Math Problem Set 4
Open Source Macroeconomics Laboratory Boot Camp

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6.1 Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n, a, b \in \mathbb{R}^n$ and $A \in M_n(\mathbb{R})$, choose $\mathbf{w} \in \mathbb{R}^n$ such that

$$\begin{aligned} \text{minimize } & -e^{-\mathbf{x}^T\mathbf{w}} \\ \text{subject to } & (A\mathbf{w} - A\mathbf{y} - \mathbf{x})^T\mathbf{w} \leq a \\ & (\mathbf{y} - \mathbf{x})^T\mathbf{w} = b \end{aligned}$$

6.5 Letting the vector $\mathbf{x} = [k \quad m]^T \in \mathbb{R}^2$ denote the amount of knobs and milk cartons respectively, the optimization problem in standard form is choosing $\mathbf{x} \in \mathbb{R}^2$ such that:

$$\begin{aligned} \text{minimize } & -[0.05 \quad 0.07]\mathbf{x} \\ \text{subject to } & \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}\mathbf{x} \leq \begin{bmatrix} 240000 \\ 6000 \end{bmatrix} \end{aligned}$$

6.6 Let us find the Jacobian matrix of $f(x, y)$ to pinpoint the critical points:

$$\begin{aligned} Df(x, y) &= [6xy + 4y^2 + y \quad 3x^2 + 8xy + x] \\ &= [y(6x + 4y + 1) \quad x(3x + 1)] \end{aligned}$$

From solving a basic systems of equations, we have that the points where $Df(x, y) = 0$ are: $(0, 0), (-\frac{1}{3}, 0), (0, -\frac{1}{4}), (-\frac{1}{9}, -\frac{1}{12})$. To see whether they are local maxima, minima, or points, we must find the Hessian matrix and see if it's positive definite at the critical point:

$$\begin{aligned} D^2f(x, y) &= \begin{bmatrix} 6y & 6x + 8y + 1 \\ 6x + 8y + 1 & 8x \end{bmatrix} \\ D^2f(0, 0) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ D^2f(-\frac{1}{3}, 0) &= \begin{bmatrix} 0 & -1 \\ -1 & -\frac{8}{3} \end{bmatrix} \\ D^2f(0, -\frac{1}{4}) &= \begin{bmatrix} -\frac{3}{2} & -1 \\ -1 & 0 \end{bmatrix} \\ D^2f(-\frac{1}{9}, -\frac{1}{12}) &= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{8}{9} \end{bmatrix} \end{aligned}$$

Based on the definition of positive definite matrices, note that $D^2f(0,0), D^2f(-\frac{1}{3},0), D^2f(0,-\frac{1}{4})$ have negative determinants, so all those points are saddle points. However, $\det(D^2f(-\frac{1}{9},-\frac{1}{12})) > 0$ and $D_{11} < 0$ so $(-\frac{1}{9},-\frac{1}{12})$ is a local maximum.

6.11 Using Newton's method and initial starting point x_0, we have that the next point is:

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = x_0 - \frac{2ax_0 + b}{2a} = -\frac{b}{2a}$$

Plugging x_1 back into f, we see that $f'(x_1) = -b + b = 0$ and $f''(x_1) = 2a > 0$. Therefore, x_1 is a local minimum (and the unique one since only x_1 satisfies $f'(x) = 0$).

6.14 Please refer to jupyter notebook rzhang15_week4_6.14.ipynb for the implementation of Newton's method.