Math Problem Set 1
Open Source Macroeconomics Laboratory Boot Camp

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1. Exercises from the book:

   3.6 Since $A \in \mathscr{F}$, which is the power set of Ω, then $A \subset \Omega = \cup_{i \in I} B_i$. Therefore, $A = \cup_{i \in I} A \cap B_i$. By the definition of a probability space, P is countably additive on $\mathscr{F}$ so $P(\cup_{i \in I} A \cap B_i) = P(A) = \sum_{i \in I} P(A \cap B_i)$.

   3.8 First, we will prove that if E_1 and E_2 are independent events, then E_1^c and E_2^c are also independent events. Note that:

$$\begin{aligned}
P(E_1^c \cap E_2^c) &= P((E_1 \cup E_2)^c) \\
&= 1 - (P(E_1) + P(E_2) - P(E_1 \cap E_2)) \\
&= 1 - P(E_1) - P(E_2) + P(E_1) \times P(E_2) \\
&= (1 - P(E_1)) \times (1 - P(E_2)) \\
&= P(E_1^c) \times P(E_2^c)
\end{aligned}$$

   From the rule of unions and intersections, we have that $(\cup_{k=1}^n E_k)^c = \cap_{k=1}^n E_k^c$. In addition, since $\{E_k\}_{k=1}^n$ is a collection of independent events, so by extending the previous proof inductively we have that $\{E_k^c\}_{k=1}^n$ is also a collection of independent events. Then we have that:

$$\begin{aligned}
P(\cup_{k=1}^n E_k) &= 1 - P((\cup_{k=1}^n E_k)^c) \\
&= 1 - P(\cap_{k=1}^n E_k^c) \\
&= 1 - \prod_{k=1}^n P(E_k^c) \\
&= 1 - \prod_{k=1}^n (1 - P(E_k))
\end{aligned}$$

3.11 From Bayes' Rule, we have that

$$\begin{aligned} P(s = \text{ crime}|s \text{ tested } +) &= \frac{P(s \text{ tested } + |s = \text{ crime})P(s = \text{ crime})}{P(s \text{ tested } +)} \\ &= \frac{1 \times \frac{1}{250{,}000{,}000}}{\frac{1}{3{,}000{,}000}} \\ &= \frac{3}{250} \end{aligned}$$

3.12 Without loss of generality, suppose the contestant picked door A_1 and Monty opened door A_2, which contains a goat. We want to show that the contestant is better off picking door A_3. Since the contestant chose the first door with no prior information, then $P(A_1) = 1/3$. Therefore, $P(A_2 \cup A_3) = 2/3$. However, if A_2 contains a goat, then $P(A_3) = 0$ and we know that $P(A_2 \cap A_3)$ since the car cannot be behind both. Thus, $P(A_2 \cup A_3) = P(A_3) = 2/3$. The contestant would have double the chance of winning if they switched doors versus sticking with their original decision.

In a similar situation with 10 doors, you would have a 1/10 probability of winning if you stuck with your original decision, but you have a 9/10 probability of winning if you switched to the remaining door.

3.16 We want to show that $E[(X-\mu)^2] = E[X^2] - \mu^2$. From the definition of variance and the fact that expectation is the weighted average and is thus additive, we have that:

$$\begin{aligned} Var[X] &= E[(X-\mu)^2] \\ &= E[(X - E[X])^2] \\ &= E[X^2 - 2XE[X] + E[X]^2] \\ &= E[X^2] - E[2XE[X]] + E[E[X]^2] \\ &= E[X^2] - 2E[X]^2 + E[X^2] \text{ since } E[X]^2 \text{ is a constant} \\ &= E[X^2] - E[X]^2 \\ &= E[X^2] - \mu^2 \end{aligned}$$

3.33 For a binomial random variable B, we have that $B = \sum_{i=1}^n B_i$, where all B_i's are independently Bernoulli distributed random variables such that $E[B_i] = p$ and $Var(B_i) = p(1-p)$. Thus, using the weak law of large numbers, we have that for all $\epsilon > 0$,

$$P(|\frac{\sum_{i=1}^n B_i}{n} - p| \geq \epsilon) = P(|\frac{B}{n} - p| \geq \epsilon) \leq \frac{p(1-p)}{n\epsilon^2}$$

3.36 Let $S_n = \sum_{i=1}^{6242} X_i$, where X_i is the probability that student i will enroll at the school, so X_i's are independently Bernoulli distributed random variables. Thus, $E[X_i] = \mu = 0.801$ and $Var[X_i] = 0.801 \times 0.199 = 0.1594$.

We want to estimate $P(S_n \geq 5500) = 1 - P(S_n \leq 5500)$. Using the Central Limit Theorem, we have the estimation that:

$$\begin{aligned} P(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq y) &= P(S_n \leq \sigma\sqrt{n}y + n\mu) \\ &\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{\frac{-x^2}{2}} dx \end{aligned}$$

We can calculate y as follows:

$$\begin{aligned} \sigma\sqrt{n}y + n\mu &= 5500 \\ y &= \frac{5500 - 6242 \times 0.801}{\sqrt{0.1594 \times 6242}} \\ &= 15.8563 \end{aligned}$$

Computing the integral using wolfram alpha, we thus have that:

$$P(S_n \geq 5500) = 1 - P(S_n \leq 5500) = 1 - 1 = 0$$

2. (a) Suppose we toss two independent coins. Let A be the event that coin 1 is heads and B be the event that coin 2 is heads. Let C be the event if there is exactly one head amongst the two coin tosses. The four possible outcomes are: $\Omega = \{HH, TT, HT, TH\}$, thus we have that $P(B) = P(C) = P(A) = \frac{1}{2}$. Note the following observations:

$$\begin{aligned} P(A \cap B) &= P(\{HH\}) = \frac{1}{4} = P(A)P(B) \\ P(A \cap C) &= P(\{HT\}) = \frac{1}{4} = P(A)P(C) \\ P(B \cap C) &= P(\{TH\}) = \frac{1}{4} = P(B)P(C) \\ P(A \cap B \cap C) &= P(\{\}) = 0 \neq P(A)P(B)P(C) \end{aligned}$$

(b) Suppose we have a fair 8-sided die. Let $B = \{1, 2, 3, 4\}, C = \{1, 5, 6, 7\}$ and $A = \{1, 4, 7, 8\}$ denote events that consist of possible die rolls. Then $P(B) = P(C) = P(A) = \frac{1}{2}$. Note the following observations:

$$\begin{aligned} P(A \cap B) &= P(\{1, 4\}) = \frac{1}{4} = P(A)P(B) \\ P(A \cap C) &= P(\{1, 7\}) = \frac{1}{4} = P(A)P(C) \\ P(A \cap B \cap C) &= P(\{1\}) = \frac{1}{8} = P(A)P(B)P(C) \\ P(B \cap C) &= P(\{1\}) = \frac{1}{8} \neq P(B)P(C) \end{aligned}$$

3. To prove that Benson's Law is a well-defined discrete probability distribution, we must show that the probability of the entire space of outcomes is 1:

$$\begin{aligned} P(\Omega) &= \sum_{d=1}^{9} \log_{10}(1+\frac{1}{d}) \\ &= \log_{10}(\sum_{d=1}^{9} \frac{d+1}{d}) \text{ this is a telescoping sum} \\ &= \log_{10} 10 \\ &= 1 \end{aligned}$$

4. (a) The probability that the person wins $\$2^n$ is if they flip $n-1$ heads in a row then tails on the nth flip. The probability of that happening is $1/2^n$. Thus, for any given winning $x_n = \$2^n$, $p_n x_n = 1$. Since $n = \mathbb{N}$, then $E[X] = \sum_{n=1}^{\infty} p_n x_n = \sum_{n=1}^{\infty} 1 = +\infty$.

   (b) Since the player has log utility, for a given winning $x_n = \$2^n$, the utility $u_n = n \log 2$. The probability has not changed, thus we have that $E[\log X] = \sum_{n=1}^{\infty} p_n u_n = \log 2 \sum_{n=1}^{\infty} \frac{n}{2^n} = 2\log 2$, which was found using Wolfram Alpha.

5. If the U.S. investor invests a dollar in the Swiss currency instead of U.S. currency this year, there is $\frac{1}{2}$ chance that next year he will receive \$1.25 USD. and $\frac{1}{2}$ chance that next year he will receive 0.80 USD. Thus, the expected value is $\frac{1.25+0.8}{2} = \$1.025$ USD. Thus, the U.S. investor should invest in Swiss currency. If the Swiss investor invests a dollar in U.S. currency instead of Swiss currency, there is $\frac{1}{2}$ chance that next year he will receive \$1.25 CHF and $\frac{1}{2}$ chance that next year he will receive 0.80 CHF. Thus, the expected value is $\frac{1.25+0.8}{2} = \$1.025$ CHF. Similarly, the Swiss investor should invest in U.S. currency.

6. (a) Let the probability density function of X be the following:

$$f_X(x) = \begin{cases} \frac{2}{x^3} & x \geq 1 \\ 0 & otherwise \end{cases}$$

   Note that $\int_{-\infty}^{\infty} f_X(x)dx = \int_1^{\infty} \frac{2}{x^3}dx = 1$, thus this is a valid continuous random variable. Note that:

$$\begin{aligned} E[X] &= \int_1^{\infty} x\frac{2}{x^3}dx \\ &= [-\frac{2}{x}]_1^{\infty} \\ &= 2 \\ E[X^2] &= \int_1^{\infty} x^2\frac{2}{x^3}dx \\ &= [2\log x]_1^{\infty} \\ &= \infty \end{aligned}$$

(b) Let $X \sim U(0,1)$ and $Y \sim exp(1.99)$. Thus, $E[X] = \frac{1}{2} < E[Y] = \frac{1}{1.99}$. Now we must show that $P(X > Y) > 1/2$:

$$\begin{aligned}
P(X > Y) &= \int_{-\infty}^{\infty} P(X > Y | Y = y) f_Y(y) dy \\
&= \int_{-\infty}^{\infty} P(X > y) f_Y(y) dy \\
&= \int_{-\infty}^{\infty} (1 - F_X(y)) f_Y(y) dy \\
&= 1 - \int_{-\infty}^{\infty} F_X(y) f_Y(y) dy \\
&= 1 - (\int_0^1 y 1.99 e^{-1.99y} dy + \int_1^{\infty} 1.99 e^{-1.99y} dy) \\
&= 0.566
\end{aligned}$$

(c) Let $X \sim U(-1,1), Y \sim U(-2,2)$, and $Z \sim U(-3,3)$. From part b), we have that $P(X > Y) = 1 - \int_{-\infty}^{\infty} F_X(y) f_Y(y) dy$. Using the pdfs and cdfs of the uniform distribution, we have that:

$$\begin{aligned}
P(X > Y) &= 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2} \\
P(X > Z) &= 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2} \\
P(Z > Y) &= 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}
\end{aligned}$$

Thus, $P(X > Y)P(X > Z)P(Z > Y) = \frac{1}{8} > 0$ and $E[X] = E[Y] = E[Z] = 0$.

7. (a) If $Z = 1$, then $Y = X \sim N(0,1)$. If $Z = -1$, then $Y = -X \sim N(0,1)$ due to the symmetry of $N(0,1)$ about 0. Then for a given y, we have that:

$$\begin{aligned}
P(Y < y) &= P(Y < y | Z = 1)P(Z = 1) + P(Y < y | Z = -1)P(Z = -1) \\
&= \frac{P(X < y) + P(-X < y)}{2} \\
&= \frac{P(X < y) + P(X > -y)}{2} \\
&= P(X < y) \text{ due to symmetry of } N(0,1)
\end{aligned}$$

Since this holds for all y, then $Y \sim N(0,1)$.

(b) For any given value $X = x$, we have that $Y \in -x, x$, thus $|X| = |Y|$ with certainty. $P(|X| = |Y|) = 1$ then follows.

(c) Note that $f_{XY}(1,2) = 0$ since $Y \in -1, 1$. However, $f_X(1) f_Y(2) = f_X(1) f_X(2) \neq 0$. Thus, X and Y are not independent.

(d) We have that $Cov[X,Y] = E[XY] - E[X]E[Y] = E[XY] = E[X^2Z] = \frac{E[X^2]-E[X^2]}{2} = 0$.

(e) If X, Y are two independent, normally distributed random variables, then $E[XY] = E[X]E[Y]$ so $Cov[X,Y] = E[XY] - E[X]E[Y] = 0$. Therefore, the statement is false.

8. First, let us state the cumulative distribution function of $X_i \sim U[0,1]$:

$$F_{X_i}(x) = \begin{cases} 1 & x > 1 \\ x & x \in [0,1] \\ 0 & x < 0 \end{cases}$$

First let us deal with the random variable $m = \min\{X_1, ..., X_n\}$:

$$\begin{aligned} F_m(x) &= P(m \leq x) = 1 - P(X_i > x)^n \\ &= 1 - (1 - P(X_i \leq x))^n = 1 - (1 - F_{X_i}(x))^n \end{aligned}$$

$$\therefore F_m(x) = \begin{cases} 1 & x > 1 \\ 1 - (1-x)^n & x \in [0,1] \\ 0 & x < 0 \end{cases}$$

$$\therefore f_m(x) = F'_m(x) = \begin{cases} n(1-x)^{n-1} & x \in [0,1] \\ 0 & otherwise \end{cases}$$

$$\therefore E[m] = \int_0^1 xn(1-x)^{n-1}dx = \frac{1}{n+1}$$

Now let us deal with the random variable $M = \max\{X_1, ..., X_n\}$:

$$F_M(x) = P(M \leq x) = P(X_i \leq x)^n = F_{X_i}(x))^n$$

$$\therefore F_M(x) = \begin{cases} 1 & x > 1 \\ x^n & x \in [0,1] \\ 0 & x < 0 \end{cases}$$

$$\therefore f_M(x) = F'_M(x) = \begin{cases} nx^{n-1} & x \in [0,1] \\ 0 & otherwise \end{cases}$$

$$\therefore E[M] = \int_0^1 xnx^{n-1}dx = \frac{n}{n+1}$$

9. (a) Letting $S_n = \sum_{i=1}^{1000} X_i$, where X_i is the probability of a good state in period i. Since the X_i's are independent Bernoulli distributed random

variables, $E[X_i] = \mu = 0.5$ and $\sigma^2 = 0.25$. We want to estimate $P(S_n \leq 510) - P(S_n \leq 490)$. Thus, using the Central limit theorem, we have that:

$$\begin{aligned} P(S_n \leq 510) &\approx \Phi(\frac{510 - n\mu}{\sigma\sqrt{n}}) \\ &= \Phi(\frac{2}{\sqrt{10}}) \\ P(S_n \leq 490) &\approx \Phi(\frac{490 - n\mu}{\sigma\sqrt{n}}) \\ &= \Phi(-\frac{2}{\sqrt{10}}) \\ \therefore P(S_n \leq 510) - P(S_n \leq 490) &\approx 0.47 \end{aligned}$$

Therefore, there is a 47% chance that the number of good states over 1000 periods differs from 500 by at most 2%.

(b) We want to find n such that $P(|\frac{S_n}{n} - \mu| \geq 0.005) \leq 0.01$. From the weak law of large numbers, we have that $\frac{\sigma^2}{n\epsilon^2} = \frac{0.25}{0.005n} = 0.01$. Therefore, $n = 5000$.

10. Since $E[X]$ is finite, we can use Jensen's inequality. Since $f(x) = e^{\theta x}$ is a differentiable convex function, according to Jensen's inequality, $E[e^{\theta X}] = 1 \geq e^{\theta E[X]}$. Since $E[X] < 0$, then $\theta > 0$ or else $1 \leq e^{\theta E[X]}$, which would contradict Jensen's inequality.