Math Problem Set 2
Open Source Macroeconomics Laboratory Boot Camp

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3.1 (i)

$$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{1}{4}(\langle x+y, x+y \rangle - \langle x-y, x-y \rangle)$$

$$= \frac{1}{4}(\langle x+y, x \rangle + \langle x+y, y \rangle - (\langle x-y, x \rangle - \langle x-y, y \rangle))$$

$$= \frac{1}{4}(\langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle - (\langle x, x \rangle - \langle y, x \rangle - \langle x, y \rangle + \langle y, y \rangle))$$

$$= \frac{1}{4}(4\langle x, y \rangle)$$

$$= \langle x, y \rangle$$

(ii)

$$\frac{1}{2}(\|x+y\|^2 + \|x-y\|^2) = \frac{1}{4}(\langle x+y, x+y \rangle - \langle x-y, x-y \rangle)$$

$$= \frac{1}{2}(\langle x+y, x \rangle + \langle x+y, y \rangle + \langle x-y, x \rangle - \langle x-y, y \rangle)$$

$$= \frac{1}{2}(\langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle + \langle x, x \rangle - \langle y, x \rangle - \langle x, y \rangle + \langle y, y \rangle)$$

$$= \frac{1}{2}(2\langle x, x \rangle + 2\langle y, y \rangle)$$

$$= \langle x, x \rangle + \langle y, y \rangle$$

$$= \|x\|^2 + \|y\|^2$$

3.2

$$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2)$$

$$= \frac{1}{4}(\langle x+y, x+y \rangle - \langle x-y, x-y \rangle + i\langle x-iy, x-iy \rangle - i\langle x+iy, x+iy \rangle)$$

$$= \frac{1}{4}(2\langle x, y \rangle + 2\langle y, x \rangle - i(2\langle x, iy \rangle + 2\langle iy, x \rangle)))$$

$$= \frac{1}{4}(2\langle x, y \rangle + 2\langle y, x \rangle - i(2i\langle x, y \rangle - 2i\langle y, x \rangle)))$$

$$= \frac{1}{4}(2\langle x, y \rangle + 2\langle y, x \rangle + 2\langle x, y \rangle - 2\langle y, x \rangle)$$

$$= \frac{1}{4}(4\langle x, y \rangle)$$

$$= \langle x, y \rangle$$

3.3 (i)

$$\cos\theta = \frac{\langle x, x^5 \rangle}{\|x\|\|x^5\|} = \frac{\int_0^1 x^6 dx}{\sqrt{\int_0^1 x^2 dx \int_0^1 x^{10} dx}} = \frac{\frac{1}{7}}{\sqrt{\frac{1}{3}\frac{1}{11}}}$$

$$\therefore \theta = \arccos\frac{\sqrt{33}}{7} = 0.608$$

(ii)

$$\cos\theta = \frac{\langle x^2, x^4 \rangle}{\|x^2\|\|x^4\|} = \frac{\int_0^1 x^6 dx}{\sqrt{\int_0^1 x^4 dx \int_0^1 x^8 dx}} = \frac{\frac{1}{7}}{\sqrt{\frac{1}{5}\frac{1}{9}}}$$

$$\therefore \theta = \arccos\frac{\sqrt{45}}{7} = 0.29$$

3.8 (i) We must prove that $S = \{\cos(t), \sin(t), \cos(2t), \sin(2t)\}$ is an orthnormal set. First let us prove

that all pairs of the basis are orthogonal:

$$\langle \cos(t), \sin(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t) \cos(t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t) d\sin(t)$$

$$= \frac{1}{2\pi} \sin^2(t)]_{-\pi}^{\pi} = 0$$

$$\langle \cos(t), \cos(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(2t) \cos(t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos(2t+t) + \cos(2t-t)}{2} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(3t) + \cos(t) dt = \frac{1}{2\pi} [\frac{1}{3} \sin(3t) + \sin(t)]_{-\pi}^{\pi} = 0$$

$$\langle \cos(t), \sin(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(2t) \cos(t) dt = \frac{2}{\pi} \int_{-\pi}^{\pi} \sin(t) \cos^2(t) dt$$

$$= \frac{-2}{\pi} \int_{-\pi}^{\pi} \cos^2(t) d\cos(t) = \frac{-2}{3\pi} [\cos^3(t)]_{-\pi}^{\pi} = 0$$

$$\langle \cos(2t), \sin(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t) \cos(2t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin(2t+t) - \sin(2t-t)}{2} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(3t) - \sin(t) dt = \frac{1}{2\pi} [-\frac{1}{3} \cos(3t) + \cos(t)]_{-\pi}^{\pi} = 0$$

$$\langle \sin(2t), \sin(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(t) \sin(2t) dt = \frac{2}{\pi} \int_{-\pi}^{\pi} \sin^2(t) \cos(t) dt$$

$$= \frac{2}{\pi} \int_{-\pi}^{\pi} \sin^2(t) d\sin(t) = \frac{2}{3\pi} [\sin^3(t)]_{-\pi}^{\pi} = 0$$

$$\langle \sin(2t), \cos(2t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(2t) \sin(2t) dt = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(2t) d\cos(2t)$$

$$= -\frac{1}{4\pi} [\cos^2(2t)]_{-\pi}^{\pi} = 0$$

Now we will prove that the norm of each basis element is equal to 1:

$$\langle\cos(t),\cos(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1+cos(2t)}{2}dt$$

$$= \frac{1}{2\pi}[t + \frac{1}{2}\sin(2t)]_{-\pi}^{\pi} = \frac{1}{2\pi}2\pi = 1$$

$$\langle\sin(t),\sin(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1-cos(2t)}{2}dt$$

$$= \frac{1}{2\pi}[t - \frac{1}{2}\sin(2t)]_{-\pi}^{\pi} = \frac{1}{2\pi}2\pi = 1$$

$$\langle\cos(2t),\cos(2t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(2t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1+cos(4t)}{2}dt$$

$$= \frac{1}{2\pi}[t + \frac{1}{4}\sin(4t)]_{-\pi}^{\pi} = \frac{1}{2\pi}2\pi = 1$$

$$\langle\sin(2t),\sin(2t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2(2t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1-cos(4t)}{2}dt$$

$$= \frac{1}{2\pi}[t - \frac{1}{4}\sin(4t)]_{-\pi}^{\pi} = \frac{1}{2\pi}2\pi = 1$$

(ii)

$$\|t\| = \sqrt{\langle t,t\rangle} = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi}t^2dt} = \sqrt{\frac{2\pi^2}{3}} = \sqrt{\frac{2}{3}}\pi$$

(iii)

$$\text{proj}_X(\cos(3t)) = \langle\cos(3t),\cos(t)\rangle\cos(t) + \langle\cos(3t),\sin(t)\rangle\sin(t)$$
$$+ \langle\cos(3t),\cos(2t)\rangle\cos(2t) + \langle\cos(3t),\sin(2t)\rangle\sin(2t)$$
$$= \frac{1}{\pi}(\cos(t)\int_{-\pi}^{\pi}\cos(3t)\cos(t)dt + \sin(t)\int_{-\pi}^{\pi}\cos(3t)\sin(t)dt$$
$$+ \cos(2t)\int_{-\pi}^{\pi}\cos(3t)\cos(2t)dt + \sin(2t)\int_{-\pi}^{\pi}\cos(3t)\sin(2t)dt)$$
$$= 0$$

(iv)

$$\text{proj}_X(t) = \langle t, \cos(t) \rangle \cos(t) + \langle t, \sin(t) \rangle \sin(t)$$
$$+ \langle t, \cos(2t) \rangle \cos(2t) + \langle t, \sin(2t) \rangle \sin(2t)$$
$$= \frac{1}{\pi} \left(\cos(t) \int_{-\pi}^{\pi} t \cos(t) dt + \sin(t) \int_{-\pi}^{\pi} t \sin(t) dt \right.$$
$$\left. + \cos(2t) \int_{-\pi}^{\pi} t \cos(2t) dt + \sin(2t) \int_{-\pi}^{\pi} t \sin(2t) dt \right)$$
$$= \frac{1}{\pi} (2\pi \sin(t) - \pi \sin(2t))$$
$$= 2\sin(t) - \sin(2t)$$

3.9 Let $L_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation transformation around the origina counterclockwise by angle θ:

$$L_\theta(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$$

Now let us calculate the inner product after the transformation for vectors $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$:

$$\langle L_\theta \mathbf{u}, L_\theta \mathbf{v} \rangle = (x_1 \cos\theta - y_1 \sin\theta)(x_2 \cos\theta - y_2 \sin\theta) + (x_1 \sin\theta + y_1 \cos\theta)(x_2 \sin\theta + y_2 \cos\theta)$$
$$= x_1 x_2 (\cos^2\theta + \sin^2\theta) + y_1 y_2 (\sin^2\theta + \cos^2\theta)$$
$$- (x_1 y_2 + x_2 y_1) \cos\theta \sin\theta + (x_1 y_2 + x_2 y_1) \cos\theta \sin\theta$$
$$= x_1 x_2 + y_1 y_2$$
$$= \langle \mathbf{u}, \mathbf{v} \rangle$$

Therefore, the rotation transformation is an orthonormal transformation.

3.10 (i) $\Longrightarrow$

$$\langle Qx, Qy \rangle = (Qx)^H (Qy) = (x^H Q^H) Qy = x^H (Q^H Qy) = \langle x, Q^H Qy \rangle$$

Since Q is an orthonormal transformation, we know that $\langle x, Q^H Qy \rangle = \langle x, y \rangle = \langle x, Iy \rangle$. Therefore, $Q^H Q = I$.

$\Longleftarrow$

We know that $Q^H Q = I$, thus:

$$\langle Qx, Qy \rangle = (Qx)^H (Qy) = (x^H Q^H) Qy = x^H (Q^H Q) y = x^H y = \langle x, y \rangle$$

(ii) By the orthonormality of matrix Q, we have that

$$\|Qx\|^2 = \langle Qx, Qx \rangle = \langle x, x \rangle = \|x\|^2$$

Since the norm is nonnegative, then $\|Qx\| = \|x\|$.

(iii) Since Q is orthonormal, $QQ^H = I$ so $Q^H = Q^{-1}$. Note that $I = QQ^H = (Q^H)^H Q^H = (Q^{-1})^H Q^{-1}$. By (i), Q^{-1} is also orthnormal.

(iv) Suppose we have the orthonormal matrix $Q = [a_1, ..., a_n]$ where a_i's are column vectors. Then the element in the ith row and jth column of matrix $Q^H Q = I$ is given by the entry $a_i^H a_j = \delta_{ij}$. Therefore, $a_i^H a_j = \langle a_i, a_j \rangle = 0$ for all $i \neq j$ and 0 otherwise, which makes $\{a_i\}$ a collection of orthnormal vectors.

(v) If Q is orthogonal, then $\det^2(Q) = \det(Q)\det(Q^H) = \det(QQ^H) = \det(I) = 1$. Therefore, $\sqrt{\det^2(Q)} = |\det(Q)| = 1$. However, the converse is not true. Take the following matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Note that $\det(A) = 2 - 1 = 1$ but the columns are not orthonormal.

(vi)
$$(Q_1 Q_2)(Q_1 Q_2)^H = (Q_1 Q_2)(Q_2^H Q_1^H) = Q_1(Q_2 Q_2^H)Q_1^H) = Q_1 Q_1^H = I$$

According to (i), $Q_1 Q_2$ is orthnormal.

3.11 Suppose we have the vectors $\{v_1, ..., v_n\}$ where vector v_n is linearly dependent on the vectors $\{v_1, ..., v_{n-1}\}$, which are linearly independent. Then the process would output the 0 vector for vector q_n since $p_{n-1} = v_k$ as the projection of the vector onto the space is itself since it is linearly dependent with the basis of the space.

3.16 (i) For any diagnal matrix D and QR decomposition, we have that $QR = QIR = QDD^{-1}R = (QD)(D^{-1}R)$. Note that all diagonal matrices (and its inverse) are both orthonormal $D = D^{-1} = D^H$ and upper triangular. Since orthonormal and triangular matrices are closed under multiplication, then $Q'R'$, where $Q' = QD$ and $R' = D^{-1}R$, is another QR decomposition of the same matrix.

(ii) Suppose there exist two QR decompositions of A: $Q_1 R_1 = A = Q_2 R_2$. Then we have that $B = Q_2^H Q^1 = R_2 R_1^{-1}$. Since orthnormal and upper triangular matrices are closed under multiplication and inverses, then B is an orthonormal, upper triangular matrix. Then B must be a diagonal matrix since $B_H = B_{-1}$ which would not hold if there were nonzero, nondiagonal entries as that would produce a lower triangular matrix which violate the closing of upper triangular matrices under the inverse. In addition, since the columns of B must be orthonormal, then all the diagonal entries are ± 1. But we know that R_1, R_2 have positive diagonal entries so $B = I$. Thus, $I = Q_2^H Q^1 = R_2 R_1^{-1}$ so we have that $Q_2 = Q_1$ and $R_1 = R_2$.

3.17
$$A^H A\mathbf{x} = A^H \mathbf{b}$$
$$(\widehat{Q}\widehat{R})^H(\widehat{Q}\widehat{R})\mathbf{x} = (\widehat{Q}\widehat{R})^H \mathbf{b}$$
$$\widehat{R}^H(\widehat{Q}^H\widehat{Q})\widehat{R}\mathbf{x} = \widehat{R}^H\widehat{Q}^H \mathbf{b}$$
$$\widehat{R}^H \widehat{R}\mathbf{x} = \widehat{R}^H\widehat{Q}^H \mathbf{b}$$
$$\widehat{R}\mathbf{x} = \widehat{Q}^H \mathbf{b}$$

3.23 Note that using the triangle inequality property of the norm, we have that:

$$\|y\| = \|x + (y - x)\| \le \|x\| + \|y - x\|$$
$$\therefore \|y\| - \|x\| \le \|y - x\| = \| - 1\|\|x - y\| = \|x - y\|$$
$$\|x\| = \|y + (x - y)\| \le \|y\| + \|x - y\|$$
$$\therefore \|x\| - \|y\| \le \|x - y\|$$

$$\therefore |\|x\| - \|y\|| \le \|x - y\|$$

3.24 We must prove that each of the following satisfies positivity (and equality), scale preservation, and triangle inequality:

(i) 1. Since $|f(t)|$ is a nonnegative function, its integral is also nonnegative. If $\|f\|_{L^1} = \int_a^b |f(t)|dt = 0$, then $f(t) = 0$ on $[a, b]$ since f is continuous on $[a, b]$.

   2. $\|\alpha f\|_{L^1} = \int_a^b |\alpha f(t)|dt = |\alpha| \int_a^b |f(t)|dt = |\alpha|\|f\|_{L^1}$

   3.

$$\|f + g\|_{L^1} = \int_a^b |f(t) + g(t)|dt \le \int_a^b |f(t)| + |g(t)|dt$$
$$= \int_a^b |f(t)|dt + \int_a^b |g(t)|dt = \|f\|_{L^1} + \|g\|_{L^1}$$

(ii) 1. Since $|f(t)|^2$ is a nonnegative function, its integral and the square root of it are also nonnegative. If $\|f\|_{L^2} = (\int_a^b |f(t)|^2 dt)^{\frac{1}{2}} = 0$, then $f(t) = 0$ on $[a, b]$ since f is continuous on $[a, b]$.

   2. $\|\alpha f\|_{L^2} = (\int_a^b |\alpha f(t)|^2 dt)^{\frac{1}{2}} = (|\alpha|^2 \int_a^b |f(t)|^2 dt)^{\frac{1}{2}} = |\alpha|\|f\|_{L^2}$

   3.

$$\|f + g\|_{L^2}^2 = \int_a^b |f(t) + g(t)|^2 dt = \int_a^b |f(t)|^2 dt + \int_a^b |g(t)|^2 dt + 2\int_a^b f(t)g(t)dt$$
$$\le \int_a^b |f(t)|^2 dt + \int_a^b |g(t)|^2 dt + 2\sqrt{\int_a^b |f(t)g(t)|^2 dt} \text{ by the Schwarz inequality}$$
$$= \left(\sqrt{\int_a^b |f(t)|^2 dt} + \sqrt{\int_a^b |g(t)|^2 dt}\right)^2$$
$$\therefore \|f + g\|_{L^2} \le \sqrt{\int_a^b |f(t)|^2 dt} + \sqrt{\int_a^b |g(t)|^2 dt} = \|f\|_{L^2} + \|g\|_{L^2}$$

(iii) 1. Since $|f(t)|$ is a nonnegative function, its supremum is also nonnegative. If $\sup_{x \in [a,b]} |f(x)| = 0$, then $f(t) = 0$ on $[a, b]$ since any non-zero value of $f(t)$ on $[a, b]$ would contradict $\sup_{x \in [a,b]} |f(x)| = 0$.

   2. $\|\alpha f\|_{L^\infty} = \sup_{x \in [a,b]} |\alpha f(x)| = |\alpha| \sup_{x \in [a,b]} |f(x)| = |\alpha|\|f\|_{L^\infty}$

3.

$$\|f + g\|_{L^\infty} = \sup_{x \in [a,b]} |f(x) + g(x)| \leq \sup_{x \in [a,b]} |f(x)| + |g(x)|$$

$$\leq \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |g(x)| = \|f\|_{L^\infty} + \|g\|_{L^\infty}$$

3.26 First, we will prove that topological equivalence (denote by $\sim$) is an equivalence relation:

1. Reflexivity: We have that $0.5\|x\|_a \leq \|x\|_a \leq 2\|x\|_a$ for all $x \in X$. Thus, $\|x\|_a \sim \|x\|_a$.

2. Symmetry: Suppose $\|x\|_a \sim \|x\|_b$. Then there exists constants $0 < m \leq M$ such that $m\|x\|_a \leq \|x\|_b \leq M\|x\|_a$. We then have that $\frac{1}{M}\|x\|_b \leq \|x\|_a \leq \frac{1}{m}\|x\|_b$ where $0 < \frac{1}{M} \leq \frac{1}{m}$. Thus, $\|x\|_b \sim \|x\|_a$ as well.

3. Transitivity: Suppose $\|x\|_a \sim \|x\|_b$ and $\|x\|_b \sim \|x\|_c$. Then there exist constants $0 < m_1 \leq M_1$ and $0 < m_2 \leq M_2$ such that $m_1\|x\|_a \leq \|x\|_b \leq M_1\|x\|_a$ and $m_2\|x\|_b \leq \|x\|_c \leq M_2\|x\|_b$. Then we have that $m_1 m_2\|x\|_a \leq \|x\|_c \leq M_1 M_2\|x\|_a$ where $0 < m_1 m_2 \leq M_1 M_2$. Therefore, $\|x\|_a \sim \|x\|_c$.

Now we will show that the p-norms for $p = 1, 2, \infty$ are topologically equivalent by establishing the following inequalities:

(i) First, note that we have:

$$\left(\sum_{i=1}^n |x_i| \right)^2 = \sum_{i=1}^n |x_i|^2 + \sum_{i \neq j}^n |x_i||x_j|$$

$$\geq \sum_{i=1}^n |x_i|^2 \text{ since all terms are nonnegative}$$

$$\sum_{i=1}^n |x_i| \geq \sqrt{\sum_{i=1}^n |x_i|^2}$$

$$\therefore \|x\|_2 \leq \|x\|_1$$

From the Cauchy-Schwarz inequality, we have that $\left| \sum_{i=1}^n x_i y_i \right|^2 \leq \sum_{j=1}^n |x_j|^2 \sum_{k=1}^n |y_j|^2$. Letting $y_i = 1$ for all $1 \leq i \leq n$, we have that

$$\left(\sum_{i=1}^n 1|x_i| \right)^2 \leq \sum_{i=1}^n |x_i|^2 \sum_{i=1}^n 1 = n \sum_{i=1}^n |x_i|^2$$

$$\sum_{i=1}^n 1|x_i| \leq \sqrt{n \sum_{i=1}^n |x_i|^2}$$

$$\therefore \|x\|_1 \leq \sqrt{n}\|x\|_2$$

Therefore, $\|x\|_1 \sim \|x\|_2$ since $\|x\|_2 \leq \|x\|_1 \leq \|x\|_2$.

(ii) Note that $\|x\|_\infty = \max_i |x_i|$. Suppose $\max_i |x_i| = |x_j|$. Therefore, $|x_j|^2 \le |x_j|^2 + \sum_{i\neq j}^n |x_i|^2 \le \sum_{i=1}^n |x_j|^2 = n|x_j|^2$. Then we have that $(\max_i |x_i|)^2 \le \sum_{=1}^n |x_i|^2 \le n(\max_i |x_i|)^2$. Therefore, $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$ so $\|x\|_2 \sim \|x\|_\infty$.

Since $\sim$ is an equivalence relation, transitivity holds so $\|x\|_1 \sim \|x\|_\infty$.

3.28   (i) From question 3.26(i), we have that $\frac{1}{\|x\|_2} \le \frac{\sqrt{n}}{\|x\|_1}$ and $\frac{1}{\|x\|_1} \le \frac{1}{\|x\|_2}$ for all x. Also using the direct inequality proved in 3.26(i), we have that:

$$\frac{1}{\sqrt{n}}\|A\|_2 = \frac{1}{\sqrt{n}}\sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} \le \frac{1}{\sqrt{n}}\sup_{x\neq 0}\sqrt{n}\frac{\|Ax\|_1}{\|x\|_1}$$

$$= \|A\|_1 = \sup_{x\neq 0}\frac{\|Ax\|_1}{\|x\|_1} \le \sup_{x\neq 0}\sqrt{n}\frac{\|Ax\|_2}{\|x\|_2}$$

$$= \sqrt{n}\|A\|_2$$

$$\therefore \frac{1}{\sqrt{n}}\|A\|_2 \le \|A\|_1 \le \sqrt{n}\|A\|_2$$

(ii) Similarly, using 3.26(ii), we have that $\frac{1}{\|x\|_\infty} \le \frac{\sqrt{n}}{\|x\|_2}$ and $\frac{1}{\|x\|_2} \le \frac{1}{\|x\|_\infty}$ for all x. Also using the direct inequality proved in 3.26(ii), we have that:

$$\frac{1}{\sqrt{n}}\|A\|_\infty = \frac{1}{\sqrt{n}}\sup_{x\neq 0}\frac{\|Ax\|_\infty}{\|x\|_\infty} \le \frac{1}{\sqrt{n}}\sup_{x\neq 0}\sqrt{n}\frac{\|Ax\|_2}{\|x\|_2}$$

$$= \|A\|_2 = \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} \le \sup_{x\neq 0}\sqrt{n}\frac{\|Ax\|_\infty}{\|x\|_\infty}$$

$$= \sqrt{n}\|A\|_\infty$$

$$\therefore \frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{n}\|A\|_\infty$$

3.29 From exercise 3.10(ii), we proved that $\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{F}^n$ and orthnormal matrix Q. Therefore, $\|Q\| = \sup_{x\neq 0}\frac{\|Q\mathbf{x}\|}{\|\mathbf{x}\|} = 1$. The induced norm of $R_\mathbf{x}$ is given by: $\|R_\mathbf{x}\|_2 = \sup_{A\neq 0}\frac{\|R_\mathbf{x}A\|_2}{\|A\|_2} = \sup_{A\neq 0}\frac{\|A\mathbf{x}\|_2}{\|A\|_2}$. Since $\|A\|_2 = \sup_{\mathbf{y}\neq 0}\frac{\|A\mathbf{y}\|_2}{\|\mathbf{y}\|_2} \ge \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2}$, then $\|\mathbf{x}\|_2 \ge \frac{\|A\mathbf{x}\|_2}{\|A\|_2}$ for all matrices A so $\|\mathbf{x}\|_2 \ge \|R_\mathbf{x}\|_2$. Note that equality is possible when A is orthonormal since $\|A\mathbf{x}\|_2 = \|\mathbf{x}\|_2 = \|A\|_2\|\mathbf{x}\|_2$, thus $\frac{\|A\mathbf{x}\|_2}{\|A\|_2} = \|\mathbf{x}\|_2$. Therefore, $\|R_\mathbf{x}\|_2 = \sup_{A\neq 0}\frac{\|R_\mathbf{x}A\|_2}{\|A\|_2} = \sup_{A\neq 0}\frac{\|A\mathbf{x}\|_2}{\|A\|_2} = \|\mathbf{x}\|_2$.

3.30 We will first show that $\|\|_S$ satisfies the properties of a norm, then that it satisfies the submultiplicative property of the matrix norm.

1. Positivity is obviously satisfied since $\|\cdot\|$ is a matrix norm.

2. Scale preservation: $\|\alpha A\|_S = \|S(\alpha A)S^{-1}\| = |\alpha|\|SAS^{-1}\| = |\alpha|\|A\|_S$.

3. Triangle inequality: $\|A+B\|_S = \|S(A+B)S^{-1}\| = \|SAS^{-1} + SBS^{-1}\| \le \|A\|_S + \|B\|_S$ since matrix multiplication obeys distributive properties from both left and right.

Since $\|\cdot\|$ is a matrix norm with the submultiplicative property, we have that:

$$\|AB\|_S = \|S(AB)S^{-1}\| = \|(SAS^{-1})(SBS^{-1})\| \le \|SAS^{-1}\|\|SBS^{-1}\| = \|A\|_S\|B\|_S$$

3.37 Let $q = 180x^2 - 168x + 24$. Then for any $p = ax^2 + bx + c$, we have that

$$\langle q, p \rangle = \int_0^1 qp\,dx = \int_0^1 (180x^2 - 168x + 24)(ax^2 + bx + c)dx$$

$$= \int_0^1 180ax^4 + 180bx^3 + 180cx^2 - 168ax^3 - 168bx^2 - 168cx + 24ax^2 + 24bx + 24c$$

$$= \frac{180a}{5} + \frac{180b}{4} + \frac{180c}{3} - \frac{168a}{4} - \frac{168b}{3} - \frac{168c}{2} + \frac{24a}{3} + \frac{24b}{2} + 24c$$

$$= (36 - 42 + 8)a + (45 - 56 + 12)b + (60 - 84 + 24)c$$

$$= 2a + b = L[p]$$

3.38 Letting the basis be $[1, x, x^2]$, then the coordinates of the basis are $[1, 0, 0], [0, 1, 0], [0, 0, 1]$. Thus $p(x) = a + bx + cx^2$ can be represented as $[a, b, c]$. Then the differentiation matrix is the following:

$$D[p](x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix} = b + 2cx = p'(x)$$

The adjoint of D is the map D^* such that (using integration by parts):

$$\langle f, D^*g \rangle = \langle Df, g \rangle$$

$$\int_0^1 f(x) D^*[g](x)dx = \int_0^1 D[f](x)g(x)dx$$

$$= \left[f(x)g(x) \right]_0^1 - \int_0^1 f(x)g'(x)dx$$

Restricting to polynomials with $f(0) = f(1)$, then we have that $D^* = -D$, which gives us the following adjoint matrix:

$$D^* = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

3.39 (i)

$$\langle (S + \alpha T)^*(\mathbf{w}), \mathbf{v} \rangle_V = \langle \mathbf{w}, (S + \alpha T)(\mathbf{v}) \rangle_W$$

$$= \langle \mathbf{w}, S\mathbf{v} + \alpha T\mathbf{v} \rangle_W$$

$$= \langle \mathbf{w}, S\mathbf{v} \rangle_W + \langle \mathbf{w}, \alpha T\mathbf{v} \rangle_W$$

$$= \langle S^*\mathbf{w}, \mathbf{v} \rangle_V + \langle (T^*(\mathbf{w}), \alpha\mathbf{v} \rangle_V$$

$$= \langle S^*\mathbf{w}, \mathbf{v} \rangle_V + \langle (\bar{\alpha}T^*(\mathbf{w}), \mathbf{v} \rangle_V$$

$$= \langle (S^* + \bar{\alpha}T^*)(\mathbf{w}), \mathbf{v} \rangle_V$$

$$\therefore (S + \alpha T)^* = S^* + (\alpha T)^* = S^* + \bar{\alpha}T^*$$

(ii)
$$\langle (S^*)^*(\mathbf{w}), \mathbf{v} \rangle_V = \langle \mathbf{w}, S^* \mathbf{v} \rangle_W = \langle S(\mathbf{w}), \mathbf{v} \rangle_V$$

Therefore, $(S^*)^* = S$.

(iii)
$$\begin{aligned}
\langle (ST)^*(\mathbf{w}), \mathbf{v} \rangle_V &= \langle \mathbf{w}, (ST)(\mathbf{v}) \rangle_W \\
&= \langle \mathbf{w}, S(T\mathbf{v}) \rangle_W \\
&= \langle S^* \mathbf{w}, T\mathbf{v} \rangle_V \\
&= \langle T^* S^* \mathbf{w}, \mathbf{v} \rangle_W \\
\therefore (ST)^* &= T^* S^*
\end{aligned}$$

(iv) Since the identity matrix is its own adjoint, we have that
$$(T^*)(T^*)^{-1} = I = I^* \implies (T^*)(T^*)^{-1} = (T^{-1}T)^* = T^*(T^{-1})^* \implies (T^*)^{-1} = (T^{-1})^*$$

3.40 (i) For matrices B, C and linear operator (also a matrix) A, we have that:
$$\langle A^* B, C \rangle = \langle B, AC \rangle = \operatorname{tr}(B^H AC) = \operatorname{tr}((A^H B)^H C) = \langle A^H B, C \rangle$$

Therefore, $A^* = A^H$.

(ii)

(iii)

3.44

3.45

3.46 (i)

(ii)

(iii)

(iv)

3.47 (i)

(ii)

(iii)

3.48 (i)

(ii)

(iii)

(iv)

(v)

(vi)

3.50