Making Waves: Secondary Markets For Automobile Under Macro Shocks

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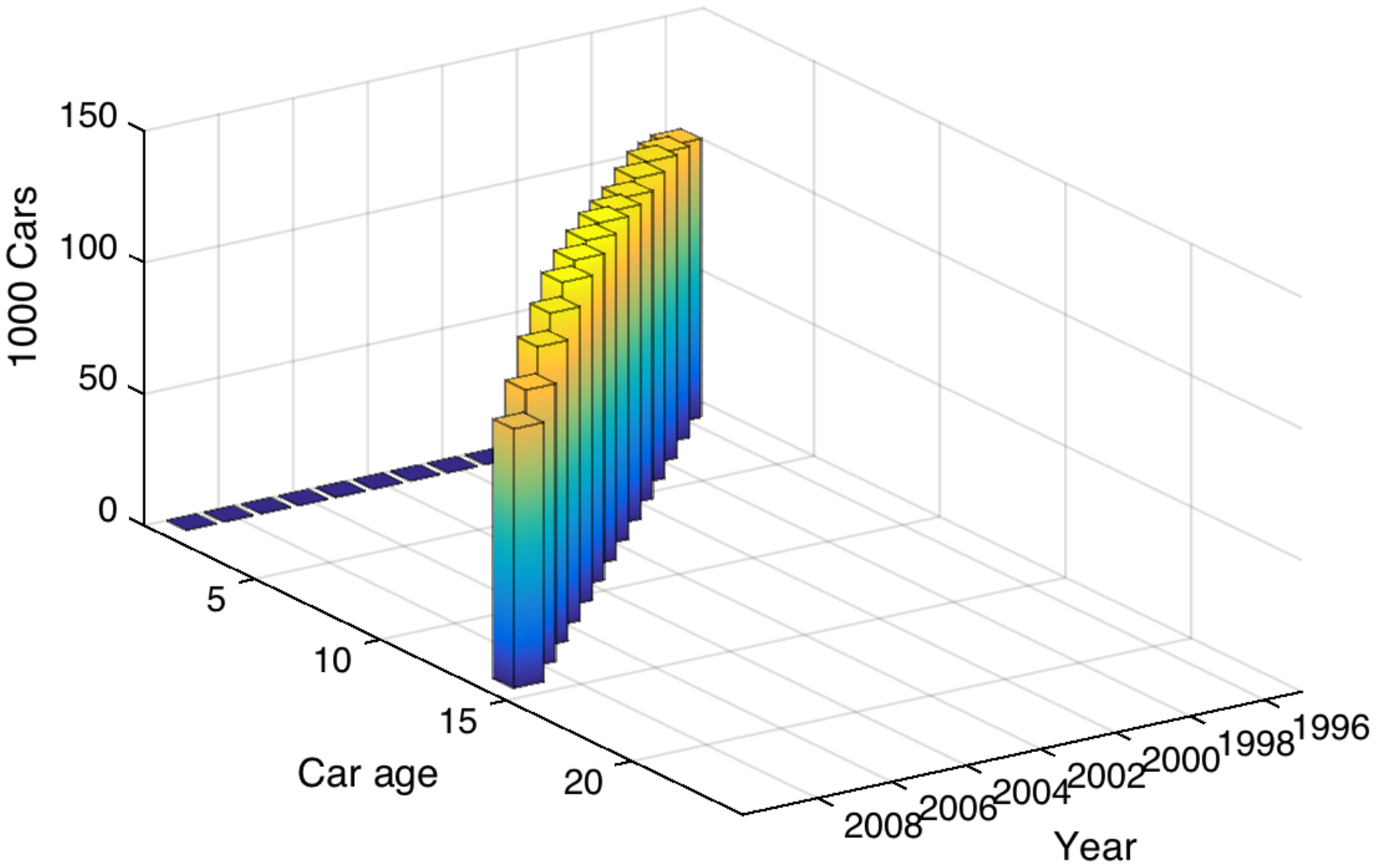
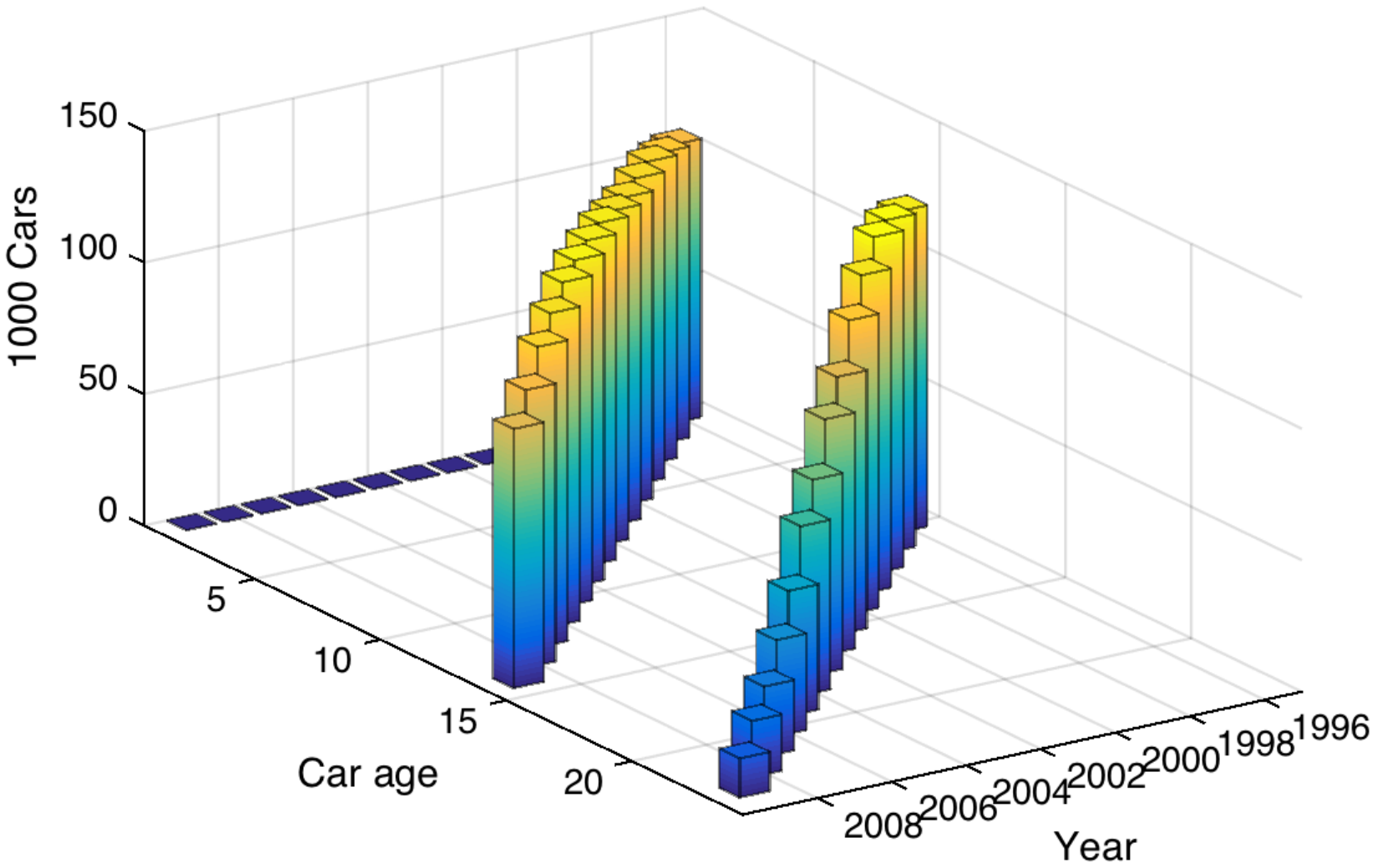
Outline

1. Introduction
2. DSE meets GDSGE
3. Toy Model
4. Quantitative Model
5. Conclusion

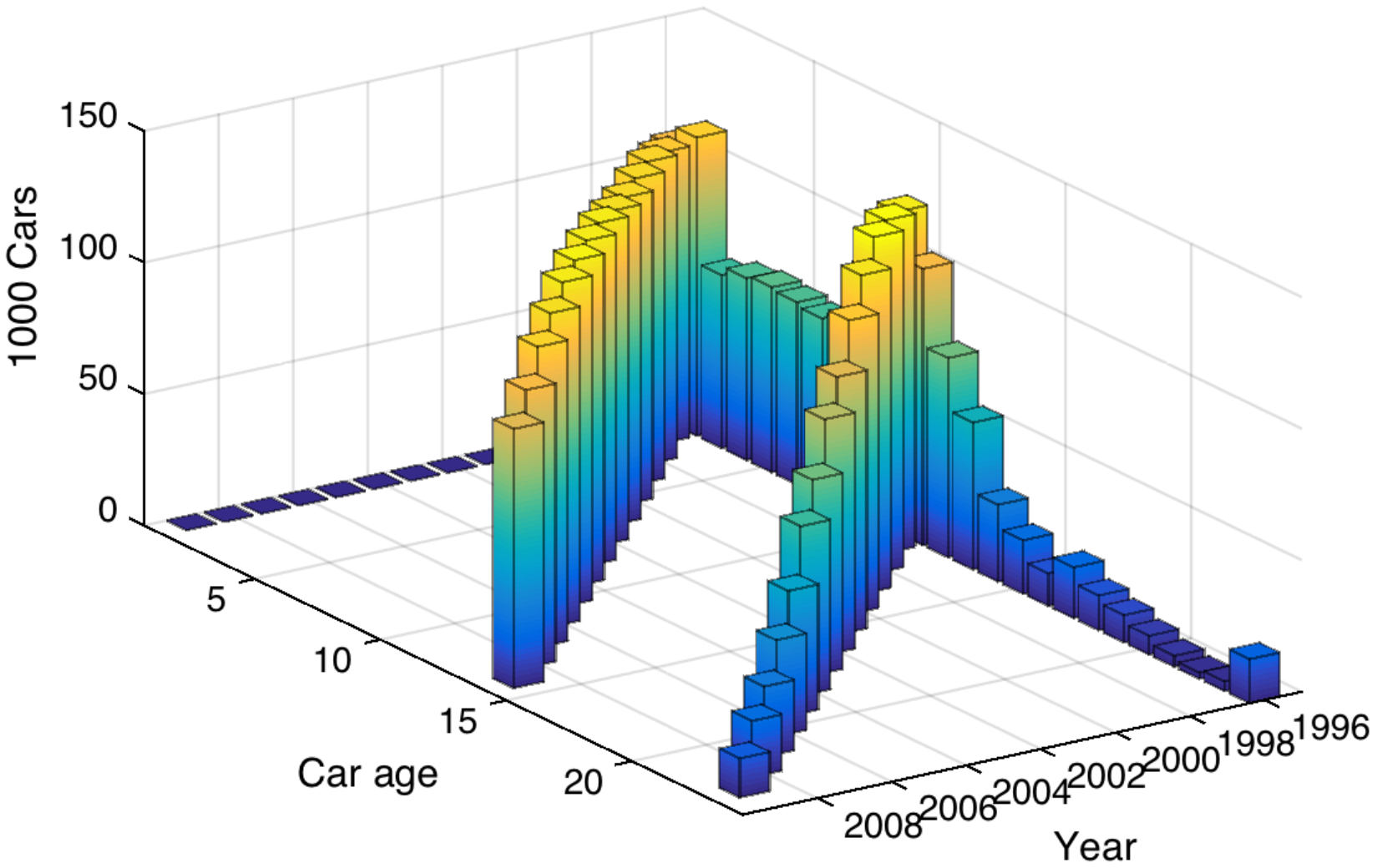
The 1995 Cohort of Cars



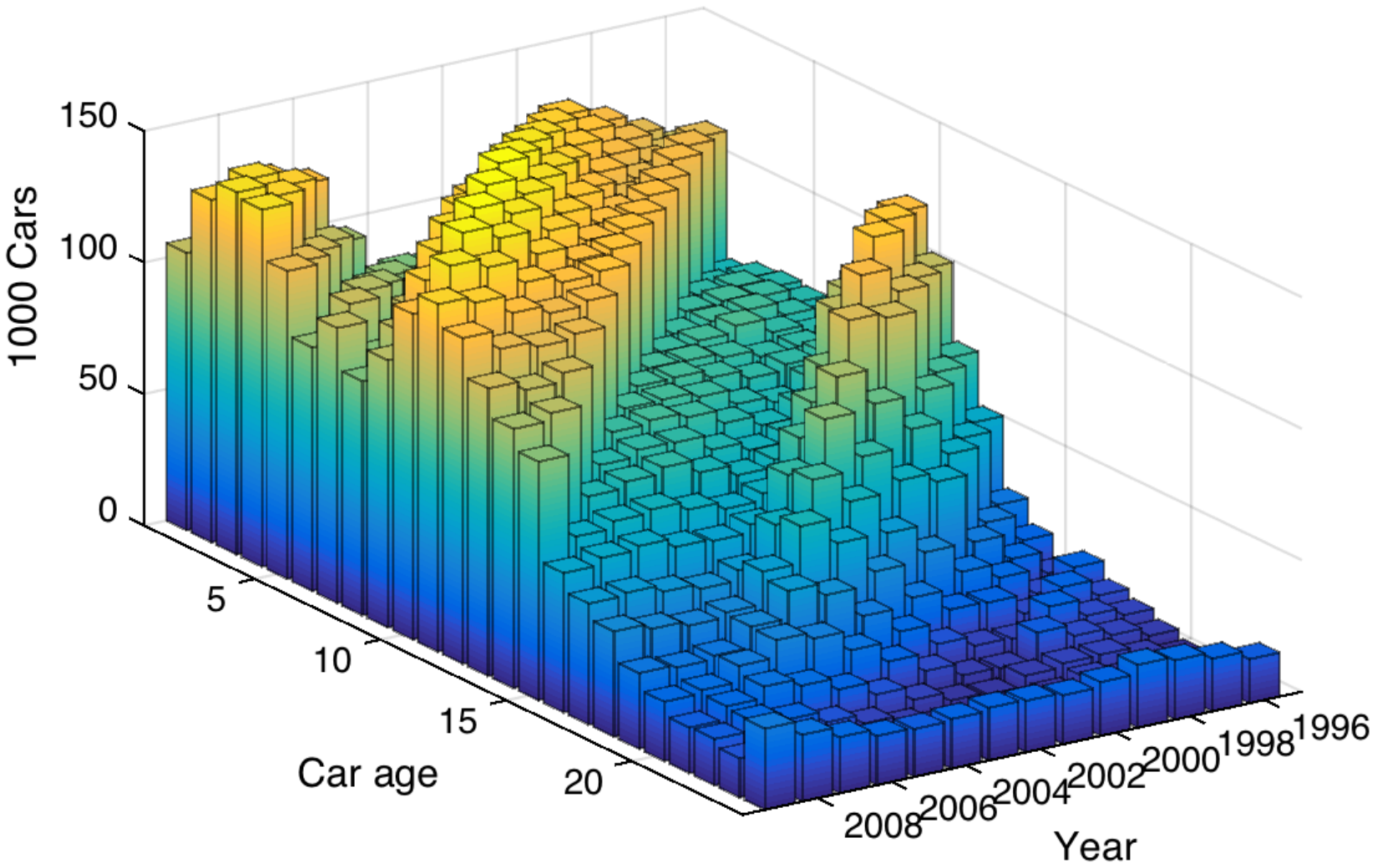
The 1995 and 1986 Cohorts of Cars



The Car Age Distribution in 1996

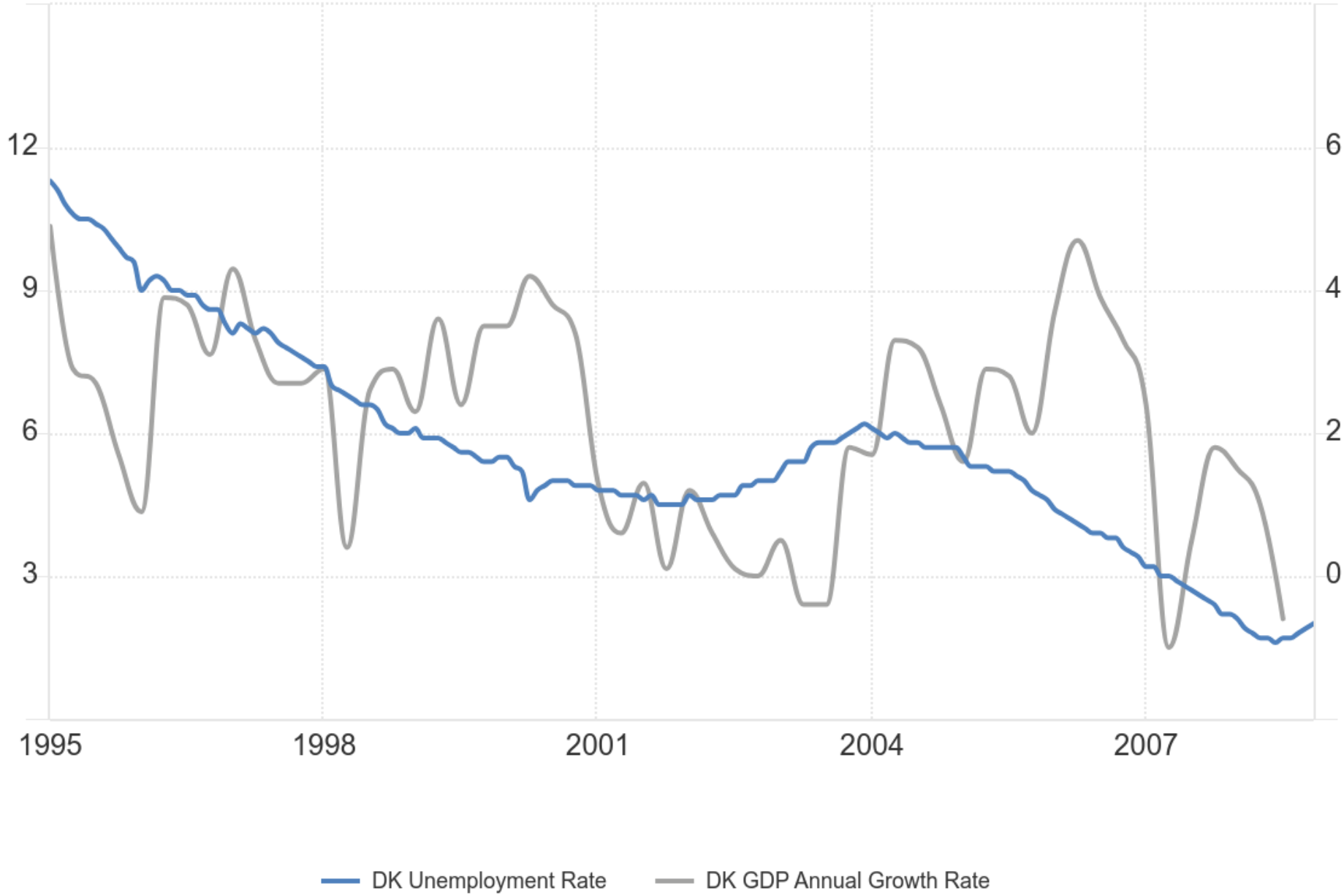


The Car Age Distribution Over Time



Macro Waves in Secondary Markets for Automobile

Source: tradingeconomics.com



Macro Waves in Secondary Markets for Automobile

- Secondary markets for automobile vary significantly over the business cycles in Denmark. What cause these business cycle variations?
- Empirical framework in Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning (2022 + WP) with aggregate demand shocks and temporary equilibrium (TE) can generate the waves
  - Rational Expectations Equilibrium (REE): Krusell-Smith (1998) + high-dimensional price vectors
- How important is the role of consumers' macro expectations?
  - Rational expectations lead to more cross-product substitution and quantity/price variation in the REE compared to naive expectations in the TE
  - However, temporary equilibrium is more robust conceptually and quantitatively:
    - ★ Utility gains from learning the macro equilibrium dynamics in TE are small
    - ★ Utility losses from ignoring the macro equilibrium dynamics in REE are small
    - ★ "It is self-evident that real-world households and firms do not forecast prices by forecasting distributions and instead solve simpler problems." (Moll, 2024)

Related Literature

- Durable good markets over the business cycles: Cooper, Haltiwanger and Power (1999)
- Impossibility of rational expectations equilibrium: Grossman and Stiglitz (1980), Broer et al (2022)
- Non-rational expectations equilibrium: Grandmont (1977), Krusell and Smith (1998), Benkard et al (2008, 2010, 2015), Ifrach and Weinstraub (2017), Fershmant and Pakes (2012), ...

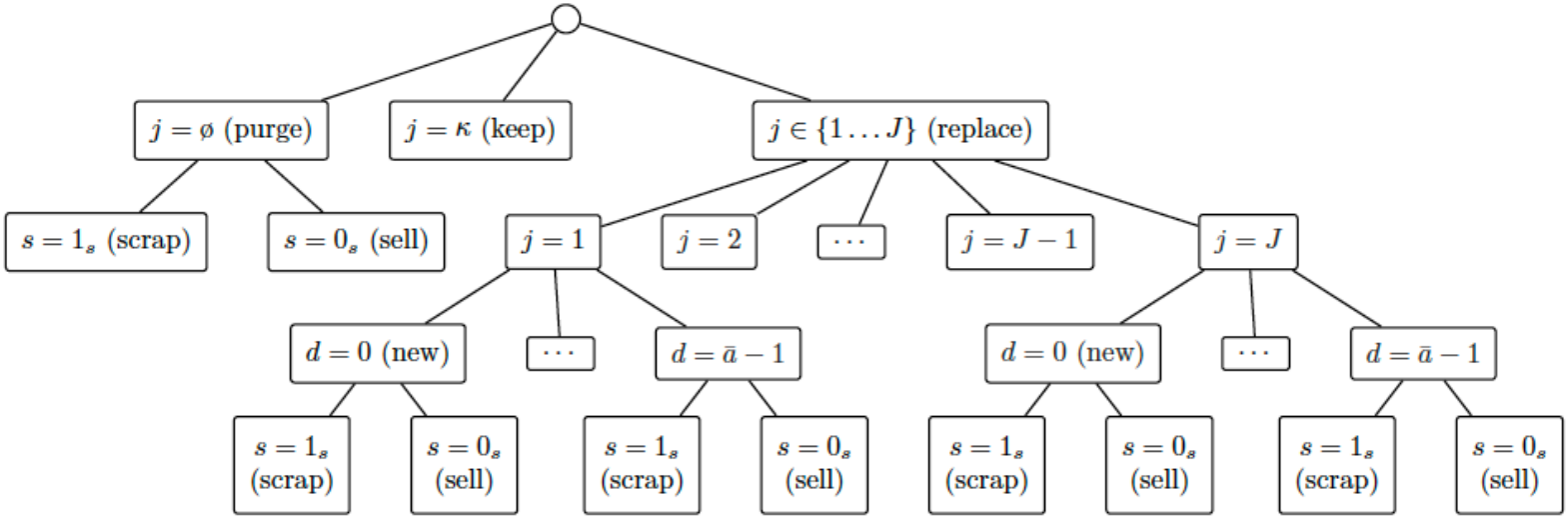
DSE + Macro Shocks

Rich mans Volvo

Poor mans Volvo

DSE + Macro Shocks

Consumer choice tree - car owner



Notes: The figure presents an example of the choice tree for a consumer who owns a car under the nested logit specification of the GEV distribution of the idiosyncratic heterogeneity term ϵ. Note three choice nests at the top level (to purge, keep or trade the existing car), two intermediate levels of nesting in the case of trading, and a additional scrappage choice of the existing car in the cases when it is traded.

DSE + Macro Shocks

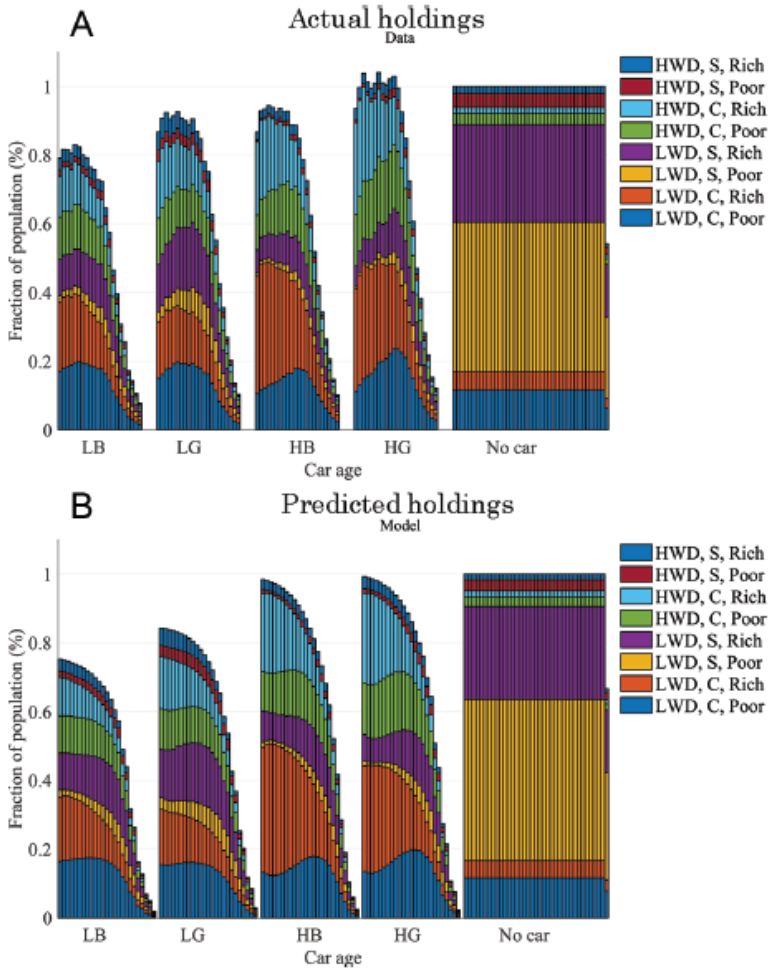


FIG. 4.—Actual and predicted holdings, by household type, car type, and car age. The graphs show the fraction of the population holding each car-age combination as well as the outside good. For the outside good, we have opted to split the bars and put them next to each other, since otherwise that option would dominate the scale of the *y*-axis. Household types: HWD = high work distance; LWD = low work distance; S = single; C = couple. Car types: LB = light, brown; LG = light, green; HB = heavy, brown; HG = heavy, green. Within each of the four car types, car ages go from 0 to 24.

DSE + Macro Shocks

- Micro states:

$$a \in \{1, 2, \ldots, \bar{a}\} \cup \{\emptyset\}$$

and ε Type-1 EV associated with different choices

- Choices

$$C(a) = \begin{cases} \{0, 1, \ldots, \bar{a} - 1\} \cup \{\emptyset\} & \text{if } a \in \{\bar{a}, \emptyset\} \\ \{0, 1, \ldots, \bar{a} - 1\} \cup \{\emptyset\} \times \{1_{\text{scr}}, 0_{\text{scrp}}\} & \text{if } a \in \{1, 2, \ldots, \bar{a} - 1\} \end{cases}$$

- Macro states: (m, q)

$$q = (q_0, q_1, \ldots, q_{\bar{a}}, q_{\emptyset})$$

and

$$v(a, d; m, q) = u(d) - \mu(m) \cdot [P_d - P_a] \ldots,$$

where

$$\mu(1 = \text{Expansion}) < \mu(2 = \text{Recession})$$

DSE + Macro Shocks

- Bellman's equation

$$\delta(a,\varepsilon;m,q) = \arg\max \begin{bmatrix} v(a,\kappa;m,q)+\sigma\varepsilon_\kappa; \\ \max_{s\in\{1_{scr},0_{scr}\}}\left[v(a,\emptyset,s;m,q)+\sigma\varepsilon_{\emptyset,s}\right]; \\ \max_{\substack{d\in\{0,1,\ldots,\bar{a}-1\},\\ s\in\{1_{scr},0_{scr}\}}}\left[v(a,d,s;m,q)+\sigma\varepsilon_{d,s}\right] \end{bmatrix}$$

where

$$\begin{aligned} &v(a,d,0_{scr};m,q) \\ &= u(d)-\mu(m)\cdot[P_d(m,q)-P_a(m,q)+T_s(a)+T_b(d)]-T^{\text{psy}}(a,d) \\ &+\beta\sum_{m'}\Big[(1-\alpha_d)\mathbb{E}^\tau V(d+1;m',q')+\alpha_d\mathbb{E}^\tau V(\bar{a};m',q')\Big]G(m'|m). \end{aligned}$$

- $\tau \in \{\text{sophisticated}, \text{naive}\}$

$$\mathbb{E}^{\text{naive}}V(.;m',q') \equiv \overline{EV}(m)$$
$$\mathbb{E}^{\text{sophis}}V(.;m',q') \equiv \mathbb{E}V(.;m',q')$$

DSE + Macro Shocks

- Choice probabilities

$$\Pi((d,s)|a) = \begin{cases} \Pi(d|a) & \text{if } a \in (\emptyset, \bar{a}) \\ \Pi(d|a) \cdot \Pi(s|d,a) & \text{if } a < \bar{a} \text{ and } d \neq \kappa \\ \Pi(\kappa|a) & \text{if } a < \bar{a} \text{ and } d = \kappa \end{cases}$$

where

$$\Pi(d|a) = \frac{\exp\left\{\frac{I(a,d;m,q)}{\sigma}\right\}}{\exp\left\{\frac{I(a,\emptyset;m,q)}{\sigma}\right\} + \exp\left\{\frac{I(a,\kappa;m,q)}{\sigma}\right\} + \sum_{d=0}^{\bar{a}-1} \exp\left\{\frac{I(a,d;m,q)}{\sigma}\right\}},$$

and

$$I(a,d;m,q) = \sigma_s \log\left(\exp\left\{\frac{v(a,d,1_{scr},m,q)}{\sigma_s}\right\} + \exp\left\{\frac{v(a,d,0_{scr},m,q)}{\sigma_s}\right\}\right)$$

DSE + Macro Shocks

- Value function

$$\mathbb{E}V^{\tau}(.;m,q) = \mathbb{E}\left[v\left(.,\delta(a,\varepsilon;m,q);m,q\right) + \sigma\varepsilon_{\delta(a,\varepsilon;m,q)}\right]$$

where

$$\begin{aligned}
&v(a,d,0_{scr};m,q) \\
&= u(d) - \mu(m)\cdot\left[P_d(m,q) - P_a(m,q) + T_s(a) + T_b(d)\right] \\
&+ \beta\sum_{m'}\left[\left(1-\alpha_d\right)\mathbb{E}V^{\tau}(d+1;m',q') + \alpha_d\mathbb{E}V^{\tau}(\bar{a};m',q')\right]G(m'|m)
\end{aligned}$$

- For sophisticated consumers

$$\begin{aligned}
&EV^{\text{sophis}}(a;m,q) \\
&= \sigma\log\left(\exp\{v(a,\kappa,m,q)/\sigma\} + \exp\{I(a,\emptyset,m,q)/\sigma\} + \sum_{d=0}^{\bar{a}-1}\exp\{I(a,d,m,q)/\sigma\}\right).
\end{aligned}$$

DSE + Macro Shocks

- Trade transition probabilities

$$\Omega(P, EV^{\tau}; m, q) =$$

$$\begin{bmatrix} \Pi(1|1)+\Pi(\kappa|1) & \dots & \Pi(\bar{a}-1|1) & \Pi(0|1) & \Pi(\emptyset|1) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \Pi(1|\bar{a}-1) & \dots & \Pi(\bar{a}-1|\bar{a}-1)+\Pi(\kappa|\bar{a}-1) & \Pi(0|\bar{a}-1) & \Pi(\emptyset|\bar{a}-1) \\ \Pi(1|\bar{a}) & \dots & \Pi(\bar{a}-1|\bar{a}) & \Pi(0|\bar{a}) & \Pi(\emptyset|\bar{a}) \\ \Pi(1|\emptyset) & \dots & \Pi(\bar{a}-1|\emptyset) & \Pi(0|\emptyset) & \Pi(\emptyset|\emptyset) \end{bmatrix}$$

- Physical transition probabilities Q

$$Q = \begin{bmatrix} 0 & 1-\alpha_1 & \dots & 0 & \alpha_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-\alpha_{\bar{a}-2} & \alpha_{\bar{a}-2} \\ 0 & 0 & \dots & 0 & 1 \\ 1-\alpha_0 & 0 & \dots & 0 & \alpha_0 \end{bmatrix}.$$

DSE + Macro Shocks: Static Block

- Demand

$$D_a(m,q) = \Pi(a|\emptyset; m, q)q_\emptyset + \sum_{d=0}^{\bar{a}} \Pi(a|d; m, q)q_d$$

- Supply

$$S_a(m,q) = (1 - \Pi(\kappa|a))(1 - \Pi(1_{\text{scr}}|a))q_a$$

- Excess demand

$$D_a(m,q) - S_a(m,q)$$

DSE + Macro Shocks: Dynamic Block

- Rational Expectations Equilibrium (REE): consistency equations

$$q' = q\Omega(P, \mathbb{E}V^{\text{sophis}}(.; m', q'); m, q)Q$$

- Temporary Equilibrium (TE):

$$\mathbb{E}^{\text{naive}} V(.; m', q') \equiv \overline{EV}(m),$$

where

$$\bar{q}(m) = \bar{q}(m)\Omega(\bar{P}(m), \overline{EV}(.; m); m, \bar{q}(m))Q.$$

$$q' = q\Omega(P, \overline{EV}(m); m, q)Q$$

DSE + Macro Shocks

- Value function

$$\mathbb{E}V^{\tau}(.;m,q) = \mathbb{E}\left[v\left(.,\delta(a,\varepsilon;m,q);m,q\right) + \sigma\varepsilon_{\delta(a,\varepsilon;m,q)}\right]$$

Proposition.

For both sophisticated and naive consumers

$$\begin{aligned}
&\mathbb{E}V^{\tau}(a;m,q) \\
&= \sum_{d\in C(\hat{a})} \Pi^{\tau}(d|\hat{a})\Pi^{\tau}(s|d,\hat{a})\left[v(\hat{a},d,s,m,q) + \sigma\mathbb{E}\left[\varepsilon_{d,s}|\delta^{\tau}=d,s,\hat{a}\right]\right] \\
&+ \sum_{d\in C(\check{a})} \Pi^{\tau}(d|\check{a})\left[v(\check{a},d,m,q) + \sigma\mathbb{E}\left[\varepsilon_{d}|\delta^{\tau}=d,\check{a}\right]\right],
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}\left[\varepsilon_{d,s}|\delta=d,s,\hat{a}\right] &= \gamma - \log\left(\Pi^{\tau}(d|\hat{a})\right) - \sigma_s \log\left(\Pi^{\tau}(s|d,\hat{a})\right) \\
\mathbb{E}\left[\varepsilon_{d}|\delta=d,\check{a}\right] &= \gamma - \log\left(\Pi^{\tau}(d|\hat{a})\right)
\end{aligned}$$

DSE + Macro Shocks

- Monetary value of macro sophistication (unilateral deviations)

$$\Delta = \frac{\mathbb{E}V^{\text{sophis}} - \mathbb{E}V^{\text{naive}}}{\mu} \geq 0$$

- REE: losses from macro naïveté, when everyone else is sophisticated
- TE: gains from macro sophistication (Ifrach and Weinstraub, 2017) when everyone else is naive
- $\mathbb{E}[\Delta] = 0$ iff TE = REE

$$\left.\frac{\partial \mathbb{E}V^{\tau}}{\partial \Pi^{\tau}}\right|_{\tau=\text{sophis}} = 0$$

GDSGE Toolbox

- GDSGE: Cao, Luo, Nie (2023) https://www.gdsge.com
- $\mathscr{Z}$: exogenous states (m);$\mathscr{S}$: endogenous states ($q_1, q_2, \ldots, q_{\bar{a}}, q_{\emptyset}$);$\mathscr{X}$: policy variables ($P, EV^{\text{sophis}}, EV^{\text{naive}}$)
- Equilibrium conditions: $z_t \in \mathscr{Z}, s_t \in \mathscr{S}_t, x_t \in \mathscr{X}_t$

$$\boldsymbol{F}\left(s_t, x_t, z_t, \{s_{t+1}(z_{t+1}), x_{t+1}(z_{t+1})\}_{z_{t+1}\in\mathscr{Z}}\right) = 0.$$

$$\boldsymbol{F} = \left(\{D_a - S_a\}, q' - q\Omega(P, \mathbb{E}V^{\tau}(.; m', q'); m, q)Q\right)$$

- Recursive equilibrium ($\mathscr{P}, \mathscr{T}$):

$$\mathscr{P} : \mathscr{Z} \times \mathscr{S} \Longmapsto \mathscr{X} \text{ and } \mathscr{T} : \mathscr{Z} \times \mathscr{S} \times \mathscr{Z} \Longmapsto \mathscr{S}$$

such that

$$\boldsymbol{F}\left(s_t, \mathscr{P}(s_t, z_t), z_t, \{\mathscr{T}(z_t, s_t, z_{t+1}), \mathscr{P}(z_{t+1}, \mathscr{T}(z_t, s_t, z_{t+1}))\}_{z_{t+1}}\right) = 0$$

- (Bellman (1952) + McFadden (1973) = Rust (1987)) + GDSGE (2023)

GDSGE Toolbox

- Discretization: $\bar{\mathscr{S}} = \{s_1, s_2, \ldots, s_K\}$
- Projection: $\bar{\mathscr{P}}(\ldots; \bar{\mathscr{S}}, b^P)$ and $\bar{\mathscr{T}}(\ldots; \bar{\mathscr{S}}, b^T)$ defined over $\mathscr{Z} \times \bar{\mathscr{S}}$ and $\mathscr{Z} \times \bar{\mathscr{S}} \times \mathscr{Z}$
- Collocation: for each z and s_k

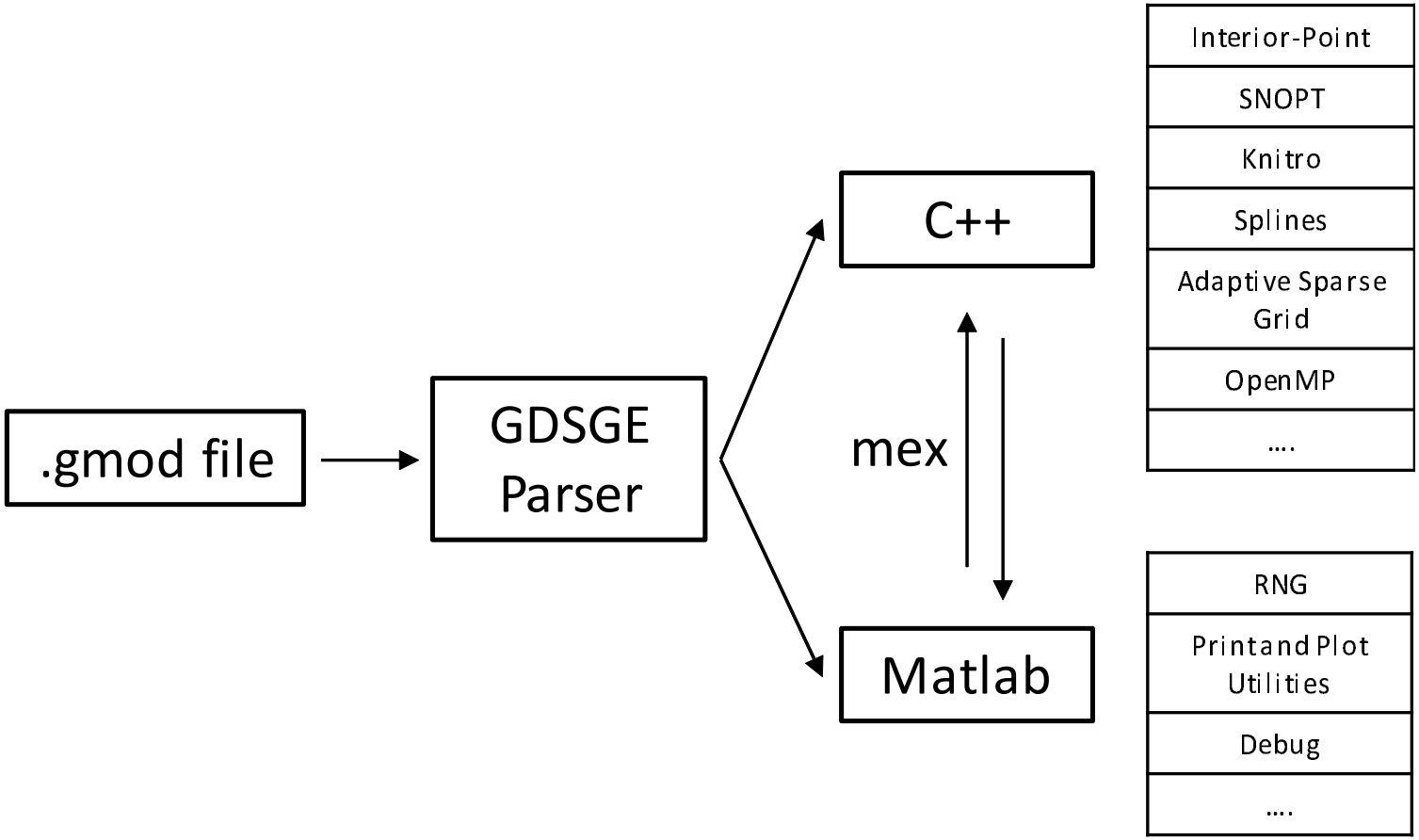
$$\boldsymbol{F}\left(z, s_k, \bar{\mathscr{P}}(z, s_k), \left\{\bar{\mathscr{T}}(z, s_k, z'), \bar{\mathscr{P}}(z', \bar{\mathscr{T}}(z, s_k, z'))\right\}_{z' \in \mathscr{Z}}\right) = 0.$$

- Time-Iteration (STPFI): A general algorithm using time iteration: Given $\bar{\mathscr{P}}^{(n)}$ - for each $z \in \mathscr{Z}$ and $s \in \mathscr{S}$, we look for s' and x' such that

$$\boldsymbol{F}\left(z, s_k, x, \left\{s'(z'), \bar{\mathscr{P}}^{(n)}(z', s'(z'))\right\}_{z' \in \mathscr{Z}}\right) = 0.$$

  - Equation solver: GDSGE constrained dogleg algorithm (Bellavia et al. ,2012)
  - Let $\bar{\mathscr{T}}^{(n+1)}(z, s, z') = s'(z')$ and $\bar{\mathscr{P}}^{(n+1)}(z, s) = x$
  - Iterate until convergence, $\left\|\bar{\mathscr{P}}^{(n+1)} - \bar{\mathscr{P}}^{(n)}\right\| < \varepsilon$
  - Initial guess: temporary equilibrium

GDSGE Toolbox



Toy Model: $\bar{a} = 2$

Parameter	Value
(u_0, u_1, u_n)	$(10, 5, 0)$
(T^s, T^b)	$(0, 1.5)$
$(\bar{P}, \underline{P})$	$(100, 10)$
$(\underline{m}, \bar{m})$	$(0.10, 0.15)$
$G(e\|e), G(r\|r)$	$(0.75, 0.75)$

Proposition.

Assume

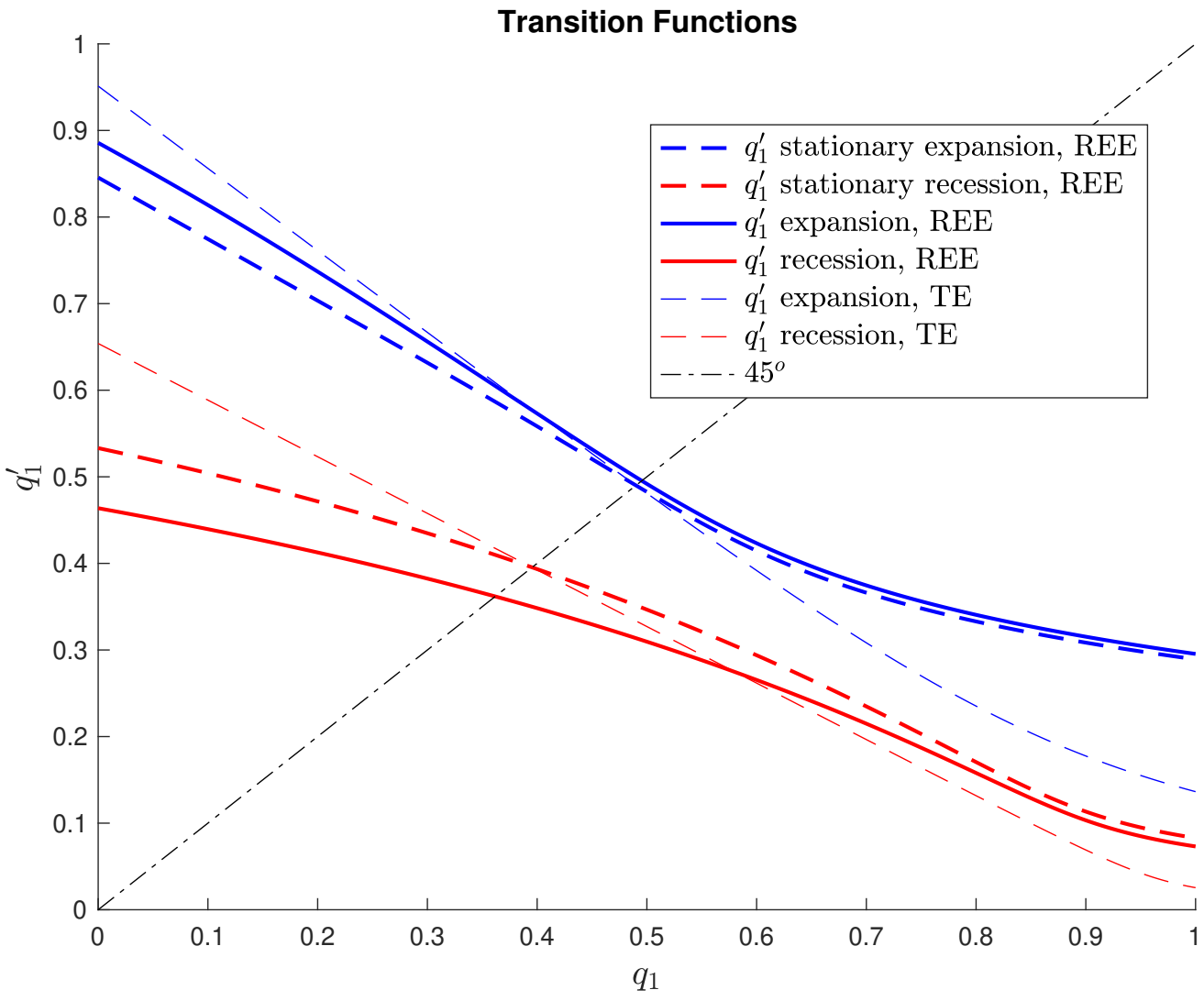
$$T^{psy}(\emptyset, d) = T^{psy}(a, d) - T^{psy}(a, \emptyset) \ \forall d.$$

In any REE and TE, consumers with no car and with a clunker $\bar{a}$ have identical choice probabilities.

- Dimension reduction

$$(q_1, \ldots, q_{\bar{a}} + q_{\emptyset})$$

Toy Model: $\bar{a} = 2$



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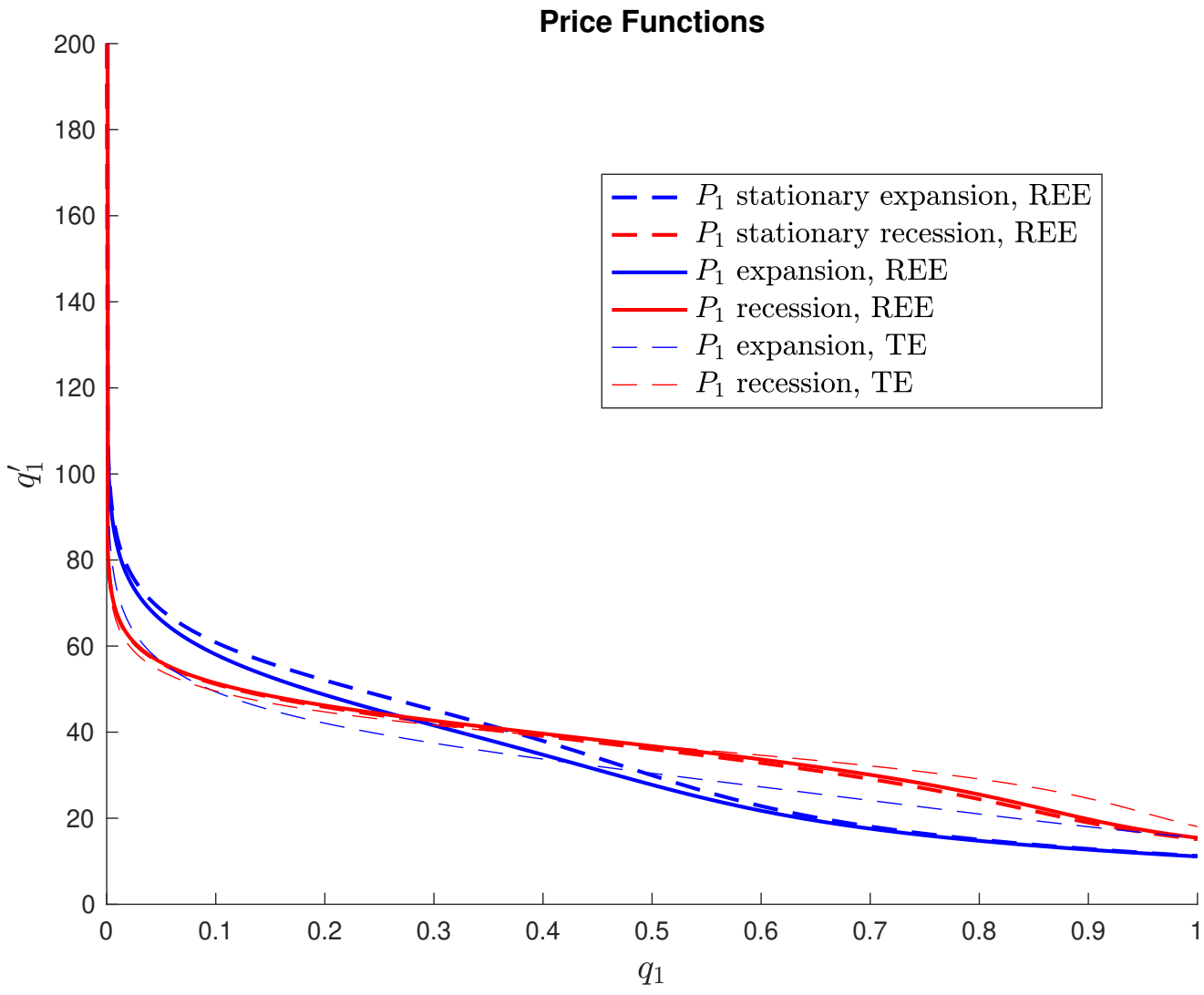

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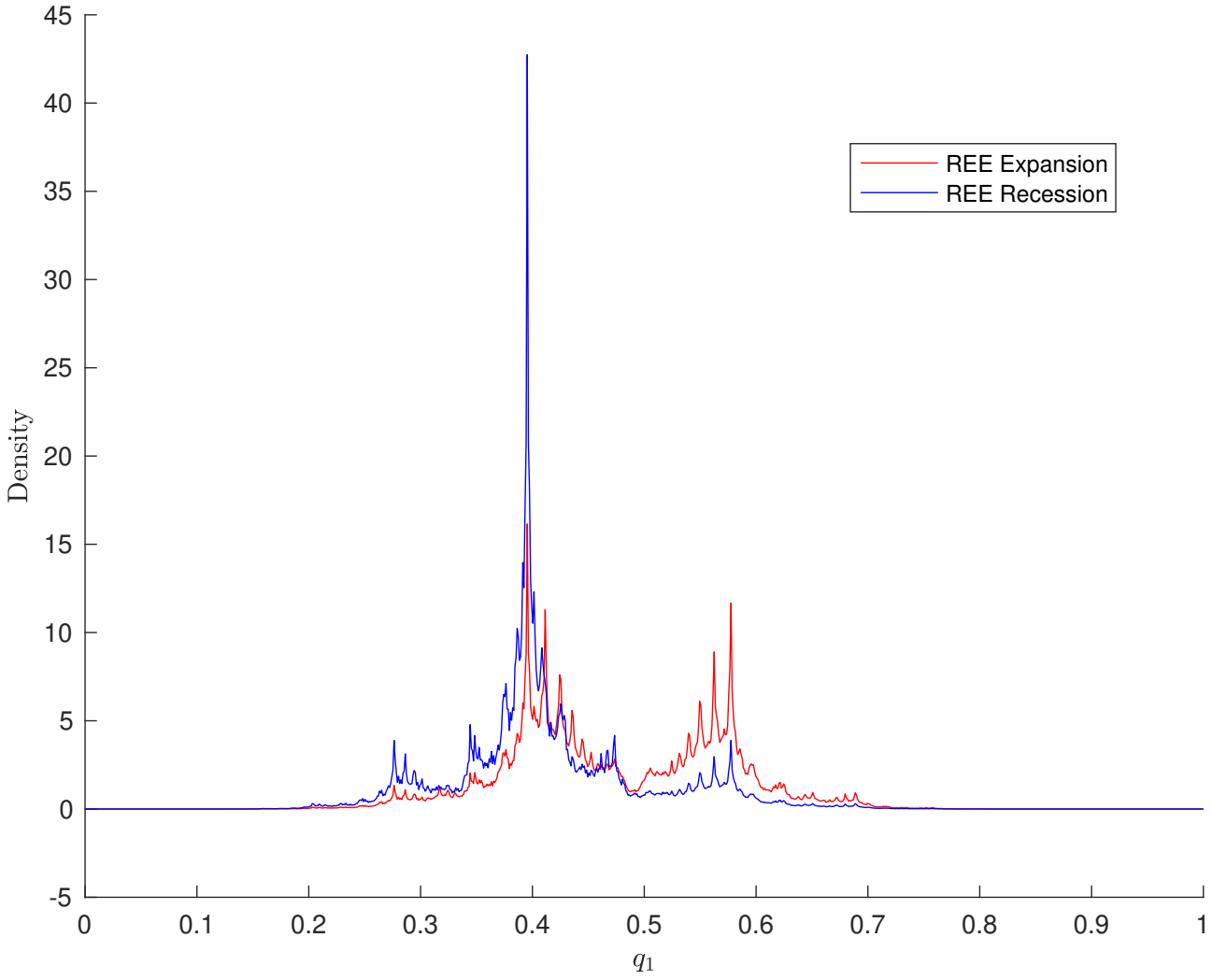
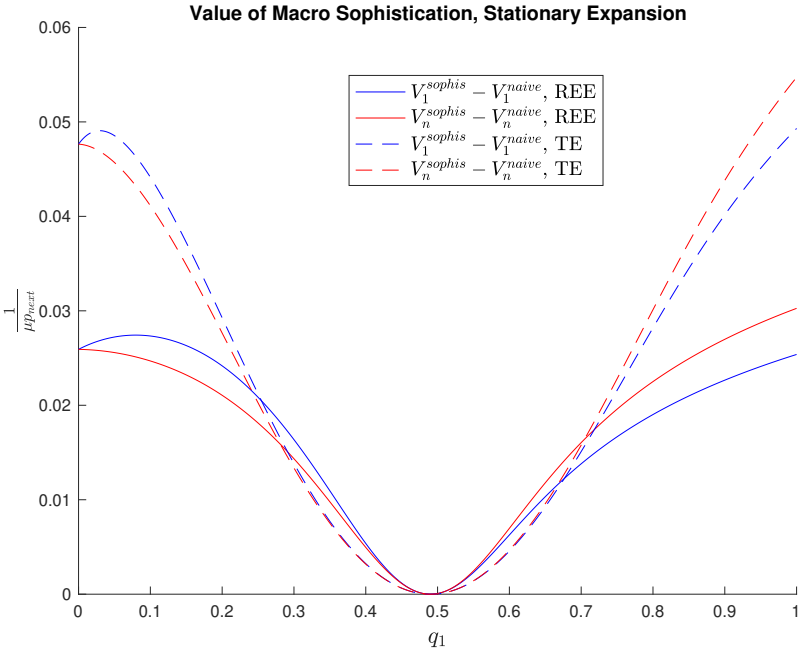
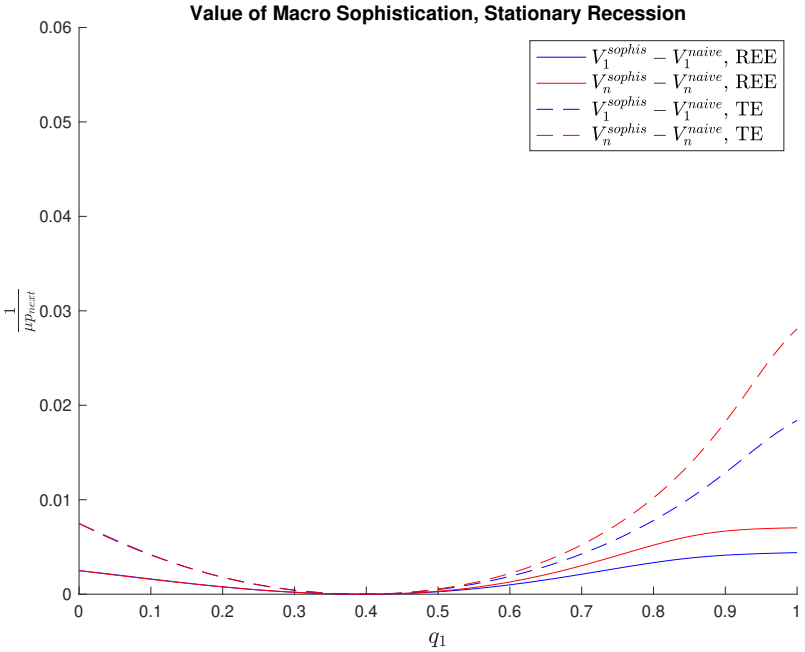


Toy Model: Value of Macro Sophistication





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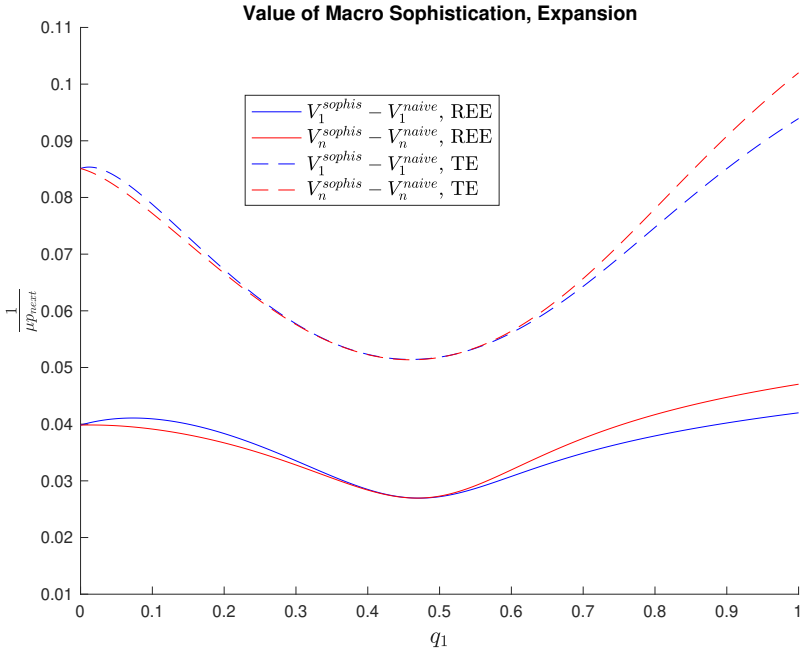
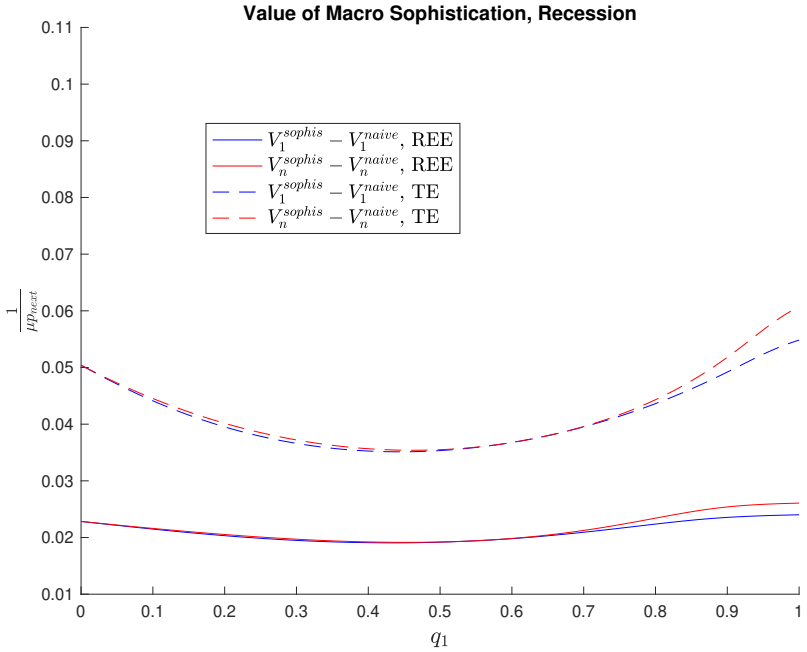
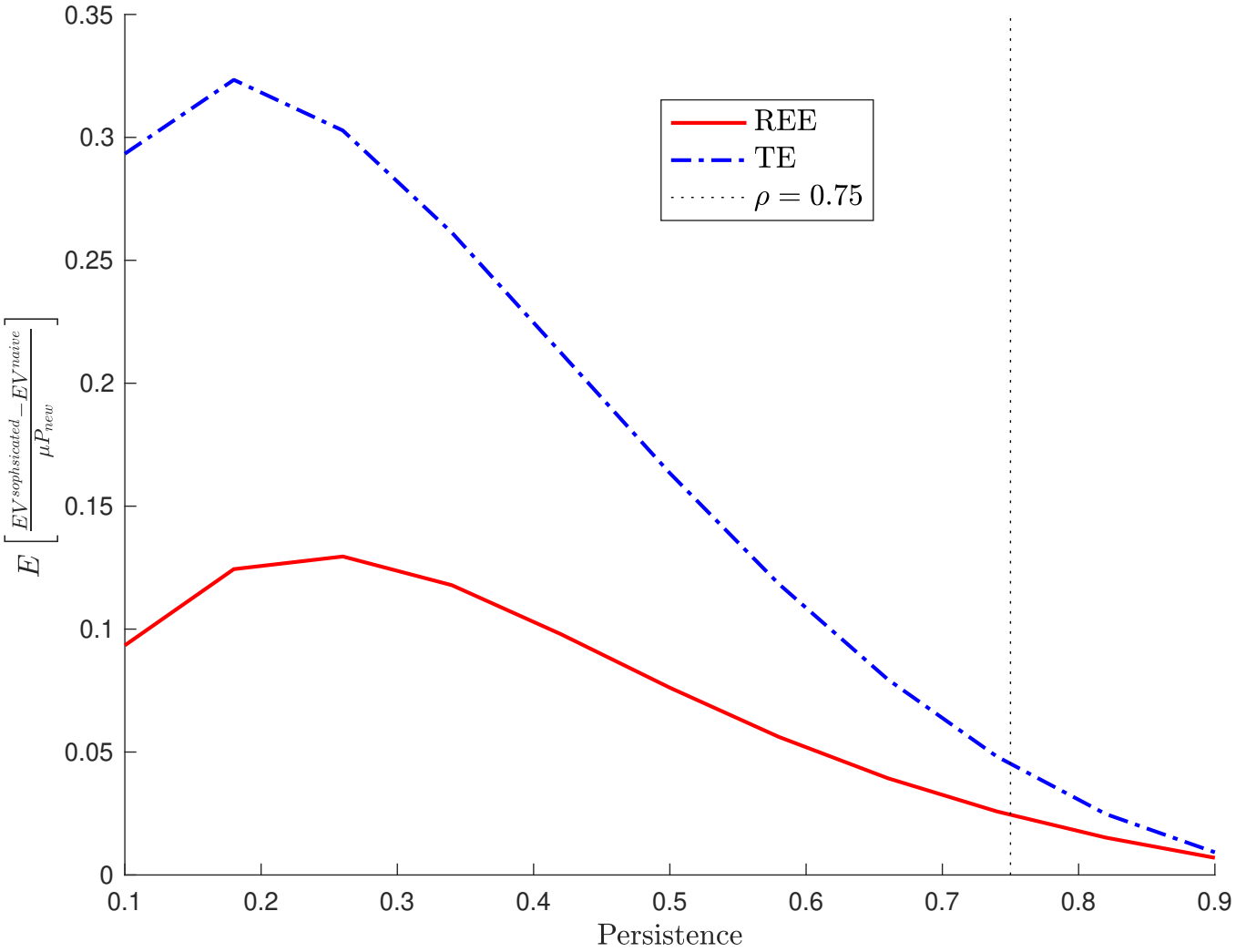




Toy Model: Value of Macro Sophistication



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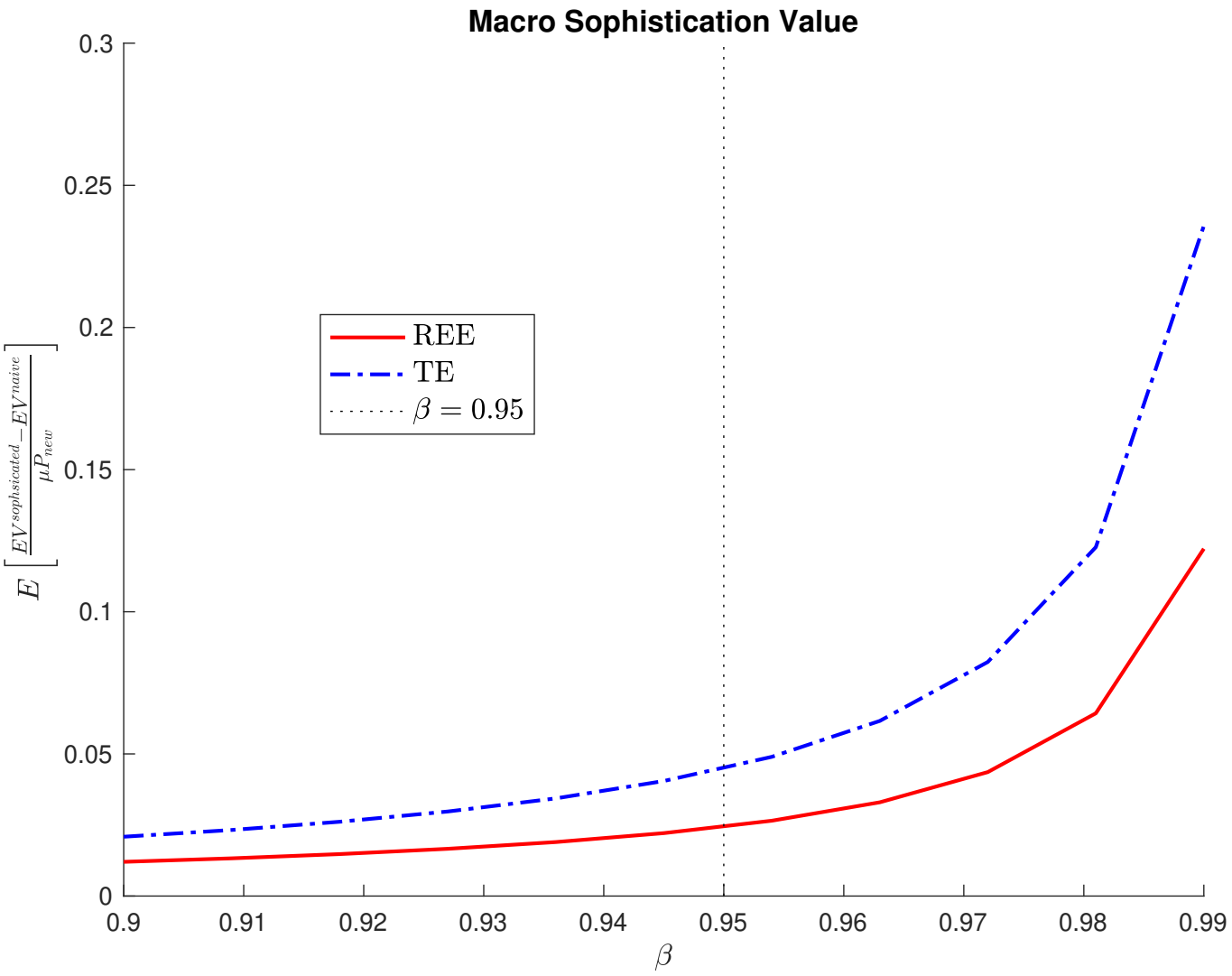
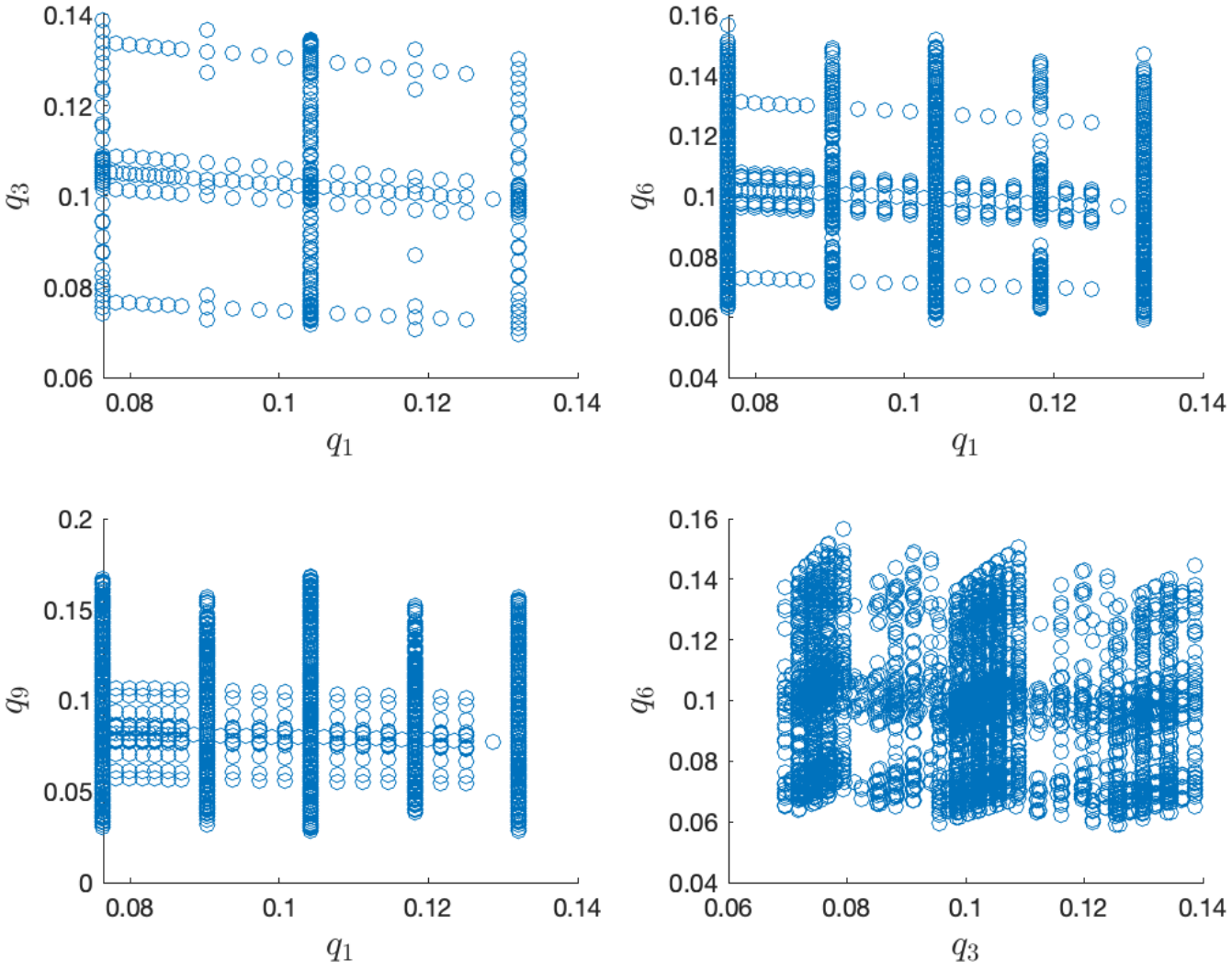


Quantitative Model

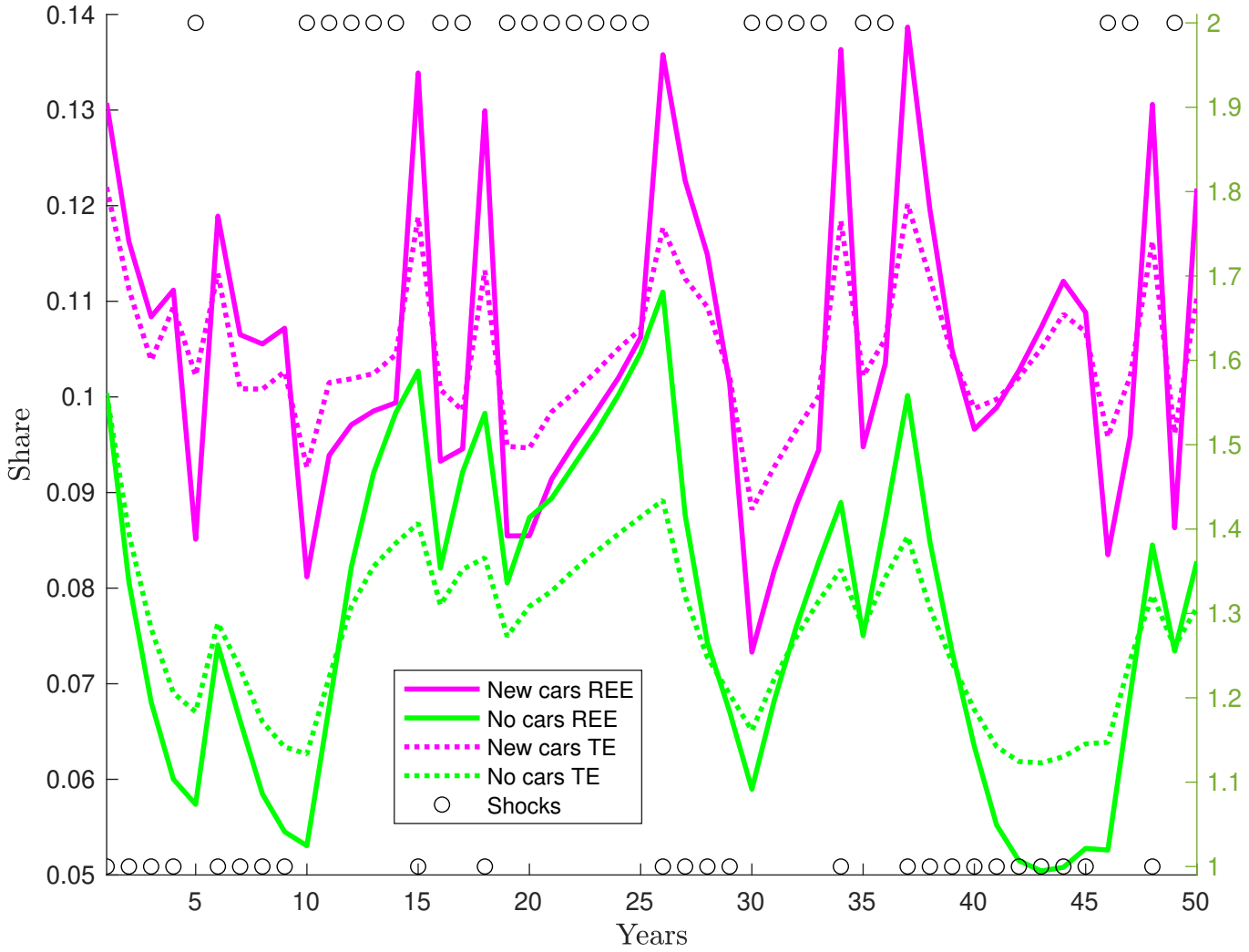
Parameter	Value
$\bar{a}$	10
$u_0 + u_1 a + u_2 a^2$	$(2.8857, -0.2427, 0.0047)$
(T^s, T^b)	$(0.046P, 0)$
T^{psy}	5.0830
T_n^{psy}	$5.0830 + 2.1604$
$(\bar{P}, \underline{P})$	$(100, 10)$
$(\underline{m}, \bar{m})$	$(0.075, 0.085)$
$G(e\|e), G(r\|r)$	$(0.75, 0.75)$

- 10-dimensional state space; 10 ($\{P_a\}$) + 10 ($\{q'_a\}$) unknowns and 10 (market clearings) + 10 (consistency) equations

Quantitative Model: ASG



Quantitative Model: Waves



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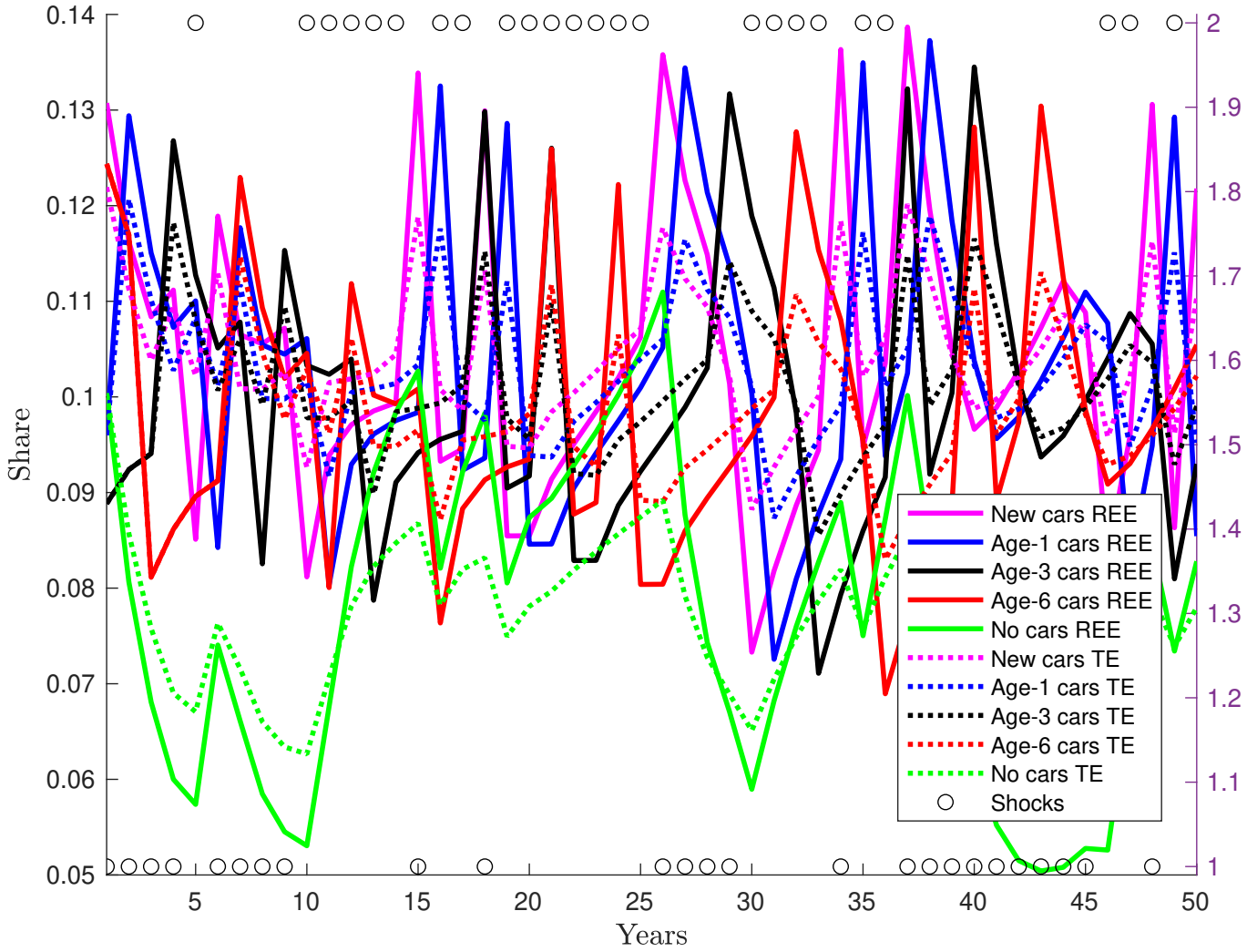

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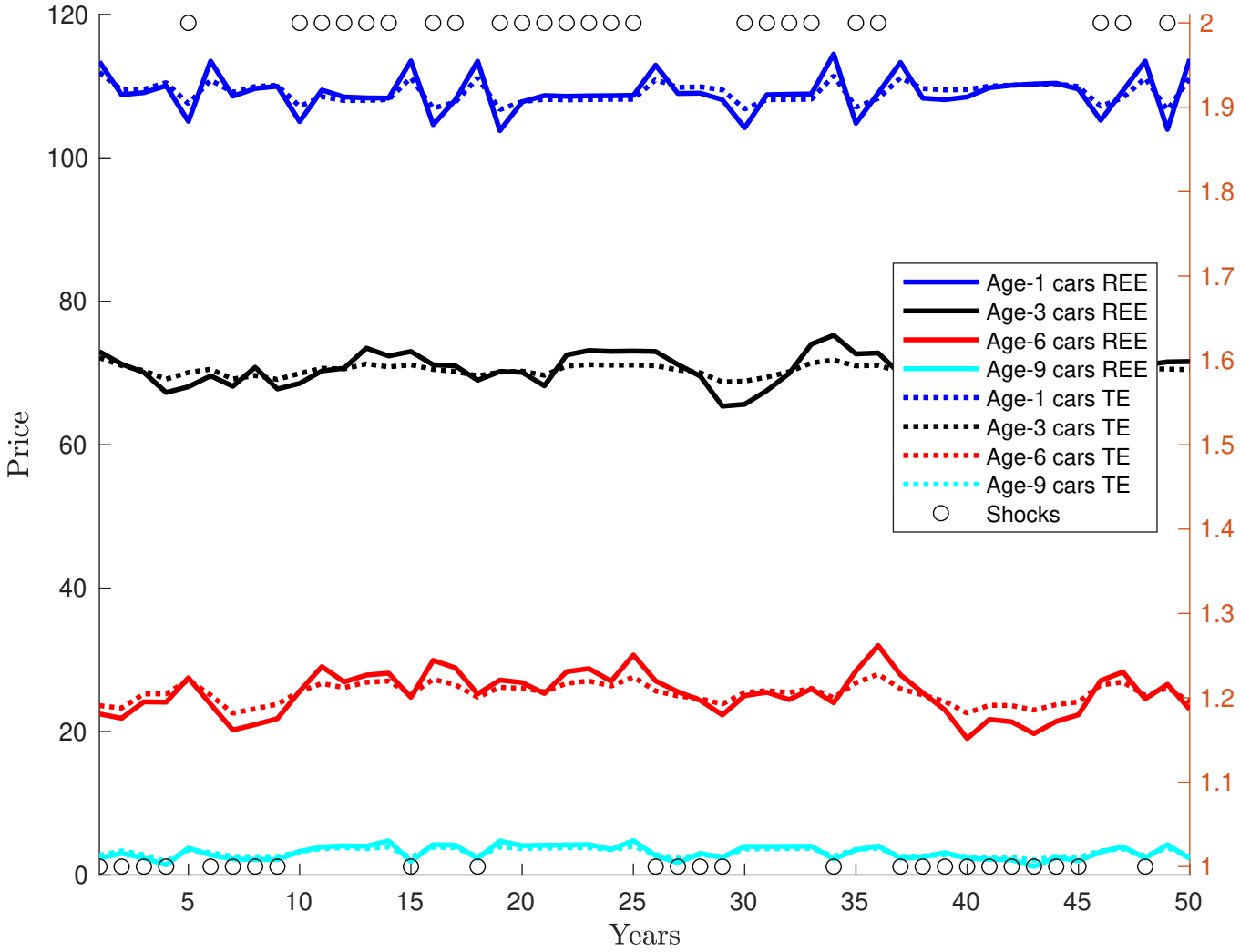
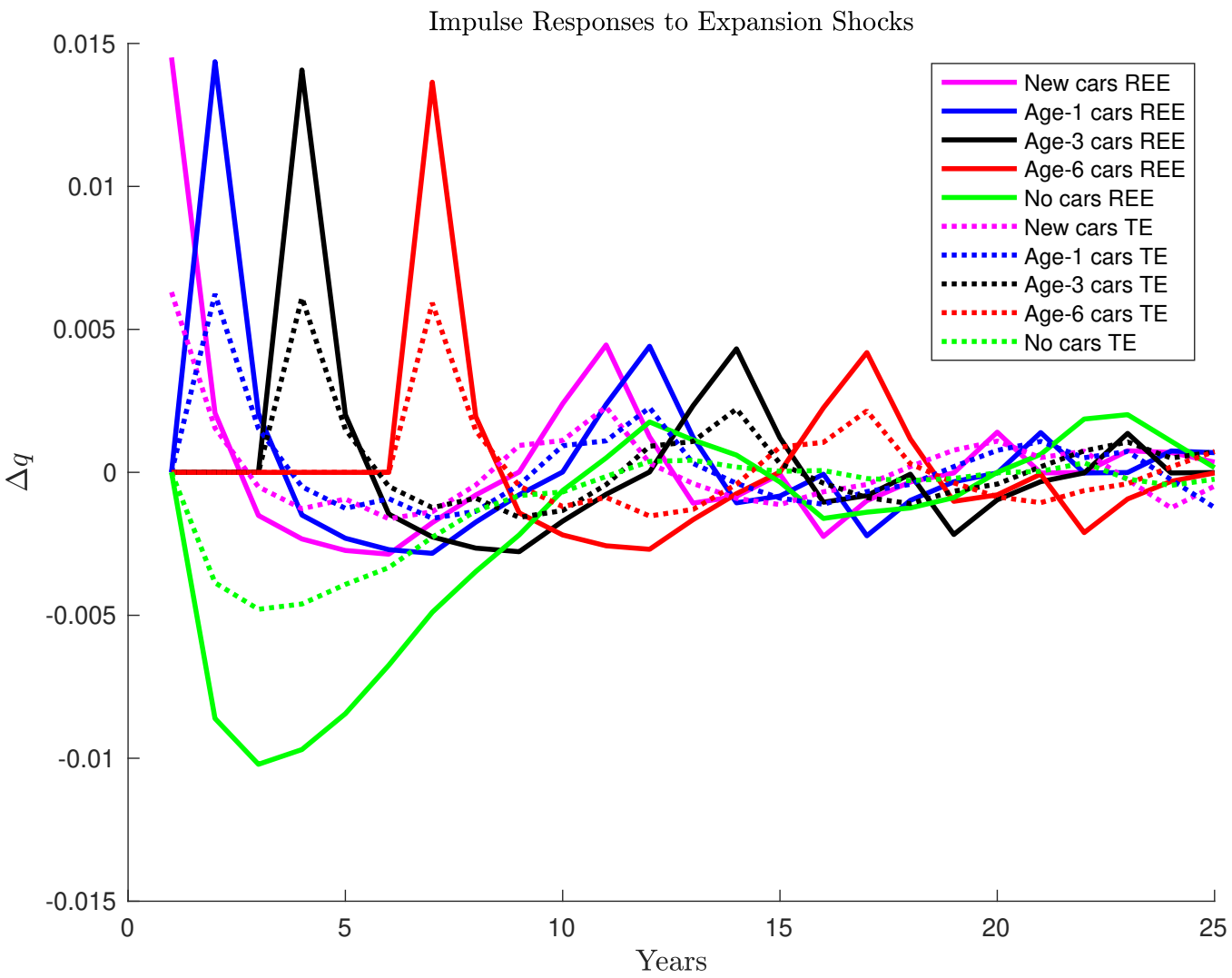
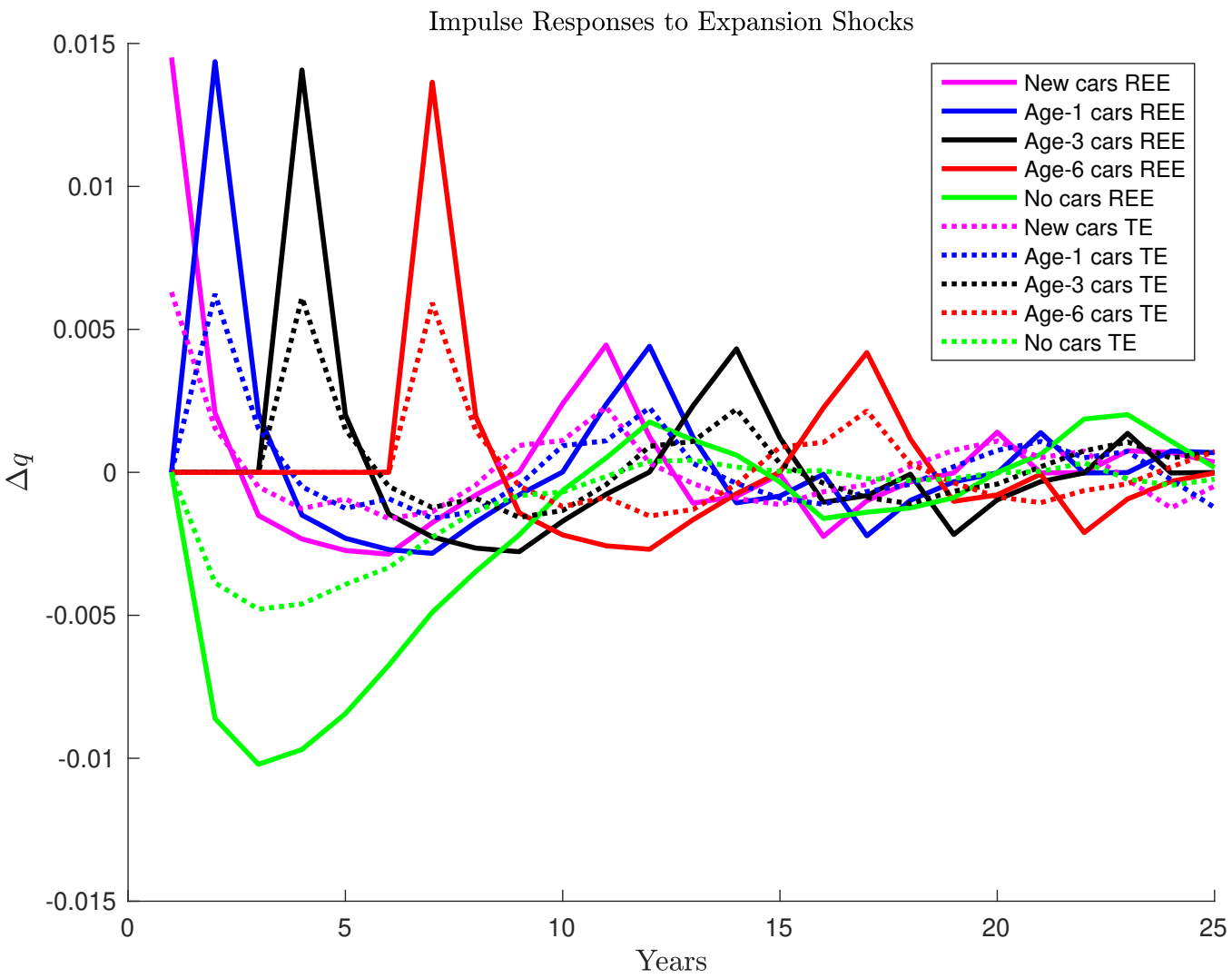


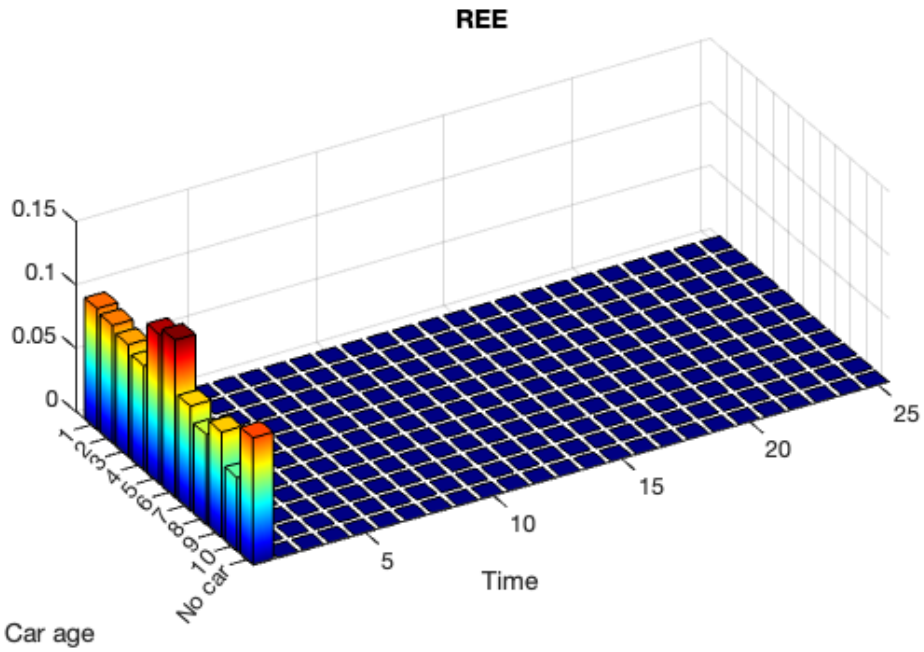
Quantitative Model: IRFs

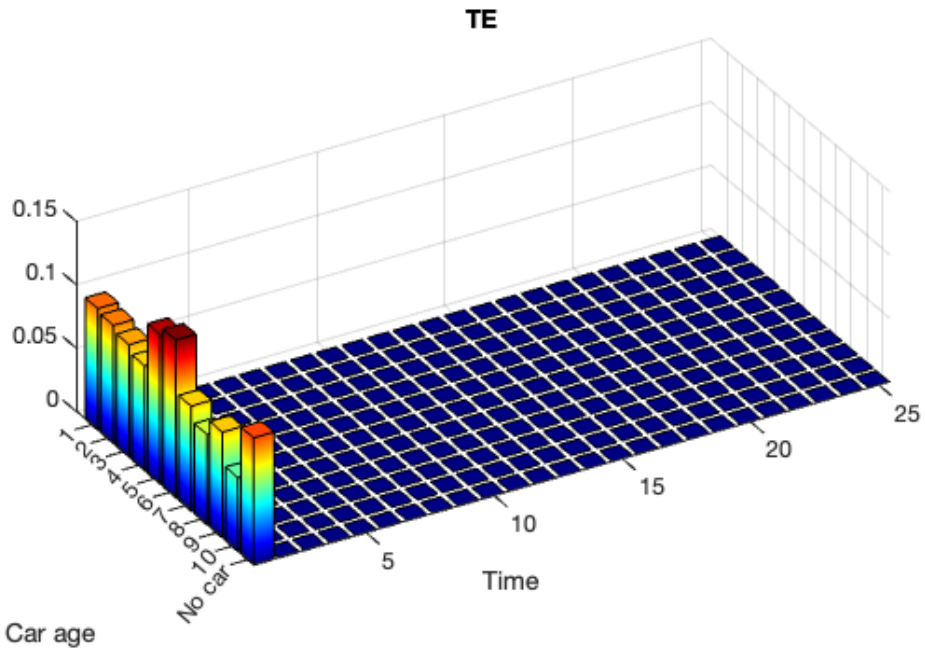


Quantitative Model: IRFs



Simulated Car Age Distribution Over Time





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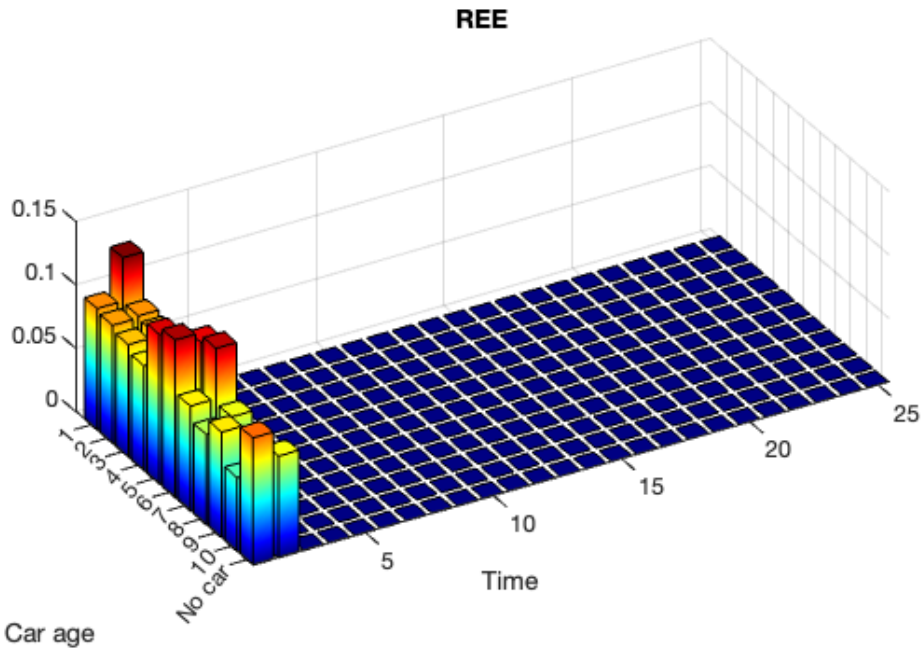

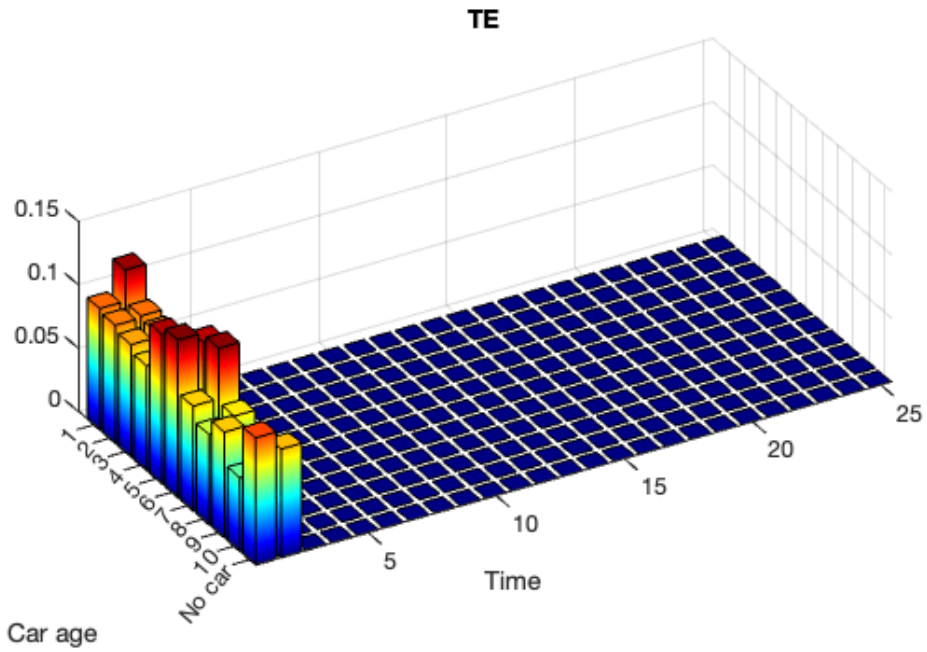

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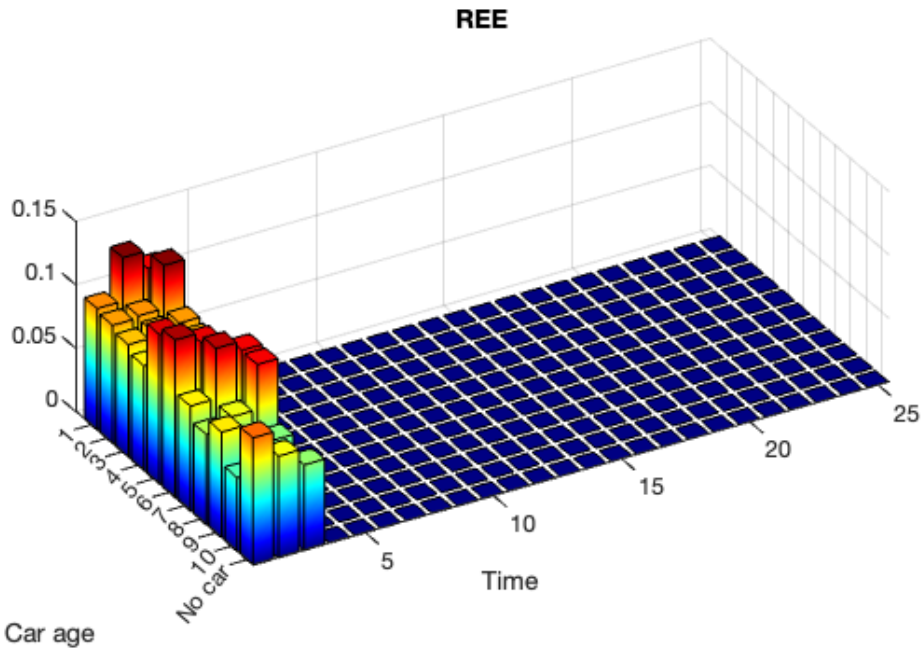

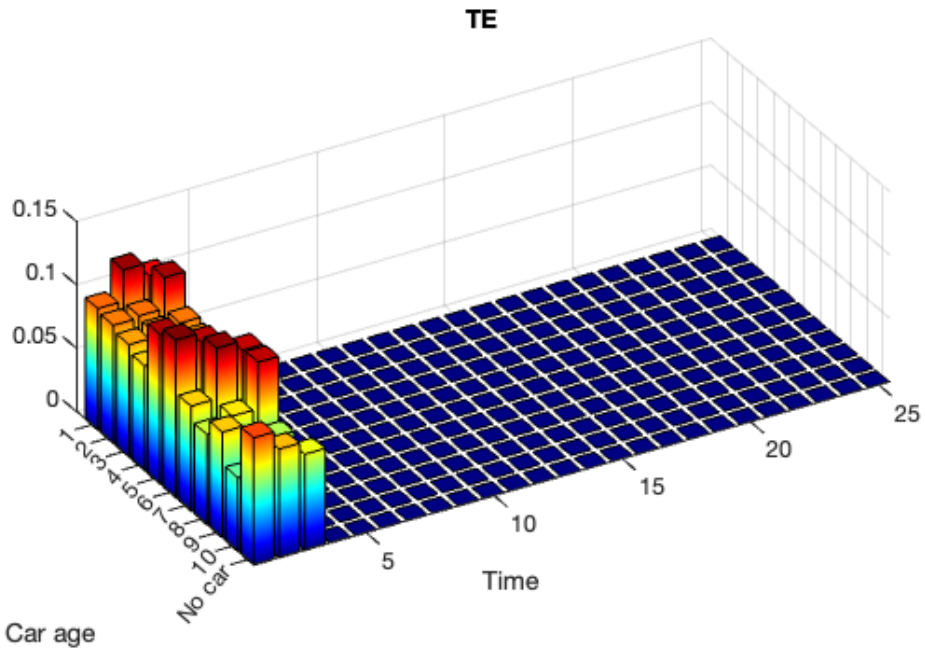

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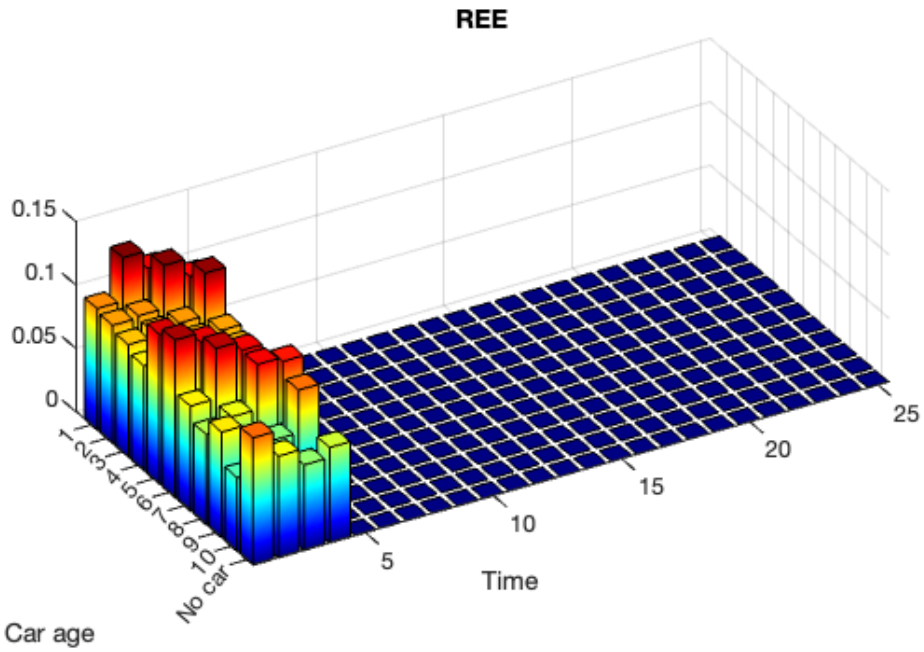

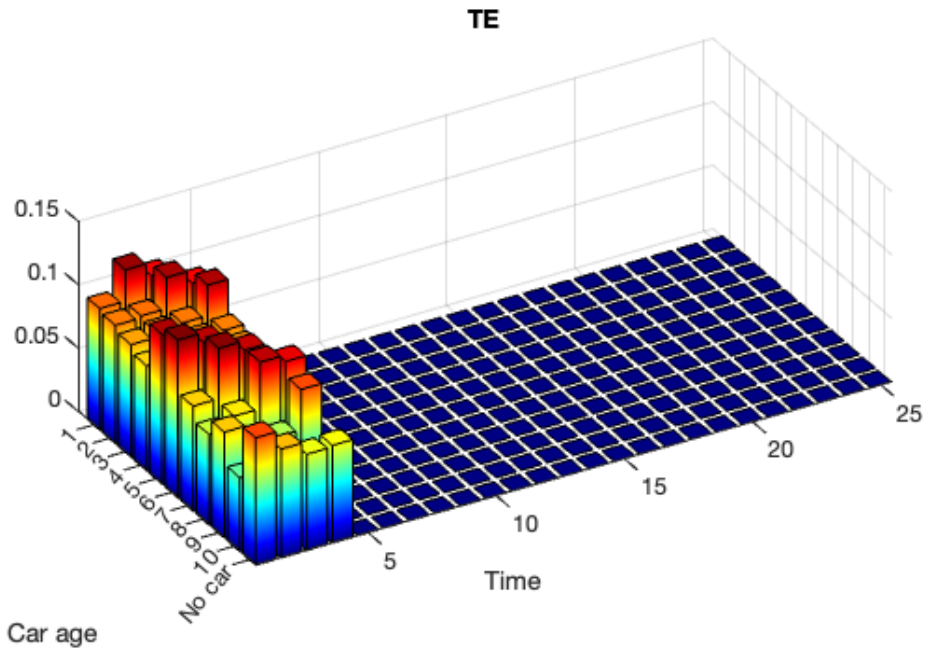

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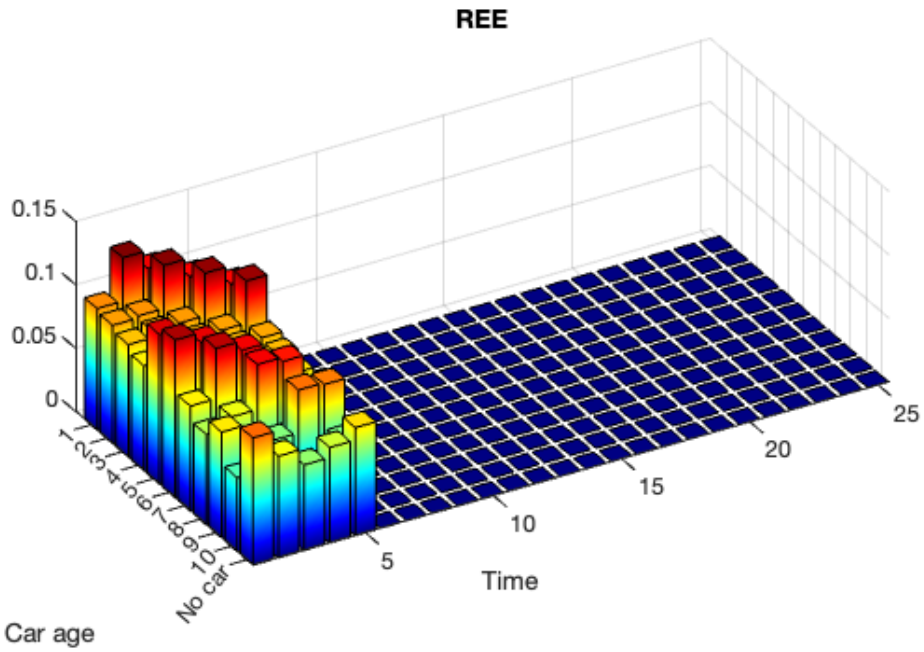

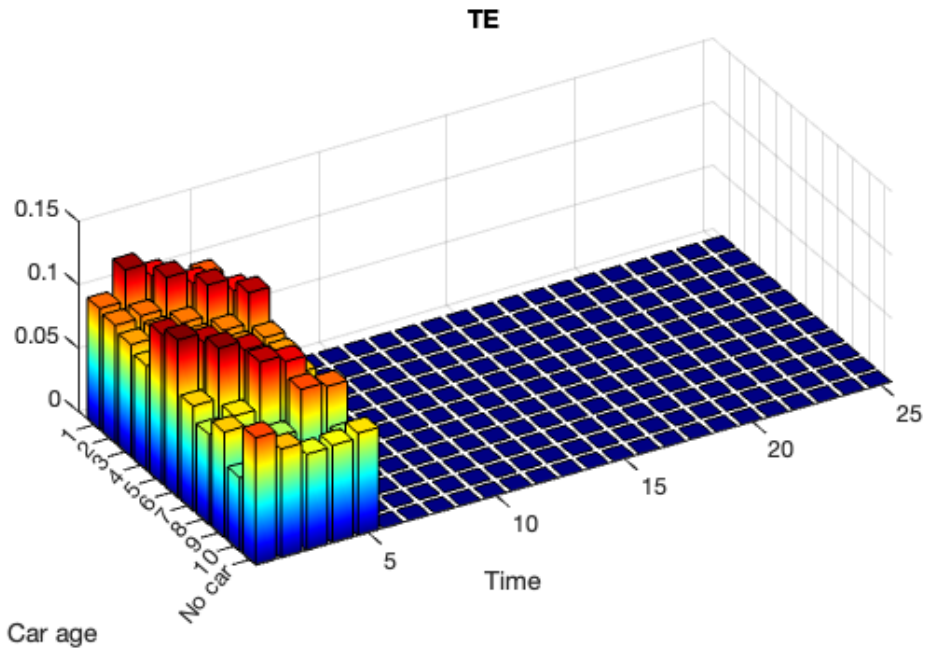

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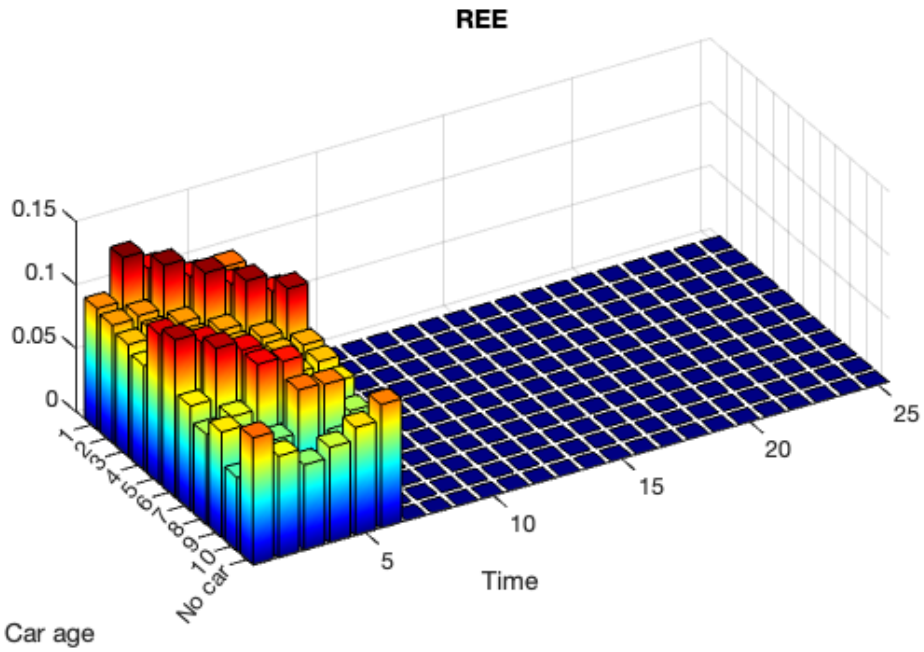

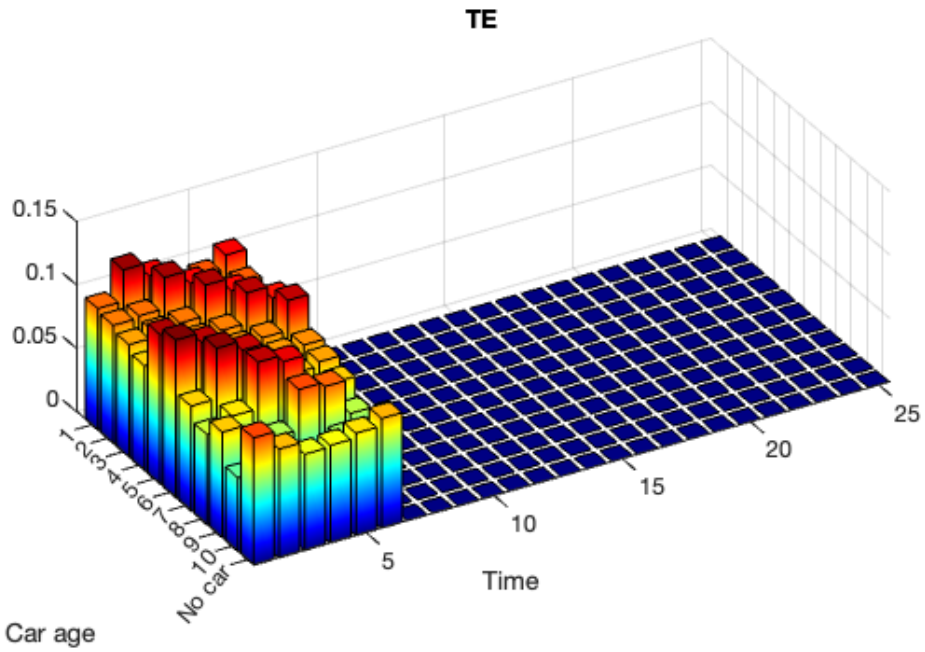

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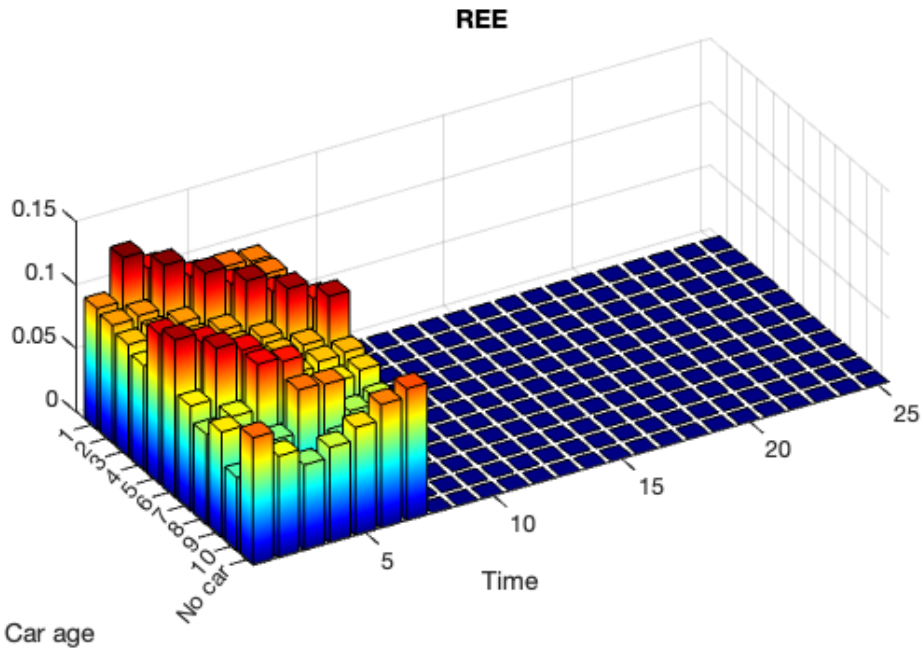

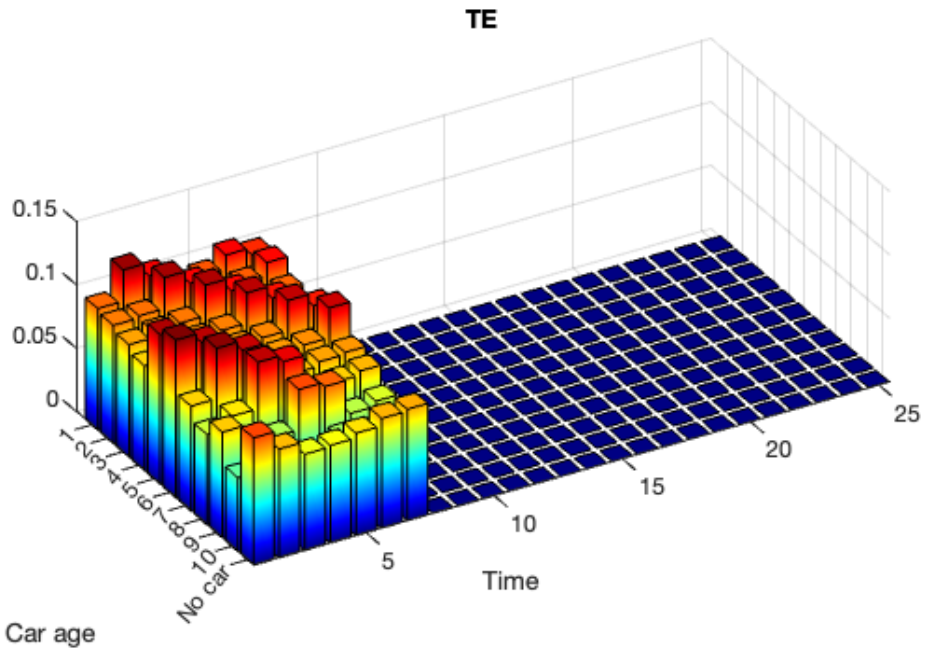

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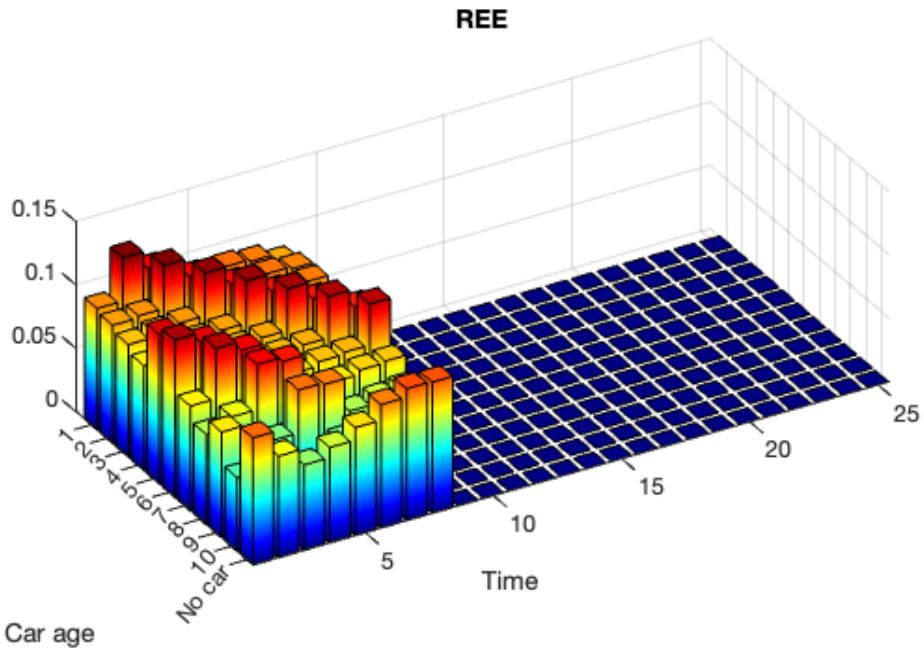

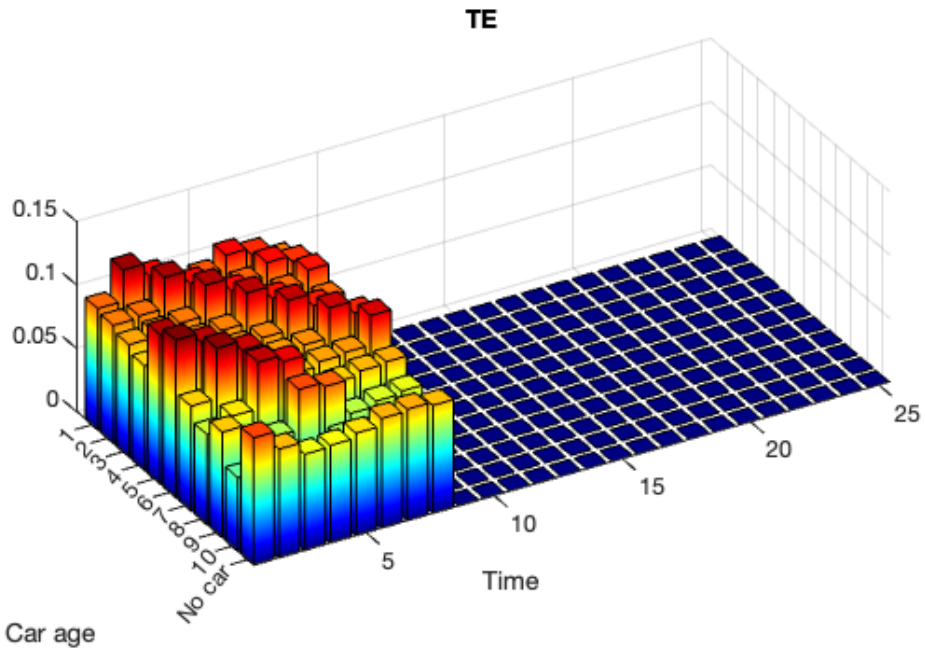

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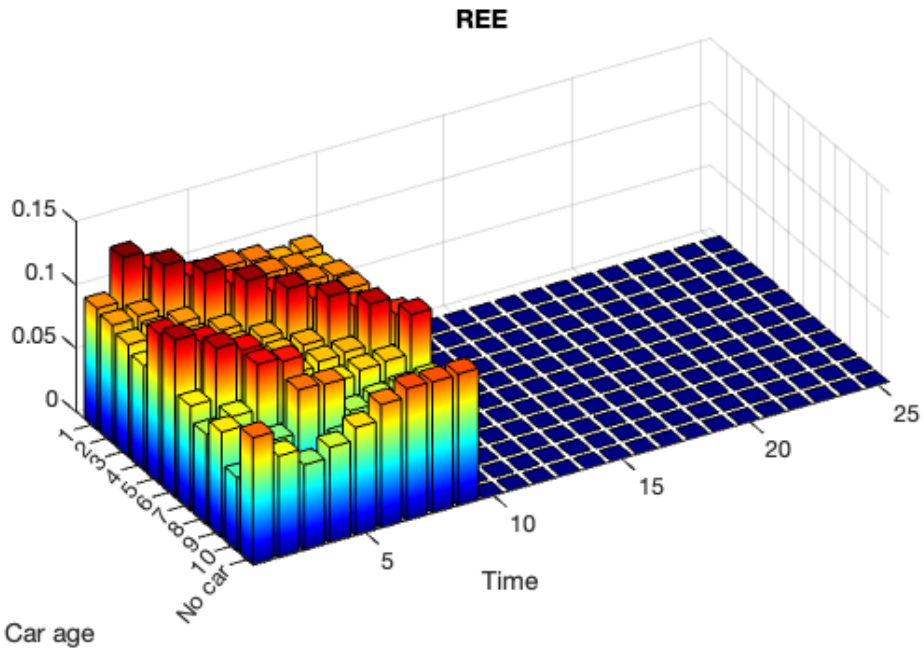

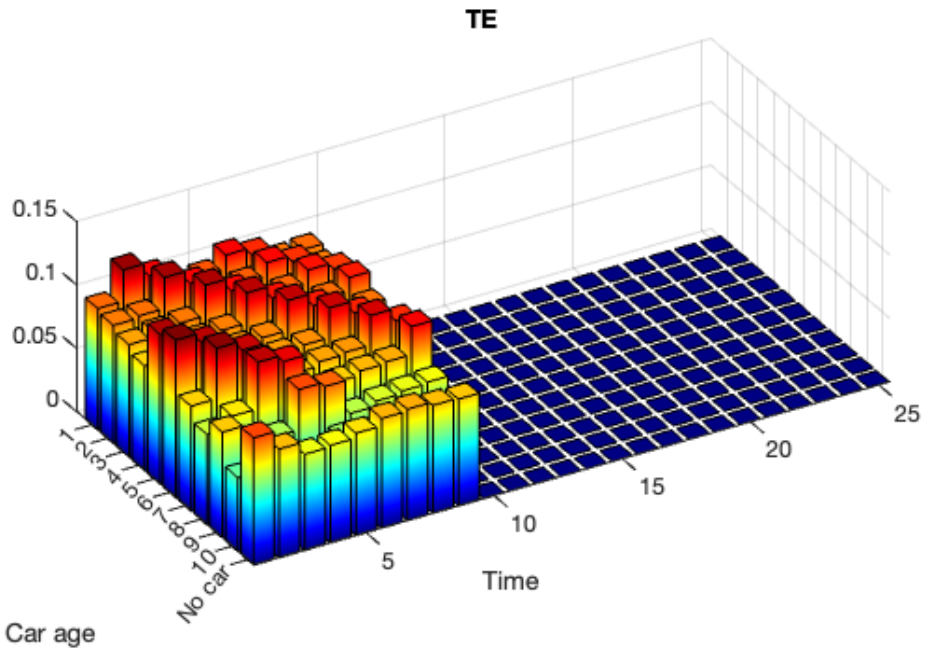

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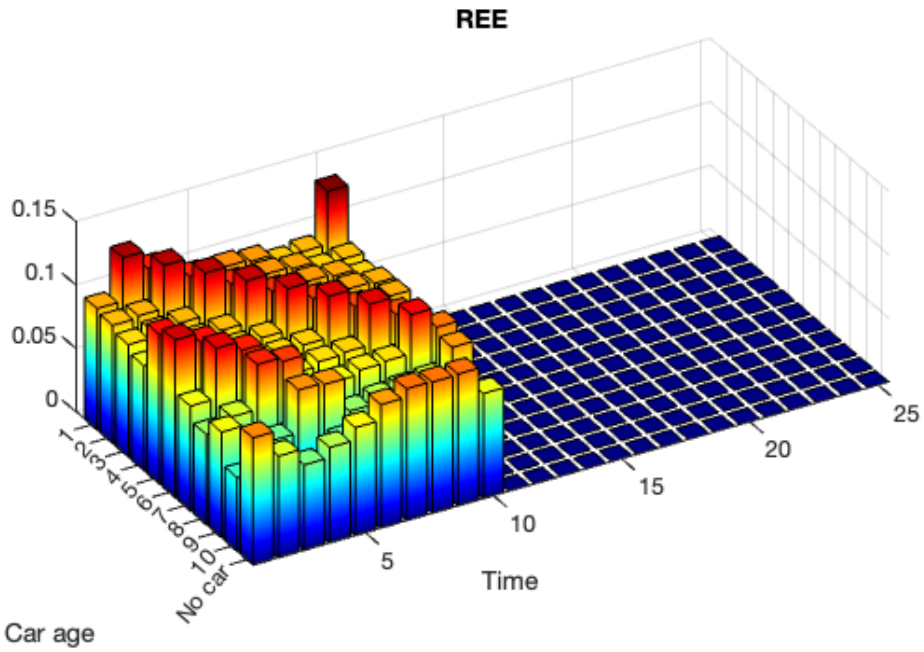

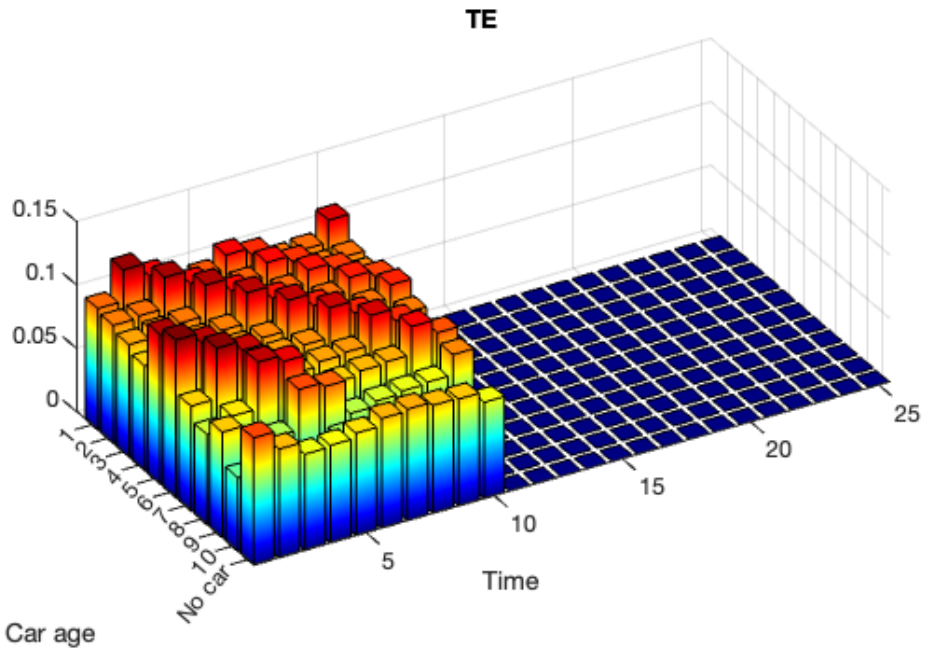

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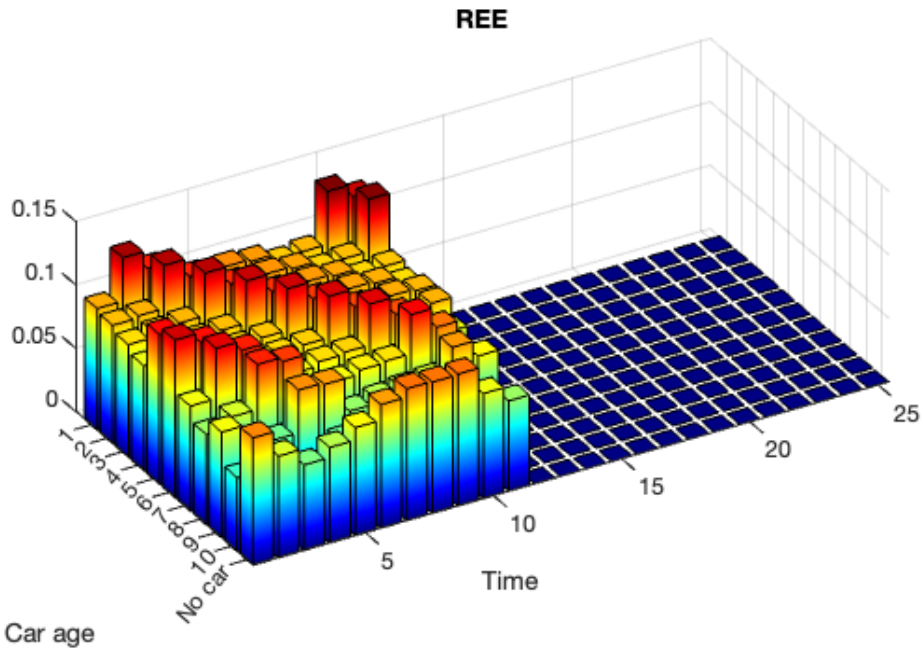

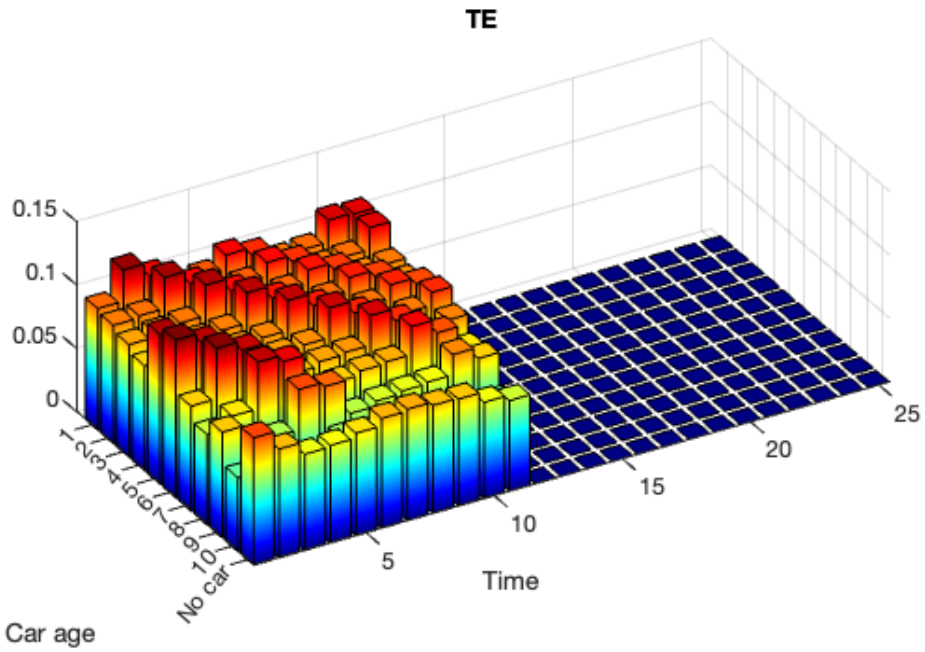

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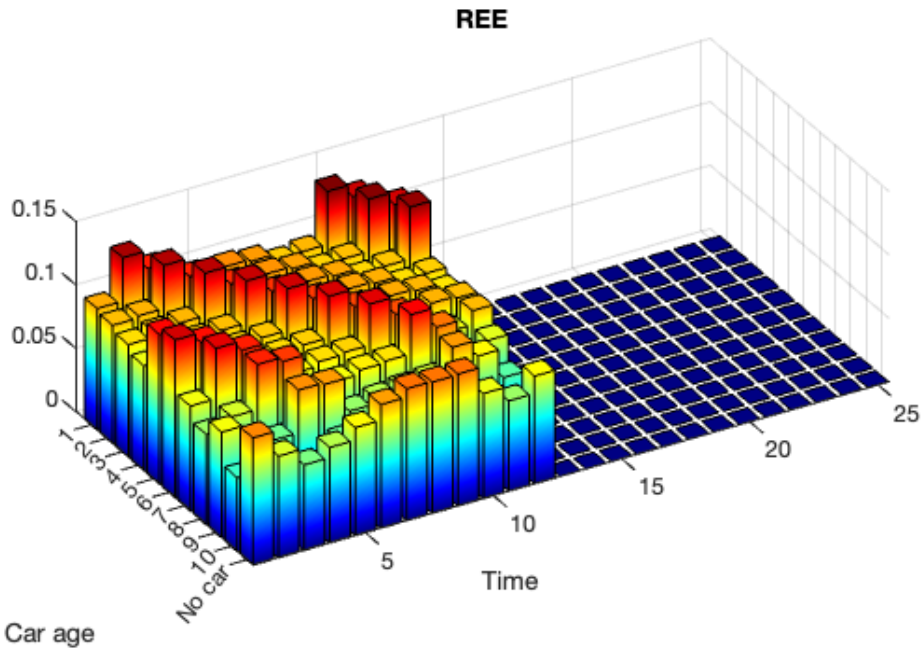

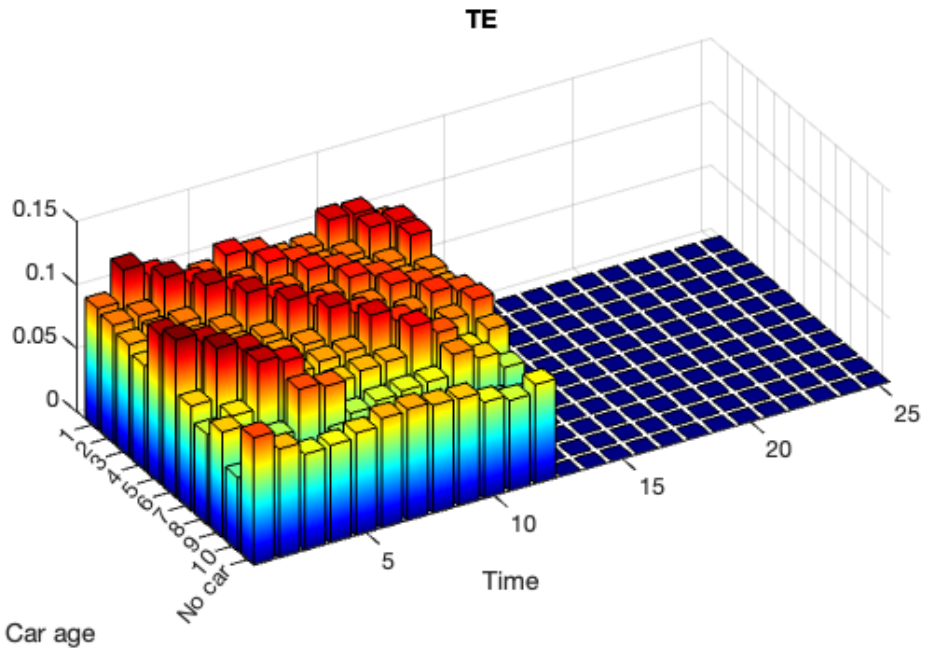

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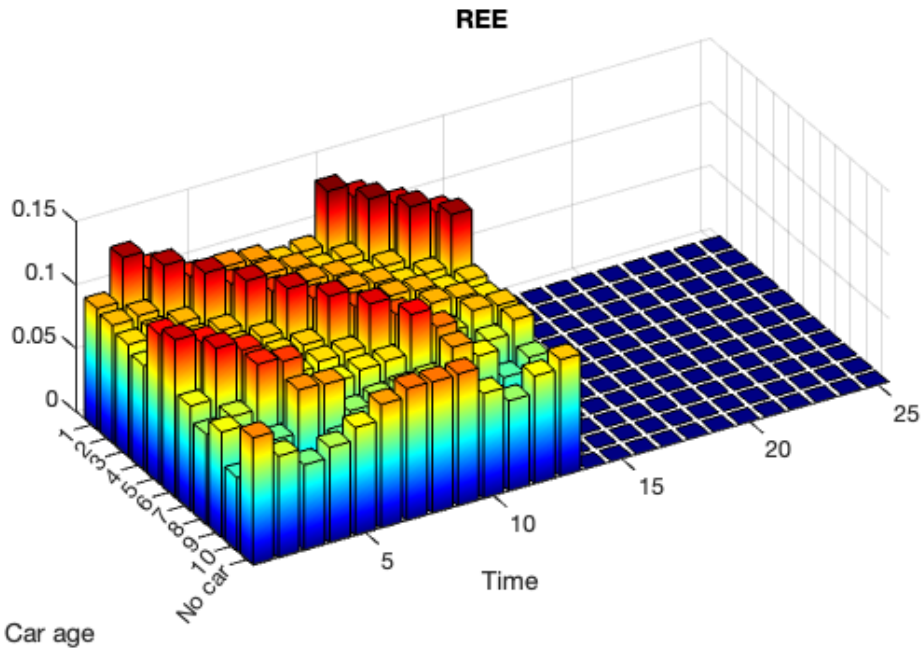

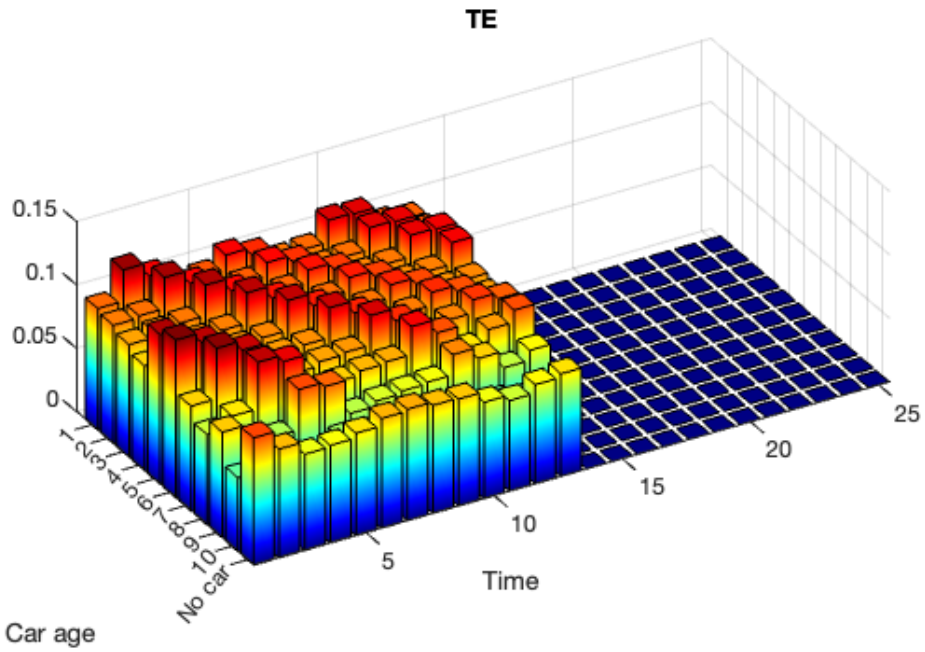

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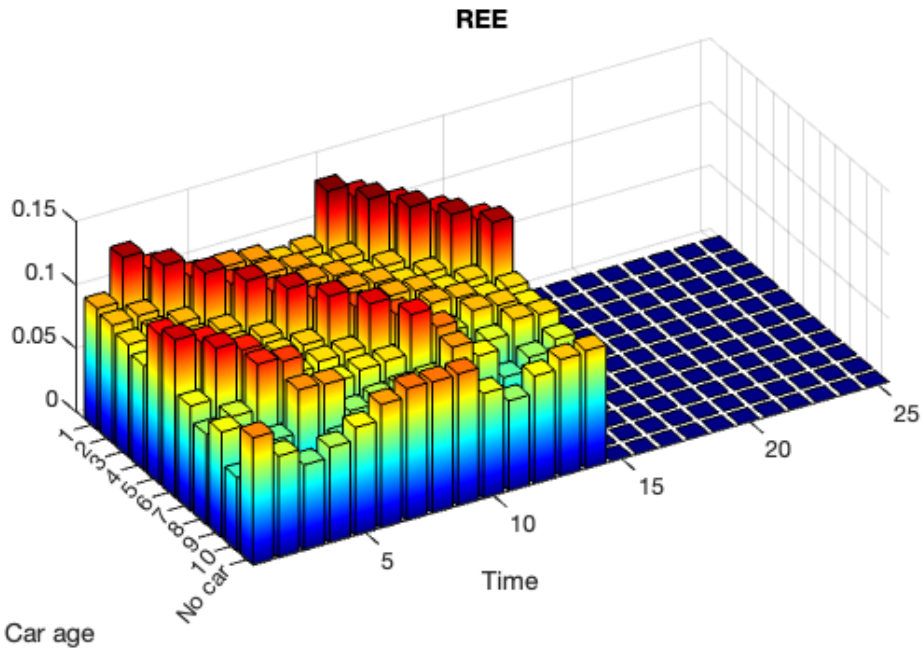

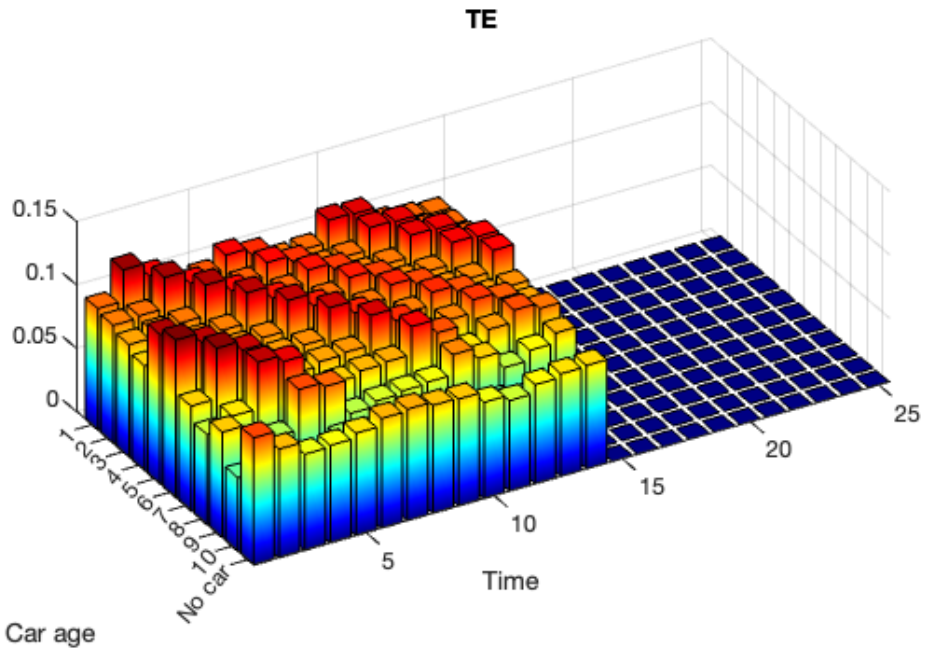

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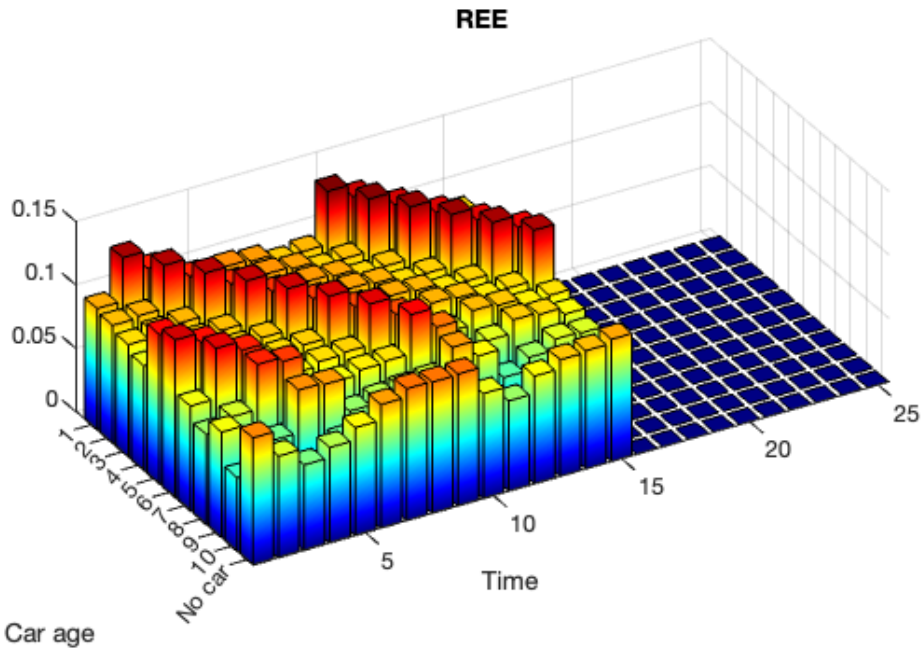

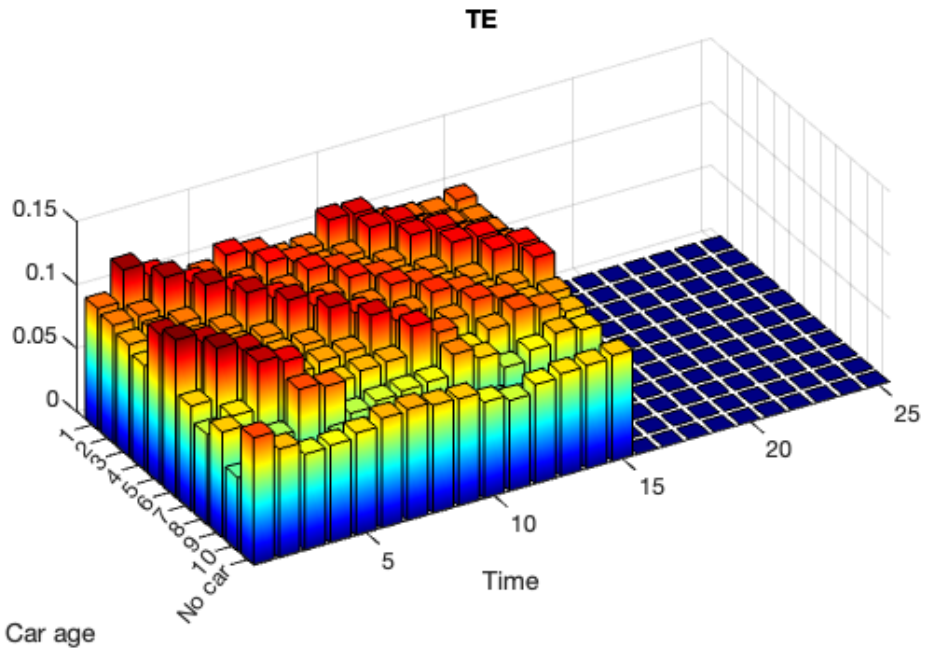

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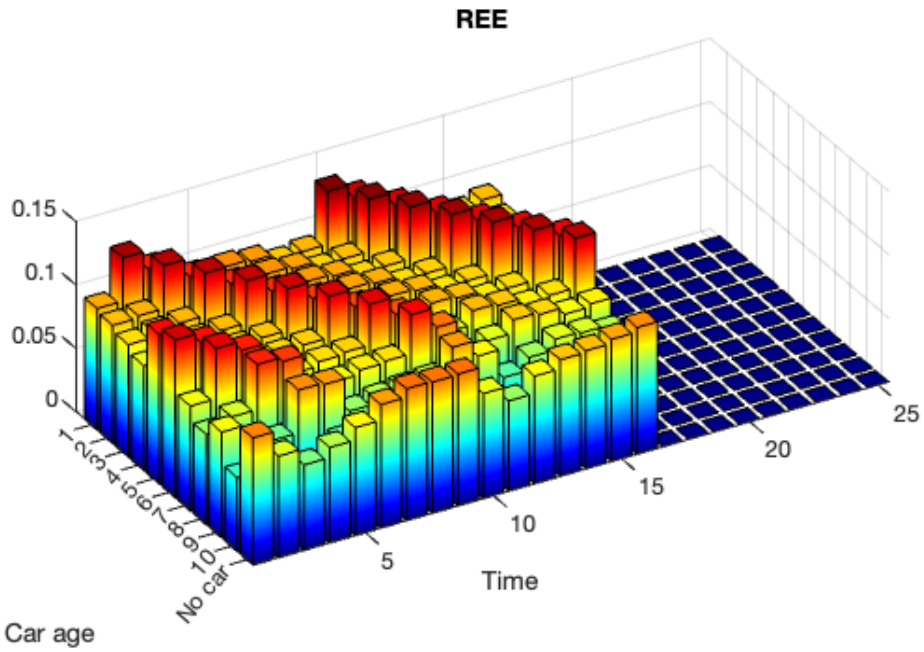

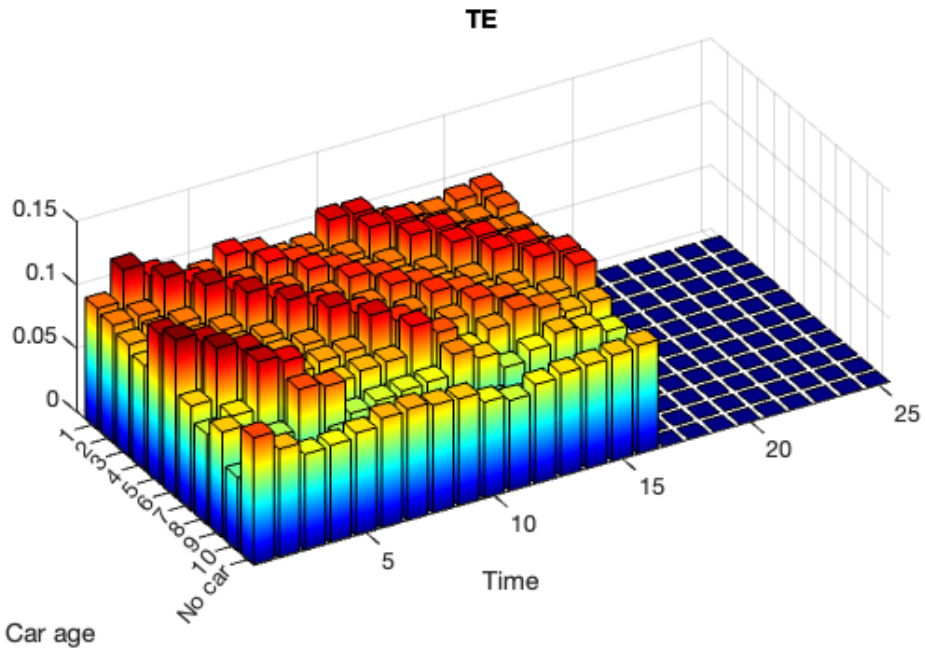

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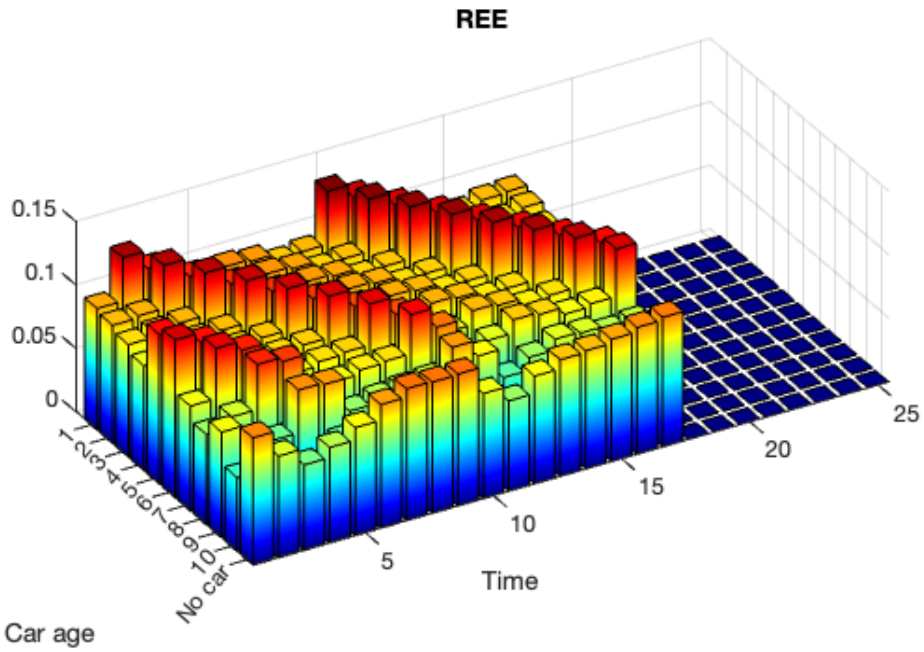

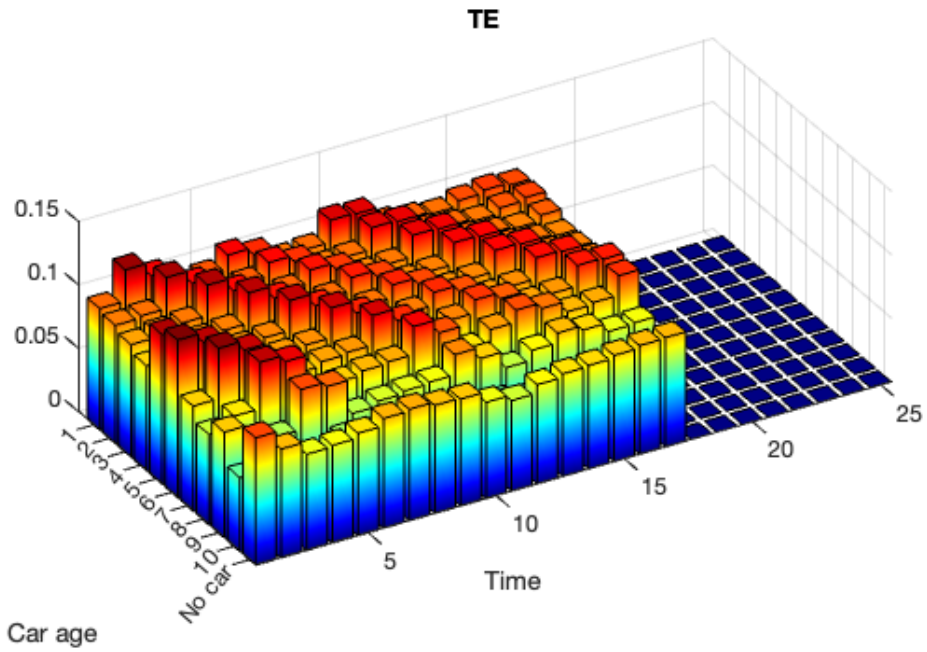

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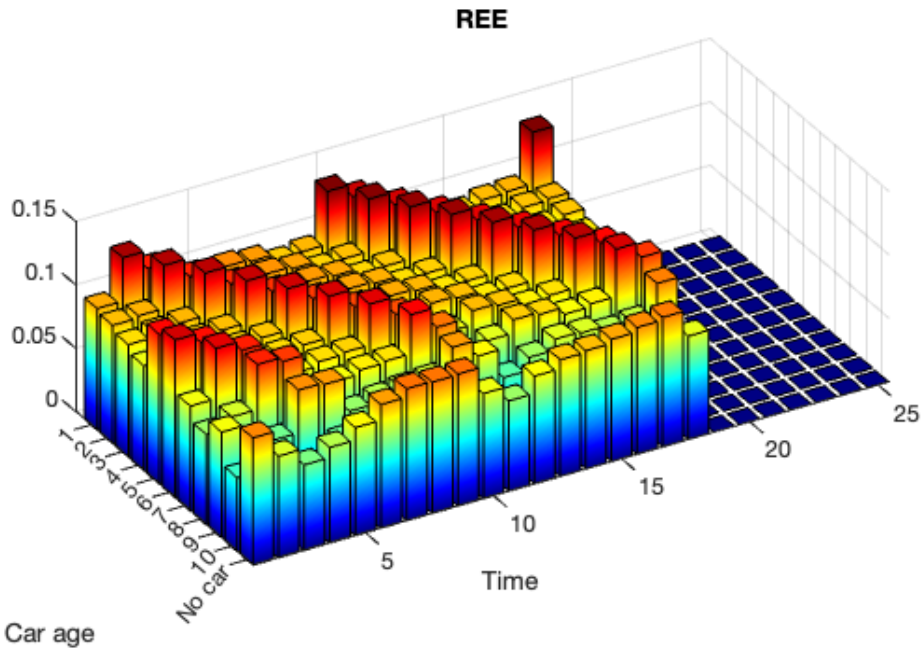

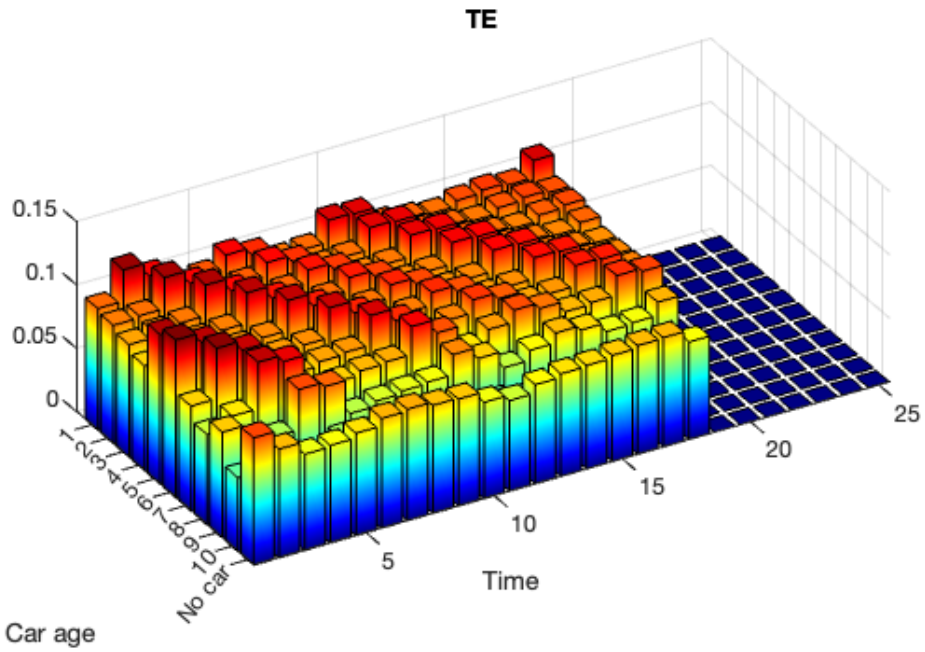

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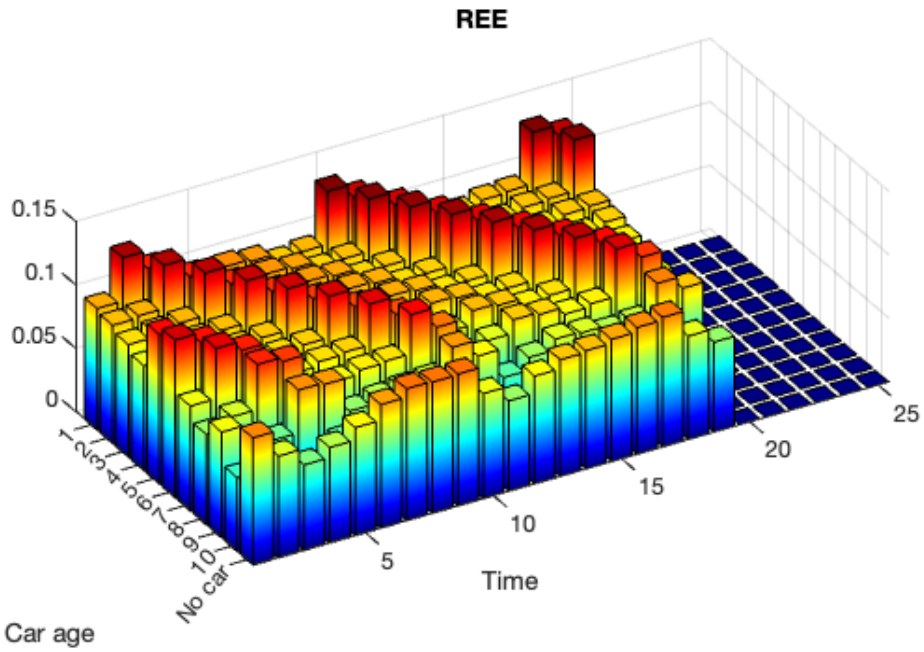

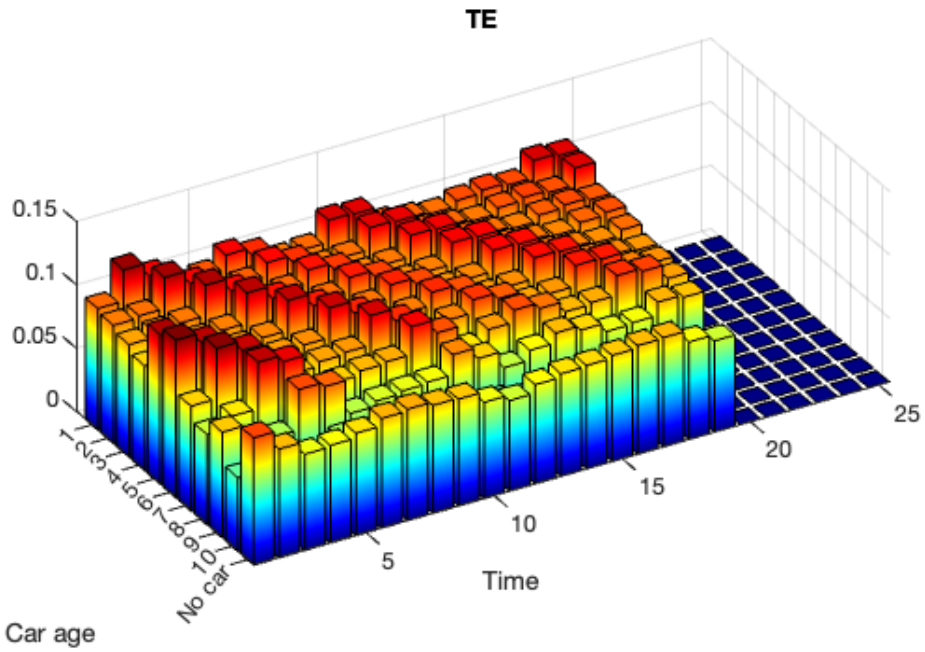

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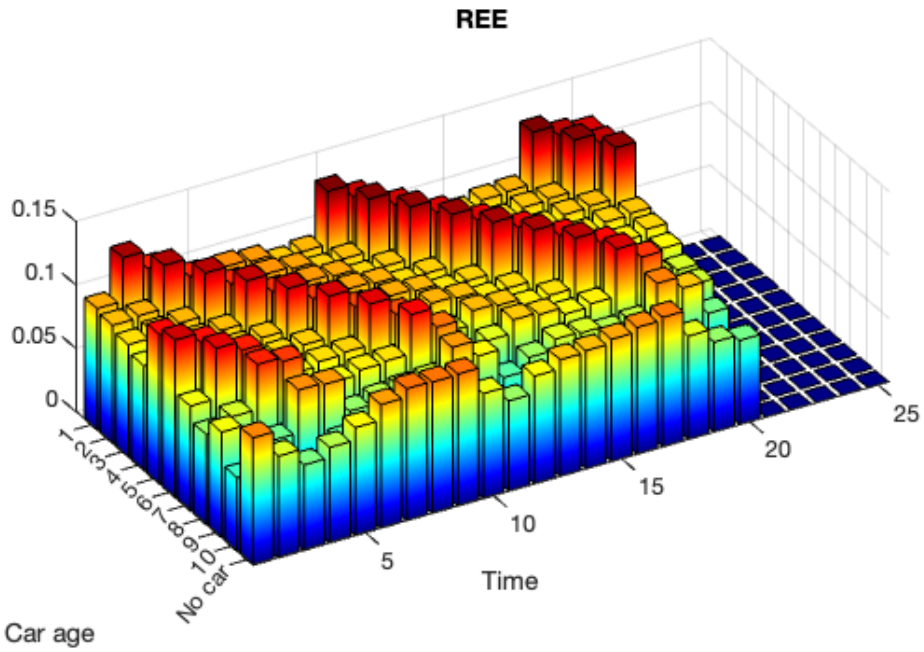

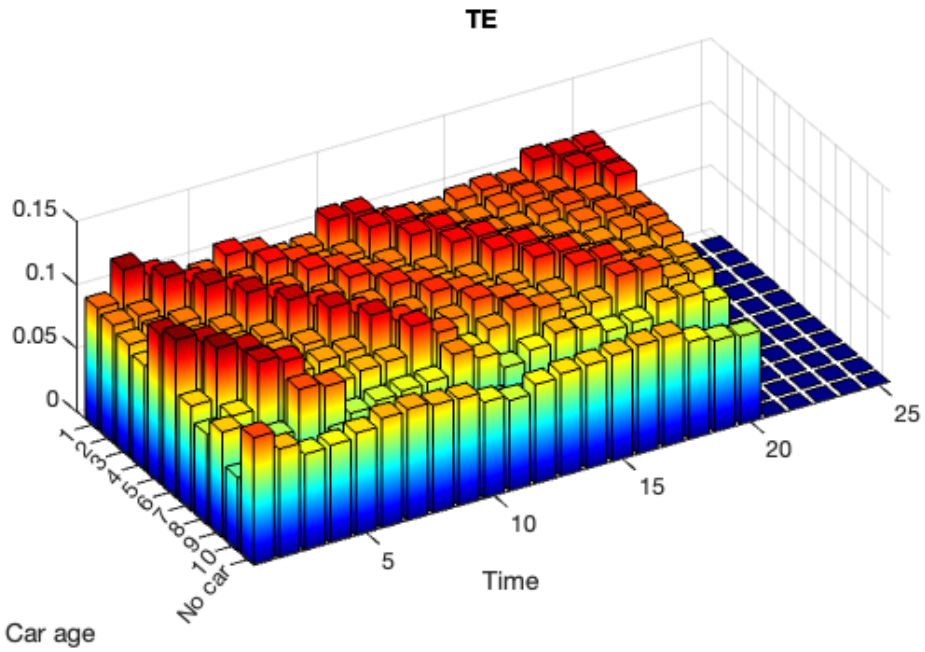

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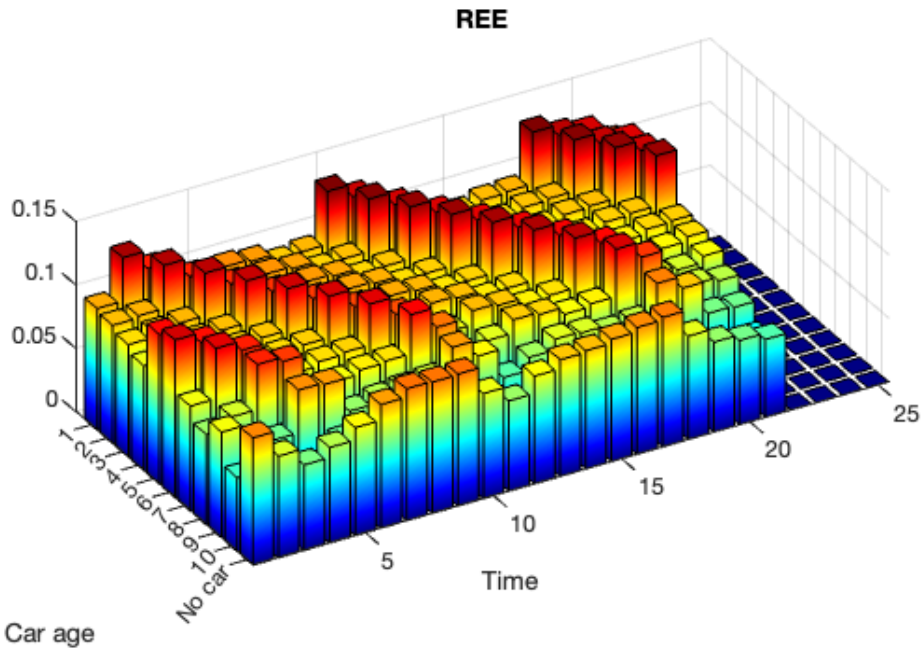

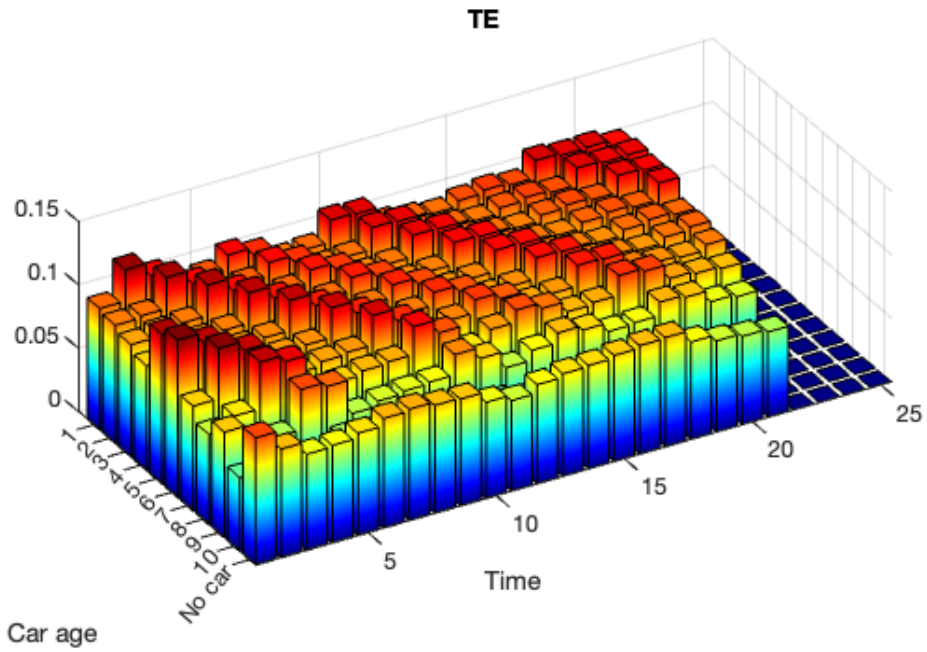

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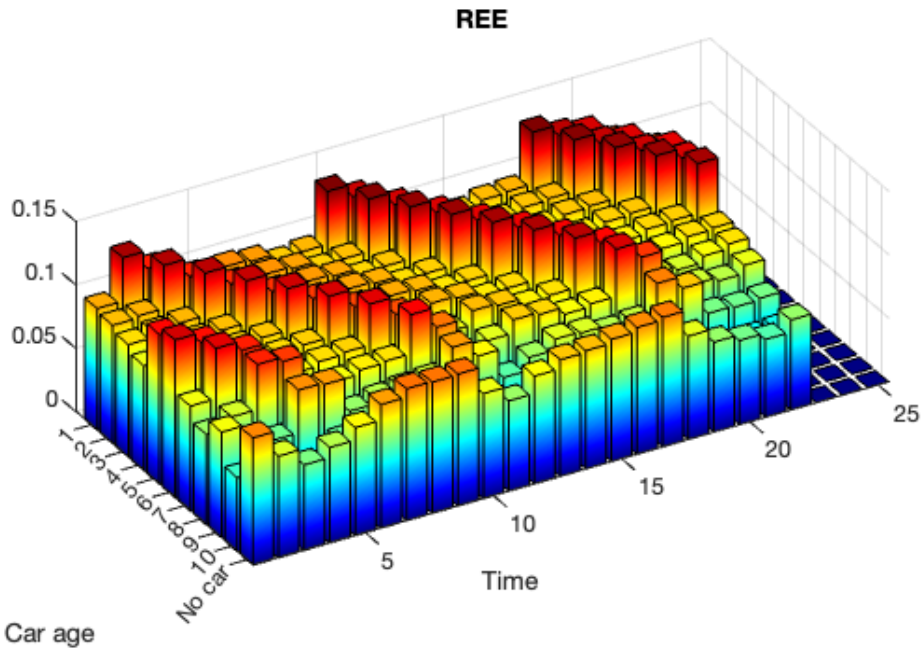

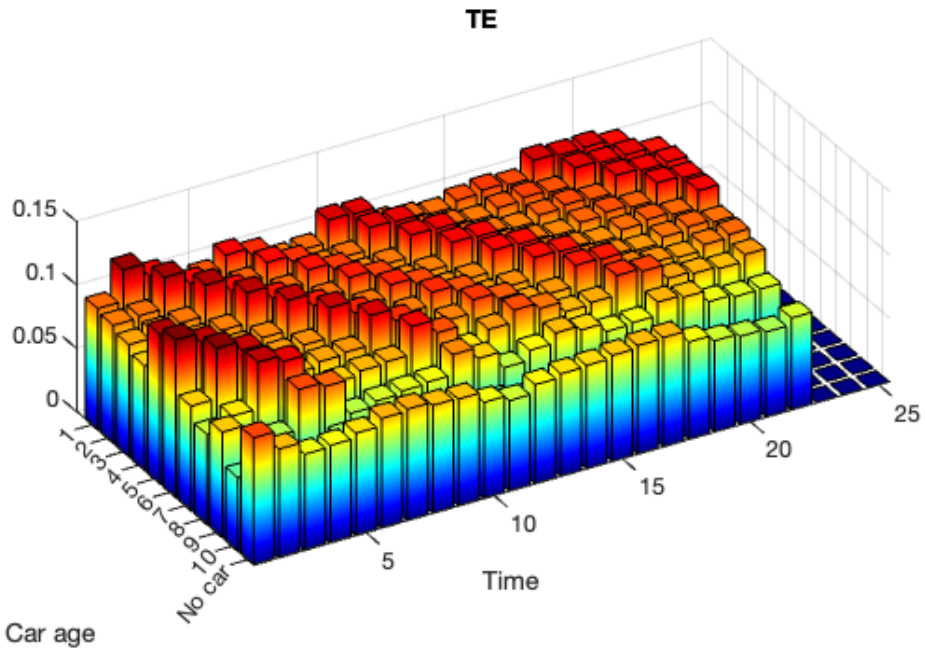

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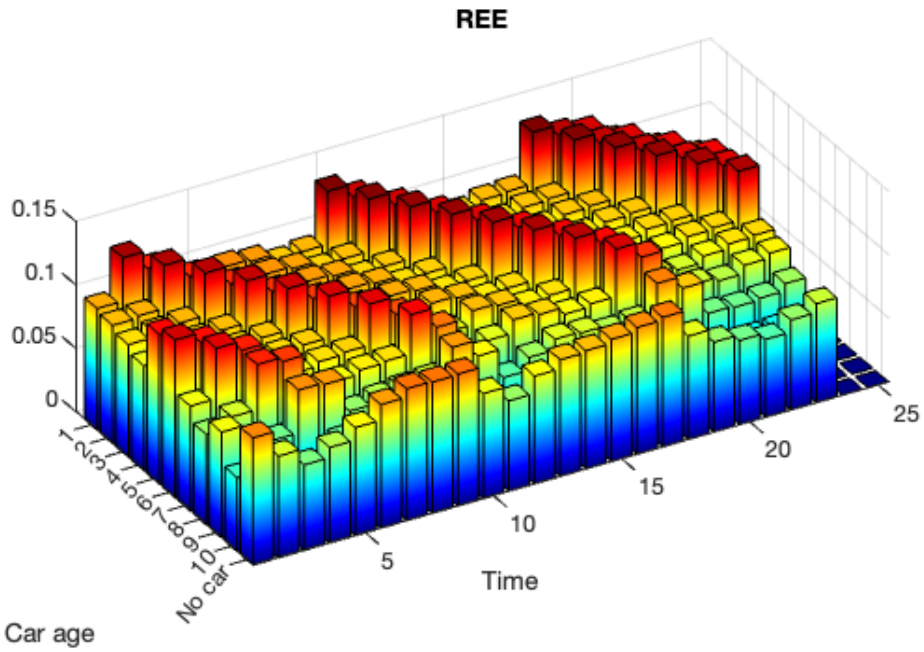

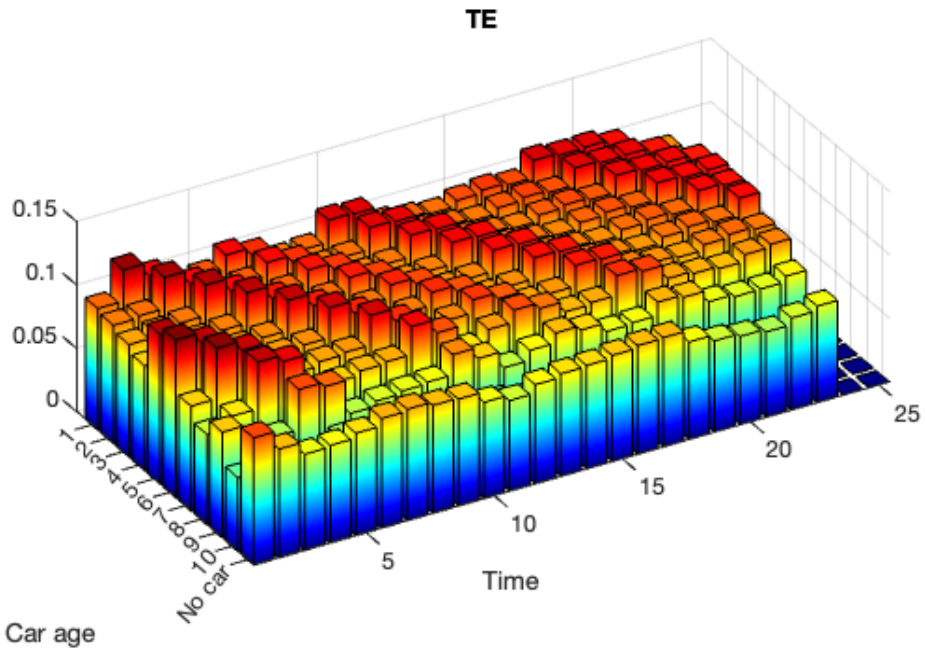

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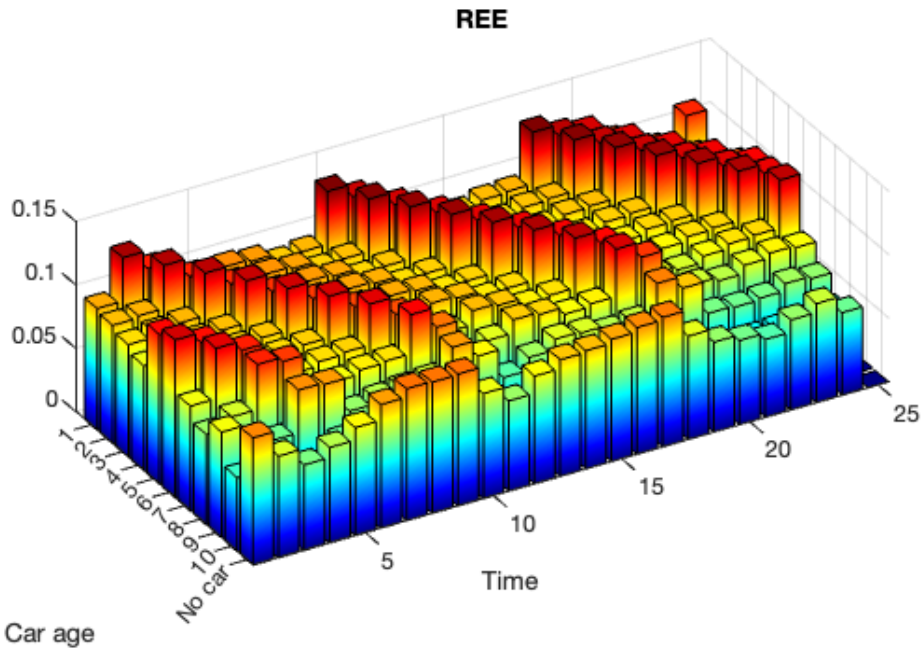

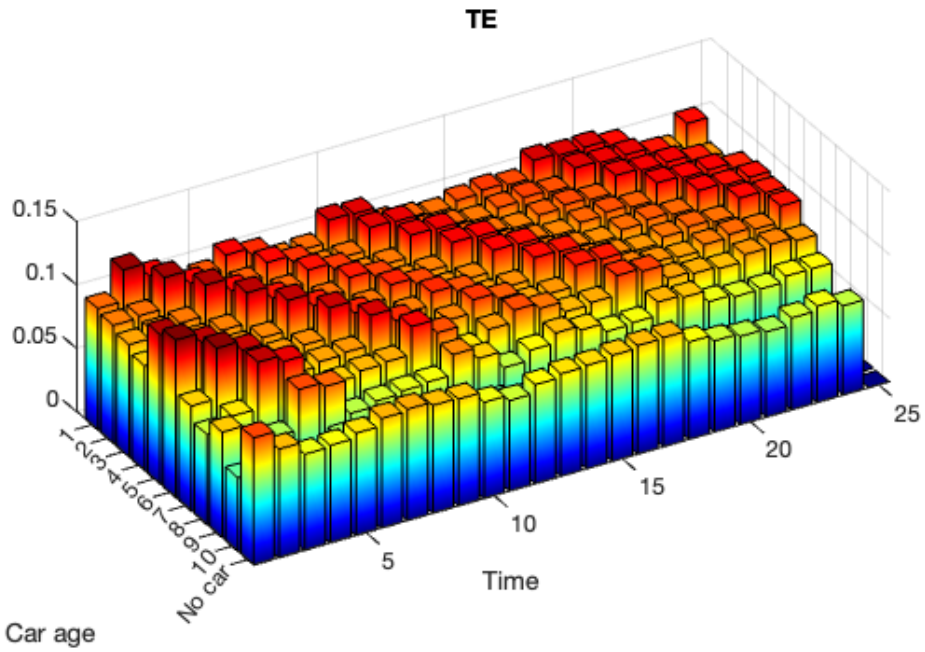

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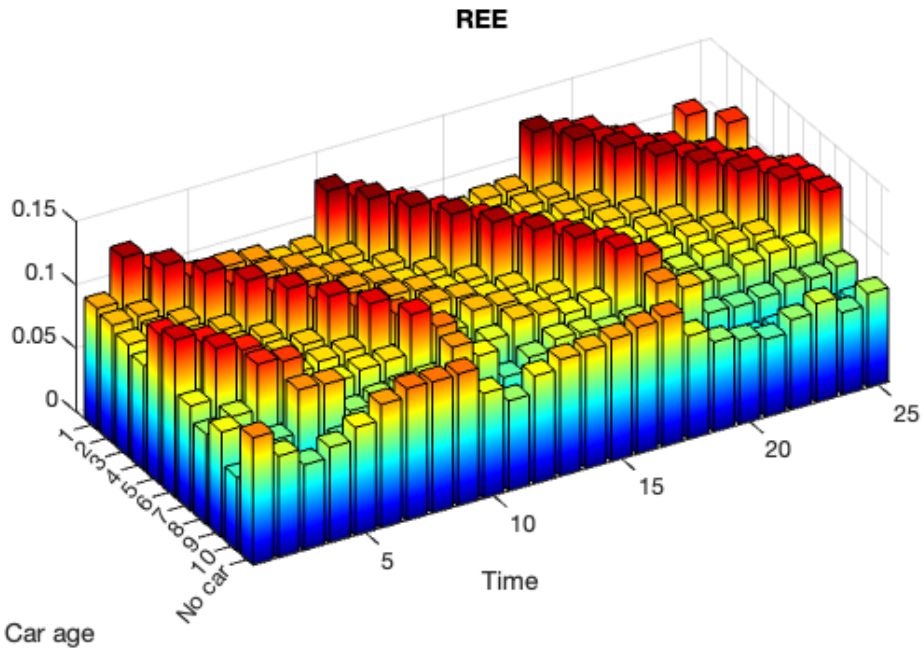
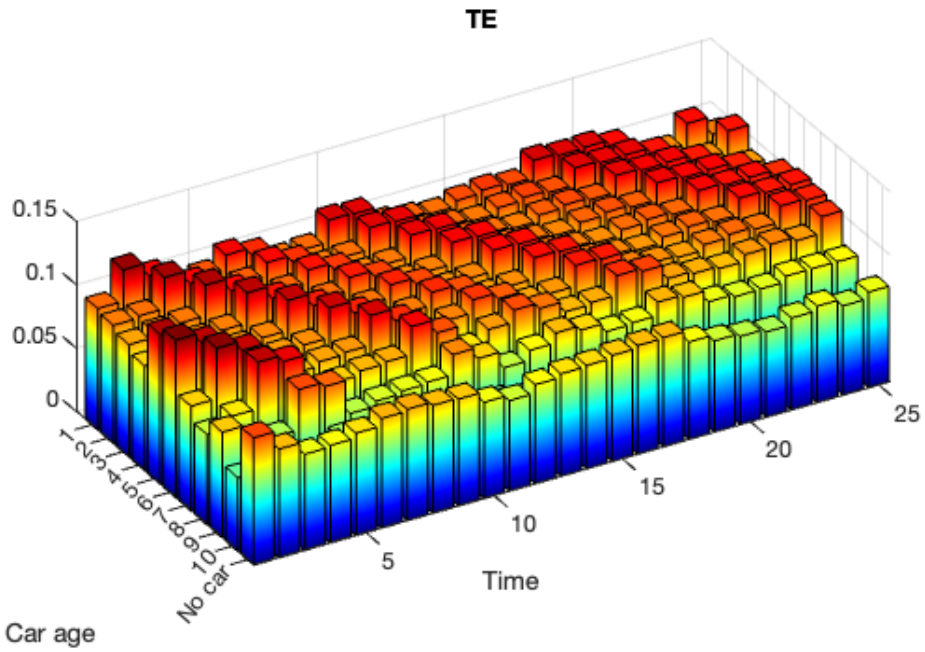
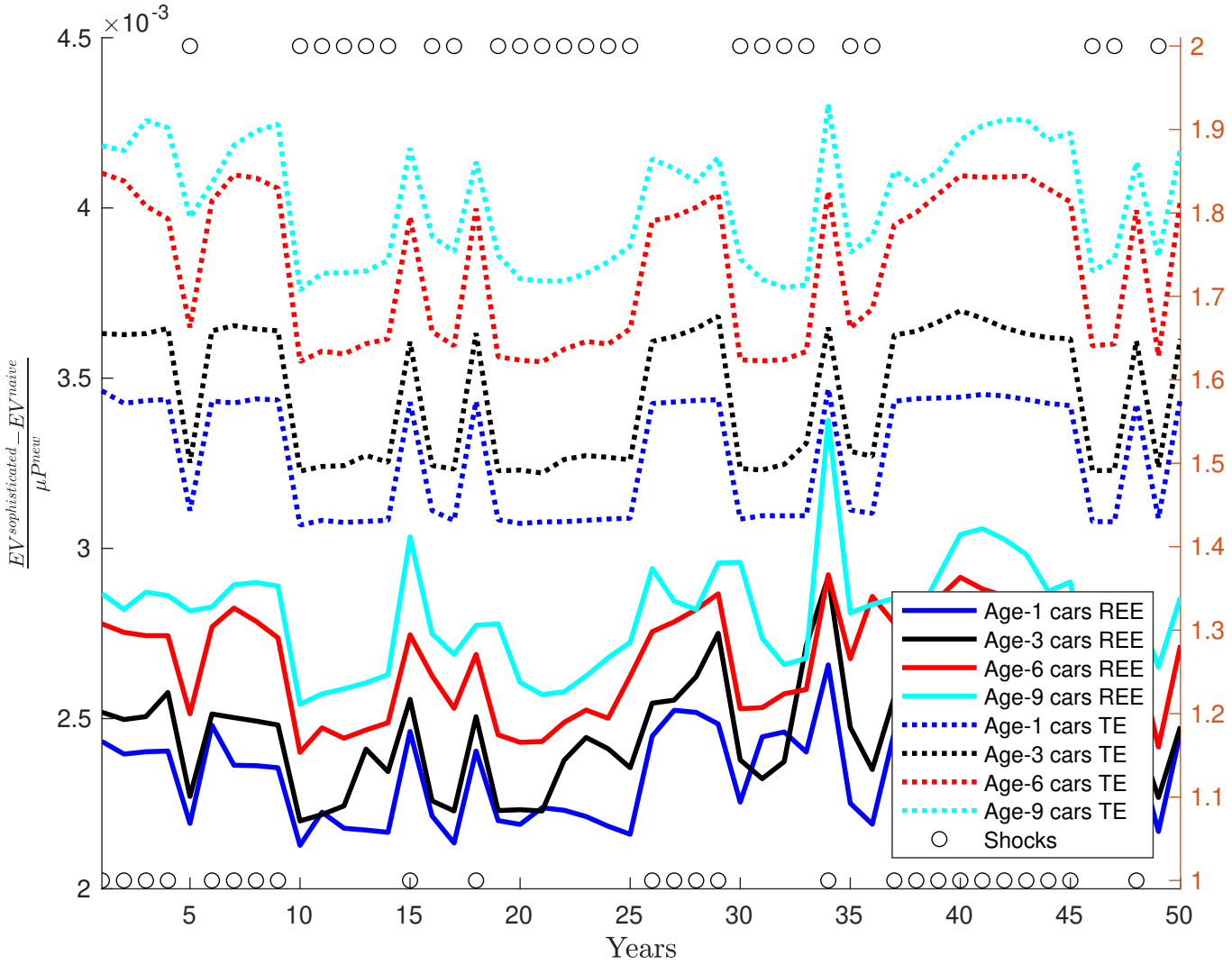




Quantitative Model: Macro Sophistication Values



Conclusion

- Macro waves in the secondary markets for automobiles
- Temporary equilibria can be more conceptually and quantitatively robust than rational expectations equilibrium
- Implications for future empirical work