# Math Problem Set 1
# Open Source Macroeconomics Laboratory Boot Camp

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1. Exercises from the book:

   3.6 Since $A \in \mathscr{F}$, which is the power set of $\Omega$, then $A \subset \Omega = \cup_{i \in I} B_i$. Therefore, $A = \cup_{i \in I} A \cap B_i$. By the definition of a probability space, $P$ is countably additive on $\mathscr{F}$ so $P(\cup_{i \in I} A \cap B_i) = P(A) = \sum_{i \in I} P(A \cap B_i)$.

   3.8 First, we will prove that if $E_1$ and $E_2$ are independent events, then $E_1^c$ and $E_2^c$ are also independent events. Note that:

$$\begin{aligned} P(E_1^c \cap E_2^c) &= P((E_1 \cup E_2)^c) \\ &= 1 - (P(E_1) + P(E_2) - P(E_1 \cap E_2)) \\ &= 1 - P(E_1) - P(E_2) + P(E_1) \times P(E_2) \\ &= (1 - P(E_1)) \times (1 - P(E_2)) \\ &= P(E_1^c) \times P(E_2^c) \end{aligned}$$

   From the rule of unions and intersections, we have that $(\cup_{k=1}^n E_k)^c = \cap_{k=1}^n E_k^c$. In addition, since $\{E_k\}_{k=1}^n$ is a collection of independent events, so from the previous proof we have that $\{E_k^c\}_{k=1}^n$ is also a collection of independent events. Then we have that:

$$\begin{aligned} P(\cup_{k=1}^n E_k) &= 1 - P((\cup_{k=1}^n E_k)^c) \\ &= 1 - P(\cap_{k=1}^n E_k^c) \\ &= 1 - \prod_{k=1}^n P(E_k^c) \\ &= 1 - \prod_{k=1}^n (1 - P(E_k)) \end{aligned}$$

3.11 From Bayes' Rule, we have that

$$
\begin{aligned}
P(s = \text{ crime}|s \text{ tested } +) &= \frac{P(s \text{ tested } + |s = \text{ crime})P(s = \text{ crime})}{P(s \text{ tested } +)} \\
&= \frac{1 \times \frac{1}{250{,}000{,}000}}{\frac{1}{3{,}000{,}000}} \\
&= \frac{3}{250}
\end{aligned}
$$

3.12 Without loss of generality, suppose the contestant picked door $A_1$ and Monty opened door $A_2$, which contains a goat. We want to show that the contestant is better off picking door $A_3$. Since the contestant chose the first door with no prior information, then $P(A_1) = 1/3$. Therefore, $P(A_2 \cup A_3) = 2/3$. However, if $A_2$ contains a goat, then $P(A_3) = 0$ and we know that $P(A_2 \cap A_3)$ since the car cannot be behind both. Thus, $P(A_2 \cup A_3) = P(A_3) = 2/3$. The contestant would have double the chance of winning if they switched doors versus sticking with their original decision.

In a similar situation with 10 doors, you would have a 1/10 probability of winning if you stuck with your original decision, but you have a 9/10 probability of winning if you switched to the remaining door.

3.16 We want to show that $E[(X-\mu)^2] = E[X^2] - \mu^2$. From the definition of variance and the fact that expectation is the weighted average and is thus additive, we have that:

$$
\begin{aligned}
Var[X] &= E[(X-\mu)^2] \\
&= E[(X - E[X])^2] \\
&= E[X^2 - 2XE[X] + E[X]^2] \\
&= E[X^2] - E[2XE[X]] + E[E[X]^2] \\
&= E[X^2] - 2E[X]^2 + E[X^2] \text{ since } E[X]^2 \text{ is a constant} \\
&= E[X^2] - E[X]^2 \\
&= E[X^2] - \mu^2
\end{aligned}
$$

3.33 For a binomial random variable $B$, we have that $B = \sum_{i=1}^{n} B_i$, where all $B_i$'s are independently Bernoulli distributed random variables such that $E[B_i] = p$ and $Var(B_i) = p(1-p)$. Thus, using the weak law of large numbers, we have that for all $\epsilon > 0$,

$$
P(|\frac{\sum_{i=1}^{n} B_i}{n} - p| \geq \epsilon) = P(|\frac{B}{n} - p| \geq \epsilon) \leq \frac{p(1-p)}{n\epsilon^2}
$$

3.36 Let $S_n = \sum_{i=1}^{6} 242X_i$, where $X_i$ is the probability that student $i$ will enroll at the school, so $X_i$'s are independently Bernoulli distributed random variables. Thus, $E[X_i] = \mu = 0.801$ and $Var[X_i] = 0.801 \times 0.199 = 0.1594$.

We want to estimate $P(S_n \geq 5500) = 1 - P(S_n \leq 5500)$. Using the Central Limit Theorem, we have the estimation that:

$$
\begin{aligned}
P(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq y) &= P(S_n \leq \sigma\sqrt{n}y + n\mu) \\
&\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{\frac{-x^2}{2}} dx
\end{aligned}
$$

We can calculate $y$ as follows:

$$
\begin{aligned}
\sigma\sqrt{n}y + n\mu &= 5500 \\
y &= \frac{5500 - 6242 \times 0.801}{\sqrt{0.1594 \times 6242}} \\
&= 15.8563
\end{aligned}
$$

Computing the integral using wolfram alpha, we thus have that:

$$
P(S_n \geq 5500) = 1 - P(S_n \leq 5500) = 1 - 1 = 0
$$

2. (a)
   (b)

3. To prove that Benson's Law is a well-defined discrete probability distribution, we must show that the probability of the entire space of outcomes is 1:

$$
\begin{aligned}
P(\Omega) &= \sum_{d=1}^{9} \log_{10}(1 + \frac{1}{d}) \\
&= \log_{10}(\sum_{d=1}^{9} \frac{d+1}{d}) \text{ this is a telescoping sum} \\
&= \log_{10} 10 \\
&= 1
\end{aligned}
$$

4. (a) The probability that the person wins $\$2^n$ is if they flip $n-1$ heads in a row then tails on the $n$th flip. The probability of that happening is $1/2^n$. Thus, for any given winning $x_n = \$2^n$, $p_n x_n = 1$. Since $n = \mathbb{N}$, then $E[X] = \sum_{n=1}^{\infty} p_n x_n = \sum_{n=1}^{\infty} 1 = +\infty$.
   (b) Since the player has log utility, for a given winning $x_n = \$2^n$, the utility $u_n = n \log 2$. The probability has not changed, thus we have that $E[\log X] = \sum_{n=1}^{\infty} p_n u_n = \log 2 \sum_{n=1}^{\infty} \frac{n}{2^n} = 2 \log 2$, which was found using Wolfram Alpha.

5.

6. (a)

(b)

(c)

7. (a)

   (b)

   (c)

   (d)

   (e)

8.

9. (a)

   (b)

10.