IDENTIFICATION & ESTIMATION OF NON-EQUILIBRIUM BELIEFS IN GAMES OF INCOMPLETE INFORMATION

LECTURE 16
Econometric Society Summer School
in Dynamic Structural Econometrics

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UCL – July 5, 2025

Games of Oligopoly Competition with Non-Equilibrium Beliefs

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1. Introduction

Firms' Information & Beliefs

- In oligopoly markets, a firm's behavior depends on its **beliefs about the behavior of other firms** in the market.
- Firms form their beliefs under **uncertainty and asymmetric information** .
- Firms are different in their **ability for collecting and processing information**, for similar reasons as they are heterogeneous in their costs of producing goods and services.
- We expect firms to be **heterogeneous in their beliefs**.
- This heterogeneity has implications on their performance and on market outcomes.

Sources of Biased Beliefs about Other Firms' Strategies

- In this lecture we focus on firms' uncertainty and potentially biased beliefs about about other competitors' strategies.

- There are different potential sources of bias in these beliefs:

1. **Limited information / attention**: Some firms do not have information about variables that are known to other firms.

2. **Bounded rationality**: Limited capacity to process information / to compute.

3. **Strategic uncertainty**: With multiple equilibria, firms may have different beliefs about the selected equilibrium. Some firms believe

   they are playing the Bertrand equilibrium with high prices, other firms believe they are playing the equilibrium with low prices, and others believe that they are playing a collusive equilibrium.

Relaxing Firms' Rational Beliefs

- Despite these arguments, in most fields in economics (and IO in particular), the status quo is **assuming Rational Expectations**.
- There are good reasons to impose assumption of equilibrium beliefs. This **restriction has identification power**:
  1. In identification and estimation of structural parameters.
  2. In counterfactual analysis: model predicts how beliefs change endogenously.
- But it can be unrealistic in some applications, and can imply serious biases in our views on firms' competition.
- In this topic, we review some recent structural empirical papers of oligopoly competition that **relax the assumption of firms' rational beliefs**.

WHERE IS MORE LIKELY TO FIND BIASED BELIEFS?

- Taking into account the sources of biased beliefs mentioned above, it seems more likely to find this phenomenon:

    1. In **young industries** / markets, where managers are still learning about competitors' strategies.

    2. After substantial **structural changes** in the industry: regulatory changes, important mergers, technological change.

    3. In markets with substantial **heterogeneity in costs/demand** across firms, such that strategies can show substantial heterogeneity.

    4. In markets where **tacit collusion** is more likely.

    5. In markets where computing an optimal (or close to optimal) strategy is not trivial and there are **not simple heuristics / rules of thumb** (e.g., auctions in electricity markets; pricing in insurance markets).

2. STATIC GAMES: MODEL

Static Game: Profit Function

- N firms competing in a market. The profit function of firm i:

$$\Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x})$$

  - a_i is firm i's action, either continuous or discrete.
  - $\boldsymbol{a}_{-i}$ is the vector with the actions of the other firms.
  - $\mathbf{x}$ represents variables that are common knowledge.
  - ε_i is private information of firm i

- Firms' types $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_N)$ are drawn from a distribution F.

- Firms choose simultaneously their actions a_i to maximize their respective expected profits.

Static Games: Beliefs

- A firm does not know the private information of its competitors and therefore it does not know (ex-ante) their actions.

- Firms form probabilistic beliefs about the actions of competitors.

- Let $b_i(\boldsymbol{a}_{-i} \mid \varepsilon_i, \mathbf{x})$ be a probability density function that represents the belief of firm i.

Static Games: Best Response

- Given its beliefs, a **firm's expected profit** is:

$$\Pi_i^e(a_i, \varepsilon_i, \mathbf{x}; b_i) \quad = \quad \int_{\boldsymbol{a}_{-i}} \Pi(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x})\; b_i(\boldsymbol{a}_{-i} | \varepsilon_i, \mathbf{x})\; d\boldsymbol{a}_{-i}$$

- A firm chooses its **strategy function** $\sigma_i(\varepsilon_i, \mathbf{x}; b_i)$, to maximize expected profits:

$$\sigma_i(\varepsilon_i, \mathbf{x}; b_i) = \arg\max_{a_i \in \mathcal{A}} \; \Pi_i^e(a_i, \varepsilon_i, \mathbf{x}; b_i)$$

Characterization of Best Response Strategies

- Let $\Delta\Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x})$ be the **marginal profit function**:

$$\begin{cases} \text{continuous action:} & \Delta\Pi_i = \partial\Pi_i / \partial a_i \\ \text{discrete action:} & \Delta\Pi_i = \Pi_i(a_i) - \Pi_i(a_i - 1) \end{cases}$$

- **ASSUMPTION 1**: $\Delta\Pi_i$ is strictly monotonic in a_i and additively separable in ε_i:

$$\Delta\Pi_i(a_i, \boldsymbol{a}_{-i}, \varepsilon_i, \mathbf{x}) = \Delta\pi_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x}) - \varepsilon_i$$

- **ASSUMPTION 2**:
  - A. Independent private values ε_i.
  - B. The CDF F is strictly increasing .
  - C. The beliefs function $b_i(a_{-i}|\varepsilon_i, x)$ does not depend on ε_i.

Characterization of Best Response Strategies [2]

- Let $\Delta\pi_i^e(a_i, \mathbf{x}; b_i)$ be the expected marginal profit up to ε_i.

$$\Delta\pi_i^e(a_i, \mathbf{x}; b_i) \equiv \int_{\boldsymbol{a}_{-i}} \Delta\pi_i(a_i, \boldsymbol{a}_{-i}, \mathbf{x}) b_i(\boldsymbol{a}_{-i}|\mathbf{x}) d\boldsymbol{a}_{-i}$$

- **[A]** A necessary & sufficient condition for best response a_i is:

$$\begin{cases} \text{continuous action:} & \Delta\pi_i^e(a_i, \mathbf{x}; b_i) - \varepsilon_i = 0 \\ \text{discrete action:} & \Delta\pi_i^e(a_i + 1, \mathbf{x}; b_i) < \varepsilon_i \leq \Delta\pi_i^e(a_i, \mathbf{x}; b_i) \end{cases}$$

Characterization of Best Response Strategies [3]

- **[B]** For any value a^0, the **cumulative choice probability function (CCP)** $P_i(a^0|\mathbf{x}) \equiv \Pr(a_i \leq a^0|\mathbf{x})$ satisfies the condition:

$$P_i(a^0|\mathbf{x}) = F\left[\Delta\pi_i^e(a^0, \mathbf{x}; b_i) \mid \mathbf{x}\right] \quad \text{for } a^0 > 0$$

- **[C]** The **quantile function** $Q_i(a^0|x) \equiv F^{-1}\left[P_i(a^0|\mathbf{x})\right]$:

$$Q_i(a^0|\mathbf{x}) = \Delta\pi_i^e(a^0, \mathbf{x}; b_i) \quad \text{for } a^0 > 0$$

Example 1: Cournot with Independent Private Costs

- $a_i \in \mathbb{R}_+$ is firm i's amount of output.
- Inverse demand function is $p = p(Q, \mathbf{x})$ where $Q = \sum_{i=1}^{N} a_i$.
- A firm's marginal cost function is $c_i(a_i, \mathbf{x}) + \varepsilon_i$.
- Then, the quantile condition (= expected marginal profit up to ε_i):

$$
\begin{aligned}
Q_i(a^0|\mathbf{x}) &= -c_i(a^0, \mathbf{x}) \\
&\quad + \int_{\boldsymbol{a}_{-i}} p\left(a^0 + \sum_{j \neq i} a_j, \mathbf{x}\right) + p'\left(a^0 + \sum_{j \neq i} a_j, \mathbf{x}\right) a^0 \, b_i(\boldsymbol{a}_{-i}|\mathbf{x})
\end{aligned}
$$

Example 2: Market Entry with IP Entry Costs

- $a_i \in \{0, 1\}$ indicator of "firm i is active in the market".
- A firm's profit **if not active** is zero, $\Pi_i(0) = 0$.
- A firm's profit **if active** is: $\Pi_i(1) = v_i(\boldsymbol{a}_{-i}, \mathbf{x}) - ec_i(\mathbf{x}) - \varepsilon_i$.
- Then, the quantile condition is:

$$Q_i(1|\mathbf{x}) \quad = \quad -ec_i(\mathbf{x}) + \sum_{\boldsymbol{a}_{-i}} v_i(\boldsymbol{a}_{-i}, \mathbf{x}) \, b_i(\boldsymbol{a}_{-i}|\mathbf{x})$$

Example 3: Procurement Auction with IP Costs

- $a_i \in \mathbb{R}$ represents firm i's bid.
- Profit function: $\Pi_i = (a_i - c_i(\mathbf{x}) - \varepsilon_i)\ 1\{a_j > a_i\ \forall j \neq i\}$
- The expected profit function is:

$$\Pi_i^e(a_i, \varepsilon_i, \mathbf{x}; b_i) \quad = \quad (a_i - c_i(\mathbf{x}) - \varepsilon_i)\ \ W(a_i, \mathbf{x}, b_i)$$

where $W(a_i, \mathbf{x}, b_i) \equiv_{\boldsymbol{a}_{-i}} 1\{a_j > a_i\ \forall j \neq i\}\ b_i(\boldsymbol{a}_{-i}|\mathbf{x})\ d\boldsymbol{a}_{-i}$ is firm i's subjective probability of wining the auction.

Example 3: Procurement Auctions [2]

- The expected marginal profit function is:

$$\Delta\Pi_i^e(a_i, \varepsilon_i, \mathbf{x}; b_i) \quad = \quad W(a_i, \mathbf{x}, b_i) + (a_i - c_i(\mathbf{x}) - \varepsilon_i)\ \Delta W(a_i, \mathbf{x}, b_i),$$

where $\Delta W(a_i, \mathbf{x}, b_i) = \partial W(a_i, \mathbf{x}, b_i) / \partial a_i$.

- We have that $\sigma_i(\varepsilon_i, \mathbf{x}, b_i) = a^0$ if and only if

$$W(a^0, \mathbf{x}, b_i) + (a^0 - c_i(\mathbf{x}) - \varepsilon_i)\Delta W(a^0, \mathbf{x}, b_i) = 0$$

- Then, $\sigma_i(\varepsilon_i, \mathbf{x}, b_i) \leq a^0$ iff $\varepsilon_i \leq a^0 - c_i(\mathbf{x}) + \frac{W(a^0, \mathbf{x}, b_i)}{\Delta W(a^0, \mathbf{x}, b_i)}$ such that:

$$Q_i(a^0|\mathbf{x}) = a^0 - c_i(\mathbf{x}) + \frac{W(a^0, \mathbf{x}, b_i)}{\Delta W(a^0, \mathbf{x}, b_i)}$$

General Model of Firms' Beliefs

$$P_i\left(a^0|\mathbf{x}\right) = F\left[\int \Delta\pi_i(a^0, \boldsymbol{a}_{-i}, \mathbf{x})\; b_i(\boldsymbol{a}_{-i}|\mathbf{x})\; d\boldsymbol{a}_{-i}\right]$$

- These best response conditions contain all the restrictions of the model on beliefs function b_i and profit function $\Delta\pi_i$.

- Many models of competition in IO – under different types of equilibrium concepts are particular versions of this model.

- **Auctions, Bertrand competition, Cournot competition, Entry models** under different types of restrictions on firms' beliefs:
  - o **Bayesian Nash equilibrium**.
  - o **Level-K and Cognitive Hierarchy Beliefs**.
  - o **Rationalizability.**

Bayesian Nash Equilibrium

- Under Bayesian Nash Equilibrium (with independent private values):

$$b_i(\boldsymbol{a}_{-i} \mid \mathbf{x}) = \Pr\left(\boldsymbol{a}_{-i} \mid \mathbf{x}\right)$$

- This is the most commonly used solution concept in games of incomplete information in IO.

- It has received particular attention in **auction games** and in discrete choice models of **market entry**, but it has been also applied to games of quantity or price competition.

Cognitive Hierarchy & Level-K Models

- Equilibrium concepts where firms have biased beliefs, that is, $b_i(\boldsymbol{a}_{-i} \mid \mathbf{x}) \neq \Pr(\boldsymbol{a}_{-i} \mid \mathbf{x})$.

- There is a finite number K of belief types that correspond to different levels of strategic sophistication.

- Believes for Level-0 can be arbitrary, $b^{(0)}$ (or CDF $B^{(0)}$).

- Level-1 firms believe that all the other firms are level 0: (CDF $B^{(1)}$)

$$B^{(1)}(\boldsymbol{a}_{-i} \mid \mathbf{x}) = \prod_{j \neq i} F\left[\int \Delta\pi(a_j, \boldsymbol{a}_{-j}, \mathbf{x})\, dB^{(0)}(\boldsymbol{a}_{-j} | \mathbf{x})\right]$$

Cognitive Hierarchy & Level-K Models [2]

- In **Level-k model**, level-k firm believes all other firms are $k-1$:

$$B^{(k)}\left(\boldsymbol{a}_{-i} \mid \mathbf{x}\right) \quad = \quad \prod_{j \neq i} F\left[\int \Delta\pi(a_j, \boldsymbol{a}_{-j}, \mathbf{x})\, dB^{(k-1)}(\boldsymbol{a}_{-j}|\mathbf{x})\right]$$

- In **Cognitive Hierarchy model**, a level-k firm believes that all other firms come from a probability distribution over levels 0 to $k-1$.

- These models impose restrictions on beliefs.
  - o There is a finite number K of belief types (typically 2 or 3).
  - o These belief functions satisfy a hierarchical equilibrium.

Rationalizability

- The concept of *Rationalizability* (Bernheim, 1984; Pearce, 1984) imposes two restrictions on firms' beliefs and behavior.
  [A.1] Every firm is rational in the sense that it maximizes its own expected profit given beliefs.
  [A.2] This rationality is common knowledge, i.e., every firms knows that all the firms know that it knows ... that all the firms are rational.

- Aradillas-Lopez & Tamer (2008) study identification under Rationalizability.

- In a game **with multiple equilibria**, the solution concept of **Rationalizability allows for biased beliefs**.

- Each firm has beliefs that are consistent with a BNE, but these beliefs may not correspond to the same BNE.

3. STATIC GAMES: IDENTIFICATION

DATA

- The researcher has a sample of M local markets, indexed by m, where she observes firms' actions and state variables (**firms' choice data**):

$$\{a_{imt},\ \mathbf{x}_{mt} : i = 1, 2, ..., N; t = 1, 2, ..., T^{data}\}$$

- In addition to these data, the researcher may have data on some components of the profit function.

- I distinguish three cases, from better to worse case scenarios:

  1. Choice data + Revenue function + Cost function.
  2. Choice data + Revenue function
  3. Only Choice Data

Choice data + Revenue function + Cost function

- It is convenient now to distinguish between revenue and costs in the profit function:

$$\pi_i = r_i - c_i$$

- Such that:

$$\Delta\pi_i = \Delta r_i - \Delta c_i$$

- Both Δr_i and Δc_i may depend on the actions of other firms, $\boldsymbol{a}_{-i}$. It depends on the model, on the type of decision variable.
    - In an entry model or in a Cournot model, Δc_i typically does not depend on $\boldsymbol{a}_{-i}$.
    - In a model of price competition with differentiated product, Δc_i typically depends on $\boldsymbol{a}_{-i}$: Δc_i depends on the quantity produced & sold by i, this quantity depends (through demand) on the own price and the price of competitors.

Binary Choice – Two Players Game

- Game of price competition where firms choose between a low price ($a_i = 0$) and a high price ($a_i = 1$).

- Notation:

    - $\Delta r_i(a_{-i}, \mathbf{x}) \equiv r_i(1, a_{-i}, \mathbf{x}) - r_i(0, a_{-i}, \mathbf{x})$

    - $\Delta c_i(a_{-i}, \mathbf{x}) \equiv c_i(1, a_{-i}, \mathbf{x}) - c_i(0, a_{-i}, \mathbf{x})$

    - $p_i(\mathbf{x}) \equiv p_i(1|\mathbf{x}) =$ prob. choosing high price.

    - $b_i(\mathbf{x}) \equiv b_i(1|\mathbf{x}) =$ belief prob. competitor chooses high price.

- Expected marginal profit (up to ε_i):

$$\Delta\pi_i^e(\mathbf{x}) = [1 - b_i(\mathbf{x})]\, [\Delta r_i(0, \mathbf{x}) - \Delta c_i(0, \mathbf{x})] + b_i(\mathbf{x})\, [\Delta r_i(1, \mathbf{x}) - \Delta c_i(1, \mathbf{x})$$

Binary Choice – Two Players Game [2]

- Best response probability:

$$p_i(\mathbf{x}) = F\left(\begin{array}{c} \Delta r_i(0,\mathbf{x}) - \Delta c_i(0,\mathbf{x}) \\ +b_i(\mathbf{x}) \ \left[\Delta r_i(1,\mathbf{x}) - \Delta r_i(0,\mathbf{x}) - \Delta c_i(1,\mathbf{x}) + \Delta c_i(0,\mathbf{x})\right] \end{array} \right)$$

- Using quantile $Q_i(\mathbf{x}) \equiv F^{-1}\left(p_i(\mathbf{x})\right)$:

$$\begin{array}{rcl} Q_i(\mathbf{x}) & = & \Delta r_i(0,\mathbf{x}) - \Delta c_i(0,\mathbf{x}) \\ & + & b_i(\mathbf{x}) \ \left[\Delta r_i(1,\mathbf{x}) - \Delta r_i(0,\mathbf{x}) - \Delta c_i(1,\mathbf{x}) + \Delta c_i(0,\mathbf{x})\right] \end{array}$$

- For the moment, I assume that F is known to the researcher.

Identification with Revenue & Cost Data

- In the static case with two-players, beliefs are identified:

$$b_i(\mathbf{x}) = \frac{Q_i(\mathbf{x}) - \Delta r_i(0,\mathbf{x}) + \Delta c_i(0,\mathbf{x})}{\Delta r_i(1,\mathbf{x}) - \Delta r_i(0,\mathbf{x}) - \Delta c_i(1,\mathbf{x}) + \Delta c_i(0,\mathbf{x})}$$

- This belief function can be compared to the actual choice probability of the competitor to test unbiased / rational beliefs:

$$b_i(\mathbf{x}) - p_{-i}(\mathbf{x}) = 0 \quad ?$$

- We can also test other restrictions on beliefs such as level-K or Cognitive Hierarchy models.

- If panel data, we can study how beliefs evolve over time (learning).

Identification with Revenue but No Cost Data

- MR functions $\Delta r_i(0, \mathbf{x})$ and $\Delta r_i(1, \mathbf{x})$ are known to the researcher but the MC is not known.

- Without further restrictions, the system of equations

$$\begin{array}{rcl} Q_i(\mathbf{x}) & = & \Delta r_i(0, \mathbf{x}) - \Delta c_i(0, \mathbf{x}) \\ & + & b_i(\mathbf{x}) \ \left[\Delta r_i(1, \mathbf{x}) - \Delta r_i(0, \mathbf{x}) - \Delta c_i(1, \mathbf{x}) + \Delta c_i(0, \mathbf{x}) \right] \end{array}$$

  cannot identify the unknown functions $b_i(\mathbf{x})$ and Δc_i.

- Without further restrictions, any belief function (including the BNE belief) is consistent with observed behavior, $Q_i(\mathbf{x})$, given the appropriate Δc_i function.

Identification: Firm-Specific Cost Shifter

- **Exclusion Restriction (Firm specific cost shifter)**:

- The vector $\mathbf{x}$ has a firm-specific components that affect the marginal profit of a firm but not the marginal profit of other firms.

- That is, $\mathbf{x} = (\widetilde{\mathbf{x}}, \mathbf{z}_i, \mathbf{z}_{-i})$ such that:

$$\Delta\pi_i(a_i, a_{-i}, \mathbf{x}) = \Delta\pi_i(a_i, a_{-i}, \widetilde{\mathbf{x}}, \mathbf{z}_i)$$

- EXAMPLES:
  - Wage rate that a firm pays to its workers.
  - Wholesale prices that a firm pays to its suppliers.
  - The firm's installed capacity.
  - The firm's capital stock.
  - The firm's Total Factor Productivity.
  - ...

IDENTIFICATION OF BELIEFS

- The **observed change in the behavior of firm i given a change in $\mathbf{z}$ of its competitors** can identify firm i's beliefs.

- Let A, B, and C be three values for $\mathbf{z}_{-i}$. Then:

$$\left\{ \begin{array}{l} Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, B) - Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, A) \\ = [b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, B) - b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, A)] \;\; [\Delta\pi_i(1, \widetilde{\mathbf{x}}, \mathbf{z}_i) - \Delta\pi_i(0, \widetilde{\mathbf{x}}, \mathbf{z}_i)] \\ \\ Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, C) - Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, A) \\ = [b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, C) - b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, A)] \;\; [\Delta\pi_i(1, \widetilde{\mathbf{x}}, \mathbf{z}_i) - \Delta\pi_i(0, \widetilde{\mathbf{x}}, \mathbf{z}_i)] \end{array} \right.$$

- And taking the ratio between these two differences, we have that:

$$\frac{b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, B) - b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, A)}{b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, C) - b_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, A)} = \frac{Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, B) - Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, A)}{Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, C) - Q_i(\widetilde{\mathbf{x}}, \mathbf{z}_i, A)}$$

- The object in LHS depends only on beliefs. The object in RHS is identified from firm i's observed behavior.

Testing Different Models of Beliefs & Learning

- Given this identified beliefs object, we can test different models or restrictions on beliefs such that:
    - Unbiased beliefs of firm i.
    - Bayesian Nash equilibrium.
    - Rationalizability.
    - Cognitive Hierarchy model; Level-k
    - ...

- If we have panel data over several periods of time, we can also test different models of learning / beliefs updating:
    - Bayesian learning
    - Fictitious play.
    - Adaptive learning.
    - ...

EXTENSIONS OF THIS IDENTIFICATION RESULT

- This result on the identification of beliefs about other firms' strategies can be extended to incorporate:
  - More than two players.
  - Continuous choice games.
  - Ordered multinomial choice games.
  - Dynamic games (discrete and continuous choice).
  - **Nonparametric distribution of private information** F.

4. Dynamic Games: Model

Model: Dynamic Game

- N firms indexed by i. Every period t, each firm takes an action $a_{it} \in \{0, 1, ..., J\}$.

- One-period **profit function** is:

$$\Pi_{it} = \pi_{it}(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t) + \varepsilon_{it}(x_{it})$$

- $\mathbf{x}_t$ = vector of common knowledge state vars. with **transition prob.**

$$f_t\left(\mathbf{x}_{t+1} \mid a_{it},\ \boldsymbol{a}_{-it},\ \mathbf{x}_t\right)$$

- $\varepsilon'_{it}s$ are private info of player i and unobservable to researcher. It is i.i.d. over time and players.

Maintain some assumptions from MPE

- **ASSUMPTION 1 (Payoff relevant state variables):** *Players' strategy functions depend only on payoff relevant state variables:* $\mathbf{x}_t$ and ε_{it}.

- **ASSUMPTION 2 (Maximization of intertemporal payoffs):** *Players are forward looking and maximize expected intertemporal payoffs.*

- **ASSUMPTION 3 (Rational beliefs on own future behavior):** *Players have rational expectations on their own behavior in the future.*

- We **relax** the assumption that firms have **unbiased or equilibrium beliefs on other players' behaviour,**

Strategies, Choice Probabilities, and Beliefs

- Let $\sigma_{it}(\mathbf{x}_t, \varepsilon_{it})$ be the strategy function for player i at period t.
- $P_{it}(a_i|\mathbf{x}_t) \equiv \Pr(\sigma_{it}(\mathbf{x}_t, \varepsilon_{it}) = a_i|\mathbf{x}_t)$ choice probability of player i.
- $B_{it+s}^{(t)}(\boldsymbol{a}_{-i}|\mathbf{x}_{t+s})$ **beliefs** of player i at period t about the behavior of other players at period $t+s$.
- The model allows the belief functions $B_{it+s}^{(t)}$ to vary freely both over t (i.e., over the period when these beliefs are formed) and over $t+s$ (i.e., over the period of the other players' behavior).
- In particular, the model allows players to update their beliefs and learn (or not) over time t.

Sequence of Beliefs $B_{it+s}^{(t)}$

Beliefs formed (t)	Period of the opponents' behavior ($t+s$)				
	$t+s=1$	$t+s=2$	$t+s=3$	...	$t+s=T$
$t=1$	$B_{i1}^{(1)}$	$B_{i2}^{(1)}$	$B_{i3}^{(1)}$	...	$B_{iT}^{(1)}$
$t=2$	-	$B_{i2}^{(2)}$	$B_{i3}^{(2)}$	...	$B_{iT}^{(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t=T$	-	-	-	...	$B_{iT}^{(T)}$

Best Response Functions

- Given her beliefs at period t, $\mathbf{B}_i(t) = \left\{ B_{i,t+s}^{(t)} : s \geq 0 \right\}$, a player best response at period t is the solution of a single-agent Dynamic Programming problem.

- At period t, the DP problem can be described in terms of: **(1)** a sequence of **expected one-period payoff functions**:

$$\pi_{it+s}^{\mathbf{B}(t)}(a_{it+s}, \mathbf{x}_{t+s}) \equiv \sum_{\boldsymbol{a}_{-i}} B_{it+s}^{(t)}(\boldsymbol{a}_{-i}|\mathbf{x}_{t+s}) \; \pi_{it+s}(a_{it+s}, \boldsymbol{a}_{-i}, \mathbf{x}_{t+s})$$

- And **(2)** a sequence of **transition probability functions**:

$$f_{it+s}^{\mathbf{B}(t)}(\mathbf{x}_{t+s+1}|a_{it+s}, \mathbf{x}_{t+s}) \equiv \sum_{\boldsymbol{a}_{-i}} B_{it+s}^{(t)}(\boldsymbol{a}_{-i}|\mathbf{x}_{t+s}) \, f_{t+s}(\mathbf{x}_{t+s+1}|a_{it+s}, \boldsymbol{a}_{-i}, \mathbf{x}_{t+s})$$

Best Response Functions [2]

- The solution of this DP problem implies the vector of **conditional choice value functions** at period t:

$$v_{it}^{\mathbf{B}(t)}(\mathbf{x}_t) = \left\{ v_{it}^{\mathbf{B}(t)}(a_i, \mathbf{x}_t) : a_i = 0, 1, ..., J \right\}$$

- And the **best response choice probabilities**:

$$P_{it}(a_i|\mathbf{x}_t) \quad = \quad \Pr\left(v_{it}^{\mathbf{B}(t)}(a_i, \mathbf{x}_t) + \varepsilon_{it}(a_i) \geq v_{it}^{\mathbf{B}(t)}(a_i', \mathbf{x}_t) + \varepsilon_{it}(a_i') \ \forall a_i' \right)$$

- For instance, in a logit model:

$$P_{it}(a_i|\mathbf{x}_t) \quad = \quad \frac{\exp\left\{ v_{it}^{\mathbf{B}(t)}(a_i, \mathbf{x}_t) \right\}}{\sum_{j=0}^{J} \exp\left\{ v_{it}^{\mathbf{B}(t)}(j, \mathbf{x}_t) \right\}}$$

Structure of the conditional choice values

- By definition the values $v_{it}^{\mathbf{B}(t)}(a_i, \mathbf{x}_t)$ have the following structure:

$$v_{it}^{\mathbf{B}(t)}(a_i, \mathbf{x}_t) \quad = \quad \mathbf{B}_{it}^{(t)}(\mathbf{x}_t)' \left[\boldsymbol{\pi}_{it}(a_{it}, \mathbf{x}_t) + \mathbf{c}_{it}^{\mathbf{B}_{(>t)}^{(t)}}(a_{it}, \mathbf{x}_t) \right]$$

- $\mathbf{B}_{it}^{(t)}(\mathbf{x}_t)$ = vector of beliefs $[B_{it}^{(t)}(\boldsymbol{a}_{-i}|\mathbf{x}_t)$ for any value $\boldsymbol{a}_{-i}]$.

- $\mathbf{B}_{(>t)}^{(t)}$ = beliefs formed at t on rivals' behavior at $t + s > t$.

- $\boldsymbol{\pi}_{it}(a_{it}, \mathbf{x}_t)$ = vector of payoffs $[\pi_{it}(a_{it}, \boldsymbol{a}_{-i}, \mathbf{x}_t)$ for any value $\boldsymbol{a}_{-i}]$.

- $\mathbf{c}_{it}^{\mathbf{B}_{(>t)}^{(t)}}(a_{it}, \mathbf{x}_t)$ = vector of **continuation values** $[\mathbf{c}_{it}^{\mathbf{B}_{(>t)}^{(t)}}(a_{it}, \boldsymbol{a}_{-i}, \mathbf{x}_t)$ for any value $\boldsymbol{a}_{-i}]$ with:

$$\mathbf{c}_{it}^{\mathbf{B}_{(>t)}^{(t)}}(a_{it}, \boldsymbol{a}_{-i}, \mathbf{x}_t) = \beta \sum V_{it+1}^{\mathbf{B}_{(>t)}^{(t)}}(\mathbf{x}_{t+1}) \, f_t(\mathbf{x}_{t+1}|a_{it}, \boldsymbol{a}_{-i}, \mathbf{x}_t)$$

5. Identification of Beliefs

Data

- Random sample of M markets, indexed by m, where we observe

$$\{a_{imt},\ \mathbf{x}_{mt} : i = 1, 2, ..., N; t = 1, 2, ..., T^{data}\}$$

- N and T^{data} are small and M is large.

- The payoff functions $\pi_{it}(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t)$ and the beliefs functions $B^{(t)}_{it+s}(\boldsymbol{a}_{-i}|\mathbf{x}_{t+s})$ are **nonparametrically specified**.

- The distribution of the unobservables Λ is assumed known.

- I focus here in a model with two players, i and j, but the paper results can be extended to N players.

Inversion of CCPs

- The model is described by the conditions:

$$P_{it}(a_i|\mathbf{x}_t) \quad = \quad \Lambda\left(a_i\,; v_{it}^{\mathbf{B}(t)}(\mathbf{x}_t)\right)$$

- The CCPs $P_{it}(a_i|\mathbf{x}_t)$ are identified using data from M markets.

- **Hotz-Miller inversion theorem** implies that we can invert the best response mapping to obtain value differences $\widetilde{v}_{it}^{\mathbf{B}(t)}(a_i, \mathbf{x}_t) \equiv v_{it}^{\mathbf{B}(t)}(a_i, \mathbf{x}_t) - v_{it}^{\mathbf{B}(t)}(0, \mathbf{x}_t)$ as functions of CCPs:

$$\widetilde{v}_{it}^{\mathbf{B}(t)}(a_i, \mathbf{x}_t) \quad = \quad \Lambda^{-1}\left(a_i\,; \mathbf{P}_{it}(\mathbf{x}_t)\right)$$

- The identification problem is to obtain beliefs and payoff functions given that $\Lambda^{-1}\left(a_i\,; \mathbf{P}_{it}(\mathbf{x}_t)\right)$ are known.

Structure of the restrictions

- Value differences $\widetilde{v}_{it}^{\mathbf{B}(t)}(a_i, \mathbf{x}_t)$ are:

$$\widetilde{v}_{it}^{\mathbf{B}(t)}(a_i, \mathbf{x}_t) \quad = \quad \mathbf{B}_{it}^{(t)}(\mathbf{x}_t)' \left[\widetilde{\boldsymbol{\pi}}_{it}(a_{it}, \mathbf{x}_t) + \widetilde{\mathbf{c}}_{it}^{\mathbf{B}_{(>t)}^{(t)}}(a_{it}, \mathbf{x}_t) \right]$$

- $\widetilde{\boldsymbol{\pi}}_{it}(a_{it}, \mathbf{x}_t) =$ vector of payoff differences $[\pi_{it}(a_{it}, \boldsymbol{a}_{-i}, \mathbf{x}_t) - \pi_{it}(0, \boldsymbol{a}_{-i}, \mathbf{x}_t)$ for any value $\boldsymbol{a}_{-i}]$.

- $\widetilde{\mathbf{c}}_{it}^{\mathbf{B}_{(>t)}^{(t)}}(a_{it}, \mathbf{x}_t) =$ vector of **differences of continuation values** $[\widetilde{\mathbf{c}}_{it}^{\mathbf{B}_{(>t)}^{(t)}}(a_{it}, \boldsymbol{a}_{-i}, \mathbf{x}_t) - \widetilde{\mathbf{c}}_{it}^{\mathbf{B}_{(>t)}^{(t)}}(0, \boldsymbol{a}_{-i}, \mathbf{x}_t)$ for any value $\boldsymbol{a}_{-i}]$.

Identification Assumptions

- **ASSUMPTION ID-1.** A player has the same beliefs in markets with the same $\mathbf{x}$ variables.

$$B_{imt+s}^{(t)}(.|\mathbf{x}) = B_{it+s}^{(t)}(.|\mathbf{x}) \quad \text{for any market } m$$

- **ASSUMPTION ID-2 (Static Exclusion Restriction)**: $\mathbf{x}_t = (s_{it}, s_{jt}, \mathbf{w}_t)$ such that s_{it} enters in the payoff function of player i but not in the payoff of the other player.

$$\pi_{it}\left(a_{it}, a_{jt}, s_{it}, s_{jt}, \mathbf{w}_t\right) = \pi_{it}\left(a_{it}, a_{jt}, s_{it}, \mathbf{w}_t\right)$$

- **ASSUMPTION ID-3 (Dynamic Exclusion Restriction)**: The transition probability of the state variable s_{it} is such that the value of s_{it+1} does not depend on (s_{it}, s_{jt}):

$$f_t\left(s_{it+1} \mid a_{it}, s_{it}, s_{jt}, \mathbf{w}_t\right) = f_t\left(s_{it+1} \mid a_{it}, \mathbf{w}_t\right)$$

Static Exclusion Restriction (ID-2)

- The exclusion restriction ID-2 appears naturally in many applications of dynamic games of oligopoly competition.
- Incumbent status, capacity, capital stock, or product quality of a firm at period $t-1$ are state variables that enter in a firm's payoff function at period t because there are investment and adjustment costs that depend on these lagged variables.
- A firm's payoff π_{it} depends also on the competitors' values of these variables at period t, but it does not depend on the competitors' values of these variables at $t-1$.
- Importantly, the assumption does not mean that player i does not condition her behavior on those excluded variables. Each player conditions his behavior on all the (common knowledge) state variables that affect the payoff of a player in the game, even if these variables are excluded from his own payoff.

Dynamic Exclusion Restriction (ID-3)

$$f_t\left(s_{it+1} \mid a_{it}, s_{it}, s_{jt}, \mathbf{w}_t\right) = f_t\left(s_{it+1} \mid a_{it}, \mathbf{w}_t\right)$$

- An important class of models that satisfies this condition is when $s_{it} = a_{i,t-1}$, such that the transition rule is simply:

$$s_{it+1} = a_{it}$$

- Many dynamic games of oligopoly competition belong to this class, e.g., market entry/exit, technology adoption, and some dynamic games of quality or capacity competition, among others.

Example: Quality competition

- Quality ladder dynamic game (Pakes and McGuire, 1994).
- s_{it} is the firm's quality at $t-1$.
- The decision variable a_{it} is the firm's quality at period t, such that:

$$s_{it+1} = a_{it}$$

- The model is dynamic because the payoff function includes a cost of adjusting quality that depends on $a_{it} - s_{it}$:

$$AC_i \left(a_{it} - s_{it} \right)$$

- Given competitors quality at period t, a_{jt}, firm i's profit does not depend on competitors' qualities at $t-1$.

Role of the Exclusion restrictions

$$\ln\left(\frac{P_{it}\left(a_i \left| s_{it}, \mathbf{s}_{-it}\right.\right)}{P_{it}\left(0 \left| s_{it}, \mathbf{s}_{-it}\right.\right)}\right) \quad = \quad \mathbf{B}_{it}^{(t)}(s_{it}, \mathbf{s}_{-it})' \left[\widetilde{\boldsymbol{\pi}}_{it}(a_{it}, s_{it}) + \widetilde{\mathbf{c}}_{it}^{\mathbf{B}_{(>t)}^{(t)}}(a_{it}, s_{it}\right]$$

- Under the two exclusion restrictions, the state variables $\mathbf{s}_{-it}$ (the competitors s_j) do not enter in the payoffs $\widetilde{\boldsymbol{\pi}}_{it}(a_{it}, s_{it})$ and on the continuation values $\widetilde{\mathbf{c}}_{it}^{\mathbf{B}_{(>t)}^{(t)}}(a_{it}, s_{it})$.

- Note: Though $\widetilde{\mathbf{c}}_{it}^{\mathbf{B}_{(>t)}^{(t)}}(a_{it}, s_{it})$ depends on beliefs, these are beliefs at periods $t+s>t$ and therefore depend on $(s_{it+s}, \mathbf{s}_{-it+s})$ for $t+s>t$.

- Therefore, the dependence of $\ln\left(\frac{P_{it}\left(a_i \left| s_{it}, \mathbf{s}_{-it}\right.\right)}{P_{it}\left(0 \left| s_{it}, \mathbf{s}_{-it}\right.\right)}\right)$ with respect to $\mathbf{s}_{-it}$ captures the dependence of beliefs $\mathbf{B}_{it}^{(t)}(s_{it}, \mathbf{s}_{-it})$ with respect to $\mathbf{s}_{-it}$.

Identification of Beliefs

- For any player i, any period t in the data, any value of $(\boldsymbol{a}_{-i}, s_{it})$, and any combination of three values $\mathbf{s}_{-it}$, say $(\mathbf{s}_{-i}^{(a)}, \mathbf{s}_{-i}^{(b)}, \mathbf{s}_{-i}^{(c)})$, the following function of beliefs is identified:

$$\frac{B_{it}^{(t)}(\boldsymbol{a}_{-i} \mid s_{it}, \mathbf{s}_{-i}^{(c)}) - B_{it}^{(t)}(\boldsymbol{a}_{-i} \mid s_{it}, \mathbf{s}_{-i}^{(a)})}{B_{it}^{(t)}(\boldsymbol{a}_{-i} \mid s_{it}, \mathbf{s}_{-i}^{(b)}) - B_{it}^{(t)}(\boldsymbol{a}_{-i} \mid s_{it}, \mathbf{s}_{-i}^{(a)})}$$

Identification of Beliefs [2]

- For instance, in a binary choice logit with two-players:

$$\frac{B_{it}^{(t)}(1 \mid s_{it}, \mathbf{s}_{-i}^{(c)}) - B_{it}^{(t)}(1 \mid s_{it}, \mathbf{s}_{-i}^{(a)})}{B_{it}^{(t)}(1 \mid s_{it}, \mathbf{s}_{-i}^{(b)}) - B_{it}^{(t)}(1 \mid s_{it}, \mathbf{s}_{-i}^{(a)})} =$$

$$\frac{\ln\left(\dfrac{P_{it}(1 \mid s_{it}, \mathbf{s}_{-i}^{(c)})}{P_{it}(0 \mid s_{it}, \mathbf{s}_{-i}^{(c)})}\right) - \ln\left(\dfrac{P_{it}(1 \mid s_{it}, \mathbf{s}_{-i}^{(a)})}{P_{it}(0 \mid s_{it}, \mathbf{s}_{-i}^{(a)})}\right)}{\ln\left(\dfrac{P_{it}(1 \mid s_{it}, \mathbf{s}_{-i}^{(b)})}{P_{it}(0 \mid s_{it}, \mathbf{s}_{-i}^{(b)})}\right) - \ln\left(\dfrac{P_{it}(1 \mid s_{it}, \mathbf{s}_{-i}^{(a)})}{P_{it}(0 \mid s_{it}, \mathbf{s}_{-i}^{(a)})}\right)}$$

- Note that we cannot identify beliefs about competitors' behavior at future periods: $B_{it+s}^{(t)}$ for $s > 0$. However, $B_{it}^{(t)}$ can provide substantial information about learning.

EMPIRICAL APPLICATION

- Dynamic game of store location by McDonalds (MD) and Burger King (BK) using data for United Kingdom during the period 1990-1995.
- Panel of 422 local markets (districts) and six years, 1990-1995.
- Information on the number of stores of McDonalds (MD) and Burger King (BK) in United Kingdom.
- Information on local market characteristics such as population, density, income per capita, age distribution, average rent, local retail taxes, and distance to the headquarters of each firm in UK.

Evolution of the Number of Stores & Markets										
Data: 422 markets, 2 firms, 5 years = 4,220 observations										
	Burger King					**McDonalds**				
	1991	1992	1993	1994	1995	1991	1992	1993	1994	1995
# Markets	98	104	118	131	150	213	220	237	248	254
Δ# Markets	17	6	14	13	19	7	7	17	11	6
# of stores	115	128	153	181	222	316	344	382	421	447
Δ# of stores	36	13	25	28	41	35	28	38	39	26
stores per market	1.17	1.23	1.30	1.38	1.48	1.49	1.56	1.61	1.70	1.76

Model

- $k_{imt} \in \{0, 1, ..., |\mathcal{K}|\}$ number of stores of firm i in market m at period $t-1$.
- $a_{imt} \in \{0, 1\}$ decision of firm i to open a new store.
- $a_{imt} + k_{imt}$ = # stores of firm i at period t.
- S_{imt} = Distance of the centroid of marker m to the centroid of the closest market where firm i had stores at year $t-1$.
- Firm i's total profit function is equal to:

$$\Pi_{imt} = VP_{imt} - EC_{imt} - FC_{imt}$$

Model [2]

- Variable profit function:

$$VP_{imt} = (\mathbf{W}_m \, \gamma) \, (a_{imt} + k_{imt}) \left[\begin{array}{c} \theta_{0i}^{VP} + \theta_{can,i}^{VP}(k_{imt} + a_{imt}) \\ +\theta_{com,i}^{VP}(a_{jmt} + k_{jmt}) \end{array} \right]$$

- Entry cost:

$$EC_{imt} \quad = \quad 1\{a_{imt} > 0\} \, \left[\theta_{0i}^{EC} + \theta_{K,i}^{EC} \, 1\{k_{imt} > 0\} + \theta_{S,i}^{EC} \, S_{imt} + \varepsilon_{it}\right]$$

- Fixed cost:

$$FC_{imt} \quad = \quad 1\{(k_{imt} + a_{imt}) > 0\} \left[\begin{array}{c} \theta_{0i}^{FC} + \theta_{lin,i}^{FC}(k_{imt} + a_{imt}) \\ +\theta_{qua,i}^{FC}(k_{imt} + a_{imt})^2 \end{array} \right]$$

Tests of Unbiased Beliefs

Data: 422 markets, 5 years = 2,110 observations

BK: $\widehat{D}$ (p-value)	66.841 (0.00029)
MD: $\widehat{D}$ (p-value)	42.838 (0.09549)

- We can reject hypothesis that BK beliefs are unbiased (p-value 0.00029).
- Restriction is more clearly rejected for large values of the state variable (distance to chain network) S_{MD}.

Where to impose unbiased beliefs?

- We propose three different criteria:
  [1] Minimize distance $\left\|B_i - P_j\right\|$
  [2] Impose unbiased beliefs for smallest values of S_j.
  [3] Most visited values of S_j.
- In this empirical application, the three criteria have the same implication: impose unbiased beliefs at the lowest values for the distance S_j.

Estimation of Dynamic Game

Data: 422 markets, 2 firms, 5 years = 4,220 observations

	$\beta = 0.95$ (not estimated)			
	Equilibrium Beliefs		Biased Beliefs	
	BK	MD	BK	MD
Var Profits:				
θ_0^{VP}	0.5413	0.8632	0.4017	0.8271
	$(0.1265)^*$	$(0.2284)^*$	$(0.2515)^*$	$(0.4278)^*$
θ_{can}^{VP} cannibalization	-0.2246	0.0705	-0.2062	0.0646
	$(0.0576)^*$	$(0.0304)^*$	$(0.1014)^*$	(0.0710)
θ_{com}^{VP} competition	**-0.0541**	-0.0876	**-0.1133**	-0.0856
	(0.0226)*	(0.0272)	**(0.0540)***	(0.0570)
Log-Likelihood	-848.4		-840.4	

Estimation of Dynamic Game

Data: 422 markets, 2 firms, 5 years = 4,220 observations

	$\beta = 0.95$ (not estimated)			
	Equilibrium Beliefs		Biased Beliefs	
	BK	MD	BK	MD
Fixed Costs:				
θ_0^{FC} fixed	0.0350	0.0374	0.0423	0.0307
	(0.0220)	(0.0265)	(0.0478)	(0.0489)
θ_{lin}^{FC} linear in k	0.0687	0.0377	0.0829	0.0467
	$(0.0259)^*$	$(0.0181)^*$	$(0.0526)^*$	(0.0291)
θ_{qua}^{FC} quadratic in k	-0.0057	0.0001	-0.0007	0.0002
	(0.0061)	(0.0163)	(0.0186)	(0.0198)

Estimation of Dynamic Game Data: 422 markets, 2 firms, 5 years = 4,220 observations				
	$\beta = 0.95$ (not estimated)			
	Equilibrium Beliefs		Biased Beliefs	
	BK	MD	BK	MD
Entry Cost:				
θ_0^{EC} fixed	0.2378	0.1887	0.2586	0.1739
	(0.0709)*	(0.0679)*	(0.1282)*	(0.0989)*
θ_K^{EC} (K¿0)	-0.0609	**-0.1070**	-0.0415	**-0.1190**
	(0.043)	(0.0395)*	(0.096)	(0.0628)*
θ_S^{EC} (linear in S)	0.0881	**0.0952**	0.1030	**0.1180**
	(0.0368)*	(0.0340)*	(0.0541)*	(0.0654)*

Implications of biased beliefs on BK's profits

- We compare the value of BK's profits during years 1991 to 1994 given its actual entry decisions with this firm's profits if its entry decisions were based on unbiased beliefs on MD's behaviour.
- **Having unbiased would increase BK's total profits** in UK by:
  2.78% in 1991;
  2.11% in 1992;
  1.20% in 1993;
  0.87% in 1994.
- Biased beliefs occur in markets which are relatively far away from the firm's network of stores. These markets are relatively smaller, and biased beliefs decline over time in the sample period as the result of geographic expansion.

Summary and Conclusions

- Strategic uncertainty can be important for competition in oligopoly markets. Under these conditions, the assumption of equilibrium beliefs can be too restrictive.

- We present sufficient conditions for the NP identification of preferences and beliefs.

- We apply these ideas to actual data and find that bias beliefs can be useful to explain a puzzle in the data.