aligned}$$

From the rule of unions and intersections, we have that $(\cup_{k=1}^n E_k)^c = \cap_{k=1}^n E_k^c$. In addition, since $\{E_k\}_{k=1}^n$ is a collection of independent events, so by extending the previous proof inductively we have that $\{E_k^c\}_{k=1}^n$ is also a collection of independent events. Then we have that:

$$\begin{aligned} P(\cup_{k=1}^n E_k) &= 1 - P((\cup_{k=1}^n E_k)^c) \\ &= 1 - P(\cap_{k=1}^n E_k^c) \\ &= 1 - \prod_{k=1}^n P(E_k^c) \\ &= 1 - \prod_{k=1}^n (1 - P(E_k)) \end{aligned}$$

3.11 From Bayes' Rule, we have that

$$\begin{aligned} P(s = \text{crime} | s \text{ tested } +) &= \frac{P(s \text{ tested } + | s = \text{crime})P(s = \text{crime})}{P(s \text{ tested } +)} \\ &= \frac{1 \times \frac{1}{250,000,000}}{\frac{1}{3,000,000}} \\ &= \frac{3}{250} \end{aligned}$$

3.12 Without loss of generality, suppose the contestant picked door A_1 and Monty opened door A_2 , which contains a goat. We want to show that the contestant is better off picking door A_3 . Since the contestant chose the first door with no prior information, then $P(A_1) = 1/3$. Therefore, $P(A_2 \cup A_3) = 2/3$. However, if A_2 contains a goat, then $P(A_3) = 0$ and we know that $P(A_2 \cap A_3)$ since the car cannot be behind both. Thus, $P(A_2 \cup A_3) = P(A_3) = 2/3$. The contestant would have double the chance of winning if they switched doors versus sticking with their original decision. In a similar situation with 10 doors, you would have a $1/10$ probability of winning if you stuck with your original decision, but you have a $9/10$ probability of winning if you switched to the remaining door.

3.16 We want to show that $E[(X - \mu)^2] = E[X^2] - \mu^2$. From the definition of variance and the fact that expectation is the weighted average and is thus additive, we have that:

$$\begin{aligned} \text{Var}[X] &= E[(X - \mu)^2] \\ &= E[(X - E[X])^2] \\ &= E[X^2 - 2XE[X] + E[X]^2] \\ &= E[X^2] - E[2XE[X]] + E[E[X]^2] \\ &= E[X^2] - 2E[X]^2 + E[X^2] \text{ since } E[X]^2 \text{ is a constant} \\ &= E[X^2] - E[X]^2 \\ &= E[X^2] - \mu^2 \end{aligned}$$

3.33 For a binomial random variable B , we have that $B = \sum_{i=1}^n B_i$, where all B_i 's are independently Bernoulli distributed random variables such that $E[B_i] = p$ and $\text{Var}(B_i) = p(1 - p)$. Thus, using the weak law of large numbers, we have that for all $\epsilon > 0$,

$$P\left(\left|\frac{\sum_{i=1}^n B_i}{n} - p\right| \geq \epsilon\right) = P\left(\left|\frac{B}{n} - p\right| \geq \epsilon\right) \leq \frac{p(1 - p)}{n\epsilon^2}$$

3.36 Let $S_n = \sum_{i=1}^{6242} X_i$, where X_i is the probability that student i will enroll at the school, so X_i 's are independently Bernoulli distributed random variables. Thus, $E[X_i] = \mu = 0.801$ and $\text{Var}[X_i] = 0.801 \times 0.199 = 0.1594$.

We want to estimate $P(S_n \geq 5500) = 1 - P(S_n \leq 5500)$. Using the Central Limit Theorem, we have the estimation that:

$$\begin{aligned} P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq y\right) &= P(S_n \leq \sigma\sqrt{n}y + n\mu) \\ &\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{x^2}{2}} dx \end{aligned}$$

We can calculate y as follows:

$$\begin{aligned} \sigma\sqrt{n}y + n\mu &= 5500 \\ y &= \frac{5500 - 6242 \times 0.801}{\sqrt{0.1594 \times 6242}} \\ &= 15.8563 \end{aligned}$$

Computing the integral using wolfram alpha, we thus have that:

$$P(S_n \geq 5500) = 1 - P(S_n \leq 5500) = 1 - 1 = 0$$

2. (a) Suppose we toss two independent coins. Let A be the event that coin 1 is heads and B be the event that coin 2 is heads. Let C be the event if there is exactly one head amongst the two coin tosses. The four possible outcomes are: $\Omega = \{HH, TT, HT, TH\}$, thus we have that $P(B) = P(C) = P(A) = \frac{1}{2}$. Note the following observations:

$$\begin{aligned} P(A \cap B) &= P(\{HH\}) = \frac{1}{4} = P(A)P(B) \\ P(A \cap C) &= P(\{HT\}) = \frac{1}{4} = P(A)P(C) \\ P(B \cap C) &= P(\{TH\}) = \frac{1}{4} = P(B)P(C) \\ P(A \cap B \cap C) &= P(\{\}) = 0 \neq P(A)P(B)P(C) \end{aligned}$$

- (b) Suppose we have a fair 8-sided die. Let $B = \{1, 2, 3, 4\}$, $C = \{1, 5, 6, 7\}$ and $A = \{1, 4, 7, 8\}$ denote events that consist of possible die rolls. Then $P(B) = P(C) = P(A) = \frac{1}{2}$. Note the following observations:

$$\begin{aligned} P(A \cap B) &= P(\{1, 4\}) = \frac{1}{4} = P(A)P(B) \\ P(A \cap C) &= P(\{1, 7\}) = \frac{1}{4} = P(A)P(C) \\ P(A \cap B \cap C) &= P(\{1\}) = \frac{1}{8} = P(A)P(B)P(C) \\ P(B \cap C) &= P(\{1\}) = \frac{1}{8} \neq P(B)P(C) \end{aligned}$$

3. To prove that Benson's Law is a well-defined discrete probability distribution, we must show that the probability of the entire space of outcomes is 1:

$$\begin{aligned}
 P(\Omega) &= \sum_{d=1}^9 \log_{10}\left(1 + \frac{1}{d}\right) \\
 &= \log_{10}\left(\sum_{d=1}^9 \frac{d+1}{d}\right) \text{ this is a telescoping sum} \\
 &= \log_{10} 10 \\
 &= 1
 \end{aligned}$$

4. (a) The probability that the person wins $\$2^n$ is if they flip $n - 1$ heads in a row then tails on the n th flip. The probability of that happening is $1/2^n$. Thus, for any given winning $x_n = \$2^n$, $p_n x_n = 1$. Since $n = \mathbb{N}$, then $E[X] = \sum_{n=1}^{\infty} p_n x_n = \sum_{n=1}^{\infty} 1 = +\infty$.
- (b) Since the player has log utility, for a given winning $x_n = \$2^n$, the utility $u_n = n \log 2$. The probability has not changed, thus we have that $E[\log X] = \sum_{n=1}^{\infty} p_n u_n = \log 2 \sum_{n=1}^{\infty} \frac{n}{2^n} = 2 \log 2$, which was found using Wolfram Alpha.
5. If the U.S. investor invests a dollar in the Swiss currency instead of U.S. currency this year, there is $\frac{1}{2}$ chance that next year he will receive \$1.25 USD. and $\frac{1}{2}$ chance that next year he will receive 0.80 USD. Thus, the expected value is $\frac{1.25+0.8}{2} = \$1.025$ USD. Thus, the U.S. investor should invest in Swiss currency. If the Swiss investor invests a dollar in U.S. currency instead of Swiss currency, there is $\frac{1}{2}$ chance that next year he will receive \$1.25 CHF and $\frac{1}{2}$ chance that next year he will receive 0.80 CHF. Thus, the expected value is $\frac{1.25+0.8}{2} = \$1.025$ CHF. Similarly, the Swiss investor should invest in U.S. currency.
6. Note that for all the following questions, the random variable are all valid since they are continuous (and thus map $[0, 1]$ to \mathbb{R}).

Let the probability density function of X be the following:

$$f_X(x) = \begin{cases} \frac{2}{x^3} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that $\int_{-\infty}^{\infty} f_X(x) dx = \int_1^{\infty} \frac{2}{x^3} dx = 1$, thus this is a valid continuous

random variable. Note that:

$$\begin{aligned}
 E[X] &= \int_1^{\infty} x \frac{2}{x^3} dx \\
 &= \left[-\frac{2}{x}\right]_1^{\infty} \\
 &= 2 \\
 E[X^2] &= \int_1^{\infty} x^2 \frac{2}{x^3} dx \\
 &= [2 \log x]_1^{\infty} \\
 &= \infty
 \end{aligned}$$

(b) Let $X \sim U(0, 1)$ and $Y \sim \exp(1.99)$. Thus, $E[X] = \frac{1}{2} < E[Y] = \frac{1}{1.99}$. Now we must show that $P(X > Y) > 1/2$:

$$\begin{aligned}
 P(X > Y) &= \int_{-\infty}^{\infty} P(X > Y | Y = y) f_Y(y) dy \\
 &= \int_{-\infty}^{\infty} P(X > y) f_Y(y) dy \\
 &= \int_{-\infty}^{\infty} (1 - F_X(y)) f_Y(y) dy \\
 &= 1 - \int_{-\infty}^{\infty} F_X(y) f_Y(y) dy \\
 &= 1 - \left(\int_0^1 y 1.99 e^{-1.99y} dy + \int_1^{\infty} 1.99 e^{-1.99y} dy \right) \\
 &= 0.566
 \end{aligned}$$

(c) Let $X \sim U(-1, 1)$, $Y \sim U(-2, 2)$, and $Z \sim U(-3, 3)$. From part b), we have that $P(X > Y) = 1 - \int_{-\infty}^{\infty} F_X(y) f_Y(y) dy$. Using the pdfs and cdfs of the uniform distribution, we have that:

$$\begin{aligned}
 P(X > Y) &= 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2} \\
 P(X > Z) &= 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2} \\
 P(Z > Y) &= 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}
 \end{aligned}$$

Thus, $P(X > Y)P(X > Z)P(Z > Y) = \frac{1}{8} > 0$ and $E[X] = E[Y] = E[Z] = 0$.

7. (a) If $Z = 1$, then $Y = X \sim N(0, 1)$. If $Z = -1$, then $Y = -X \sim N(0, 1)$ due

to the symmetry of $N(0, 1)$ about 0. Then for a given y , we have that:

$$\begin{aligned} P(Y < y) &= P(Y < y|Z = 1)P(Z = 1) + P(Y < y|Z = -1)P(Z = -1) \\ &= \frac{P(X < y) + P(-X < y)}{2} \\ &= \frac{P(X < y) + P(X > -y)}{2} \\ &= P(X < y) \text{ due to symmetry of } N(0, 1) \end{aligned}$$

Since this holds for all y , then $Y \sim N(0, 1)$.

- (b) For any given value $X = x$, we have that $Y \in -x, x$, thus $|X| = |Y|$ with certainty. $P(|X| = |Y|) = 1$ then follows.
- (c) Note that $f_{XY}(1, 2) = 0$ since $Y \in -1, 1$. However, $f_X(1)f_Y(2) = f_X(1)f_X(2) \neq 0$. Thus, X and Y are not independent.
- (d) We have that $Cov[X, Y] = E[XY] - E[X]E[Y] = E[XY] = E[X^2Z] = \frac{E[X^2] - E[X^2]}{2} = 0$.
- (e) As seen from the previous parts, X, Y are normally distributed random variables with covariance zero, but are dependent. Therefore, the statement is false since there is a counterexample.

8. First, let us state the cumulative distribution function of $X_i \sim U[0, 1]$:

$$F_{X_i}(x) = \begin{cases} 1 & x > 1 \\ x & x \in [0, 1] \\ 0 & x < 0 \end{cases}$$

First let us deal with the random variable $m = \min \{X_1, \dots, X_n\}$:

$$\begin{aligned} F_m(x) &= P(m \leq x) = 1 - P(X_i > x)^n \\ &= 1 - (1 - P(X_i \leq x))^n = 1 - (1 - F_{X_i}(x))^n \end{aligned}$$

$$\therefore F_m(x) = \begin{cases} 1 & x > 1 \\ 1 - (1 - x)^n & x \in [0, 1] \\ 0 & x < 0 \end{cases}$$

$$\therefore f_m(x) = F'_m(x) = \begin{cases} n(1 - x)^{n-1} & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore E[m] = \int_0^1 xn(1 - x)^{n-1}dx = \frac{1}{n+1}$$

Now let us deal with the random variable $M = \max \{X_1, \dots, X_n\}$:

$$F_M(x) = P(M \leq x) = P(X_i \leq x)^n = F_{X_i}(x)^n$$

$$\therefore F_M(x) = \begin{cases} 1 & x > 1 \\ x^n & x \in [0, 1] \\ 0 & x < 0 \end{cases}$$

$$\therefore f_M(x) = F'_M(x) = \begin{cases} nx^{n-1} & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore E[M] = \int_0^1 xnx^{n-1}dx = \frac{n}{n+1}$$

9. (a) Letting $S_n = \sum_{i=1}^{1000} X_i$, where X_i is the probability of a good state in period i . Since the X_i 's are independent Bernoulli distributed random variables, $E[X_i] = \mu = 0.5$ and $\sigma^2 = 0.25$. We want to estimate $P(S_n \leq 510) - P(S_n \leq 490)$. Thus, using the Central limit theorem, we have that:

$$\begin{aligned} P(S_n \leq 510) &\approx \Phi\left(\frac{510 - n\mu}{\sigma\sqrt{n}}\right) \\ &= \Phi\left(\frac{2}{\sqrt{10}}\right) \\ P(S_n \leq 490) &\approx \Phi\left(\frac{490 - n\mu}{\sigma\sqrt{n}}\right) \\ &= \Phi\left(-\frac{2}{\sqrt{10}}\right) \end{aligned}$$

$$\therefore P(S_n \leq 510) - P(S_n \leq 490) \approx 0.47$$

Therefore, there is a 47% chance that the number of good states over 1000 periods differs from 500 by at most 2%.

- (b) We want to find n such that $P(|\frac{S_n}{n} - \mu| \geq 0.005) \leq 0.01$. From the weak law of large numbers, we have that $\frac{\sigma^2}{n\epsilon^2} = \frac{0.25}{0.005n} = 0.01$. Therefore, $n = 5000$.
10. Since $E[X]$ is finite, we can use Jensen's inequality. Since $f(x) = e^{\theta x}$ is a differentiable convex function, according to Jensen's inequality, $E[e^{\theta X}] = 1 \geq e^{\theta E[X]}$. Since $E[X] < 0$, then $\theta > 0$ or else $1 \leq e^{\theta E[X]}$, which would contradict Jensen's inequality.