# Heterogeneity, Uncertainty and Learning: Semiparametric Identification and Estimation

Jackson Bunting
U. Washington

Paul Diegert
TSE

Arnaud Maurel
Duke U. and NBER

DSE Conference, UCL

July 3, 2025

# Motivation

- Panel data with continuous outcomes and discrete choices $(Y_{it}, D_{it})$.
- $(Y_{it}, D_{it})$ depend on multidimensional (time invariant) latent variable, $X_i^*$.
- Some components of $X_i^*$ initially unknown by the agents.
- Arises frequently in applications.
- E.g.: productivity partly unknown to workers when they enter the workforce.
- Workers update their beliefs about their productivity as they observe their wages.

- Panel data with continuous outcomes and discrete choices $(Y_{it}, D_{it})$.
- $(Y_{it}, D_{it})$ depend on multidimensional (time invariant) latent variable, $X_i^*$.
- Some components of $X_i^*$ initially unknown by the agents.
- Arises frequently in applications.
- E.g.: productivity partly unknown to workers when they enter the workforce.
- Workers update their beliefs about their productivity as they observe their wages.

## Motivation (Cont'd)

- Learning models are popular in economics, empirical micro in particular.
- Used in various fields, including Labor (Miller, 1984; Antonovics and Golan, 2012; Pastorino, 2024), Education (Arcidiacono, 2004; Stinebrickner and Stinebrickner, 2012; Arcidiacono et al., 2025); IO/Health (Ackerberg, 2003; Crawford and Shum, 2005; Aguirregabiria and Jeon, 2020).
- However, much remains to be known about identification.
- Standard to assume parametric outcome model (Gaussian), and specific learning process (RE with Bayesian updating).
- Even with these assumptions, no general identification result in the literature.

## Motivation (Cont'd)

- Learning models are popular in economics, empirical micro in particular.
- Used in various fields, including Labor (Miller, 1984; Antonovics and Golan, 2012; Pastorino, 2024), Education (Arcidiacono, 2004; Stinebrickner and Stinebrickner, 2012; Arcidiacono et al., 2025); IO/Health (Ackerberg, 2003; Crawford and Shum, 2005; Aguirregabiria and Jeon, 2020).
- However, much remains to be known about identification.
- Standard to assume parametric outcome model (Gaussian), and specific learning process (RE with Bayesian updating).
- Even with these assumptions, no general identification result in the literature.

## This paper

- We provide conditions for identification of a learning model (with private information) from sequence of individual choices and outcomes:
  - Maintain some structure on the outcome model (linearity and normality of the errors and unknown factor).
  - Very few restrictions on choice process and learning rule.
  - Do not rely on measurements of unobserved heterogeneity.
- In a learning model with only unknown heterogeneity, we establish identification without assuming normality.
- Computationally tractable semi-nonparametric estimator.
- Application to ability learning in the context of occupational choice.

## This paper

- We provide conditions for identification of a learning model (with private information) from sequence of individual choices and outcomes:
  - Maintain some structure on the outcome model (linearity and normality of the errors and unknown factor).
  - Very few restrictions on choice process and learning rule.
  - Do not rely on measurements of unobserved heterogeneity.
- In a learning model with only unknown heterogeneity, we establish identification without assuming normality.
- Computationally tractable semi-nonparametric estimator.
- Application to ability learning in the context of occupational choice.

# This paper

- We provide conditions for identification of a learning model (with private information) from sequence of individual choices and outcomes:
  - Maintain some structure on the outcome model (linearity and normality of the errors and unknown factor).
  - Very few restrictions on choice process and learning rule.
  - Do not rely on measurements of unobserved heterogeneity.
- In a learning model with only unknown heterogeneity, we establish identification without assuming normality.
- Computationally tractable semi-nonparametric estimator.
- Application to ability learning in the context of occupational choice.

## This paper

- We provide conditions for identification of a learning model (with private information) from sequence of individual choices and outcomes:
  - Maintain some structure on the outcome model (linearity and normality of the errors and unknown factor).
  - Very few restrictions on choice process and learning rule.
  - Do not rely on measurements of unobserved heterogeneity.
- In a learning model with only unknown heterogeneity, we establish identification without assuming normality.
- Computationally tractable semi-nonparametric estimator.
- Application to ability learning in the context of occupational choice.

# This paper

- We provide conditions for identification of a learning model (with private information) from sequence of individual choices and outcomes:
  - Maintain some structure on the outcome model (linearity and normality of the errors and unknown factor).
  - Very few restrictions on choice process and learning rule.
  - Do not rely on measurements of unobserved heterogeneity.
- In a learning model with only unknown heterogeneity, we establish identification without assuming normality.
- Computationally tractable semi-nonparametric estimator.
- Application to ability learning in the context of occupational choice.

# Outline

# Setup

- We observe a short panel of discrete choices, continuous outcomes and covariates:

$$(D_{it}, Y_{it}, X_{it})_{t=1,2,\dots,T}$$

- Two types of latent variables:
  - $X^*_{i,k} \in \mathbb{R}$: *known* by the agent.
  - $X^*_{i,u} \in \mathbb{R}^p$: initially *unknown* by the agent.
- $\dim(X^*_{i,u}) = p \geq 2$: non-diagonal covariance matrix $\rightarrow$ correlated learning.
- Idiosyncratic shocks affecting the outcomes, $\epsilon_{it}$.

## Setup

- We observe a short panel of discrete choices, continuous outcomes and covariates:

$$(D_{it}, Y_{it}, X_{it})_{t=1,2,\dots,T}$$

- Two types of latent variables:
  - $X^*_{i,k} \in \mathbb{R}$: *known* by the agent.
  - $X^*_{i,u} \in \mathbb{R}^p$: initially *unknown* by the agent.

- $\dim(X^*_{i,u}) = p \geq 2$: non-diagonal covariance matrix $\rightarrow$ correlated learning.

- Idiosyncratic shocks affecting the outcomes, $\epsilon_{it}$.

# Setup

- We observe a short panel of discrete choices, continuous outcomes and covariates:

$$(D_{it}, Y_{it}, X_{it})_{t=1,2,...,T}$$

- Two types of latent variables:
  - $X^*_{i,k} \in \mathbb{R}$: *known* by the agent.
  - $X^*_{i,u} \in \mathbb{R}^p$: initially *unknown* by the agent.
- $\dim(X^*_{i,u}) = p \geq 2$: non-diagonal covariance matrix $\rightarrow$ correlated learning.
- Idiosyncratic shocks affecting the outcomes, $\epsilon_{it}$.

- Interactive fixed effect model for potential outcomes (omitting individual subscript $i$):

$$Y_t(d) = X_t^\intercal \beta_{t,d} + X_k^* \lambda_{t,d}^k + (X_u^*)^\intercal \lambda_{t,d}^u + \epsilon_t(d).$$

- Key assumption on choices: do not directly depend on $X_u^*$. Specifically:

$$D_t \perp\!\!\!\perp X_u^* \mid X^t, Y^{t-1}, D^{t-1}, X_k^*.$$

- Idiosyncratic shocks $\epsilon_t(d)$ are independent of $(X^t, Y^{t-1}, D^{t-1}, X^*)$ (with $X^* = (X_k^*, X_u^*)$).

- Interactive fixed effect model for potential outcomes (omitting individual subscript $i$):

$$Y_t(d) = X_t^{\intercal}\beta_{t,d} + X_k^* \lambda_{t,d}^k + (X_u^*)^{\intercal}\lambda_{t,d}^u + \epsilon_t(d).$$

- Key assumption on choices: do not directly depend on $X_u^*$.

  Specifically:

$$D_t \perp\!\!\!\perp X_u^* \mid X^t, Y^{t-1}, D^{t-1}, X_k^*.$$

- Idiosyncratic shocks $\epsilon_t(d)$ are independent of $(X^t, Y^{t-1}, D^{t-1}, X^*)$ (with $X^* = (X_k^*, X_u^*)$).

Together with a law of motion for $X_t$, this can be summarized in the following assumption:

**Assumption 1 (Conditional Independence)**

*For any* $t \geq 2$ *and* $d \in Supp(D_t)$,

$$F_{\epsilon_t(d),D_t,X_t|Y^{t-1},D^{t-1},X^{t-1},X^*} = F_{\epsilon_t(d)} F_{D_t|X^t,Y^{t-1},D^{t-1},X_k^*} F_{X_t|Y^{t-1},D^{t-1},X^{t-1}}.$$

*For any* $d \in Supp(D_1)$, $F_{\epsilon_1(d),D_1,X_1|X^*} = F_{\epsilon_1(d)} F_{D_1|X_1,X_k^*} F_{X_1|X^*}$.

- Denote by $\mathcal{I}_t$ the agent's information set in period $t$. For $t > 1$:

$$\mathcal{I}_t = (Y_1, \ldots, Y_{(t-1)}, D_1, \ldots D_{(t-1)}, X_1, \ldots, X_t, X_k^*)$$

and, for $t = 1$,

$$\mathcal{I}_1 = (X_1, X_k^*)$$

- Model consistent with agents forming their beliefs about $X_u^*$ based on (subsets of) $\mathcal{I}_t$.
- Accommodates Bayesian updating under rational expectations.
- Also allows for various other models of expectations formation, including biased beliefs, or myopic expectations.

- Denote by $\mathcal{I}_t$ the agent's information set in period $t$. For $t > 1$:

$$\mathcal{I}_t = (Y_1, \ldots, Y_{(t-1)}, D_1, \ldots D_{(t-1)}, X_1, \ldots, X_t, X_k^*)$$

and, for $t = 1$,

$$\mathcal{I}_1 = (X_1, X_k^*)$$

- Model consistent with agents forming their beliefs about $X_u^*$ based on (subsets of) $\mathcal{I}_t$.
- Accommodates Bayesian updating under rational expectations.
- Also allows for various other models of expectations formation, including biased beliefs, or myopic expectations.

# Outline

Model parameters ($\theta$):

- Outcome equation parameters $(\beta, \lambda)$.
- Distribution of the unobservables $(X_k^*, X_u^*, \epsilon_t)$.
- Conditional choice probabilities $\mathbb{P}(D_t = d | \mathcal{I}_t)$ (CCP).
- Covariate process $(X_t | Y^{t-1}, D^{t-1}, X^{t-1})$.

Two cases:

1. Unknown unobserved heterogeneity only (*Pure Learning*).
2. Known and unknown unobserved heterogeneity (*Learning with Private Information*).

# Identification

Model parameters ($\theta$):

- Outcome equation parameters $(\beta, \lambda)$.
- Distribution of the unobservables $(X_k^*, X_u^*, \epsilon_t)$.
- Conditional choice probabilities $\mathbb{P}(D_t = d | \mathcal{I}_t)$ (CCP).
- Covariate process $(X_t | Y^{t-1}, D^{t-1}, X^{t-1})$.

Two cases:

1. Unknown unobserved heterogeneity only (*Pure Learning*).
2. Known and unknown unobserved heterogeneity (*Learning with Private Information*).

# Intuition

- Selection problem: identify the interactive fixed effects model

$$Y_t(d) = \lambda_{t,d}^{\intercal} X^* + \epsilon_t(d)$$

  from the distribution of $(Y^T, D^T)$.

- Key idea: use CCPs to adjust for selection.
- Suppose $\text{supp}(D_t) = \{0, 1\}$, and consider the distribution of $Y^T(1) \equiv (Y_1(1), Y_2(1), \ldots, Y_T(1))$.

## Intuition

- Selection problem: identify the interactive fixed effects model

$$Y_t(d) = \lambda_{t,d}^{\intercal} X^* + \epsilon_t(d)$$

  from the distribution of $(Y^T, D^T)$.

- Key idea: use CCPs to adjust for selection.
- Suppose $\mathrm{supp}(D_t) = \{0, 1\}$, and consider the distribution of $Y^T(1) \equiv (Y_1(1), Y_2(1), \ldots, Y_T(1))$.

## Intuition (Cont'd)

- First, note that

$$f_{Y^T|D^T}(y^T|1)\frac{f_{D^T}(1)}{f_{D^T|Y^T(1)}(1|y^T)} = f_{Y^T(1)}(y^T).$$

- With no covariates and $X^* = X_u^*$, it follows from Assumption 1 that the inverse selection weight, $f_{D^T|Y^T(1)}$, is identified as follows:

$$f_{D^T|Y^T(1)}(1|y^T) = f_{D_t|Y^{t-1},D^{t-1}}(1|y^{t-1},1)f_{D^{t-1}|Y^{t-2},D^{t-2}}(1|y^{t-2},1)...f_{D_1}(1).$$

- Identifies in turn the distribution of $Y^T(1)$.
- Back to standard interactive fixed effects setup (Freyberger, 2018).

# Intuition (Cont'd)

- First, note that

$$f_{Y^T|D^T}(y^T|1)\frac{f_{D^T}(1)}{f_{D^T|Y^T(1)}(1|y^T)} = f_{Y^T(1)}(y^T).$$

- With no covariates and $X^* = X_u^*$, it follows from Assumption 1 that the inverse selection weight, $f_{D^T|Y^T(1)}$, is identified as follows:

$$f_{D^T|Y^T(1)}(1|y^T) = f_{D_t|Y^{t-1},D^{t-1}}(1|y^{t-1},1)f_{D^{t-1}|Y^{t-2},D^{t-2}}(1|y^{t-2},1)...f_{D_1}(1).$$

- Identifies in turn the distribution of $Y^T(1)$.
- Back to standard interactive fixed effects setup (Freyberger, 2018).

- Generalize the previous ideas to include covariates and multiple potential outcomes.
- Since $X^* = X^*_u$, $\mathcal{I}_t = \{Y^{t-1}, D^{t-1}, X^t\}$ and $f_{D_t,X_t|\mathcal{I}_t}$ and $f_{D_1|X_1}$ are identified from the data (selection on obs.).
- Assume for simplicity that $X_t \perp\!\!\!\perp X^* | X_{t-1}$ (relaxed in the paper).
- It follows that $f_{Y^T(d^T)|X^T}(y^T; x^T)$ is identified:

$$\frac{f_{Y^T,D^T,X^T}(y^T, d^T, x^T)}{f_{D_1,X_1}(d_1, z_1)\prod_{t=2}^{T} f_{D_t,X_t|\mathcal{I}_t}(d_t, x_t; h_t)} = f_{Y^T(d^T)|X^T}(y^T; x^T)$$

- One can then use results from the interactive fixed effect literature (e.g., Freyberger 2018) to identify the outcome eq. parameters and the distributions of $(X^*, \epsilon_{it})$.

# Pure Learning

- Generalize the previous ideas to include covariates and multiple potential outcomes.
- Since $X^* = X_u^*$, $\mathcal{I}_t = \{Y^{t-1}, D^{t-1}, X^t\}$ and $f_{D_t,X_t|\mathcal{I}_t}$ and $f_{D_1|X_1}$ are identified from the data (selection on obs.).
- Assume for simplicity that $X_t \perp\!\!\!\perp X^* | X_{t-1}$ (relaxed in the paper).
- It follows that $f_{Y^T(d^T)|X^T}(y^T; x^T)$ is identified:

$$\frac{f_{Y^T,D^T,X^T}(y^T, d^T, x^T)}{f_{D_1,X_1}(d_1, z_1) \prod_{t=2}^{T} f_{D_t,X_t|\mathcal{I}_t}(d_t, x_t; h_t)} = f_{Y^T(d^T)|X^T}(y^T; x^T)$$

- One can then use results from the interactive fixed effect literature (e.g., Freyberger 2018) to identify the outcome eq. parameters and the distributions of $(X^*, \epsilon_{it})$.

- With private information, $X^* = (X^*_k, X^*_u)$, $\mathcal{I}_t = \{Y^{t-1}, D^{t-1}, X^t, X^*_k\}$.
- Key difference: CCPs are no longer identified directly from the data.
- Under a normality assumption, we show identification of the joint distribution of $(Y^T, D^T, X^T, X^*_k)$.
- Having recovered the distribution of $\mathcal{I}_t$, identification follows from similar arguments as in the pure learning case.

- With private information, $X^* = (X_k^*, X_u^*)$, $\mathcal{I}_t = \{Y^{t-1}, D^{t-1}, X^t, X_k^*\}$.
- Key difference: CCPs are no longer identified directly from the data.
- Under a normality assumption, we show identification of the joint distribution of $(Y^T, D^T, X^T, X_k^*)$.
- Having recovered the distribution of $\mathcal{I}_t$, identification follows from similar arguments as in the pure learning case.

## Learning with private information - Normality

**Assumption 2 (Normality)**

$(X_u^*, \epsilon)$ *are distributed according to*

$$X_u^* \mid (X_1 = x_1, X_k^* = x_k^*) \sim \mathcal{N}\left(0, \Sigma_u(x_1)\right)$$
$$\epsilon_t(d) \sim \mathcal{N}(0, \sigma_t(d)^2)$$

- Standard in applied learning papers, which also use this assumption to specify a learning and choice model (e.g. Thomas, 2019; Arcidiacono et al., 2025).
- We maintain the restriction on the outcome model, but:
  - Remain flexible on the learning and choice process.
  - Do not assume a specific distribution for the known heterogeneity component ($X_k^*$).

# Learning with private information - Normality

**Assumption 2 (Normality)**

*$(X_u^*, \epsilon)$ are distributed according to*

$$X_u^* \mid (X_1 = x_1, X_k^* = x_k^*) \sim \mathcal{N}\left(0, \Sigma_u(x_1)\right)$$
$$\epsilon_t(d) \sim \mathcal{N}(0, \sigma_t(d)^2)$$

- Standard in applied learning papers, which also use this assumption to specify a learning and choice model (e.g. Thomas, 2019; Arcidiacono et al., 2025).
- We maintain the restriction on the outcome model, but:
  - Remain flexible on the learning and choice process.
  - Do not assume a specific distribution for the known heterogeneity component ($X_k^*$).

**Assumption 2 (Normality)**

*$(X_u^*, \epsilon)$ are distributed according to*

$$X_u^* \mid (X_1 = x_1, X_k^* = x_k^*) \sim \mathcal{N}(0, \Sigma_u(x_1))$$
$$\epsilon_t(d) \sim \mathcal{N}(0, \sigma_t(d)^2)$$

- Standard in applied learning papers, which also use this assumption to specify a learning and choice model (e.g. Thomas, 2019; Arcidiacono et al., 2025).
- We maintain the restriction on the outcome model, but:
  - Remain flexible on the learning and choice process.
  - Do not assume a specific distribution for the known heterogeneity component ($X_k^*$).

**Assumption 3 (Compact support)**

*The support of $X_k^*$ is compact.*

- Structural models typically assume the existence of a finite (and known) number of unobserved heterogeneity types.
- Allows for discrete or continuous distribution of $X_k^*$.
- Compactness plays an important role to identify $f_{X_k^*}$.

**Assumption 3 (Compact support)**

*The support of $X_k^*$ is compact.*

- Structural models typically assume the existence of a finite (and known) number of unobserved heterogeneity types.
- Allows for discrete or continuous distribution of $X_k^*$.
- Compactness plays an important role to identify $f_{X_k^*}$.

## Additional assumptions

(C) *Rank conditions*: For any $d^T$, all $p \times p$ submatrices of $[\lambda^u_{1,d_1} \cdots \lambda^u_{T,d_T}]$ are full rank; $f_{X_k^* | Y^{t-1}, D^t, X^t} > 0$ almost surely.

(R) *Regularity conditions*, mostly ruling out knife-edge cases because of linearity, e.g., for all $d$, $\lambda^k_{t,d} \neq (\lambda^u_{t,d})^\intercal \Sigma_t \sum_{s=1}^{t-1} \lambda^k_{s,d_s} \frac{\lambda^k_{s,d_s}}{\sigma^2_{s,d_s}}$. $\rightarrow$ Aggregate effect of $X_k^*$ on outcomes is non-zero.

(N) *Normalizations*: there is a $d$ s.t. $\lambda^k_{1,d} = 1$; there is a sequence $d^p$ such that $[\lambda^u_{t_1,d_1} \cdots \lambda^u_{t_p,d_p}] = I_p$.

▸ Assumptions

# Main Identification Result

**Theorem 1**

*Suppose the distribution of $(Y_t, D_t, X_t)_{t=1}^{T}$ is known for $T = 2p + 1$ and Assumptions 1-3, and (C), (R), (N) hold. Then $\theta$ is point identified.*

- As in the model without private information, no assumption placed on the learning process.
- Existing identification results (Hotz and Miller, 1993; Arcidiacono and Miller, 2011) can then be applied to establish identification of particular choice models from the CCPs ($f_{D_t|\mathcal{I}_t}$).
- Proof uses implication of Assumptions 1-2 that cross-sectional wages are a Gaussian mixture conditional on history; use Bruni and Koch (1985) to identify conditional distributions of $X_k^*$. Proof Sketch

## Main Identification Result

**Theorem 1**

*Suppose the distribution of $(Y_t, D_t, X_t)_{t=1}^{T}$ is known for $T = 2p + 1$ and Assumptions 1-3, and (C), (R), (N) hold. Then $\theta$ is point identified.*

- As in the model without private information, no assumption placed on the learning process.
- Existing identification results (Hotz and Miller, 1993; Arcidiacono and Miller, 2011) can then be applied to establish identification of particular choice models from the CCPs ($f_{D_t|\mathcal{I}_t}$).
- Proof uses implication of Assumptions 1-2 that cross-sectional wages are a Gaussian mixture conditional on history; use Bruni and Koch (1985) to identify conditional distributions of $X_k^*$. Proof Sketch

# Main Identification Result

**Theorem 1**

*Suppose the distribution of $(Y_t, D_t, X_t)_{t=1}^{T}$ is known for $T = 2p + 1$ and Assumptions 1-3, and (C), (R), (N) hold. Then $\theta$ is point identified.*

- As in the model without private information, no assumption placed on the learning process.
- Existing identification results (Hotz and Miller, 1993; Arcidiacono and Miller, 2011) can then be applied to establish identification of particular choice models from the CCPs ($f_{D_t|\mathcal{I}_t}$).
- Proof uses implication of Assumptions 1-2 that cross-sectional wages are a Gaussian mixture conditional on history; use Bruni and Koch (1985) to identify conditional distributions of $X_k^*$. Proof Sketch

**Theorem 1**

*Suppose the distribution of $(Y_t, D_t, X_t)_{t=1}^{T}$ is known for $T = 2p + 1$ and Assumptions 1-3, and (C), (R), (N) hold. Then $\theta$ is point identified.*

- As in the model without private information, no assumption placed on the learning process.
- Existing identification results (Hotz and Miller, 1993; Arcidiacono and Miller, 2011) can then be applied to establish identification of particular choice models from the CCPs ($f_{D_t|\mathcal{I}_t}$).
- Proof uses implication of Assumptions 1-2 that cross-sectional wages are a Gaussian mixture conditional on history; use Bruni and Koch (1985) to identify conditional distributions of $X_k^*$. Proof Sketch

# Outline

- We focus here on learning under private information.
- Given an i.i.d. sample of data $(Y_{it}, D_{it}, X_{it} : t = 1, 2, \ldots, T)_{i=1}^{N}$, we estimate the model parameters via sieve MLE. Likelihood
- Nonparametric objects include the distribution of $X_k^*$ and the CCPs.
- Establish consistency for fixed $T \geq 2p + 1$.

# Sieve MLE estimation

- We focus here on learning under private information.
- Given an i.i.d. sample of data $(Y_{it}, D_{it}, X_{it} : t = 1, 2, \ldots, T)_{i=1}^{N}$, we estimate the model parameters via sieve MLE. Likelihood
- Nonparametric objects include the distribution of $X_k^*$ and the CCPs.
- Establish consistency for fixed $T \geq 2p + 1$.

- We consider a sieve space for $F_{X_k^*}$ based on Koenker and Mizera (2014):

$$\mathcal{F}_n = \left\{ v \mapsto \sum_{s=1}^{q_n} \omega_s \mathbf{1}\{v \leq \bar{v}_{sn}\} \,\middle|\, \sum_{s}^{q_n} \omega_s = 1 \right\}$$

where $\mathcal{S}_n = \{\bar{v}_{1n}, \ldots, \bar{v}_{q_n n}\}$, for some $q_n$, as a grid of support points for $X_k^*$.

- Profile likelihood estimation, where we maximize over $F_{X_k^*}$ given $\theta \setminus F_{X_k^*}$ in an inner step.
- Fixing the other parameters, this is a convex optimization problem that can be solved very efficiently.
- Can be implemented using our Python package `spmlex`.

## Sieve MLE estimation - Implementation

- We consider a sieve space for $F_{X_k^*}$ based on Koenker and Mizera (2014):

$$\mathcal{F}_n = \left\{ v \mapsto \sum_{s=1}^{q_n} \omega_s \mathbf{1}\{v \leq \bar{v}_{sn}\} \,\middle|\, \sum_{s}^{q_n} \omega_s = 1 \right\}$$

  where $\mathcal{S}_n = \{\bar{v}_{1n}, \ldots, \bar{v}_{q_n n}\}$, for some $q_n$, as a grid of support points for $X_k^*$.
- Profile likelihood estimation, where we maximize over $F_{X_k^*}$ given $\theta \setminus F_{X_k^*}$ in an inner step.
- Fixing the other parameters, this is a convex optimization problem that can be solved very efficiently.
- Can be implemented using our Python package `spmlex`.

# Sieve MLE estimation (Cont'd)

- Often interested in particular functions of the model parameters.
- Example: decomposition of discounted lifetime earnings into a predictable and unpredictable components (Cunha et al., 2005; Cunha and Heckman, 2008, 2016).
- We consider a plug-in sieve estimator, and provide sufficient conditions for consistency and asymptotic normality for a family of functionals of the model parameters.
- Special case: variances of the predictable and unpredictable components of outcomes.

Family of functionals

## Sieve MLE estimation (Cont'd)

- Often interested in particular functions of the model parameters.
- Example: decomposition of discounted lifetime earnings into a predictable and unpredictable components (Cunha et al., 2005; Cunha and Heckman, 2008, 2016).
- We consider a plug-in sieve estimator, and provide sufficient conditions for consistency and asymptotic normality for a family of functionals of the model parameters.
- Special case: variances of the predictable and unpredictable components of outcomes.

Family of functionals

# Outline

# Simulation design

- $T = 3$, $\dim(X_u^*) = 1$, binary choice $d \in \mathcal{D}_{it} = \{1, 2\}$, 2 covariates.
- Biased beliefs:
$$\mathcal{E}_i(Y_{it}(d)|\mathcal{I}_{it}) = \mathbb{E}(Y_{it}(d)|\mathcal{I}_{it}) + \gamma_d X_{k,i}^*$$
- Choice process:
$$D_{it} = \text{argmax}_d u_{it}(d|\mathcal{I}_{it}) + \eta_{it}(d)$$
where $u_{it}(d|\mathcal{I}_{it}) = \rho \mathcal{E}_i\left(Y_{it}(d)|\mathcal{I}_{it}\right)$ and $\eta_{it}(d)$ is i.i.d. Type 1 EV.
- $X_{k,i}^*$ is a finite mixture of truncated normal r.v.'s., while $X_{u,i}^* \sim N(0, \sigma_u^2)$ and $\epsilon_{it}(d) \sim N(0, \sigma^2(d))$.

## Simulation design

- $T = 3$, $\dim(X_u^*) = 1$, binary choice $d \in \mathcal{D}_{it} = \{1, 2\}$, 2 covariates.
- Biased beliefs:

$$\mathcal{E}_i(Y_{it}(d)|\mathcal{I}_{it}) = \mathbb{E}(Y_{it}(d)|\mathcal{I}_{it}) + \gamma_d X_{k,i}^*$$

- Choice process:

$$D_{it} = \text{argmax}_d u_{it}(d|\mathcal{I}_{it}) + \eta_{it}(d)$$

where $u_{it}(d|\mathcal{I}_{it}) = \rho\mathcal{E}_i\left(Y_{it}(d)|\mathcal{I}_{it}\right)$ and $\eta_{it}(d)$ is i.i.d. Type 1 EV.

- $X_{k,i}^*$ is a finite mixture of truncated normal r.v.'s., while $X_{u,i}^* \sim N(0, \sigma_u^2)$ and $\epsilon_{it}(d) \sim N(0, \sigma^2(d))$.

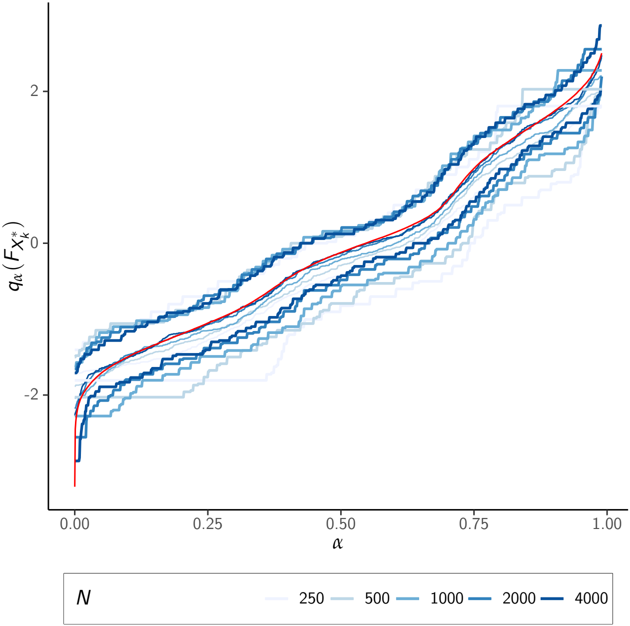
## Simulation design

- $T = 3$, $\dim(X_u^*) = 1$, binary choice $d \in \mathcal{D}_{it} = \{1, 2\}$, 2 covariates.
- Biased beliefs:
$$\mathcal{E}_i(Y_{it}(d)|\mathcal{I}_{it}) = \mathbb{E}(Y_{it}(d)|\mathcal{I}_{it}) + \gamma_d X_{k,i}^*$$
- Choice process:
$$D_{it} = \text{argmax}_d u_{it}(d|\mathcal{I}_{it}) + \eta_{it}(d)$$
where $u_{it}(d|\mathcal{I}_{it}) = \rho \mathcal{E}_i\left(Y_{it}(d)|\mathcal{I}_{it}\right)$ and $\eta_{it}(d)$ is i.i.d. Type 1 EV.
- $X_{k,i}^*$ is a finite mixture of truncated normal r.v.'s., while $X_{u,i}^* \sim N(0, \sigma_u^2)$ and $\epsilon_{it}(d) \sim N(0, \sigma^2(d))$.

# Monte Carlo results: squared bias and variance ($\times 1,000$)

| | N = 250 | | N = 500 | | N = 1,000 | | N = 2,000 | | N = 4,000 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | sq bias | var | sq bias | var | sq bias | var | sq bias | var | sq bias | var |
| $\lambda^k_{1,1}$ | 2.746 | 27.521 | 1.701 | 12.890 | 0.624 | 7.268 | 0.009 | 3.684 | 0.001 | 1.468 |
| $\lambda^k_{2,1}$ | 1.148 | 25.977 | 0.558 | 10.825 | 0.226 | 4.777 | 0.004 | 2.594 | 0.000 | 1.089 |
| $\lambda^k_{2,2}$ | 0.869 | 10.978 | 0.254 | 5.825 | 0.073 | 2.654 | 0.007 | 1.383 | 0.000 | 0.743 |
| $\lambda^k_{3,1}$ | 3.986 | 33.663 | 0.873 | 13.719 | 0.178 | 5.683 | 0.003 | 3.069 | 0.000 | 1.330 |
| $\lambda^k_{3,2}$ | 5.702 | 36.861 | 0.674 | 12.556 | 0.224 | 5.305 | 0.011 | 2.408 | 0.005 | 1.080 |
| $\lambda^u_{1,2}$ | 0.979 | 13.945 | 0.306 | 4.733 | 0.170 | 2.438 | 0.015 | 1.330 | 0.000 | 0.605 |
| $\lambda^u_{2,1}$ | 0.040 | 8.317 | 0.027 | 5.139 | 0.036 | 1.947 | 0.014 | 1.003 | 0.002 | 0.481 |
| $\lambda^u_{2,2}$ | 1.478 | 14.880 | 0.494 | 6.218 | 0.130 | 3.324 | 0.009 | 1.522 | 0.004 | 0.643 |
| $\lambda^u_{3,1}$ | 0.446 | 9.912 | 0.093 | 5.003 | 0.062 | 2.187 | 0.030 | 0.968 | 0.023 | 0.469 |
| $\lambda^u_{3,2}$ | 0.106 | 21.919 | 0.101 | 8.903 | 0.112 | 4.148 | 0.004 | 2.140 | 0.005 | 0.936 |
| $\sigma^2(1)$ | 0.453 | 2.477 | 0.091 | 1.241 | 0.030 | 0.672 | 0.007 | 0.298 | 0.001 | 0.136 |
| $\sigma^2(2)$ | 1.228 | 4.449 | 0.242 | 2.237 | 0.027 | 1.058 | 0.016 | 0.701 | 0.008 | 0.332 |

# Quantiles of distribution of $X_k^*$



# Outline

# Occupational choice and learning

- We apply our framework to study occupational choice.
- Focus on the role of uncertainty vs. heterogeneity, in a context where agents may learn over time.
- We do not rely on an auxiliary measurement system, unlike much of the empirical literature (e.g. Cunha and Heckman, 2008; Arcidiacono et al., 2025).
- Data: National Longitudinal Survey of Youth 1997 (NLSY97).
  - Sample of 2,453 white men born between 1980 and 1984.
  - Restrict to full-time workers between ages 27 and 32.
  - Outcome variable ($Y$): Log hourly wage; choice variable ($D$): high- or low-skill occupation.

- We apply our framework to study occupational choice.
- Focus on the role of uncertainty vs. heterogeneity, in a context where agents may learn over time.
- We do not rely on an auxiliary measurement system, unlike much of the empirical literature (e.g. Cunha and Heckman, 2008; Arcidiacono et al., 2025).
- Data: National Longitudinal Survey of Youth 1997 (NLSY97).
    - Sample of 2,453 white men born between 1980 and 1984.
    - Restrict to full-time workers between ages 27 and 32.
    - Outcome variable ($Y$): Log hourly wage; choice variable ($D$): high- or low-skill occupation.

# Occupational choice and learning

- We apply our framework to study occupational choice.
- Focus on the role of uncertainty vs. heterogeneity, in a context where agents may learn over time.
- We do not rely on an auxiliary measurement system, unlike much of the empirical literature (e.g. Cunha and Heckman, 2008; Arcidiacono et al., 2025).
- Data: National Longitudinal Survey of Youth 1997 (NLSY97).
  - Sample of 2,453 white men born between 1980 and 1984.
  - Restrict to full-time workers between ages 27 and 32.
  - Outcome variable ($Y$): Log hourly wage; choice variable ($D$): high- or low-skill occupation.

# Set-up and implementation

- $T = 3$, $\dim(X_u^*) = 1$, $\mathcal{D}_t = \{0, 1\}$ for $t \in \{1, 2, 3\}$ (low- or high-skill sector).
- Potential outcomes ($Y_t(d)$): log hourly wages in period t (two-year average).
- Where, for $d \in \{0, 1\}$:

$$Y_t(d) = \beta_{t,d} + X_k^* \lambda_{t,d}^k + X_u^* \lambda_{t,d}^u + \epsilon_t(d).$$

- CCPs:
  - $h_t(X_k^*, Y^{t-1}, D^{t-1}) := P(D_t = 1 | X_k^*, Y^{t-1}, D^{t-1})$
- Estimation via sieve MLE.
- We use a flexible logit for the CCPs $h_t$, and the sieve space $\mathcal{F}_n$ for $F_{X_k^*}$ (with 56 eq. spaced support points).

## Set-up and implementation

- $T = 3$, $\dim(X_u^*) = 1$, $\mathcal{D}_t = \{0, 1\}$ for $t \in \{1, 2, 3\}$ (low- or high-skill sector).
- Potential outcomes ($Y_t(d)$): log hourly wages in period t (two-year average).
- Where, for $d \in \{0, 1\}$:

$$Y_t(d) = \beta_{t,d} + X_k^* \lambda_{t,d}^k + X_u^* \lambda_{t,d}^u + \epsilon_t(d).$$

- CCPs:
  - $h_t(X_k^*, Y^{t-1}, D^{t-1}) := P(D_t = 1 | X_k^*, Y^{t-1}, D^{t-1})$
- Estimation via sieve MLE.
- We use a flexible logit for the CCPs $h_t$, and the sieve space $\mathcal{F}_n$ for $F_{X_k^*}$ (with 56 eq. spaced support points).

## Set-up and implementation

- $T = 3$, $\dim(X_u^*) = 1$, $\mathcal{D}_t = \{0, 1\}$ for $t \in \{1, 2, 3\}$ (low- or high-skill sector).
- Potential outcomes ($Y_t(d)$): log hourly wages in period t (two-year average).
- Where, for $d \in \{0, 1\}$:

$$Y_t(d) = \beta_{t,d} + X_k^* \lambda_{t,d}^k + X_u^* \lambda_{t,d}^u + \epsilon_t(d).$$

- CCPs:
  - $h_t(X_k^*, Y^{t-1}, D^{t-1}) := P(D_t = 1 | X_k^*, Y^{t-1}, D^{t-1})$
- Estimation via sieve MLE.
- We use a flexible logit for the CCPs $h_t$, and the sieve space $\mathcal{F}_n$ for $F_{X_k^*}$ (with 56 eq. spaced support points).

## Set-up and implementation

- $T = 3$, $\dim(X_u^*) = 1$, $\mathcal{D}_t = \{0,1\}$ for $t \in \{1,2,3\}$ (low- or high-skill sector).
- Potential outcomes ($Y_t(d)$): log hourly wages in period t (two-year average).
- Where, for $d \in \{0,1\}$:

$$Y_t(d) = \beta_{t,d} + X_k^* \lambda_{t,d}^k + X_u^* \lambda_{t,d}^u + \epsilon_t(d).$$

- CCPs:
  - $h_t(X_k^*, Y^{t-1}, D^{t-1}) := P(D_t = 1 | X_k^*, Y^{t-1}, D^{t-1})$
- Estimation via sieve MLE.
- We use a flexible logit for the CCPs $h_t$, and the sieve space $\mathcal{F}_n$ for $F_{X_k^*}$ (with 56 eq. spaced support points).

# Model fit

| | $Y_1$ Est. | $Y_1$ Data | $Y_2$ Est. | $Y_2$ Data | $Y_3$ Est. | $Y_3$ Data |
|---|---|---|---|---|---|---|
| A. No periods in high-skill occupation | | | | | | |
| *Mean* | | | | | | |
| | 2.45 | 2.45 | 2.51 | 2.52 | 2.57 | 2.57 |
| *Covariance Matrix* | | | | | | |
| $Y_1$ | 0.18 | 0.17 | 0.15 | 0.14 | 0.14 | 0.13 |
| $Y_2$ | — | — | 0.18 | 0.19 | 0.17 | 0.17 |
| $Y_3$ | — | — | — | — | 0.22 | 0.21 |
| B. Some periods in high-skill occupation | | | | | | |
| *Mean* | | | | | | |
| | 2.58 | 2.58 | 2.65 | 2.68 | 2.82 | 2.80 |
| *Covariance Matrix* | | | | | | |
| $Y_1$ | 0.18 | 0.21 | 0.12 | 0.14 | 0.13 | 0.12 |
| $Y_2$ | — | — | 0.18 | 0.20 | 0.15 | 0.13 |
| $Y_3$ | — | — | — | — | 0.22 | 0.19 |
| C. All periods in high-skill occupation | | | | | | |
| *Mean* | | | | | | |
| | 2.78 | 2.76 | 2.91 | 2.91 | 3.01 | 3.00 |
| *Covariance Matrix* | | | | | | |
| $Y_1$ | 0.24 | 0.26 | 0.16 | 0.16 | 0.16 | 0.17 |
| $Y_2$ | — | — | 0.23 | 0.21 | 0.16 | 0.19 |
| $Y_3$ | — | — | — | — | 0.25 | 0.26 |

# Selection patterns

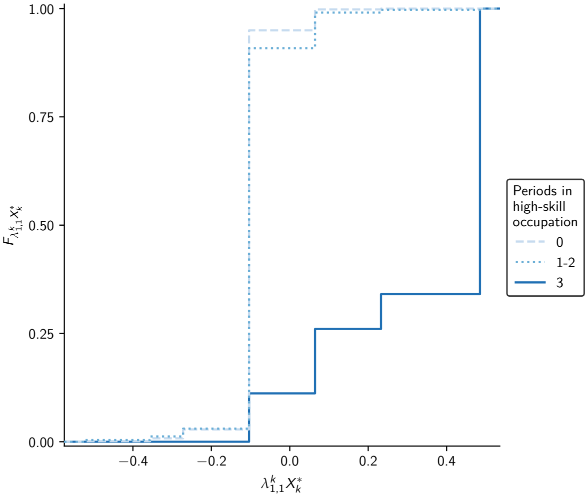



Figure: Selection into high-skill occupation.

- We estimate the share of the variance of future wages that is due to heterogeneity (forecastable) vs. uncertainty (unforecastable).
- Focus on the discounted value of wages of the later two periods: $\overline{Y}(d_2) = \sum_{t=2}^{3}(1-\rho)^{t-2}Y_t(d_2)$, for $d_2 \in \{0,1\}$, where $\rho = .05$
- Compute the share of variance that is forecastable before ($t = 0$) and after ($t = 1$) the first period of labor market experience.

- We estimate the share of the variance of future wages that is due to heterogeneity (forecastable) vs. uncertainty (unforecastable).
- Focus on the discounted value of wages of the later two periods: $\overline{Y}(d_2) = \sum_{t=2}^{3}(1-\rho)^{t-2} Y_t(d_2)$, for $d_2 \in \{0, 1\}$, where $\rho = .05$
- Compute the share of variance that is forecastable before ($t = 0$) and after ($t = 1$) the first period of labor market experience.

| | $\overline{Y}(1)$ | | $\overline{Y}(0)$ | |
|---|---|---|---|---|
| $(t, D^t)$ | Total Variance | Share Forecastable | Total Variance | Share Forecastable |
| $(0, \emptyset)$ | 0.66 | 0.12 | 1.07 | 0.43 |
| | (0.52, 0.77) | (0.07, 0.23) | (0.73, 3.93) | (0.20, 0.85) |
| $(1, 0)$ | 0.57 | 0.65 | 0.64 | 0.80 |
| | (0.44, 0.69) | (0.59, 0.74) | (0.56, 0.75) | (0.76, 0.85) |
| $(1, 1)$ | 0.70 | 0.53 | 1.27 | 0.76 |
| | (0.57, 0.81) | (0.41, 0.69) | (0.80, 5.40) | (0.64, 0.96) |

Table: Decomposition of variance of wages into forecastable and unforecastable components. Note: Each row reports the variance decomposition conditional on a sequence of prior choices. The first row is the variance decomposition at period 0 before the first choices are made. The second and third rows are the variance decomposition conditional on the first occupational choice. The total variance is the variance of $\overline{Y}(d_2)$, conditional having made the choice $D^t$, which can therefore be a selected sample. The share forecastable is the ratio of the forecastable variance (including both the variance coming from $X_k^*$ and the posterior mean of $X_u^*$ after observing $D^t$) to the total variance. Bootstrap 95% confidence intervals are given in parentheses.

## Concluding remarks

- We establish identification of a potential outcomes model where a portion of the unobserved heterogeneity may be unknown at the time of decisions.
  - Setup with normally distributed errors and unknown heterogeneity, but distribution-free on the known heterogeneity component.
  - Very few restrictions on decision process and learning rule.
  - No auxiliary measurements.
- When individuals do not have private information, relax normality.
- Computationally tractable semi-nonparametric estimation procedure that performs well in finite samples (Python package `spmlex` available on GitHub).
- Illustration to uncertainty and learning in occupational choice.
- Evidence of fast ability revelation based on realized wages.

**Thank you!**

## Concluding remarks

- We establish identification of a potential outcomes model where a portion of the unobserved heterogeneity may be unknown at the time of decisions.
  - Setup with normally distributed errors and unknown heterogeneity, but distribution-free on the known heterogeneity component.
  - Very few restrictions on decision process and learning rule.
  - No auxiliary measurements.
- When individuals do not have private information, relax normality.
- Computationally tractable semi-nonparametric estimation procedure that performs well in finite samples (Python package `spmlex` available on GitHub).
- Illustration to uncertainty and learning in occupational choice.
- Evidence of fast ability revelation based on realized wages.

**Thank you!**

## Concluding remarks

- We establish identification of a potential outcomes model where a portion of the unobserved heterogeneity may be unknown at the time of decisions.
  - Setup with normally distributed errors and unknown heterogeneity, but distribution-free on the known heterogeneity component.
  - Very few restrictions on decision process and learning rule.
  - No auxiliary measurements.
- When individuals do not have private information, relax normality.
- Computationally tractable semi-nonparametric estimation procedure that performs well in finite samples (Python package `spmlex` available on GitHub).
- Illustration to uncertainty and learning in occupational choice.
- Evidence of fast ability revelation based on realized wages.

**Thank you!**

## Concluding remarks

- We establish identification of a potential outcomes model where a portion of the unobserved heterogeneity may be unknown at the time of decisions.
  - Setup with normally distributed errors and unknown heterogeneity, but distribution-free on the known heterogeneity component.
  - Very few restrictions on decision process and learning rule.
  - No auxiliary measurements.
- When individuals do not have private information, relax normality.
- Computationally tractable semi-nonparametric estimation procedure that performs well in finite samples (Python package `spmlex` available on GitHub).
- Illustration to uncertainty and learning in occupational choice.
- Evidence of fast ability revelation based on realized wages.

**Thank you!**

# Assumptions

KL1 For any $d \in \text{Supp}(D_t)$

$$F_{\epsilon_t(d),D_t,Z_t|Y^{t-1},D^{t-1},Z^{t-1}X^*} = F_{\epsilon_t(d)} F_{D_t|Y^{t-1},D^{t-1},Z^t,X_k^*} F_{Z_t|Y^{t-1},D^{t-1},Z_{t-1}}.$$

KL2 $(\lambda_u \mid Z_1 = z_1, X_k^* = v_k) \sim N\left(0, \Sigma_u(z_1, v_k)\right)$ and $\epsilon_t(d) \sim N(0, \sigma_t(d)^2)$.

# Assumptions (Cont'd)

KL3 (A) For some $d_1$, $\alpha_1(d_1) = 0$, $F_{k1}(d_1) = 1$. (B) For some $(d_1, d_2, \ldots, d_p)$, $(F_{u1}(d_1)F_{u2}(d_2)\ldots F_{up}(d_p)) = I_{p\times p}$.

KL4 (A) $\Theta_1$ is a compact set. (B) Supp($X_k^*$) is a compact set. (C) For each $t$, $F_{ut}^{\mathsf{T}}(d_t)\Sigma_t F_{ut}(d_t) + \sigma_t^2(d_t) \neq 0$, $\sigma_t(d_t) \neq 0$ and $\Sigma_u(z_1, v_k)$ is non-singular. (D) $dF_{X_k^*|Y^{t-1},Z^t,D^t}(v_k; y^{t-1}, z^t, d^t) > 0$ for all for all $t$ and $v_k$ in the support of $X_k^*$. (E) For each $t$, the variance-covariance matrix of $(1_n, Z_{it})$ is non-singular.

# Assumptions (Cont'd)

KL5 (A) For each $d_t$ there are sequences $d^{t-1}, \tilde{d}^{t-1}$ such that $F_{ut}(d_t)^\intercal \Sigma_t \sum_{s=1}^{t-1} \left( F_{us}(d_s) \frac{F_{ks}(d_s)}{\sigma_s^2(d_s)} - F_{us}(\tilde{d}_s) \frac{F_{ks}(\tilde{d}_s)}{\sigma_s^2(\tilde{d}_s)} \right) \neq 0$. (B) For all $d_t$, $F_{kt}(d_t) \neq 0$. (C) For all $d_t$, $F_{kt}(d_t) - F_{ut}(d_t)^\intercal \Sigma_t \sum_{s=1}^{t-1} F_{us}(d_s) \frac{F_{ks}(d_s)}{\sigma_s^2(d_s)} \neq 0$. (D) For each $(d_2, d_1)$, $F_{u2}(d_2)^\intercal \Sigma_2(\lambda_{ui}) F_{u1}(d_1) \frac{F_{k1}(d_1)}{\sigma_1^2(d_1)} \neq 0$ (E) There are sets $\{d_{2,i} : i = 1, 2, \ldots, k\}$, $\{\tilde{d}_{2,i} : i = 1, 2, \ldots, k\}$ which are subsets of $\mathrm{Supp}(D_2)$ and satisfy

$$\begin{aligned}
&(F_{u2}(d_{2,1}) F_{u2}(d_{2,2}) \ldots F_{u2}(d_{2,k}))^{-\intercal} \,\mathrm{vec}(F_{k2}(d_{2,1}), \ldots, F_{k2}(d_{2,k})) \\
&\neq \left(F_{u2}(\tilde{d}_{2,1}) F_{u2}(\tilde{d}_{2,2}) \ldots F_{u2}(\tilde{d}_{2,k})\right)^{-\intercal} \mathrm{vec}(F_{k2}(\tilde{d}_{2,1}), \ldots, F_{k2}(\tilde{d}_{2,k})).
\end{aligned}$$

(F) Any $p \times p$ submatrix of $(F_{u1}(d_1) F_{u2}(d_2) \ldots F_{uT}(d_T))$ has full rank Back

# Unknown factor only: assumptions

L1 For any $d \in \mathrm{Supp}(D_t)$

$$F_{\epsilon_t(d), D_t, Z_t | Y^{t-1}, D^{t-1}, Z^{t-1}, X_u^*} = F_{\epsilon_t(d)} F_{D_t | Y^{t-1}, D^{t-1}, Z^t} F_{Z_t | Y^{t-1}, D^{t-1}, Z^{t-1}}.$$

L2 (A) The joint PDF of $(Y, X_u^*)$ conditional on $Z$ is bounded and continuous, as are all its marginal and conditional densities. (B) $X_u^* \mid Z$ has full support. (C) The characteristic function of $\epsilon_t(d)$ is non-vanishing, and $\mathrm{E}[\epsilon_t | Z, X_u^*] = 0$.

# Unknown factor only: assumptions (Cont'd)

L3 For some choice sequence $(d_t \colon t = 1, 2, \ldots, p)$, (A) $(F_1(d_1) \ldots F_p(d_p)) = I_{p \times p}$ and (B) $\alpha_t(d_t) = 0$ for each $t = 1, 2, \ldots, p$.

L4 (A) $f_{Y^{t-1}, Z^t, D^t}(y^{t-1}, z^t, d^t) > 0$ for all $t$. (B) The variance-covariance matrix of $X_u^* \mid Z$ is full rank.

L5 Any $p \times p$ sub-matrix of $F(d) = (F_1(d_1) F_2(d_2) \ldots F_T(d_T))$ is full rank.

Back

# Proof outline

1. $Y_t | Y^{t-1}, D^t, X^t$ is a mixture of normal distributions, with weights $f_{X_k^* | D^t, Y^{t-1}, X^t}(x_k)$ (Lemma 1).

2. Under compact support and regularity conditions, rely on Bruni and Koch (1985, Th.3) to identify the distribution $Y_t | Y^{t-1}, D^t, X^t, X_k^*$ and mixture weights, up to a one-to-one transformation $\pi$ of $X_k^*$.

3. $E(Y_t | Y^{t-1}, D^t, X^t, X_k^*)$ is linear in $X_k^* \to \pi$ linear.

4. Rank conditions and factor normalizations $\to \pi = \mathrm{Id} \Rightarrow$ Identification of the distribution of $(Y^T, D^T, X^T, X_k^*)$.

5. Outcome eq. parameters, distribution of unknown component $X_u^*$ and id. shocks: from the distribution of $(Y^T, D^T, X^T | X_k^*)$, weighted by conditional choice probability.

Return

## Proof outline

1. $Y_t | Y^{t-1}, D^t, X^t$ is a mixture of normal distributions, with weights $f_{X_k^* | D^t, Y^{t-1}, X^t}(x_k)$ (Lemma 1).
2. Under compact support and regularity conditions, rely on Bruni and Koch (1985, Th.3) to identify the distribution $Y_t | Y^{t-1}, D^t, X^t, X_k^*$ and mixture weights, up to a one-to-one transformation $\pi$ of $X_k^*$.
3. $E(Y_t | Y^{t-1}, D^t, X^t, X_k^*)$ is linear in $X_k^* \rightarrow \pi$ linear.
4. Rank conditions and factor normalizations $\rightarrow \pi = \mathrm{Id} \Rightarrow$ Identification of the distribution of $(Y^T, D^T, X^T, X_k^*)$.
5. Outcome eq. parameters, distribution of unknown component $X_u^*$ and id. shocks: from the distribution of $(Y^T, D^T, X^T | X_k^*)$, weighted by conditional choice probability.

Return

# Proof outline

1. $Y_t | Y^{t-1}, D^t, X^t$ is a mixture of normal distributions, with weights $f_{X_k^* | D^t, Y^{t-1}, X^t}(x_k)$ (Lemma 1).
2. Under compact support and regularity conditions, rely on Bruni and Koch (1985, Th.3) to identify the distribution $Y_t | Y^{t-1}, D^t, X^t, X_k^*$ and mixture weights, up to a one-to-one transformation $\pi$ of $X_k^*$.
3. $E(Y_t | Y^{t-1}, D^t, X^t, X_k^*)$ is linear in $X_k^* \to \pi$ linear.
4. Rank conditions and factor normalizations $\to \pi = \mathrm{Id} \Rightarrow$ Identification of the distribution of $(Y^T, D^T, X^T, X_k^*)$.
5. Outcome eq. parameters, distribution of unknown component $X_u^*$ and id. shocks: from the distribution of $(Y^T, D^T, X^T | X_k^*)$, weighted by conditional choice probability.

Return

## Proof outline

1. $Y_t|Y^{t-1}, D^t, X^t$ is a mixture of normal distributions, with weights $f_{X_k^*|D^t,Y^{t-1},X^t}(x_k)$ (Lemma 1).

2. Under compact support and regularity conditions, rely on Bruni and Koch (1985, Th.3) to identify the distribution $Y_t|Y^{t-1}, D^t, X^t, X_k^*$ and mixture weights, up to a one-to-one transformation $\pi$ of $X_k^*$.

3. $E(Y_t|Y^{t-1}, D^t, X^t, X_k^*)$ is linear in $X_k^* \to \pi$ linear.

4. Rank conditions and factor normalizations $\to \pi = \mathrm{Id} \Rightarrow$ Identification of the distribution of $(Y^T, D^T, X^T, X_k^*)$.

5. Outcome eq. parameters, distribution of unknown component $X_u^*$ and id. shocks: from the distribution of $(Y^T, D^T, X^T|X_k^*)$, weighted by conditional choice probability.

Return

# Likelihood

The individual log-likelihood contribution is:

$$\log \int \prod_{t=1}^{T} \left( \frac{1}{\sigma_t(d_t)} \phi_1 \left( \frac{y_t - \alpha_t(d_t) - z_t' \beta_t(d_t) - v_u' F_{ut}(d_t) - v_k F_{kt}(d_t)}{\sigma_t(d_t)} \right) \times \bar{h}_t(d_t, z_t, v_k, x_t) \right) \times \prod_{t=1}^{T-1} g_t(z_{t+1} \mid z_t, y_t, d_t) \times \frac{1}{\sqrt{\left|\Sigma_u(z_1, v_k)\right|}} \phi_p \left( \Sigma_u^{-\frac{1}{2}}(z_1, v_k) v_u \right) \times dF_{X_k^*}(v_k; z_1) \, dv_u$$

⏩ Back

# CCP sieve space

Utility:

- Linear index in $\left(Y_1, \ldots, Y_{t-1}, X_k^*\right)$ where coefficients are conditional on full choice sequence.
- Nonlinear transformation of the linear index by a polynomial of order $m_n \to \infty$.

$$u_t(d|h) = f^{m_n}\left(\pi_{0,t}(d^{t-1}) + \pi_{1,t}(d^{t-1})y_1 + \ldots + \pi_{t-1,t}(d^{t-1})y_{t-1} + \pi_{k,t}(d^{t-1})x_{ik}^*\right.$$

$$f^{m_n}(y) = \sum_{s=1}^{m_n} f_s y^{s-1}$$

Probabilities obtained by assuming an EV1 additive choice shock

$$p_t(d|h) = \frac{\exp(u_t(d|h))}{\sum_{d'} \exp(u_t(d'|h))}$$

Back

# Sieve MLE estimation - functionals

- Simple example with static choice (e.g. occupational choice). For a rate of time preference $\rho$, the present value of lifetime earnings is:

$$\widetilde{Y}_{t_0}(d) = \sum_{t=t_0}^{T} \frac{Y_t(d)}{(1+\rho)^{t-t_0}}$$

- Predictable component is given by, denoting by $\mathcal{I}_{t_0}$ the information set at time $t = t_0$:

$$E(\widetilde{Y}_{t_0}(d)|\mathcal{I}_{t_0})$$

where we assume that $\mathcal{I}_{t_0} = \{X_k^*, W^{t_0}\}$ with $W^t = (Y^{t-1}, D^{t-1}, Z^t)$.

# Sieve MLE estimation - functionals

- Simple example with static choice (e.g. occupational choice). For a rate of time preference $\rho$, the present value of lifetime earnings is:

$$\widetilde{Y}_{t_0}(d) = \sum_{t=t_0}^{T} \frac{Y_t(d)}{(1+\rho)^{t-t_0}}$$

- Predictable component is given by, denoting by $\mathcal{I}_{t_0}$ the information set at time $t = t_0$:

$$E(\widetilde{Y}_{t_0}(d)|\mathcal{I}_{t_0})$$

where we assume that $\mathcal{I}_{t_0} = \{X_k^*, W^{t_0}\}$ with $W^t = (Y^{t-1}, D^{t-1}, Z^t)$.

# Sieve MLE estimation - functionals (Cont'd)

Variance of the predictable and unpredictable components of $\widetilde{Y}_{t_0}(d)$ are given by:

$$\sigma^2_{k,t_0}(d) = \int \left( E(\widetilde{Y}_{t_0}(d)|\mathcal{I}_{t_0}) - E(\widetilde{Y}_{t_0}(d)) \right)^2 dF_{X_k^*, W^{t_0}}(x_k^*, w^{t_0})$$

$$\sigma^2_{u,t_0}(d) = \int \mathrm{Var}(\widetilde{Y}_{t_0}(d)|\mathcal{I}_{t_0}) dF_{X_k^*, W^{t_0}}(x_k^*, w^{t_0})$$

- As agents learn and update their beliefs about $X_u^* \Rightarrow$ Evolution over time of share of predictable/unpredictable earnings variance.
- We wish to estimate and conduct inference on these types of parameters.

# Sieve MLE estimation - functionals (Cont'd)

- We provide in the paper general results for the following class of functionals. Namely, consider a function $f_1$ which maps $(\theta, w, x_k^*)$ to $\mathbb{R}$ such that $f_1(\theta, w, x_k^*)$ is given by:

$$g\left(\mathrm{E}\left[\sum_{i \in I_t} \omega_i Y_{t_i}(d_i) \mid W^t = w, X_k^* = x_k^*\right], \mathrm{Var}\left[\sum_{i \in I_t} \omega_i Y_{t_i}(d_i) \mid W^t = w, X_k^* = x_k^*\right]\right)$$

- We define the functional of $\theta$ as

$$f(\theta) = \int f_1(\theta, w, x_k^*) dF_{W^t, X_k^*}(w, x_k^*).$$

$\Rightarrow$ We propose to estimate $f(\theta^*)$ via plug-in sieve MLE, and establish consistency and asymptotic normality.

Back

# Sieve MLE estimation - functionals (Cont'd)

- We provide in the paper general results for the following class of functionals. Namely, consider a function $f_1$ which maps $(\theta, w, x_k^*)$ to $\mathbb{R}$ such that $f_1(\theta, w, x_k^*)$ is given by:

$$g\left(\mathrm{E}\left[\sum_{i\in I_t}\omega_i Y_{t_i}(d_i) \mid W^t = w, X_k^* = x_k^*\right], \mathrm{Var}\left[\sum_{i\in I_t}\omega_i Y_{t_i}(d_i) \mid W^t = w, X_k^* = x_k^*\right]\right)$$

- We define the functional of $\theta$ as

$$f(\theta) = \int f_1(\theta, w, x_k^*) dF_{W^t, X_k^*}(w, x_k^*).$$

$\Rightarrow$ We propose to estimate $f(\theta^*)$ via plug-in sieve MLE, and establish consistency and asymptotic normality.

Back