5. Finite Dependence and Unobserved Heterogeneity

Robert A. Miller

DSE at UCL

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Introduction

Motivation

- Estimation of dynamic discrete choice models is complicated by the calculation of expected future payoffs.
- These complications are mitigated when finite dependence holds.
- *Intuitively,* ρ *period dependence holds when two sequences of weighted (nonoptimal) choices leading off from different initial choices generate the same distribution of state variables* $\rho + 1$ *periods later.*
- Exploiting the finite dependence property reduces the number of CCPs to estimate.
- Most empirical applications have the finite dependence property.
- Under the conditional independence assumption, finite dependence has empirical content (can be tested) without specifying utilities.
- The finite dependence property depends on the state transition functions but not on optimizing behavior.

Introduction

Framework

- $t \in \{1, \ldots, T\}$ stands for time, where $T \leq \infty$.
- $x_t \in \{1, \ldots, X\} \equiv \mathbb{X}$ is the state at t, where $\mathbb{X}$ is a finite set.
- $j \in \{1, \ldots, J\}$ is a (mutually exclusive) choice.
- $d_{jt} = 1$ if j is picked at t and otherwise $d_{jt} = 0$.
- $f_{jt}(x_{t+1}|x_t)$ is the probability of x_{t+1} occurring in period $t+1$ conditional on x_t and $d_{jt} = 1$.
- $\epsilon_t \equiv (\epsilon_{1t}, \ldots, \epsilon_{Jt})$ has continuous support and is IID over t with PDF $g(\epsilon_t)$ satisfying $E[\max\{\epsilon_{1t}, \ldots, \epsilon_{Jt}\}] \leq \overline{\epsilon} < \infty$.
- For some $\beta \in (0, 1)$, the individual sequentially chooses the vector $d_t \equiv (d_{1t}, \ldots, d_{Jt})$ to maximize:

$$E\left\{\sum_{t=1}^{T}\sum_{j=1}^{J}\beta^{t-1}d_{jt}\left[u_{jt}(x_t) + \epsilon_{jt}\right]\right\} \tag{1}$$

- $d_t^o(x_t, \epsilon_t)$ is the optimal decision rule with j^{th} element $d_{jt}^o(x_t, \epsilon_t)$.
- $p_t(x_t) \equiv (p_{1t}(x_t), \ldots, p_{Jt}(x_t))$ are the CCPs, where:

$$p_{jt}(x_t) \equiv \int d_{jt}^o\left(x_t, \epsilon_t\right) g\left(\epsilon_t\right) d\epsilon_t \tag{2}$$

- $V_t(x_t)$ is the ex-ante value function defined as :

$$V_t(x_t) \equiv E\left\{\sum_{\tau=t}^{T}\sum_{j=1}^{J}\beta^{\tau-t} d_{j\tau}^o\left(x_\tau, \epsilon_\tau\right)\left(u_{j\tau}(x_\tau) + \epsilon_{j\tau}\right)\right\}$$

- The conditional value function for action j defined as:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x_{t+1}=1}^{X} V_{t+1}(x_{t+1}) f_{jt}\left(x_{t+1} | x_t\right) \tag{3}$$

- The conditional value function correction for action j is defined:

$$\psi_j[p_t(x)] \equiv V_t(x) - v_{jt}(x) \tag{4}$$

- We defined a weight sequence $\{\omega_{k\tau}(x_\tau, j, t)\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)}$ satisfying:

$$|\omega_{k\tau}(x_\tau, j, t)| < \infty \text{ and } \sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j, t) = 1$$

- To emphasize the potential dependence of the weight sequence on (j, x_t, t) we could have written:

$$\omega_{k\tau}(x_\tau, j, x_t, t) \text{ rather than } \omega_{k\tau}(x_\tau, j, t)$$

- Similarly recall that $\kappa_{t+1}(x_{t+1}|j, x_t, t) \equiv f_{jt}(x_{t+1}|x_t)$ and:

$$\kappa_{\tau+1}(x_{\tau+1}|j, x_t, t) \equiv \sum_{x_\tau=1}^{X} \sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j, t) f_{k\tau}(x_{\tau+1}|x_\tau)\kappa_\tau(x_\tau|j, x_t, t) \tag{5}$$

- To be more explicit we could have written:

$$\kappa_{\tau+1}^{(\omega)}(x_{\tau+1}|j, x_t, t) \text{ rather than } \kappa_{\tau+1}(x_{\tau+1}|j, x_t, t)$$

- The representation given in the second lecture implies:

$$v_{jt}(x_t) = V_t(x) - \psi_j[p_t(x_t)] = \tag{6}$$

$$\left\{ \begin{array}{l} u_{jt}(x_t) + \sum_{x=1}^{X} \beta^{\rho+1} V_{t+\rho+1}(x) \kappa_{t+\rho+1}(x|j, x_t, t) \\ + \sum_{\tau=t+1, k=1, x_\tau=1}^{t+\rho, J, X} \beta^{\tau-t} \left[\begin{array}{l} u_{k\tau}(x_\tau) \\ +\psi_k[p_\tau(x_\tau)] \end{array} \right] \left[\begin{array}{l} \omega_{k\tau}(x_\tau, j, t) \\ \times \kappa_\tau(x_\tau|j, x_t, t) \end{array} \right] \end{array} \right\}$$

- In this formulation:
  - $V_{t+\rho+1}(x)$ captures the value from $t+\rho+1$ having reached x.
  - $\kappa_{t+\rho+1}(x|j, x_t, t) \equiv \kappa_{t+\rho+1}^{(\omega)}(x|j, x_t, t)$ is the probability of reaching x by following $\{\omega_{k\tau}(x_\tau, j, t)\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)}$ from (t, x_t).
  - $\psi_k[p_\tau(x_\tau)]$ are correction terms for nonoptimal behavior.

Finite Dependence

Definition

- The pair of choices $\{i, j\}$ exhibits ρ*-period dependence* at (t, x_t) if there exist a pair of sequences of decision weights:

$$\{\omega_{k\tau}(x_\tau, i, t)\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)} \text{ and } \{\omega_{k\tau}(x_\tau, j, t))\}_{(k,\tau)=(1,t+1)}^{(J,t+\rho)}$$

such that for all $x_{t+\rho+1} \in \{1, \ldots, X\}$:

$$\kappa_{t+\rho+1}(x_{t+\rho+1} | i, x_t, t) = \kappa_{t+\rho+1}(x_{t+\rho+1} | j, x_t, t)$$

- Finite dependence:
  1. trivially holds for $\rho = T - t$ when $T < \infty$, but only merits attention when $\rho < T - t$.
  2. extends to games by conditioning on the player as well.
  3. might hold for some choice pairs but not others, and for certain states but not others.
  4. could be defined for mixed choices to start the sequence, not just deterministic moves; this analysis extends to the more general case.

- If there is finite dependence for (i, j, x_t, t), then:

$$u_{jt}(x_t) + \psi_j[p_t(x_t)] - u_{it}(x_t) - \psi_i[p_t(x_t)] =$$

$$\sum_{(k,\tau,x_\tau)=(1,t+1,1)}^{(J,t+\rho,X)} \beta^{\tau-t} \left\{ \begin{array}{l} u_{k\tau}(x_\tau) \\ +\psi_k[p_\tau(x_\tau)] \end{array} \right\} \left[\begin{array}{l} \omega_{k\tau}(x_\tau, i, t)\kappa_\tau(x_\tau|i, x_t, t) \\ -\omega_{k\tau}(x_\tau, j, t)\kappa_\tau(x_\tau|j, x_t, t) \end{array} \right] \tag{7}$$

- To derive this equation:
  - replace $v_{jt}(x)$ with $V_t(x) - \psi_j[p_t(x_t)]$ in (6)
  - form an analogous equation for i
  - difference the two resulting equations
  - note the $V_t(x)$ and $V_{t+\rho+1}(x)\kappa_{t+\rho+1}(x|x_t, j, t)$ terms cancel.

- A *terminal choice* ends the optimization problem or the game for that particular player.
- They are quite common in empirical applications.
- Supposing the first choice a terminal choice, then $u_{1t}(x_t)$ can be interpreted as the final payoff at (t, x_t) if that choice is made then.
- Setting $\omega_{k,t+1}(t, x_{t+1}, i) = 0$ for all (x, i) and $k \neq 1$, Equation (7) reduces to:

$$
\begin{aligned}
& u_{1t}(x_t) + \psi_1[p_t(x_t)] - u_{jt}(x_t) - \psi_j[p_t(x_t)] \\
= \;& \textstyle\sum_{x_{t+1}=1}^{X} \beta \left\{ u_{1,t+1}(x_{t+1}) + \psi_1 \left[p_{t+1}(x_{t+1}) \right] \right\} f_{jt}(x_{t+1}|x_t)
\end{aligned}
$$

Simple Examples of Finite Dependence

Renewal choices are also quite common in empirical applications

- Similarly a *renewal choice* yields a probability distribution of the state variable next period that does not depend on the current state.
- If the first choice is a renewal choice, then for all $j \in \{1, \ldots, J\}$:

$$\begin{aligned}\sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}|x_{t+1}) f_{jt}(x_{t+1}|x_t) &= \sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}) f_{jt}(x_{t+1}|x_t) \\ &= f_{1,t+1}(x_{t+2}) \sum_{x_{t+1}=1}^{X} f_{jt}(x_{t+1}|x_t) \\ &= f_{1,t+1}(x_{t+2}) \qquad (8)\end{aligned}$$

- In this case Equation (7) implies:

$$\begin{aligned}& u_{1t}(x_t) + \psi_1[p_t(x_t)] - u_{jt}(x_t) - \psi_j[p_t(x_t)] \\ = & \sum_{x=1}^{X} \beta \{u_{1,t+1}(x) + \psi_1 [p_{t+1}(x)]\} [f_{jt}(x|x_t) - f_{1t}(x|x_t)]\end{aligned}$$

Simple Examples of Finite Dependence

An example of 2-period finite dependence

- How does finite dependence work when $\rho > 1$?
- Consider the following model of labor supply and human capital.
- In each of T periods an individual chooses:
  - $d_{2t} = 1$ to work
  - $d_{1t} = 1$ to stay home.
- She accumulates human capital, x_t, from working:
  - If $d_{1t} = 1$ then $x_{t+1} = x_t$.
  - If $d_{2t} = 1$ and $t > 1$ then $x_{t+1} = x_t + 1$.
  - If $d_{j=2,t=1} = 1$ then

$$x_2 = \begin{cases} 2 \text{ with probability } 0.5 \\ 1 \text{ with probability } 0.5 \end{cases}$$

- Summarizing, human capital only increases with work, by a unit, except in the first period, when it might jump to two.

Simple Examples of Finite Dependence

Establishing finite dependence in the labor supply example

- When $t > 1$, one-period dependence can be attained with $x_{t+2} = x_t + 1$ by working one period out of the next two:
  - If $d_{2t} = 1$ set $\omega_{1,t+1}(x_{t+1} = x_t + 1, j = 2, t) = 1$.
  - If $d_{1t} = 1$ set $\omega_{2,t+1}(x_{t+1} = x_t, j = 1, t) = 1$.
- When $t = 1$ two-period dependence can be attained with $x_3 = 2$:
  - Stay home at $t = 1$ (setting $d_{11} = 1$), and then work for two periods:

$$
\begin{aligned}
1 &= \omega_{k=2,\tau=2}(x_2 = 0, j = 1, t = 1) \\
&= \omega_{k=2,\tau=3}(x_3 = 1, j = 1, t = 1)
\end{aligned}
$$

  - Work at $t = 1$ (setting $d_{21} = 1$), and work in $t = 2$ only if human capital increases by one unit at $t = 1$:

$$
\begin{aligned}
1 &= \omega_{k=1,\tau=2}(x_2 = 2, j = 2, t = 1) \\
&= \omega_{k=2,\tau=2}(x_2 = 1, j = 2, t = 1) \\
&= \omega_{k=1,\tau=3}(x_2 = 3, j = 2, t = 1)
\end{aligned}
$$

Simple Examples of Finite Dependence

Nonstationary search model

- Consider a simple search model in which all jobs are temporary, lasting only one period.
- Each period $t \in \{1, \ldots, T\}$ an individual may:
  - stay home by setting $d_{1t} = 1$
  - or apply for temporary employment setting $d_{2t} = 1$.
- Job applicants are successful with probability λ_t, time varying job offer arrival rates.
- Experience $x \in \{1, \ldots, X\}$ increases by one unit with each period of work, up to X, and does not depreciate.

Simple Examples of Finite Dependence

Finite dependence in this search model

- For all (x_t, t) with $x_t < X$ set:
  - $d_{1t} = 1$ (stay home) and then "apply for employment" with weight:

$$\begin{aligned} \lambda_t / \lambda_{t+1} &= \omega_{k=2,t+1}(x_t, j=1, t) \\ &= 1 - \omega_{k=1,t+1}(x_t, j=1, t) \end{aligned}$$

  - $d_{2t} = 1$ (seek work) and then stay home:

$$\omega_{k=1,t+1}\left(x_t, j=2, t\right) = \omega_{k=1,t+1}\left(x_t + 1, j=2, t\right) = 1$$

- This pair of sequencing attains one-period dependence since:

$$\kappa_3(x_{t+3} | j=1, x_t, t) = \kappa_3(x_{t+3} | j=2, x_t, t) = \begin{cases} 1 - \lambda_t \text{ for } x_{t+3} = x_t \\ \lambda_t \text{ for } x_{t+3} = x_t + 1 \end{cases}$$

- Note that if $\lambda_t > \lambda_{t+1}$ then $\omega_{k=2,t+1}(x_t, j=1, t) > 1$ and:

$$\omega_{k=1,t+1}(x_{t+1}, j=1, t) = 1 - \lambda_t / \lambda_{t+1} < 0$$

- Suppose the data comprise N observations of the state variables and decisions denoted by $\left\{d_{nt_n}, x_{nt_n}, x_{n,t_n+1}\right\}_{n=1}^{N}$ sampled within a time frame of $t \in \{1, \ldots, S\}$.
- For expositional simplicity suppose the probability of sampling each $x \in \{1, \ldots, X\}$ in $t \in \{1, \ldots, S\}$ is strictly positive.
- M separate instances of finite dependence within that time frame.
- Say each pair of choices includes the first choice.
- Label the M paths by (j_m, x_m, t_m, ρ_m) for $m \in \{1, \ldots, M\}$.
- Assume:
  - $g(\epsilon_t)$ is known.
  - $\theta \equiv (\theta_1, \ldots, \theta_K) \in \Theta$, a closed convex set in $\mathbb{R}^K$.
  - $u_{jt}(x) \equiv \mu_{jt}(x, \theta)$, where $\mu_{jt}(x, \theta)$ is known function.
  - M instances of finite dependence suffice for identification.

- For all $(t, x, j) \in \{1, \ldots, S\} \times \{1, \ldots, X\}\, t \in \{1, \ldots, J\}$:
  - define

$$\widehat{p}_{jt}(x) \equiv \frac{\sum_{n=1}^{N} 1\left\{d_{nt_n j} = 1\right\} 1\left\{t_n = t\right\} 1\left\{x_{nt_n} = x\right\}}{\sum_{n=1}^{N} 1\left\{t_n = t\right\} 1\left\{x_{nt_n} = x\right\}}$$

  - estimate the $XJ\mathcal{T}$ CCP vector $p \equiv (p_{11}(1), \ldots, p_{JS}(X))'$ with $\widehat{p}$ formed from $\widehat{p}_{jt}(x)$.
  - Also estimate $f_{jt}(x)$ with $\widehat{f}_{jt}(x)$ in this first stage, for example with a cell estimator (similar to the CCP estimator).

Estimation

The MD estimator

- Define $y\left(p,f\right) \equiv \left(y_1\left(p,f\right),\ldots,y_M\left(p,f\right)\right)'$ where:

$$y_m\left(p,f\right) \equiv \psi_1[p_{t_m}(x_m)] - \psi_{j_m}[p_{t_m}(x_m)]$$
$$+\sum_{\tau=t_m+1}^{t_m+\rho_m}\sum_{k=1}^{J}\sum_{x_\tau=1}^{X}\beta^{\tau-t_m}\psi_k[p_\tau(x_\tau)]\left[\begin{array}{l}\omega_{k\tau}(x_\tau,1,t_m)\kappa_\tau(x_\tau|1,x_m,t_m)-\\ \omega_{k\tau}(x_\tau,j_m,t_m)\kappa_\tau(x_\tau|j_m,x_m,t_m)\end{array}\right]$$

and $Z(p,f,\theta) \equiv (Z_1(p,f,\theta),\ldots,Z_M(p,f,\theta))'$ where:

$$Z_m(p,f,\theta) \equiv \mu_{j_m,t_m}(x_m,\theta)$$
$$-\sum_{\tau=t_m+1}^{t_m+\rho_m}\sum_{k=1}^{J}\sum_{x_\tau=1}^{X}\beta_{k\tau}^{\tau-t_m}\mu_{k\tau}(x_\tau,\theta)\left[\begin{array}{l}\omega_{k\tau}(x_\tau,1,t_m)\kappa_\tau(x_\tau|1,x_m,t_m)-\\ \omega_{k\tau}(x_\tau,j_m,t_m)\kappa_\tau(x_\tau|j_m,x_m,t_m)\end{array}\right]$$

- For any M dimensional positive definite matrix W define:

$$\widehat{\theta} \equiv \arg\min_{\theta}\left[y\left(\widehat{p},\widehat{f}\right) - Z\left(\widehat{p},\widehat{f},\theta\right)\right]' W \left[y\left(\widehat{p}\right) - Z\left(\widehat{p},\widehat{f},\theta\right)\right] \quad (9)$$

One-Period Dependence in Optimization Problems

Approach

- *Guess and verify* is typically used to establish finite dependence.
- There is however a systematic way for determining finite period dependence.
- The algorithm iterates between two procedures that:
  1. checks counterfactual outcomes arising from deterministic choices that might either induce or rule out finite dependence.
  2. lists the elements of a matrix to determine its rank.
- See:
  1. Peter Arcidiacono and Robert Miller (2019 *Quantitative Economics*)
  2. See Jaepil Lee: "A Structural Analysis of Opioid Misuse: Health, Labor, Policy, and Misperception on Opioid Misuse Risk" (working paper)

- Relaxing the conditional independence assumption is tantamount to reopening the multiple integration challenge.
- Consider what happens when unobserved beliefs evolve endogenously.
- The payoff from working job $m \in \{1, 2, \ldots\}$ at time $t \in \{0, 1, \ldots\}$ is:

$$x_{mt} \equiv \psi_t + \xi_m + \sigma_m \epsilon_{mt} \tag{10}$$

  where:
  - ψ_t is a lifecycle trend shaping term that plays no role in the analysis;
  - ξ_m is a job match parameter drawn from $N\left(\gamma_m, \delta_m^2\right)$;
  - ε_{mt} is an idiosyncratic *iid* disturbance drawn from $N\left(0, 1\right)$
- Every period t the individual chooses a job $m \in \{1, 2, \ldots\}$ where:
  - $d_t = (d_{1t}, d_{2t}, \ldots)$ denotes her choice, $d_{mt} \in \{0, 1\}$ and $\sum_{m=1}^{\infty} d_{mt} = 1$.
  - her realized lifetime utility is $\sum_{t=0}^{\infty} \sum_{m=1}^{\infty} \beta^t d_{mt} x_{mt}$

- At $t = 0$ the individual sees $\left(\gamma_m, \delta_m^2\right)$ for each m.
- After making her choice, she also sees ψ_t, and $d_{mt}x_{mt}$ for all m.
- Her posterior beliefs for job m at time $t \in \{0, 1, \ldots\}$ are $N\left(\gamma_{mt}, \delta_{mt}^2\right)$ where:

$$\begin{aligned}\gamma_{mt,+1} &= \frac{\delta_m^{-2}\gamma_m + \sigma_m^{-2}\sum_{s=0}^{t}\left(x_{ms} - \psi_s\right)d_{ms}}{\delta_m^{-2} + \sigma_m^{-2}\sum_{s=0}^{t} d_{ms}} \\ &= \gamma_{mt} + \left(x_{mt} - \psi_t\right) / \left(\sigma_m^2\delta_{mt}^{-2} + 1\right) d_{mt}\end{aligned}$$

and

$$\delta_{m,t+1}^{-2} = \delta_m^{-2} + \sigma_m^{-2}\sum_{s=0}^{t} d_{ms} = \delta_{mt}^{-2} + \sigma_m^{-2} \tag{11}$$

- She maximizes the sum of expected payoffs, sequentially choosing d_t given her beliefs $N\left(\gamma_{mt}, \delta_{mt}^2\right)$ for each $m \in M$.

Corollary (from Theorem 2 in Gittens and Jones,1974)

At each $t \in \{1, 2, \ldots\}$ it is optimal to select the $m \in M$ maximizing:

$$DAI_m(\gamma_{mt}, \delta_{mt}) \equiv \sup_{\tau \geq t} \left\{ \frac{E\left[\sum_{r=t}^{\tau} \beta^{r-t} (x_{mr} - \psi_r) \,|\gamma_{mt}, \delta_{mt}\right]}{E\left[\sum_{r=t}^{\tau} \beta^{r-t} \,|\gamma_{mt}, \delta_{mt}\right]} \right\}$$

- If τ is fixed and there is perfect foresight, the fundamental ratio is:
    - the discounted sum of benefits $\sum_{r=t}^{\tau} \beta^{r-t} (x_{mr} - \psi_r)$
    - divided by the discounted sum of time $\sum_{r=t}^{\tau} \beta^{r-t}$.
- For example if project A yields 5 and takes 2 periods to complete, and B yields 3 but only takes 1 period, do A first if and only if:

$$\begin{aligned} & 5 + 3\beta^2 > 3 + 5\beta \\ \iff \quad & 5(1-\beta) > 3(1-\beta)(1+\beta) \\ \iff \quad & DAI_A \equiv 5 \big/ (1+\beta) > 3 \equiv DAI_B \end{aligned}$$

- Define $D(\sigma)$ is the (standard) DAI for a (hypothetical) job whose
  - fixed match parameter ξ is drawn from $N(0,1)$
  - whose random component in the payoff is $\sigma\varepsilon_t$.

Corollary (Proposition 4 of Miller, 1984)

In this model:

$$DAI_m(\gamma_{mt}, \delta_{mt}) = \gamma_{mt} + \delta_{mt} D\left[\left(\frac{\sigma_m}{\delta_m}\right)^2 + \sum_{s=0}^{t-1} d_{ms}\right]$$

- We can prove $D(\cdot)$ is a decreasing function, implying that $DAI_m(\gamma_{mt}, \delta_{mt}) \uparrow$ as:
  - γ_{mt} and δ_{mt} and $\beta \uparrow$
  - σ_m and $\sum_{s=0}^{t-1} d_{ms} \downarrow$.

- Assuming $(\gamma_m, \delta_m, \sigma_m) = (\gamma, \delta, \sigma)$ for all m, a world in which all differences between jobs are match specific, it suffices to only keep track of the current job match. (Why?)
- Define h_t as the discrete hazard at t periods as the probability a spell ends after t periods conditional on surviving that long. Then:

$$
\begin{aligned}
h_t &\equiv \Pr\left\{\gamma_t + \delta_t D\left[\left(\frac{\sigma}{\delta}\right)^2 + t, \beta\right] \leq \gamma + \delta D\left[\left(\frac{\sigma}{\delta}\right)^2, \beta\right]\right\} \\
&= \Pr\left\{\frac{\gamma_t - \gamma}{\sigma} \leq \frac{\delta}{\sigma} D\left[\left(\frac{\sigma}{\delta}\right)^2, \beta\right] - \frac{\delta_t}{\sigma} D\left[\left(\frac{\sigma}{\delta}\right)^2 + t, \beta\right]\right\} \\
&= \Pr\left\{\rho_t \leq \alpha^{-1/2} D(\alpha, \beta) - (\alpha + t)^{-1/2} D(\alpha + t, \beta)\right\}
\end{aligned}
$$

where $\rho_t \equiv (\gamma_t - \gamma)/\sigma$ and $\alpha \equiv (\sigma/\delta)^2$ which implies:

$$
\frac{\delta_t}{\sigma} = \frac{\left[\delta^{-2} + t\sigma^{-2}\right]^{-1/2}}{\sigma} = \left[\left(\frac{\delta}{\sigma}\right)^{-2} + t\right]^{-1/2} = (\alpha + t)^{-1/2}
$$

Probability Distribution of Spell Lengths

Relating the hazard rate to the distribution of normalized match qualities

- Define the probability distribution of transformed means of spells surviving at least t periods as:

$$\Psi_t(\rho) \equiv \Pr\{\rho_t \leq \rho\} = \Pr\left\{\sigma^{-1}(\gamma_t - \gamma) \leq \rho\right\} = \Pr\{\gamma_t \leq \gamma + \rho\sigma\}$$

- To help fix ideas note that $\Psi_0(\rho) = 0$ for all $\rho < 0$ and $\Psi_0(0) = 1$.
- From the definition of h_t and $\Psi_t(\rho)$:

$$\begin{aligned} h_t &= \Pr\left\{\rho_t \leq \alpha^{-1/2} D(\alpha, \beta) - (\alpha + t)^{-1/2} D(\alpha + t, \beta)\right\} \\ &= \Psi_t\left[\alpha^{-1/2} D(\alpha, \beta) - (\alpha + t)^{-1/2} D(\alpha + t, \beta)\right] \end{aligned}$$

- To derive the discrete hazard, we recursively compute $\Psi_t(\rho)$.

Probability Distribution of Spell Lengths

Inequalities relating to normalized match qualities after one period

- Every match survives at least one period::

$$\Psi_1(\rho) = \Pr\{\gamma_1 \leq \gamma + \rho\sigma\} = \Phi\left[\alpha^{1/2}(\alpha+1)^{1/2}\rho\right]$$

and the spell ends if:

$$\rho_1 < \alpha^{-1/2} D(\alpha,\beta) - (\alpha+1)^{-1/2} D(\alpha+1,\beta) \equiv \rho_1^*$$

- Therefore the proportion of spells ending after one period is:

$$\begin{aligned} h_1 &= \Psi_1\left[\alpha^{-1/2} D(\alpha,\beta) - (\alpha+1)^{-1/2} D(\alpha+1,\beta)\right] \\ &= \Phi\left\{\begin{array}{l} \left[\alpha^{1/2}(\alpha+1)^{1/2}\right] \\ \times\left[\alpha^{-1/2} D(\alpha,\beta) - (\alpha+1)^{-1/2} D(\alpha+1,\beta)\right] \end{array}\right\} \\ &> 1/2 \quad (\text{because } D(\cdot) \text{ is decreasing in } \alpha) \end{aligned}$$

A :new job has greater information value and half the time a higher mean.

Probability Distribution of Spell Lengths

Recursively computing the distribution of normalized match qualities

- Continuing in this line of reasoning:

$$\Psi_2(\rho) = \frac{\int_{-\infty}^{\infty} \Phi \left[\begin{array}{l} \alpha^{1/2} (\alpha+1)^{1/2} \times \\ \left(\rho - \epsilon \left[(\alpha+1)(\alpha+2) \right]^{-1/2} \right) \end{array} \right] d\Phi(\epsilon) - h_1}{1 - h_1}$$

and more generally (from page 1112 of Miller, 1984):

$$\Psi_{t+1}(\rho) \equiv \frac{\int_{-\infty}^{\infty} \Psi_t \left(\rho - \epsilon \left[(\alpha+t)(\alpha+t+1) \right]^{-1/2} \right) d\Phi(\epsilon) - h_t}{1 - h_t}$$

Maximum Likelihood Estimation

Complete and incomplete spells

- Suppose the sample comprises a cross section of spells $n \in \{1, \ldots, N\}$, some of which are completed after τ_n periods, and some of which are incomplete lasting at least τ_n periods. Let:

$$\rho\left(n\right) \equiv \begin{cases} \tau_n \text{ if spell is complete} \\ \{\tau_n, \tau_{n+1}, \ldots\} \text{ if spell is incomplete} \end{cases}$$

- Let $p_\tau\left(\alpha_n, \beta_n\right)$ denote the unconditional probability of individual n with discount factor β_n working τ periods in a new job with information factor α_n before switching to another new job in the same occupation:

$$p_\tau\left(\alpha_n, \beta_n\right) \equiv h_\tau\left(\alpha_n, \beta_n\right) \prod_{s=1}^{\tau-1}\left[1 - h_s\left(\alpha_n, \beta_n\right)\right]$$

- Then the joint probability of spell duration times observed in the sample is:

$$\prod_{n=1}^{N} \sum_{\tau \in \rho(n)} p_\tau\left(\alpha_n, \beta_n\right)$$

Motivating Example

Rust's (1987) bus engine revisited

- The integration in the job matching example is:
  1. quite cumbersome
  2. suggestive of how quickly integration becomes unmanageable if jobs differ *exante* as well as *expost*.
- CCP estimators can be exploited to ameliorate this problem.
- Recalling Mr. Zurcher's problem:
  - Replace the existing engine ($d_{1t} = 1$), or keep it for at least one more period ($d_{2t} = 1$).
  - Bus mileage x_t follows the update rule $x_{t+1} = d_{1t} + d_{2t}(x_t + 1)$.
  - Transitory iid choice-specific shocks, ϵ_{jt}, are T1EV.
  - Zurcher sequentially maximizes expected discounted sum of payoffs:

$$E\left\{\sum_{t=1}^{\infty} \beta^{t-1}\left[d_{2t}(\theta_1 x_t + \theta_2 s + \epsilon_{2t}) + d_{1t}\epsilon_{1t}\right]\right\}$$

- Now suppose s, the bus make, is unobserved.

- To show how the EM algorithm helps, consider the infeasible case where $s \in \{1, \ldots, S\}$ is unobserved but $p(x, s)$ is known.
- Let π_s denote population probability of being in unobserved state s.
- Supposing β is known the ML estimator for this "easier" problem is:

$$\{\hat{\theta}, \hat{\pi}\} = \arg\max_{\theta, \pi} \sum_{n=1}^{N} \ln \left[\sum_{s=1}^{S} \pi_s \prod_{t=1}^{T} l(d_{nt} | x_{nt}, s, p, \theta) \right]$$

where $p \equiv p(x, s)$ is a string of probabilities assigned/estimated for each (x, s) and $l(d_{nt} | x_{nt}, s_n, p, \theta)$ is derived from our representation of the conditional valuation functions and takes the form:

$$\frac{d_{1nt} + d_{2nt} \exp(\theta_1 x_{nt} + \theta_2 s + \beta \ln [p(0, s)] - \beta \ln [p(x_{nt} + 1, s)]}{1 + \exp(\theta_1 x_{nt} + \theta_2 s + \beta \ln [p(0, s)] - \beta \ln [p(x_{nt} + 1, s)])}$$

- Maximizing over the sum of a log of summed products is computationally burdensome.

Motivating Example

Why EM is attractive (when CCP's are known)

- The EM algorithm is a computationally attractive alternative to directly maximizing the likelihood.
- Denote by $d_n \equiv (d_{n1}, \ldots, d_{nT})$ and $x_n \equiv (x_{n1}, \ldots, x_{nT})$ the full sequence of choices and mileages observed in the data for bus n.
- At the m^{th} iteration:

$$\begin{aligned} q_{ns}^{(m+1)} &= \Pr\left\{ s \,\middle|\, d_n, x_n, \theta^{(m)}, \pi_s^{(m)}, p \right\} \\ &= \frac{\pi_s^{(m)} \prod_{t=1}^{T} l(d_{nt} | x_{nt}, s, p, \theta^{(m)})}{\sum_{s'=1}^{S} \pi_{s'}^{(m)} \prod_{t=1}^{T} l(d_{nt} | x_{nt}, s', p, \theta^{(m)})} \end{aligned}$$

$$\pi_s^{(m+1)} = N^{-1} \sum_{n=1}^{N} q_{ns}^{(m+1)}$$

$$\theta^{(m+1)} = \arg\max_{\theta} \sum_{n=1}^{N} \sum_{s=1}^{S} \sum_{t=1}^{T} q_{ns}^{(m+1)} \ln[l(d_{nt} | x_{nt}, s, p, \theta)]$$

Our algorithm begins by setting initial values for $\theta^{(1)}$, $\pi^{(1)}$, and $p^{(1)}(\cdot)$:

Step 1 Compute $q_{ns}^{(m+1)}$ as:

$$q_{ns}^{(m+1)} = \frac{\pi_s^{(m)} \prod_{t=1}^{T} l\left[d_{nt}|x_{nt}, s, p^{(m)}, \theta^{(m)}\right]}{\sum_{s'=1}^{S} \pi_s^{(m)} \prod_{t=1}^{T} l\left(d_{nt}|x_{nt}, s', p^{(m)}, \theta^{(m)}\right)}$$

Step 2 Compute $\pi_s^{(m+1)}$ according to:

$$\pi_s^{(m+1)} = \frac{\sum_{n=1}^{N} q_{ns}^{(m+1)}}{N}$$

Step 3 Update $p^{(m+1)}(x, s)$ using one of two rules below.

Step 4 Obtain $\theta^{(m+1)}$ from:

$$\theta^{(m+1)} = \arg\max_{\theta} \sum_{n=1}^{N} \sum_{s=1}^{S} \sum_{t=1}^{T} q_{ns}^{(m+1)} \ln\left[l\left(d_{nt}|x_{nt}, s_n, p^{(m+1)}, \theta\right)\right.$$

Motivating Example

Updating the CCP's

- Take a weighted average of decisions to replace engine, conditional on x, where weights are the conditional probabilities of being in unobserved state s.

Step 3A Update CCP's with:

$$p^{(m+1)}(x,s) = \frac{\sum_{n=1}^{N}\sum_{t=1}^{T} d_{1nt} q_{ns}^{(m+1)} I(x_{nt}=x)}{\sum_{n=1}^{N}\sum_{t=1}^{T} q_{ns}^{(m+1)} I(x_{nt}=x)}$$

- Or in a stationary infinite horizon model use identity from model that likelihood returns CCP of replacing the engine:

Step 3B Update CCP's with:

$$p^{(m+1)}(x_{nt}, s_n) = l(d_{nt1}=1|x_{nt}, s_n, p^{(m)}, \theta^{(m)})$$

- Suppose $s \in \{0, 1\}$ equally weighted.
- There are two observed state variables
  1. total accumulated mileage:

$$x_{1t+1} = \begin{cases} \Delta_t \text{ if } d_{1t} = 1 \\ x_{1t} + \Delta_t \text{ if } d_{2t} = 1 \end{cases}$$

  2. permanent route characteristic for the bus, x_2, that systematically affects miles added each period.
- We assume $\Delta_t \in \{0, 0.125, \ldots, 24.875, 25\}$ is drawn from:

$$f(\Delta_t | x_2) = \exp\left[-x_2(\Delta_t - 25)\right] - \exp\left[-x_2(\Delta_t - 24.875)\right]$$

and x_2 is a multiple 0.01 drawn from a discrete equi-probability distribution between 0.25 and 1.25.

- Let θ_{0t} be an aggregate shock (denoting cost fluctuations say).
- The difference in current payoff from retaining versus replacing the engine is:

$$u_{2t}(x_{1t}, s) - u_{1t}(x_{1t}, s) \equiv \theta_{0t} + \theta_1 \min\{x_{1t}, 25\} + \theta_2 s$$

- Denoting the observed state variables by $x_t \equiv (x_{1t}, x_2)$, this translates to:

$$\begin{aligned} v_{2t}(x_t, s) - v_{1t}(x_t, s) &= \theta_{0t} + \theta_1 \min\{x_{1t}, 25\} + \theta_2 s \\ &\quad + \beta \sum_{\Delta_t \in \Lambda} \left\{ \ln\left[\frac{p_{1t}(0, s)}{p_{1t}(x_{1t} + \Delta_t, s)} \right] \right\} f(\Delta_t | x_2) \end{aligned}$$

Table 1 of Arcidiacono and Miller (2011, page 1854)

	DGP	s Observed FIML	s Observed CCP	Ignoring s CCP	s Unobserved FIML	s Unobserved CCP	Time Effects s Observed CCP	Time Effects s Unobserved CCP
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
θ_0 (intercept)	2	2.0100	1.9911	2.4330	2.0186	2.0280		
		(0.0405)	(0.0399)	(0.0363)	(0.1185)	(0.1374)		
θ_1 (mileage)	−0.15	−0.1488	−0.1441	−0.1339	−0.1504	−0.1484	−0.1440	−0.1514
		(0.0074)	(0.0098)	(0.0102)	(0.0091)	(0.0111)	(0.0121)	(0.0136)
θ_2 (unobs. state)	1	0.9945	0.9726		1.0073	0.9953	0.9683	1.0067
		(0.0611)	(0.0668)		(0.0919)	(0.0985)	(0.0636)	(0.1417)
β (discount factor)	0.9	0.9102	0.9099	0.9115	0.9004	0.8979	0.9172	0.8870
		(0.0411)	(0.0554)	(0.0591)	(0.0473)	(0.0585)	(0.0639)	(0.0752)
Time (minutes)		130.29	0.078	0.033	275.01	6.59	0.079	11.31
		(19.73)	(0.0041)	(0.0020)	(15.23)	(2.52)	(0.0047)	(5.71)

[a]Mean and standard deviations for 50 simulations. For columns 1–6, the observed data consist of 1000 buses for 20 periods. For columns 7 and 8, the intercept (θ_0) is allowed to vary over time and the data consist of 2000 buses for 10 periods. See the text and the Supplemental Material for additional details.

- Suppose there is a finite maximum number of firms in a market at any one time denoted by I.
- If a firm exits, the next period an opening occurs to a potential entrant, who may decide to exercise this one time option, or stay out.
- At the beginning of each period every incumbent firm has the option of quitting the market or staying one more period.
- Let $d_t^{(i)} \equiv \left(d_{t1}^{(i)}, d_{t2}^{(i)}\right)$, where $d_{t1}^{(i)} = 1$ means i exits or stays out of the market in period t, and $d_{t2}^{(i)} = 1$ means i enters or does not exit.
- If $d_{t2}^{(i)} = 1$ and $d_{t-1,1}^{(i)} = 1$ then the firm in spot i at time t is an entrant, and if $d_{t-1,2}^{(i)} = 1$ the spot i at time t is an incumbent.

Entry Exit Game

State variables

- In this application there are three components to the state variables and $x_t = (x_1, x_{2t}, s_t)$.
- The first is a permanent market characteristic, denoted by x_1, and is common across firms in the market. Each market faces an equal probability of drawing any of the possible values of x_1 where $x_1 \in \{1, 2, \ldots, 10\}$.
- The second, x_{2t}, is whether or not each firm is an incumbent, $x_{2t} \equiv \{d_{t-1,2}^{(1)}, \ldots, d_{t-1,2}^{(I)}\}$. Entrants pay a start up cost, making it more likely that stayers choose to fill a slot than an entrant.
- A demand shock $s_t \in \{1, \ldots, 5\}$ follows a first order Markov chain.
- In particular, the probability that $s_{t+1} = s_t$ is fixed at $\pi \in (0, 1)$, and probability of any other state occurring is equally likely:

$$\Pr\{s_{t+1} \,|s_t\} = \begin{cases} \pi \text{ if } s_{t+1} = s_t \\ (1-\pi)/4 \text{ if } s_{t+1} \neq s_t \end{cases}$$

- Each active firm produces one unit so revenue, denoted by y_t, is just price.
- Price is determined by:
  1. the supply of active firms in the market, $\sum_{i=1}^{I} d_{t2}^{(i)}$
  2. a permanent market characteristic, x_1
  3. the Markov demand shock s_t
  4. another temporary shock, denoted by η_t, distributed *iId* standard normal distribution, revealed to each market after the entry and exit decisions are made.
- The price equation is:

$$y_t = \alpha_0 + \alpha_1 x_1 + \alpha_2 s_t + \alpha_3 \sum_{i=1}^{I} d_{t2}^{(i)} + \eta_t$$

Entry Exit Game

Expected profits conditional on competition

- We assume costs comprise a choice specific disturbance $\epsilon_{tj}^{(i)}$ that is privately observed, plus a linear function of $\left(x_t^{(i)}, s_t^{(i)}, d_t^{(-i)}\right)$.
- Net current profits for exiting incumbent firms, and potential entrants who do not enter, are $\epsilon_{1t}^{(i)}$. Thus $U_1^{(i)}\left(x_t^{(i)}, s_t^{(i)}, d_t^{(-i)}\right) \equiv 0$.
- Current profits from being active are the sum of $\left(\epsilon_{2t}^{(i)} + \eta_t\right)$ and:

$$U_2^{(i)}\left(x_t^{(i)}, s_t^{(i)}, d_t^{(-i)}\right) \equiv \theta_0 + \theta_1 x_1 + \theta_2 s_t + \theta_3 \sum_{\substack{i'=1 \\ i' \neq i}}^{I} d_{2t}^{(i')} + \theta_4 d_{1,t-1}^{(i)}$$

  where θ_4 is the startup cost that only entrants pay.
- In equilibrium $E(\eta_t) = 0$ so:

$$u_j^{(i)}(x_t, s_t) = \theta_0 + \theta_1 x_1 + \theta_2 s_t + \theta_3 \sum_{\substack{i'=1 \\ i' \neq i}}^{I} p_2^{(i')}(x_t, s_t) + \theta_4 d_{1,t-1}^{(i)}$$

Entry Exit Game

Terminal choice property

- We assume the firm's private information, $\epsilon_{jt}^{(i)}$, is distributed T1EV.
- In Lecture 5 we showed that since exiting is a terminal choice, with a payoff normalized to zero, given T1EV, the conditional value function for being active is:

$$\begin{aligned} v_2^{(i)}(x_t, s_t) &= u_2^{(i)}(x_t, s_t) \\ &\quad -\beta \sum_{x \in X} \sum_{s \in S} \left(\ln \left[p_1^{(i)}(x, s) \right] \right) f_2^{(i)}(x, s | x_t, s_t) \end{aligned}$$

- The future value term is then expressed as a function solely of the one-period-ahead probabilities of exiting and the transition probabilities of the state variables.

- The number of firms in each market is set to six and we simulated data for 3,000 markets.
- The discount factor is set to $\beta = 0.9$.
- Starting at an initial date with six potential entrants in the market, we solved the model, ran the simulations forward for twenty periods, and used the last ten periods to estimate the model.
- The key difference between this Monte Carlo and the renewal Monte Carlo is that the conditional choice probabilities have an additional effect on both current utility and the transitions on the state variables due to the effect of the choices of the firm's competitors on profits.

Entry Exit Game

Results from Monte Carlo simulations (Arcidiacono and Miller, 2011)

	DGP (1)	s_t Observed (2)	Ignore s_t (3)	CCP Model (4)	CCP Data (5)	Two-Stage (6)	No Prices (7)
Profit parameters							
θ_0 (intercept)	0	0.0207	−0.8627	0.0073	0.0126	−0.0251	−0.0086
		(0.0779)	(0.0511)	(0.0812)	(0.0997)	(0.1013)	(0.1083)
θ_1 (obs. state)	0.05	−0.0505	−0.0118	−0.0500	−0.0502	−0.0487	−0.0495
		(0.0028)	(0.0014)	(0.0029)	(0.0041)	(0.0039)	(0.0038)
θ_2 (unobs. state)	.25	0.2529		0.2502	0.2503	0.2456	0.2477
		(0.0080)		(0.0123)	(0.0148)	(0.0148)	(0.0158)
θ_3 (no. of competitors)	−0.2	−0.2061	0.1081	−0.2019	−0.2029	−0.1926	−0.1971
		(0.0207)	(0.0115)	(0.0218)	(0.0278)	(0.0270)	(0.0294)
θ_4 (entry cost)	−1.5	−1.4992	−1.5715	−1.5014	−1.4992	−1.4995	−1.5007
		(0.0131)	(0.0133)	(0.0116)	(0.0133)	(0.0133)	(0.0139)
Price parameters							
α_0 (intercept)	7	6.9973	6.6571	6.9991	6.9952	6.9946	
		(0.0296)	(0.0281)	(0.0369)	(0.0333)	(0.0335)	
α_1 (obs. state)	−0.1	−0.0998	−0.0754	−0.0995	−0.0996	−0.0996	
		(0.0023)	(0.0025)	(0.0028)	(0.0028)	(0.0028)	
α_2 (unobs. state)	0.3	0.2996		0.2982	0.2993	0.2987	
		(0.0045)		(0.0119)	(0.0117)	(0.0116)	
α_3 (no. of competitors)	−0.4	−0.3995	−0.2211	−0.3994	−0.3989	−0.3984	
		(0.0061)	(0.0051)	(0.0087)	(0.0088)	(0.0089)	
π (persistence of unobs. state)	0.7			0.7002	0.7030	0.7032	0.7007
				(0.0122)	(0.0146)	(0.0146)	(0.0184)
Time (minutes)		0.1354	0.1078	21.54	27.30	15.37	16.92
		(0.0047)	(0.0010)	(1.5278)	(1.9160)	(0.8003)	(1.6467)

Mean and standard deviations for 100 simulations. Observed data consist of 3000 markets for 10 periods with 6 firms in each market. In column 7, the CCP's are updated with the model.