Lecture 18: Solving directional dynamic games for all Markov perfect equilibria

Econometric Society Summer Schools in Dynamic Structural Econometrics

Fedor Iskhakov, Australian National University
John Rust, Georgetown University
Bertel Schjerning, University of Copenhagen

UCL, London
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ROAD MAP

1. Collusion of Australian corrugated fibre packaging (CFP) producers
   - Collusion between Amcor and Visy
   - Bertrand pricing and investment game
   - Solution concept: Markov perfect equilibrium (MPE)
2. Experiment with the model
3. State recursion algorithm
   - Theory of directional dynamic games (DDGs)
4. Recursive lexicographical search (RLS) algorithm
5. Full solution for the leapfrogging game
6. Structural estimation of directional dynamic games with Nested RLS method

Motivation: Collusion on the beach

Amcor-Visy price fixing

- Amcor Managing Director: Peter Brown
- Visy CEO: Harry Debney

Amcor-Visy collusion case

- Australian market for cardboard is essentially a duopoly
- Between 2000 and 2005 *Visy* and *Amcor* colluded to divide the market of cardboard and to fix prices
- 2007: Visy admits to have been manipulating the market, issued with $36 million fine
- July 2009: Cadbury vs. Amcor, damages estimated at $235.8 million, settles out of court
- March 2011: Class action suit against both Amcor and Visy settles out of court for $95 million

Cardboard industry in Australia

Bertrand price competition in the short run, with leapfrogging investments by both firms in the longer run

- Cardboard is a highly standardized product
- Strong incentives for Bertrand-like price cutting
- Amcor and Visy do minimal amounts of R&D themselves,
- Spend considerable amounts on cost reducing investments
  - Amcor plans to build state-of-the-art paper mill in Botany Bay before the collusion took place
  - "B9" plant finally opened on February 1, 2013
- Amcor and Visy purchase new technology from other companies that specialize in doing the R&D and reduce cost of production of cardboard

Amcor's New B9 Paper Mill

Main Mill Site, Botany Bay Road, Botany Bay NSW

Source: Amcor

B9 is an example of leapfrogging

- Amcor's existing paper plant was over 50 years old
- "The B9 paper machine, so named as it is the ninth paper machine to operate at the company's Botany site, will produce more than 400,000 tonnes of paper annually when operating at full capacity and will deliver significant environmental benefits."
- Cost: $500 million, the largest single investment in Amcor's 144 year history. "Largest and most innovative recycled paper machine of its kind in Australasia"
- "The machine is 330 metres long, and 22 metres high, and produces 1.6 km of paper per minute and reduces water consumption by 26%, energy usage by 34% and the amount of waste sent to landfill by 75%" (Nigel Garrard, Amcor CEO)

But collusion caused B9 to be *delayed*

- Amcor had planned B9 back in 1999, and at that time internal studies estimated huge rate of return for this investment because it would enable it to leapfrog Visy to become the low-cost producer of CFP in Australia.
- Amcor and Visy were locked in a *price war* that started in 1999, around the time the Amcor Board authorized the B9 investment.
- However when Visy and Amcor started to collude in 2000, the B9 project was curiously scrapped. B9 was not actually started until 2011, well after the end of the collusion in 2005. B9 only came online in February 2013.

Leapfrogging

Leapfrogging equilibrium

- Firms invest in alternating fashion and take turns in cost leadership
- Market price makes permanent downward shifts

"The Bertrand Investment Paradox"

- Should Bertrand competitors undertake cost-reducing investments?
- If both firms acquire state-of-art technology simultaneously, the following Bertrand price competition leads to zero profits for each firm
- Since both firms have access to cost reducing technology, does either of them have any incentive to invest ex ante?

Dynamic Bertrand price competition

Stochastic dynamic game

- Two Bertrand competitors, $n = 2$, no entry or exit
- Discrete time, infinite horizon ($t = 1, 2, \ldots, \infty$)
- Each firm maximizes expected discounted profits, common discount factor $\beta \in (0, 1)$
- Each firm has two choices in each period:
  1. Price for the product
  2. Whether or not to buy the state of the art technology

Static Bertrand price competition in each period

- Continuum of consumers make static purchase decision
- No switching costs: buy from the lower price supplier

Cost-reducing investments

State-of-the-art production cost c process

- Initial value c_0, lowest value 0: $0 \leq c \leq c_0$
- Discretized with n points
- Follows exogenous Markov process and only improves
- Markov transition probability $\pi(c_{t+1}|c_t)$
  $\pi(c_{t+1}|c_t) = 0$ if $c_{t+1} > c_t$

Investment choice: binary

- Investment cost of $K(c)$ to obtain marginal cost c
- One period construction time: production with technology obtained at t starts at $t + 1$

State space and information structure

Common knowledge

- State of the game: cost structure (c_1, c_2, c)
- State space is $S = (c_1, c_2, c) \subset R^3$: $c_1 \geq c$, $c_2 \geq c$
- Actions are observable

Private information

- In each period each firm incurs additive costs (benefits) from not investing and investing $\eta\epsilon_{i,I}$ and $\eta\epsilon_{i,N}$
- $\epsilon_{i,I}$ and $\epsilon_{i,N}$ are extreme value distributed, independent across choice, time and firms
- $\eta \geq 0$ is a scaling parameter
- Investment choice probabilities have logit form for $\eta > 0$

Timing of moves

Pricing decisions are made simultaneously

Expected one period profit of firm i from Bertrand game ($j \neq i$)

$$r_i(c_1, c_2) = \begin{cases} 0 & \text{if } c_i \geq c_j \\ c_j - c_i & \text{if } c_i < c_j \end{cases}$$

Two versions regarding investment decisions

1. Simultaneous moves:
   - Investment decisions are made simultaneously
2. Alternating moves:
   - The "right to move" state variable $m \in \{1, 2\}$,
   - When $m = i$, only firm i can make a cost reducing investment
   - m follows an own Markov process (deterministic alternation as a special case).

Actions and behavior strategies

Two choices in each period

- $p_i(c_1, c_2, c) = \max(c_1, c_2)$ – Bertrand pricing decision
- $P_i^I(c_1, c_2, c)$ – probability of firm i to invest in state-of-the-art production technology

  $P_i^N(c_1, c_2, c) = 1 - P_i^I(c_1, c_2, c)$ – probability not to invest

Strategy profile

- $\sigma = (\sigma_1, \sigma_2)$ – pair of Markovian *behavior* strategies

  $\sigma_i = \Big(p_i(c_1, c_2, c), P_i^I(c_1, c_2, c)\Big) \in \mathbb{R}_+ \times [0, 1]$
- Pure strategies included as special case

Definition of Markov Perfect Equilibium

Definition (Markov perfect equilibrium (MPE))

MPE of Bertrand investment stochastic game is a pair of

- strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*)$, and
- pair of *value functions* $V(s) = \Big(V_1(s), V_2(s)\Big)$, $V_i : S \to R$,

such that

1. Bellman equations (below) are satisfied for each firm, and
2. strategies σ_1^* and σ_2^* constitute mutual best responses, and assign positive probabilities only to the actions in the set of maximizers of the Bellman equations.

Bellman equations, firm $i = 1$, simultaneous moves

$$V_i(c_1, c_2, c) = \max\left[v_i^I(c_1, c_2, c) + \eta\epsilon_{i,I}, v_i^N(c_1, c_2, c) + \eta\epsilon_{i,N}\right]$$

$$\begin{aligned} v_i^N(c_1, c_2, c) &= r^i(c_1, c_2) + \beta EV_i(c_1, c_2, c|N) \\ v_i^I(c_1, c_2, c) &= r^i(c_1, c_2) - K(c) + \beta EV_i(c_1, c_2, c|I) \end{aligned}$$

With extreme value shocks, the investment probability is

$$P_i^I(c_1, c_2, c) = \frac{\exp\{v_i^I(c_1, c_2, c)/\eta\}}{\exp\{v_i^I(c_1, c_2, c)/\eta\} + \exp\{v_i^N(c_1, c_2, c)/\eta\}}$$

Bellman equations, firm $i = 1$, simultaneous moves

The expected values are given by

$$\begin{aligned} EV_i(c_1, c_2, c|N) &= \int_0^c \left[P_j^I(c_1, c_2, c)H_i(c_1, c, c') + \right. \\ &\qquad \left. P_j^N(c_1, c_2, c)H_i(c_1, c_2, c')\right] \pi(dc'|c) \\ EV_i(c_1, c_2, c|I) &= \int_0^c \left[P_j^I(c_1, c_2, c)H_i(c, c, c') + \right. \\ &\qquad \left. P_j^N(c_1, c_2, c)H_i(c, c_2, c')\right] \pi(dc'|c) \end{aligned}$$

where

$$H_i(c_1, c_2, c) = \eta \log\left[\exp\left(v_i^N(c_1, c_2, c)/\eta\right) + \exp\left(v_i^I(c_1, c_2, c)/\eta\right)\right]$$

is the "smoothed max" or logsum function

Solving and estimating directional dynamic games: Bertrand pricing and investment (leapfrogging) game

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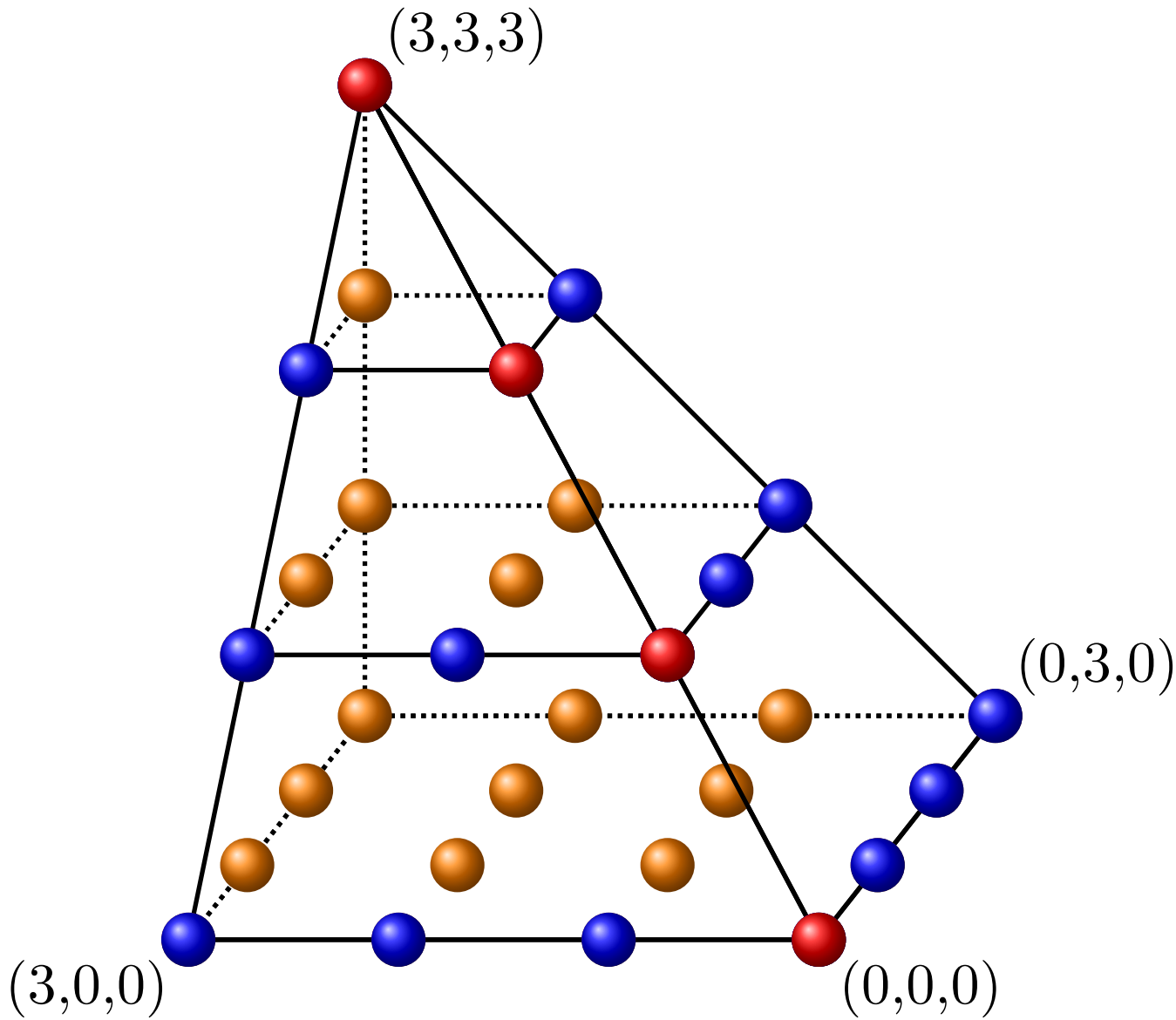
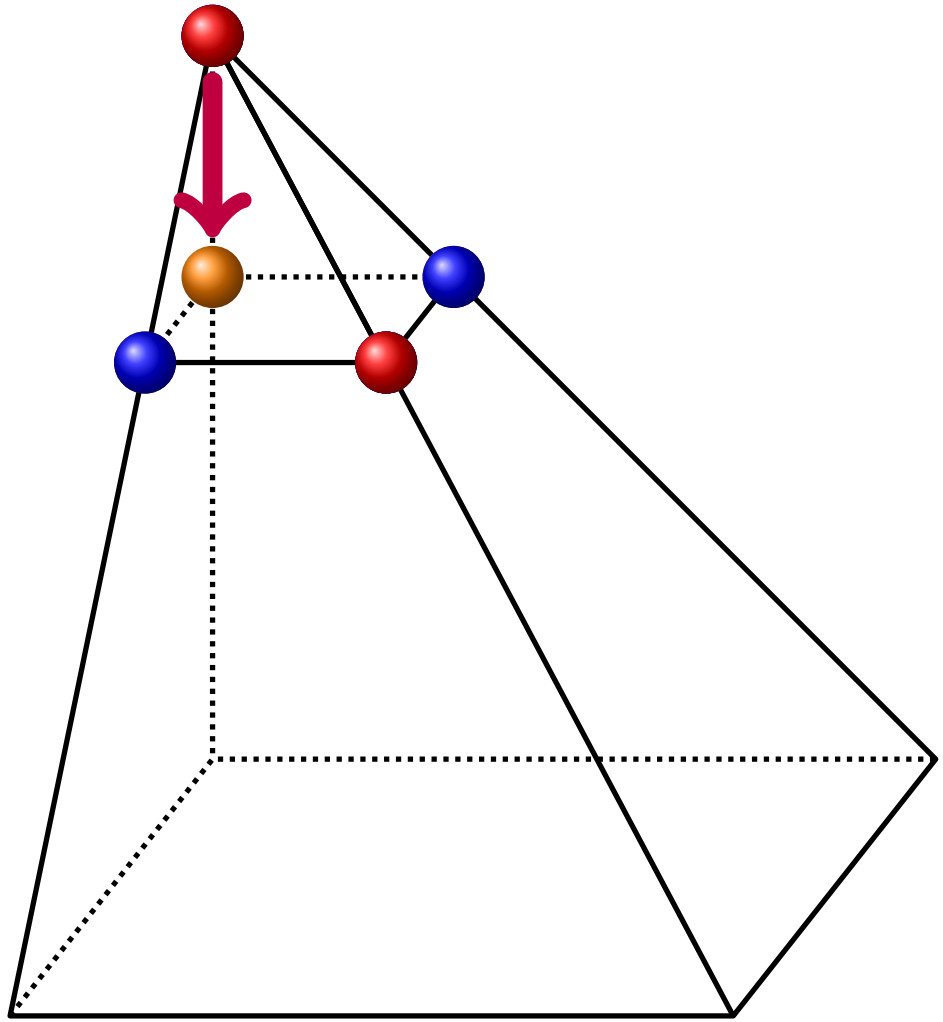
Discretized state space = a "quarter pyramid"

$S = \{(c_1, c_2, c) | c_1 \geq c,\ c_2 \geq c, c \in [0, 3]\},\ n = 4$



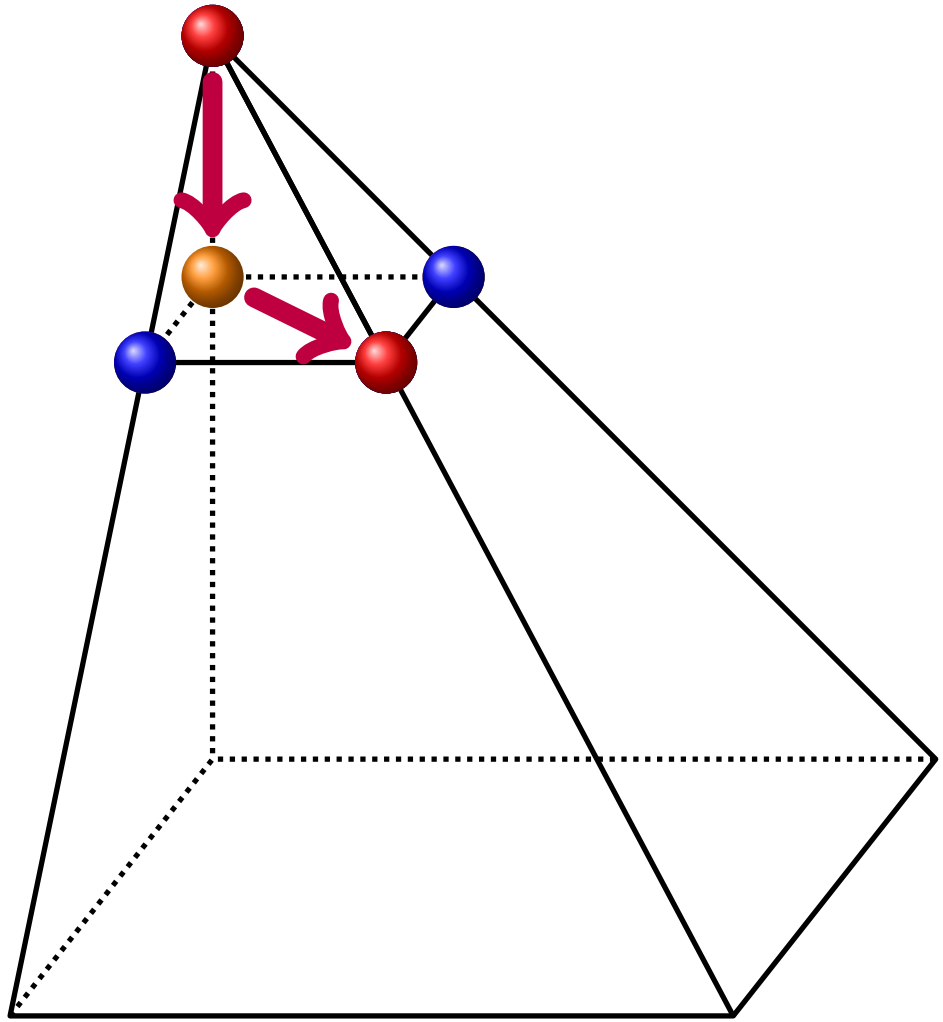
Game dynamics: example

The game starts at the apex, as some point technology improves



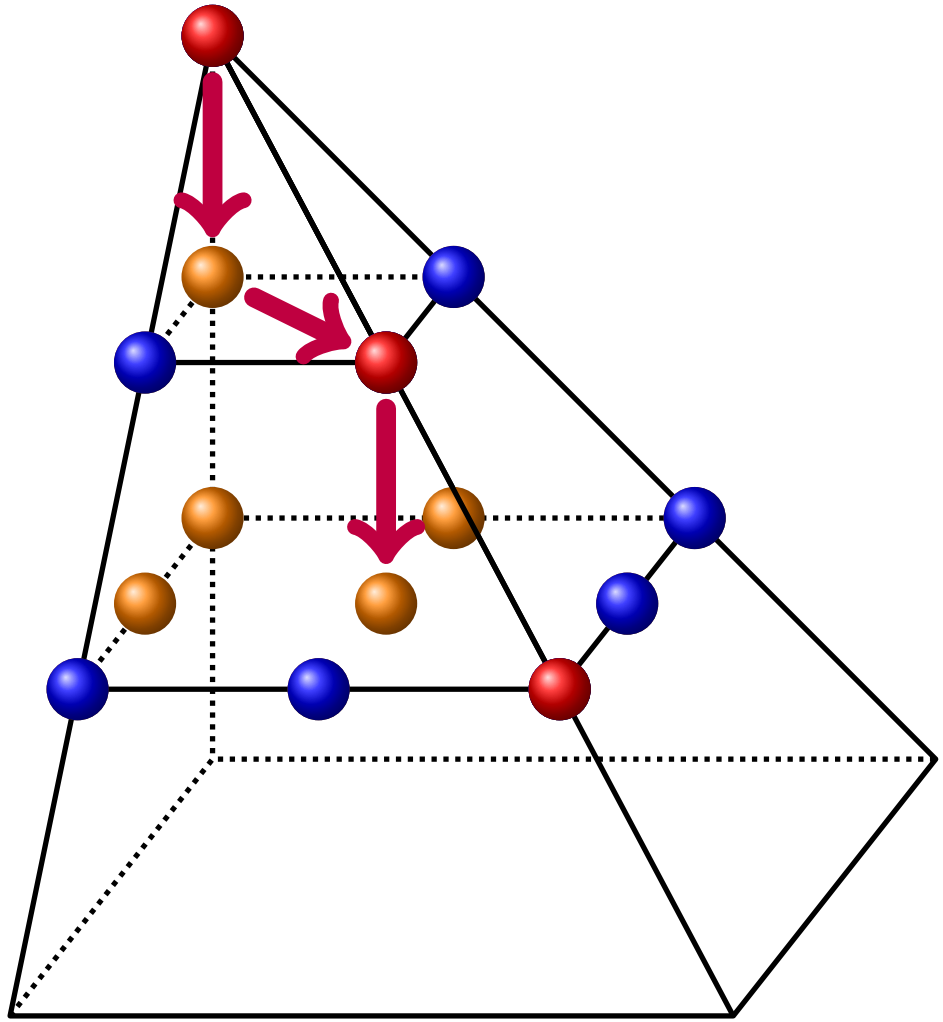
Game dynamics: example

Both firms buy new technology $c = 2 \rightsquigarrow (c_1, c_2, c) = (2, 2, 2)$



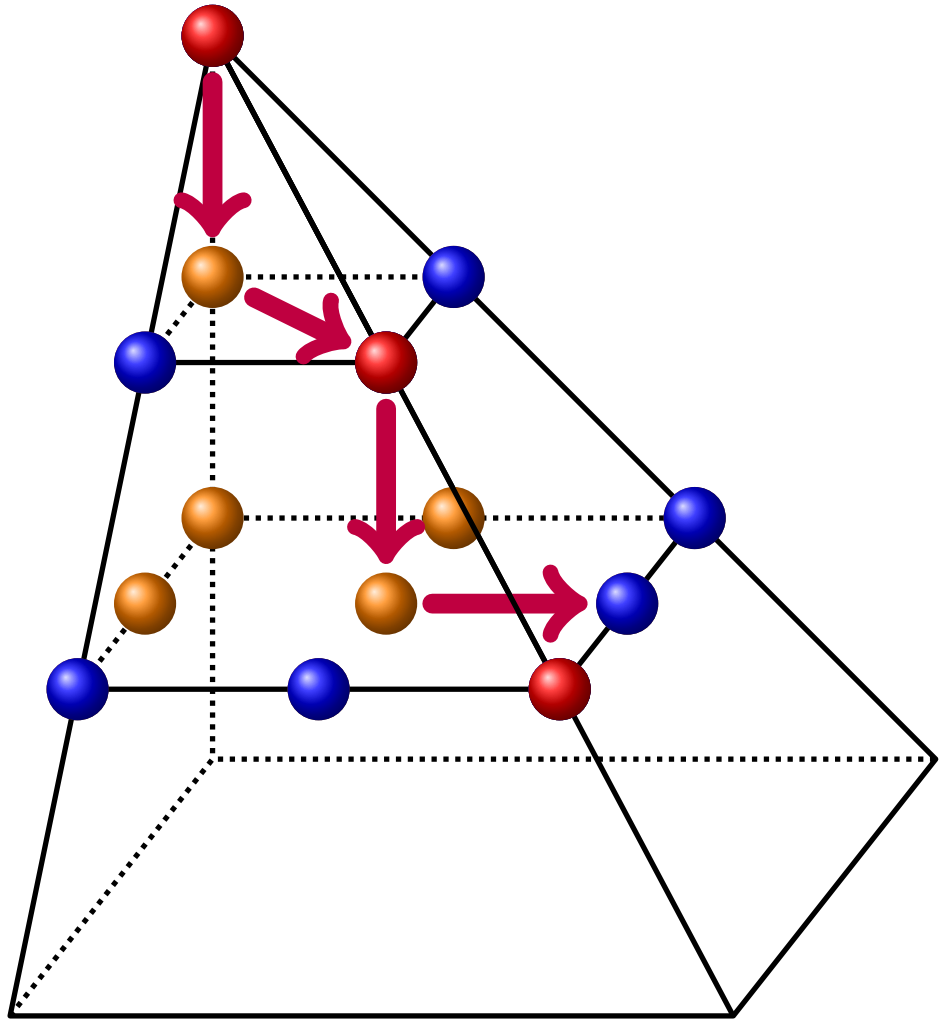
Game dynamics: example

State-of-the-art technology becomes $c = 1 \rightsquigarrow (c_1, c_2, c) = (2, 2, 1)$



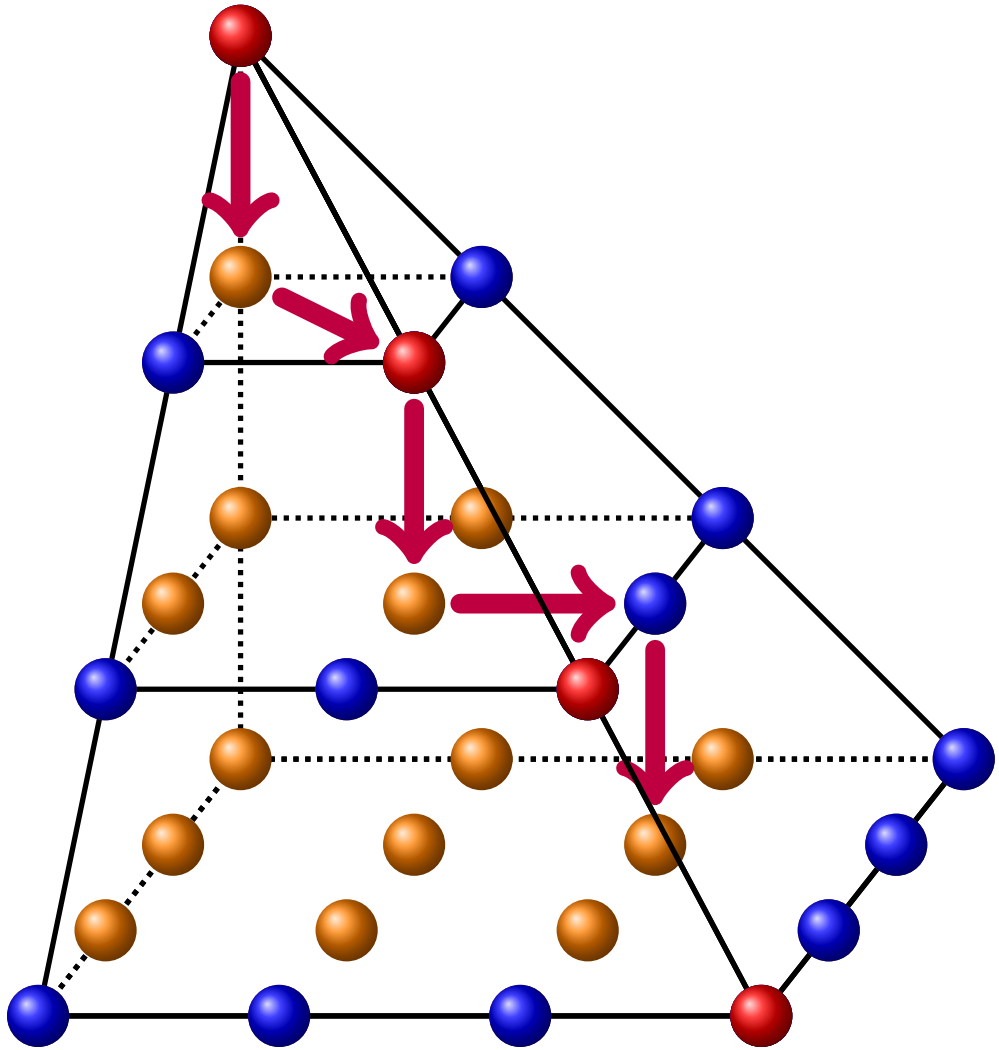
Game dynamics: example

Firm 1 invests and becomes cost leader $\rightsquigarrow (c_1, c_2, c) = (1, 2, 1)$



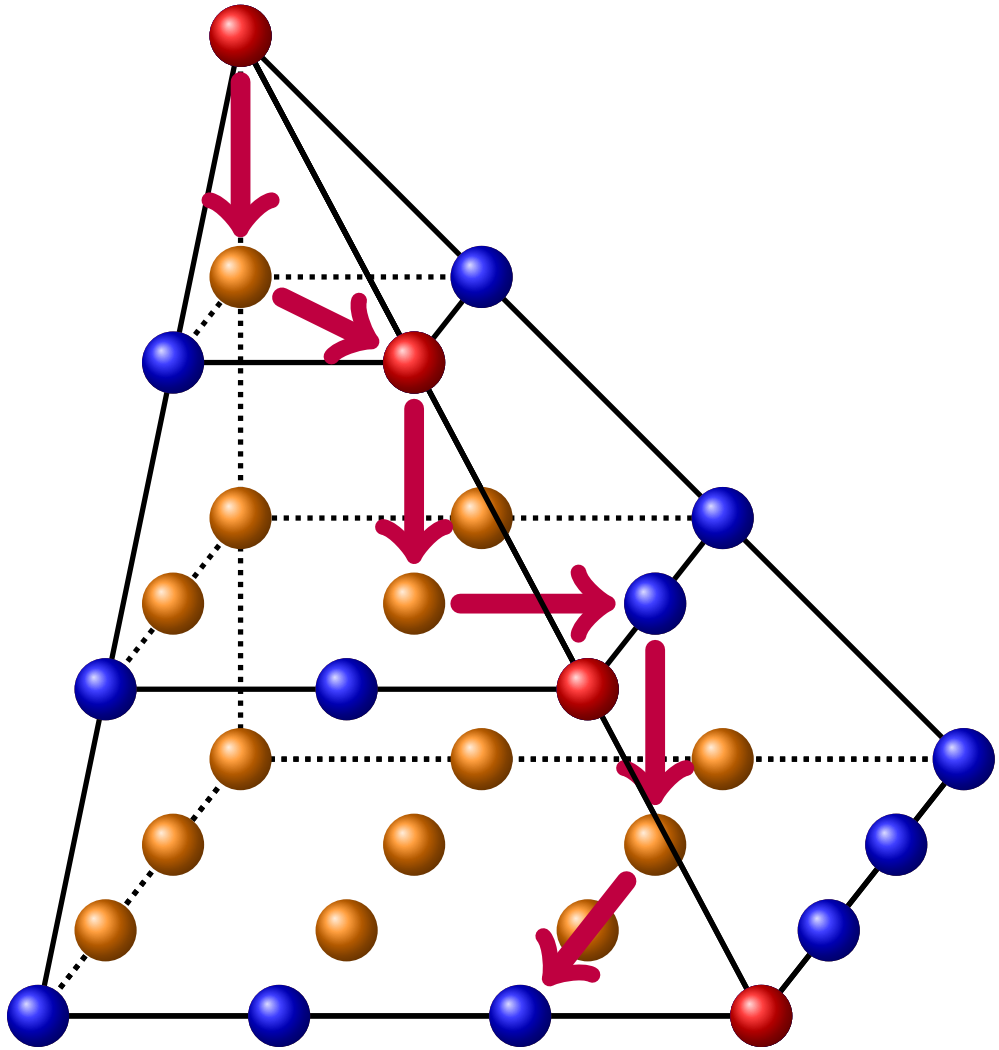
Game dynamics: example

State-of-the-art technology becomes $c = 0 \rightsquigarrow (c_1, c_2, c) = (1, 2, 0)$



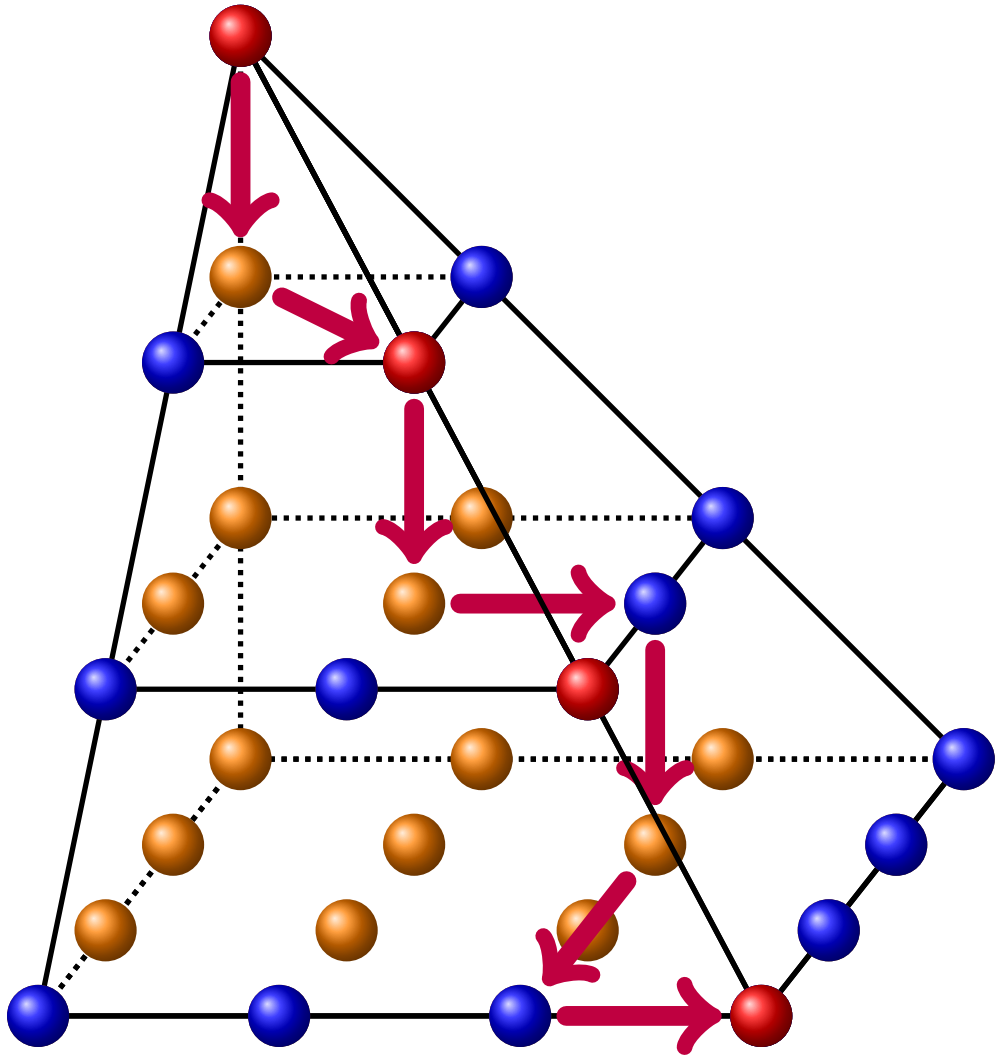
Game dynamics: example

Firm 2 leapfrogs firm 1 to become new cost leader $\rightsquigarrow (c_1, c_2, c) = (1, 0, 0)$



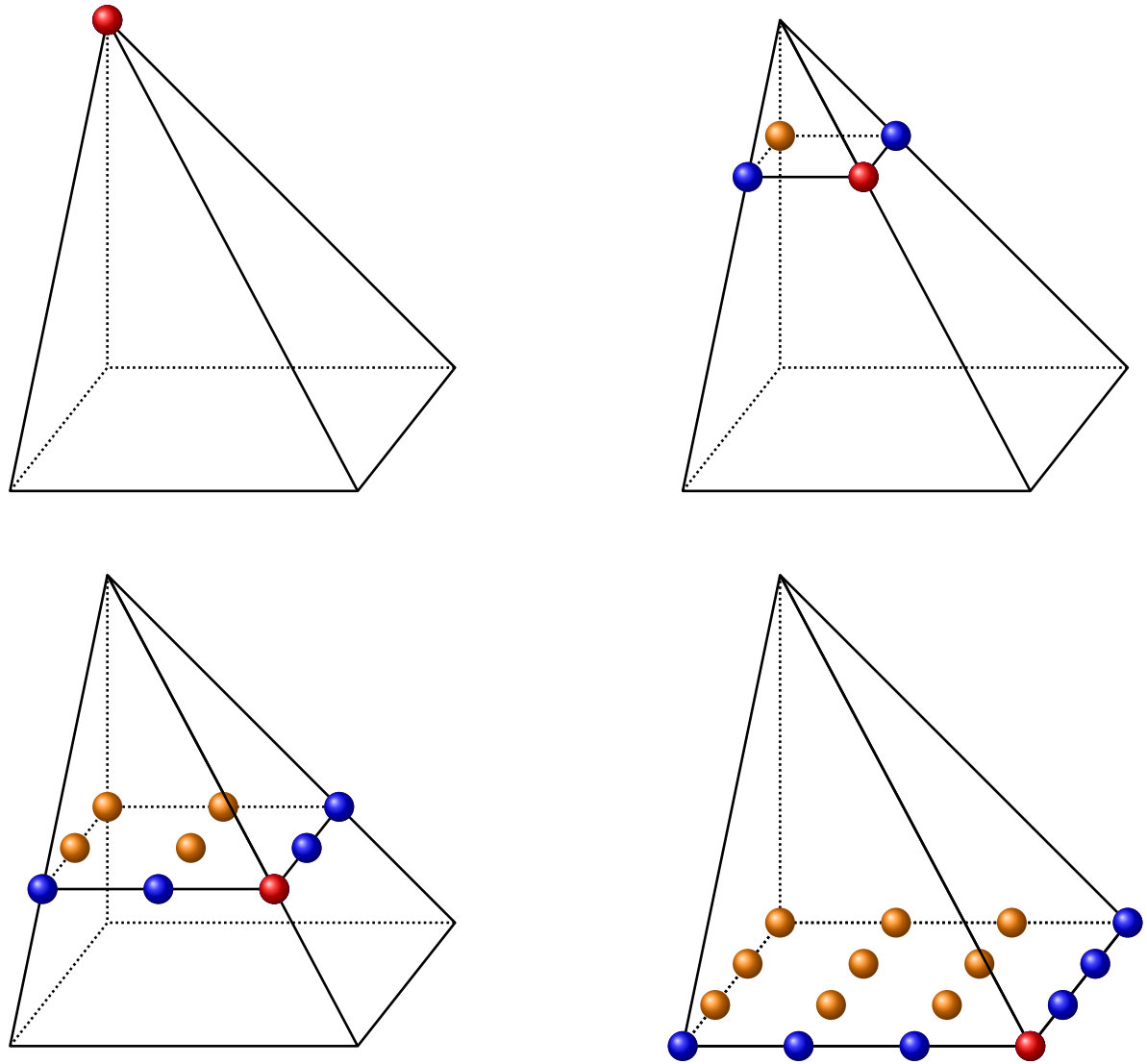
Game dynamics: example

Firm 1 invests, and the game reaches terminal state $\rightsquigarrow (c_1, c_2, c) = (0, 0, 0)$



Transitions due to technological progress

As c decreases, the game falls through the layers of the pyramid



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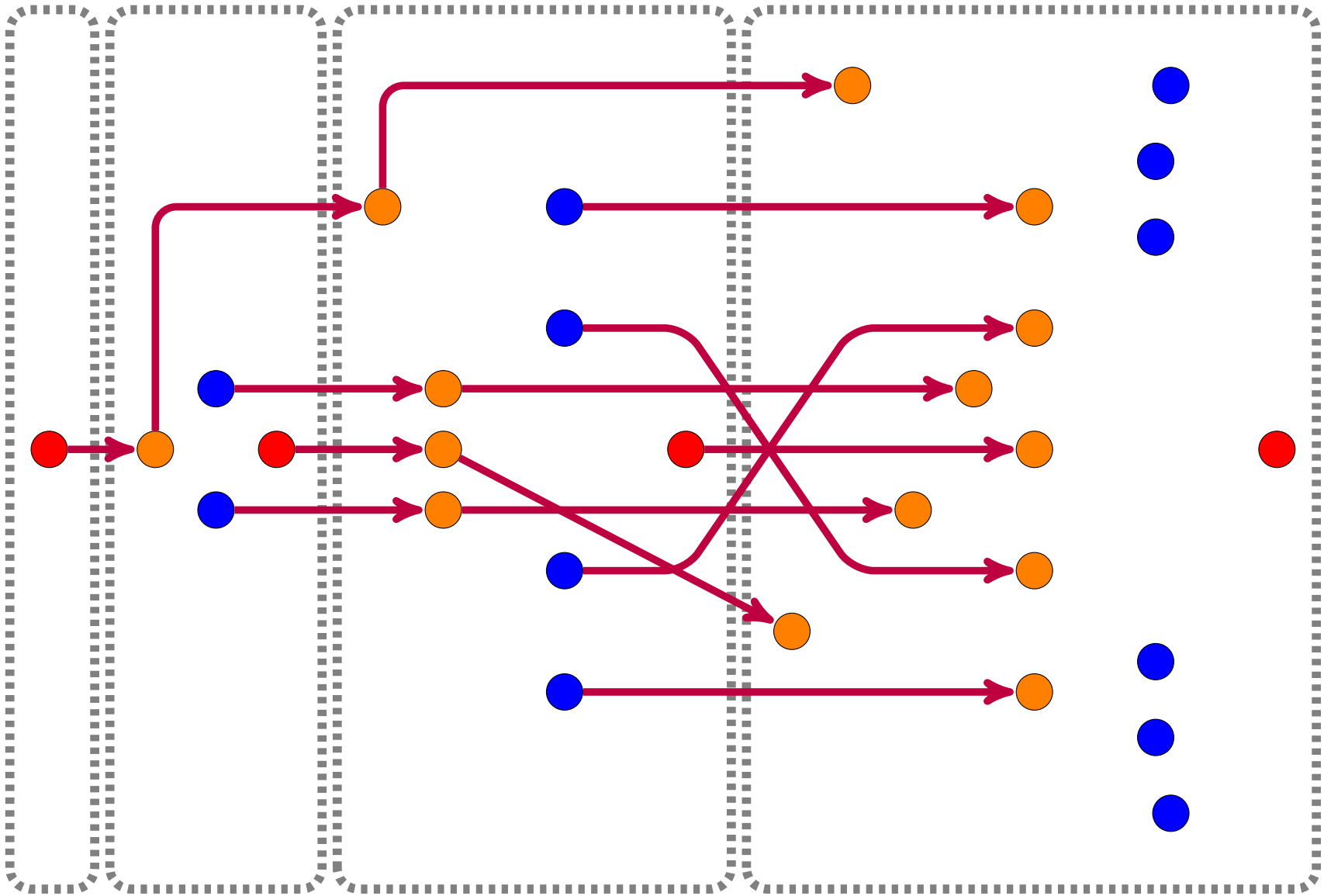
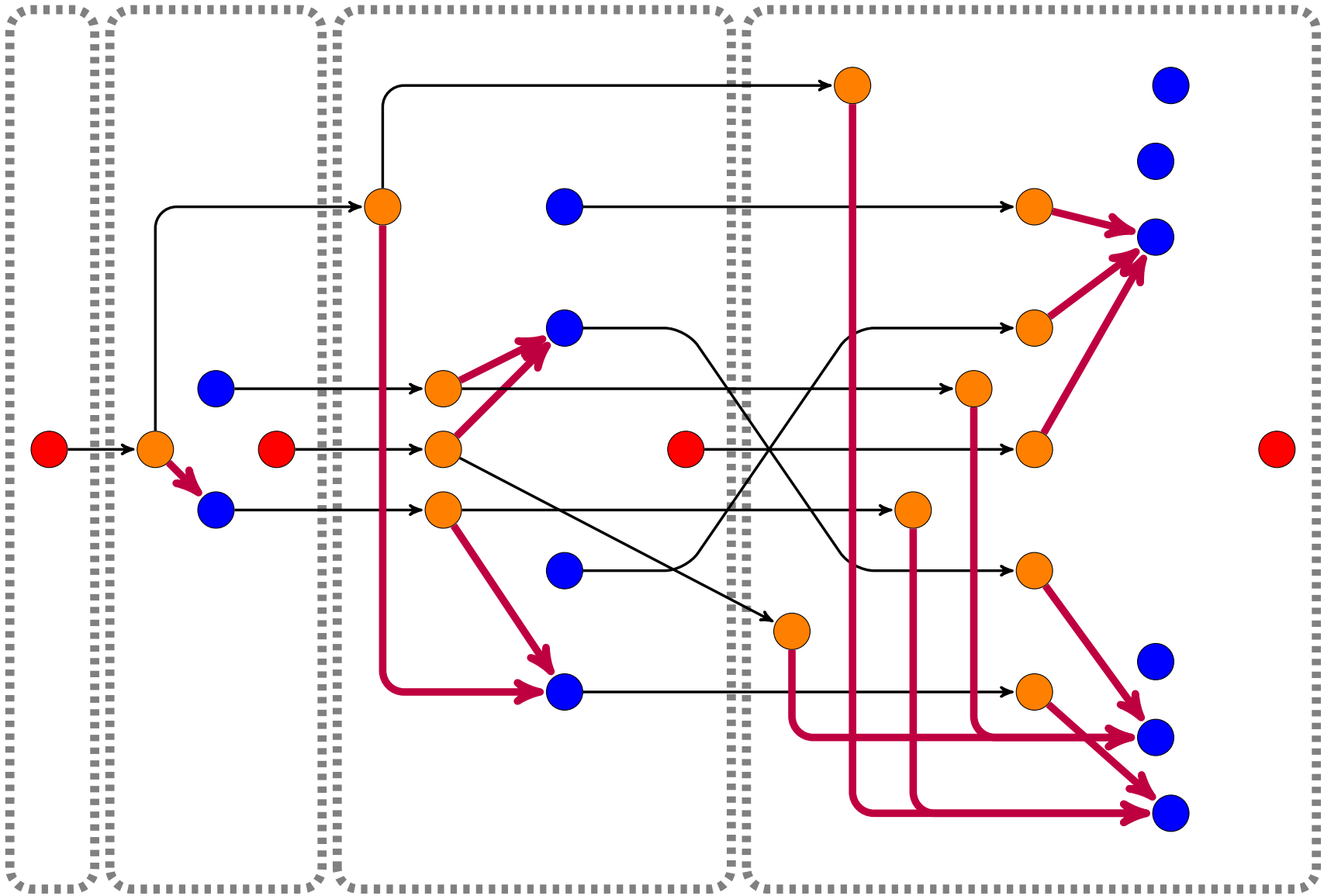


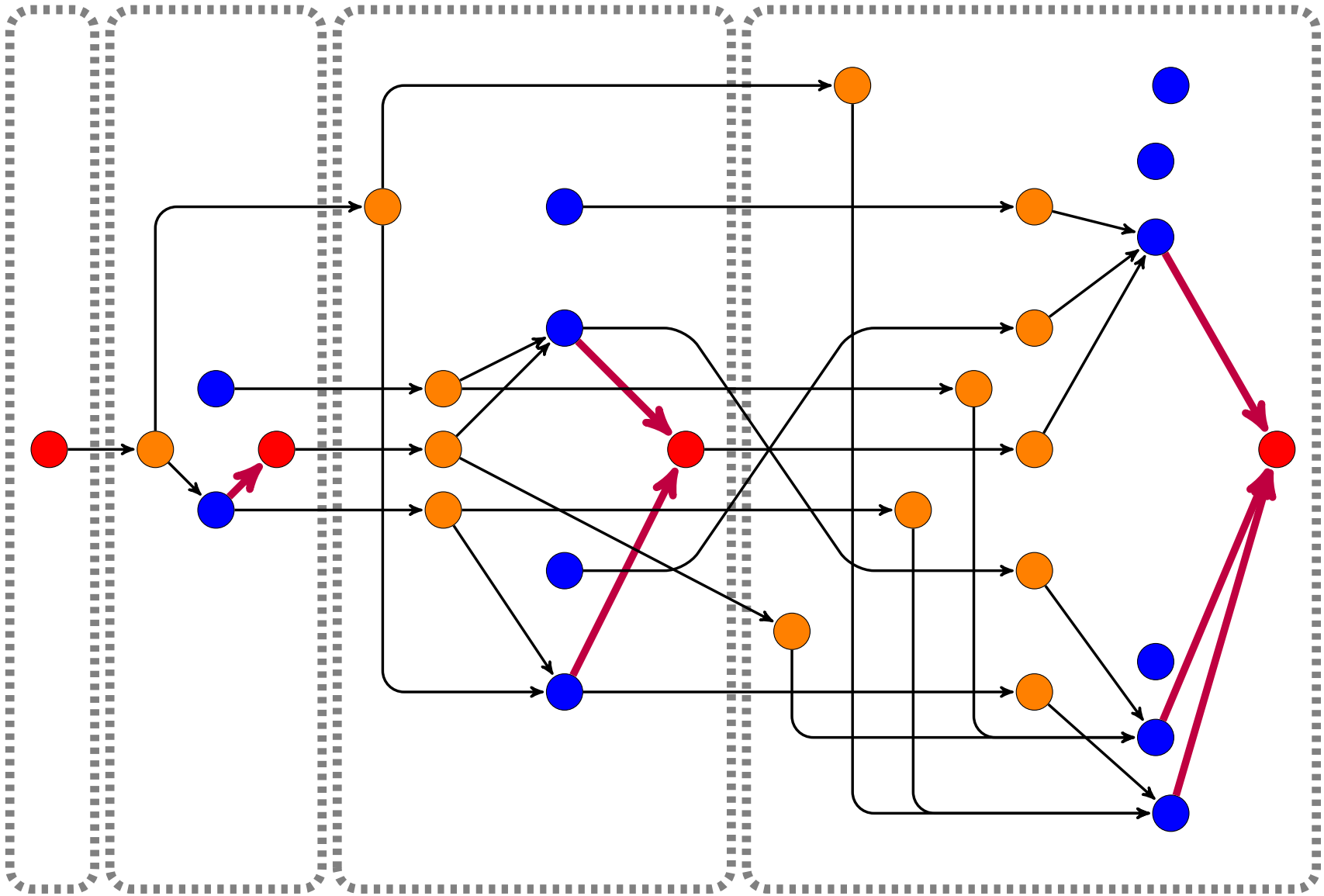
Strategy-specific partial order on S

Strategy σ_1 of firm 1: invest at all interior points



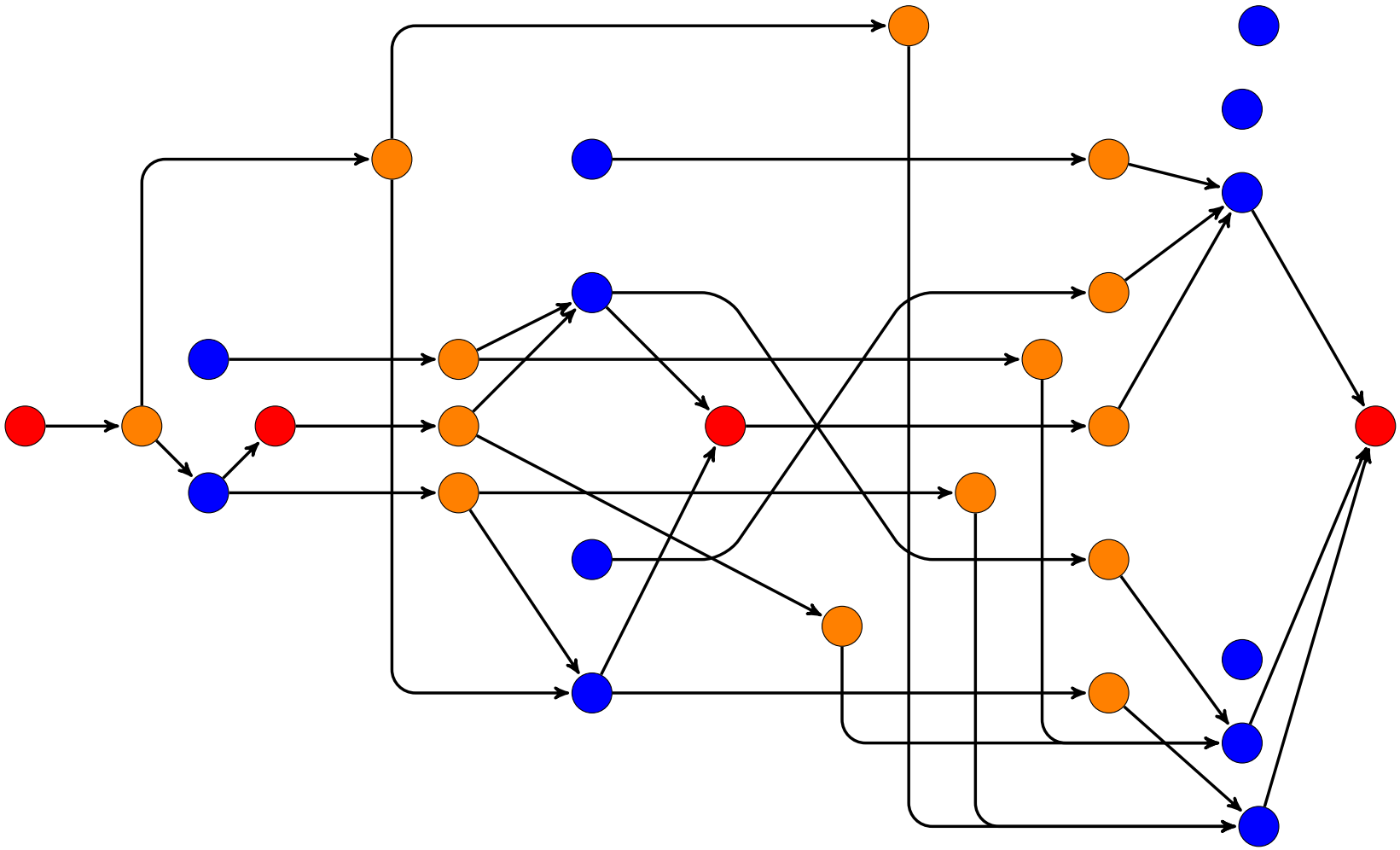
Strategy-specific partial order on S

Strategy σ_2 of firm 2: invest at all edge points



Strategy-specific partial order on S

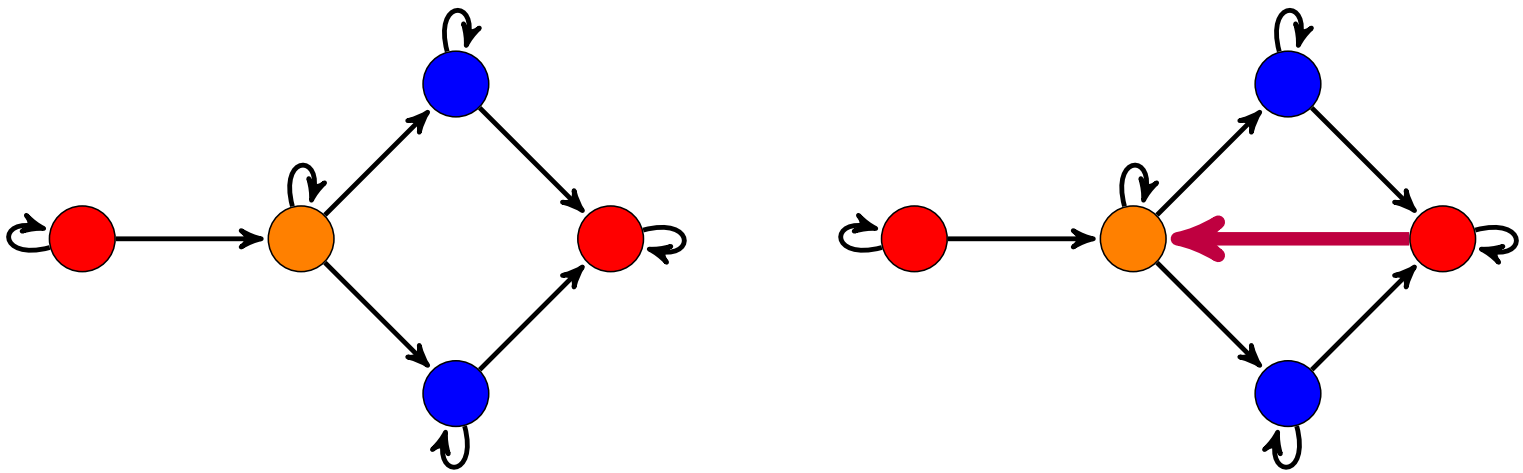
Strategy $\sigma = (\sigma_1, \sigma_2)$ of both firms



No loop (anti-cycling) condition

Hypothetical strategy profile inducing cycles

Self-loops appear when the game remains in the same state for two or more consecutive periods of time

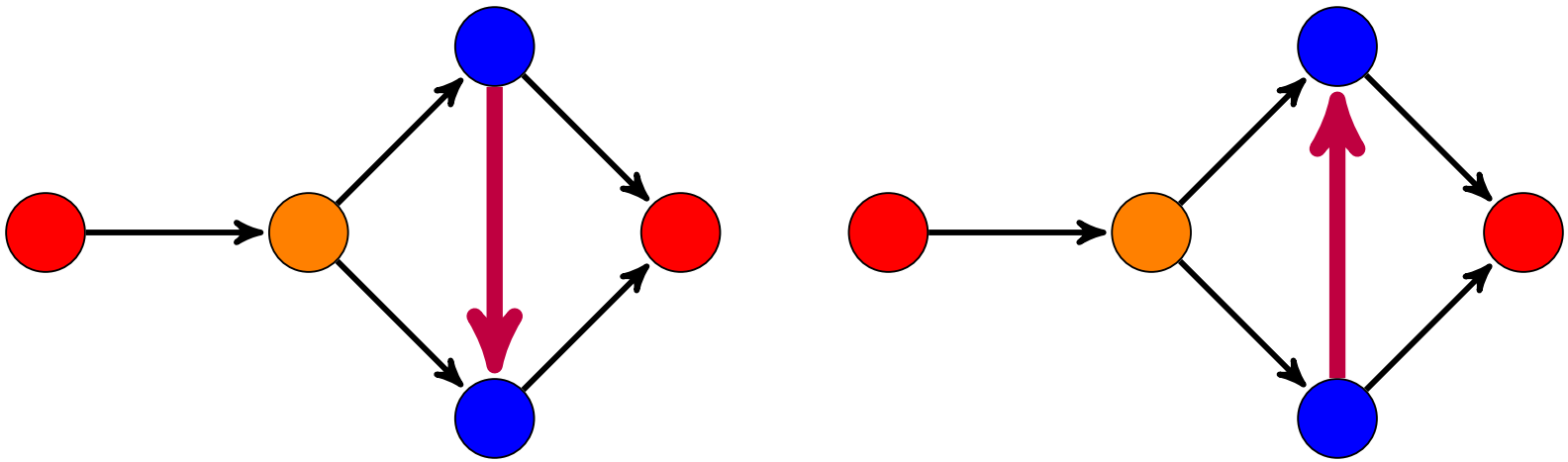


But loops between different states are not allowed

Consistency of strategy specific partial orders

Two hypothetical inconsistent strategies

Two strategies that induce opposite transitions are inconsistent



Note that in both cases the no-loop condition is satisfied

Definition of the Dynamic Directional Games

Definition (Dynamic Directional Games, DDG)

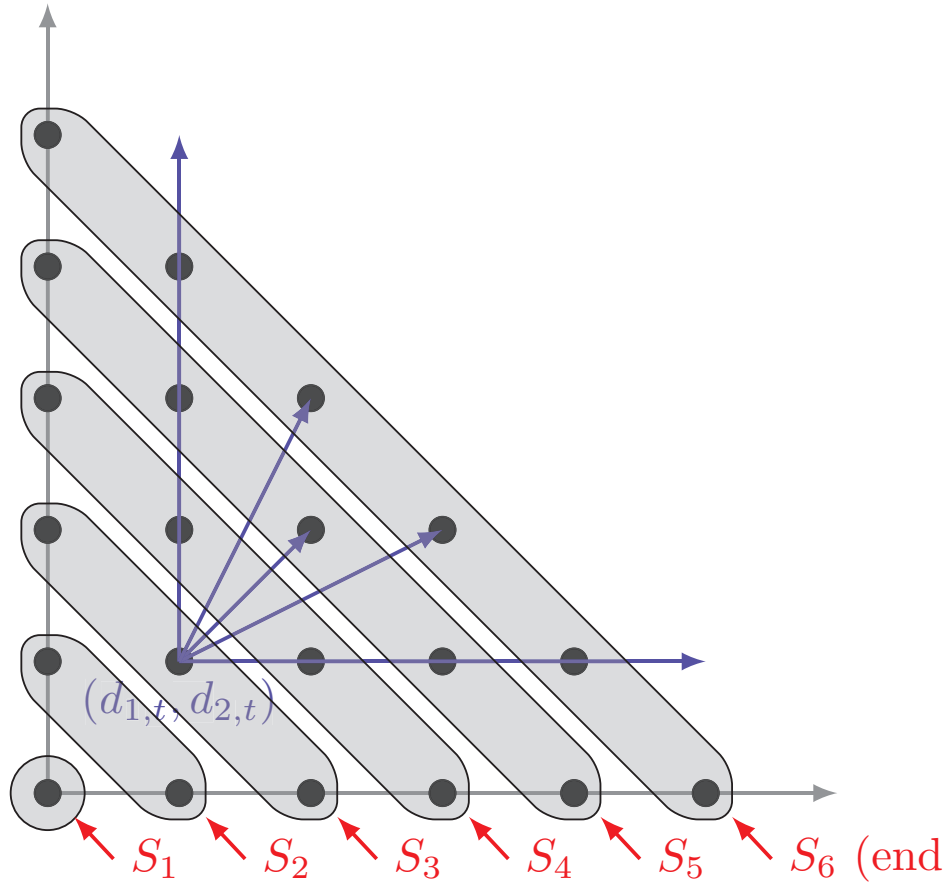
Finite state Markovian stochastic game is a DDG if it holds:

1. Every feasible Markovian strategy σ satisfies the no loop condition.
2. Every pair of feasible Markovian strategies σ and σ' induce consistent partial orders on the state space.

Example of DDG: Martket tipping game

Dubè, Hitsch and Chintagunta, 2010 *Marketing Science*
Tipping and Concentration in Markets with Indirect Network Effects

- Two competing gaming **platforms** (Xbox vs. Playstation)
- Number of games for each console depends on market share
- Consumers choose product they believe will win the war of the standards, or delay purchase
- Brand choices are absorbing: adoption base can only increase

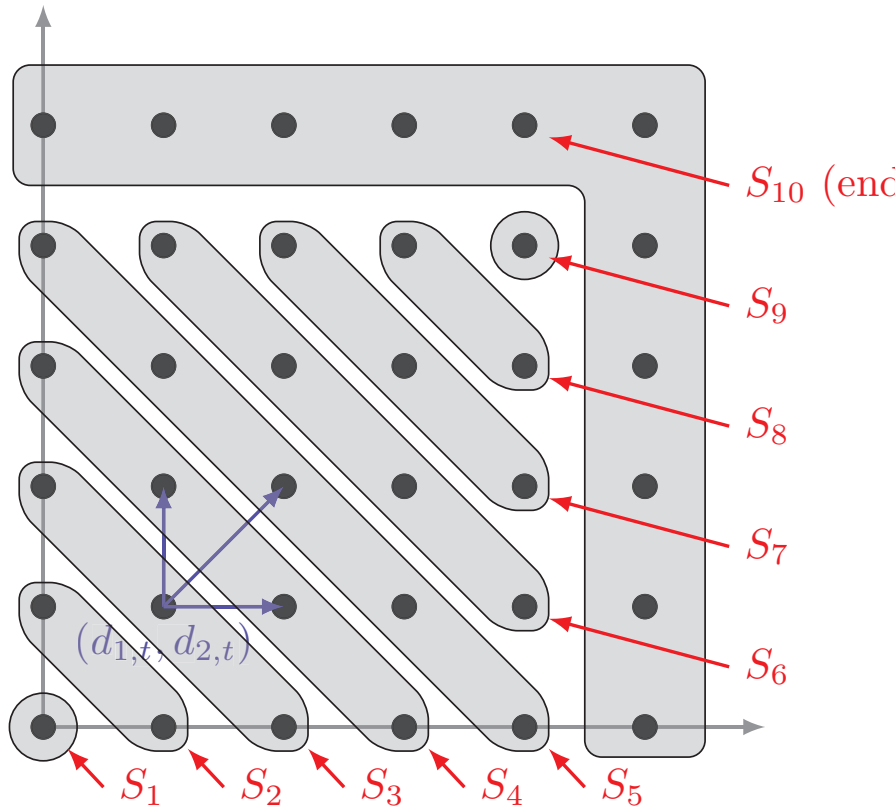


Example of DDG: Patent race game

Judd, Schmedders and Yeltekin, 2012 *IER*

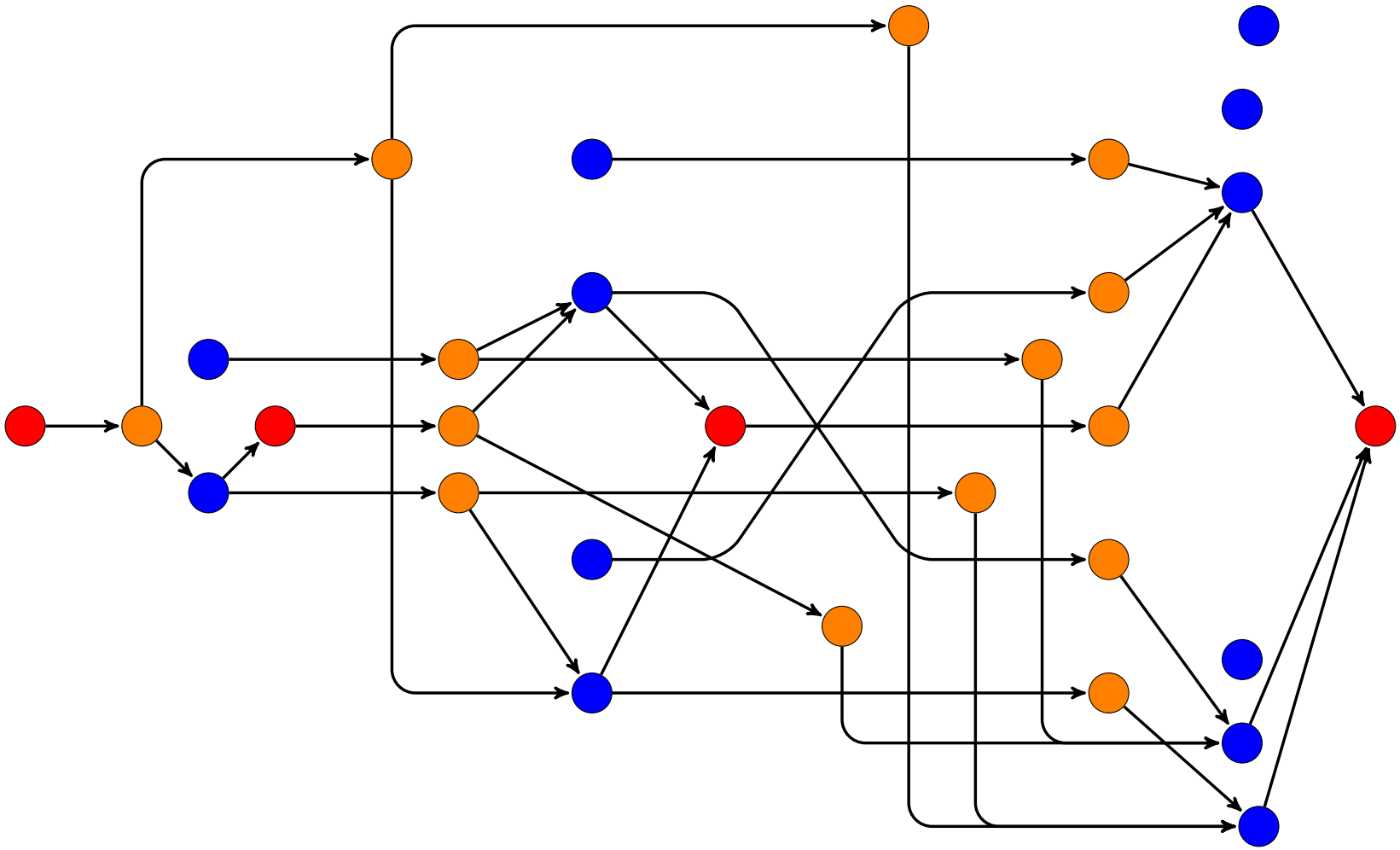
Optimal Rules for Patent Races

- Two firms are engaged in the N-steps race to acquire a patent
- Probability a step depends on the chosen R&D investment
- Only forward steps are possible, no "forgetting"
- Used state dependence structure to speed up computation of MPE by solving non-linear systems



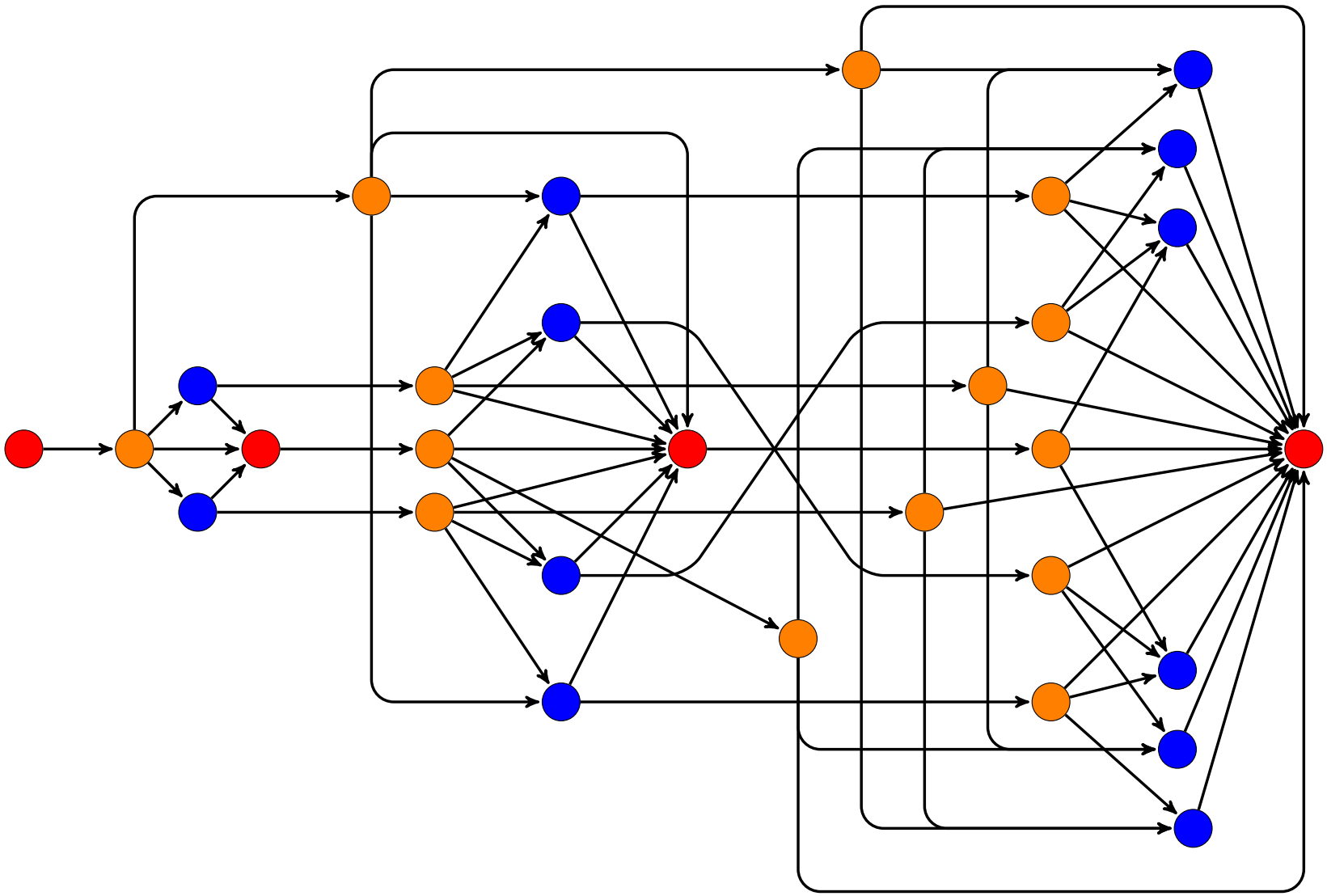
Strategy-specific partial order on S

Strategy $\sigma = (\sigma_1, \sigma_2)$ of both firms



DAG recursion to partition S into stages

Identify and then remove terminal states



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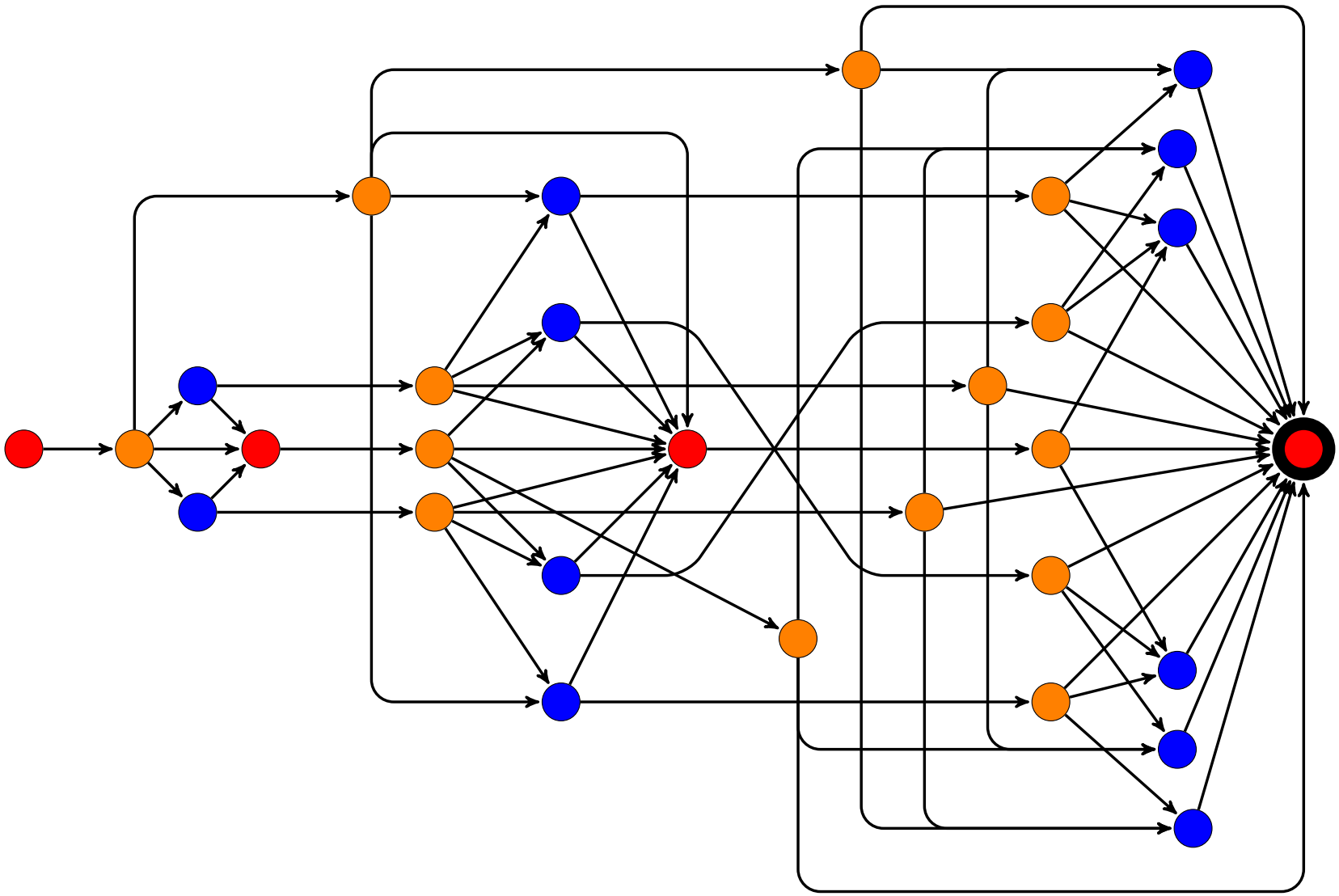

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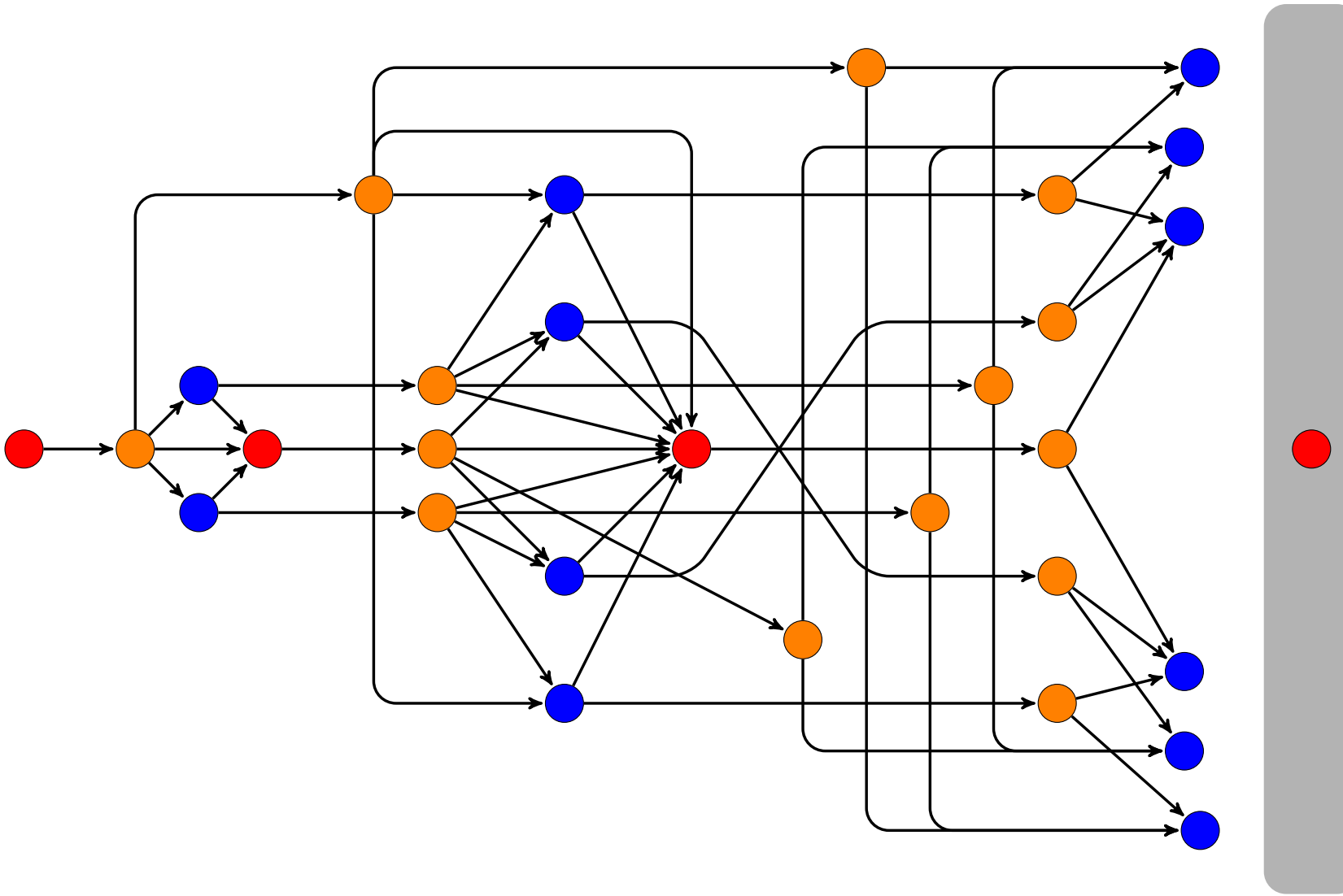

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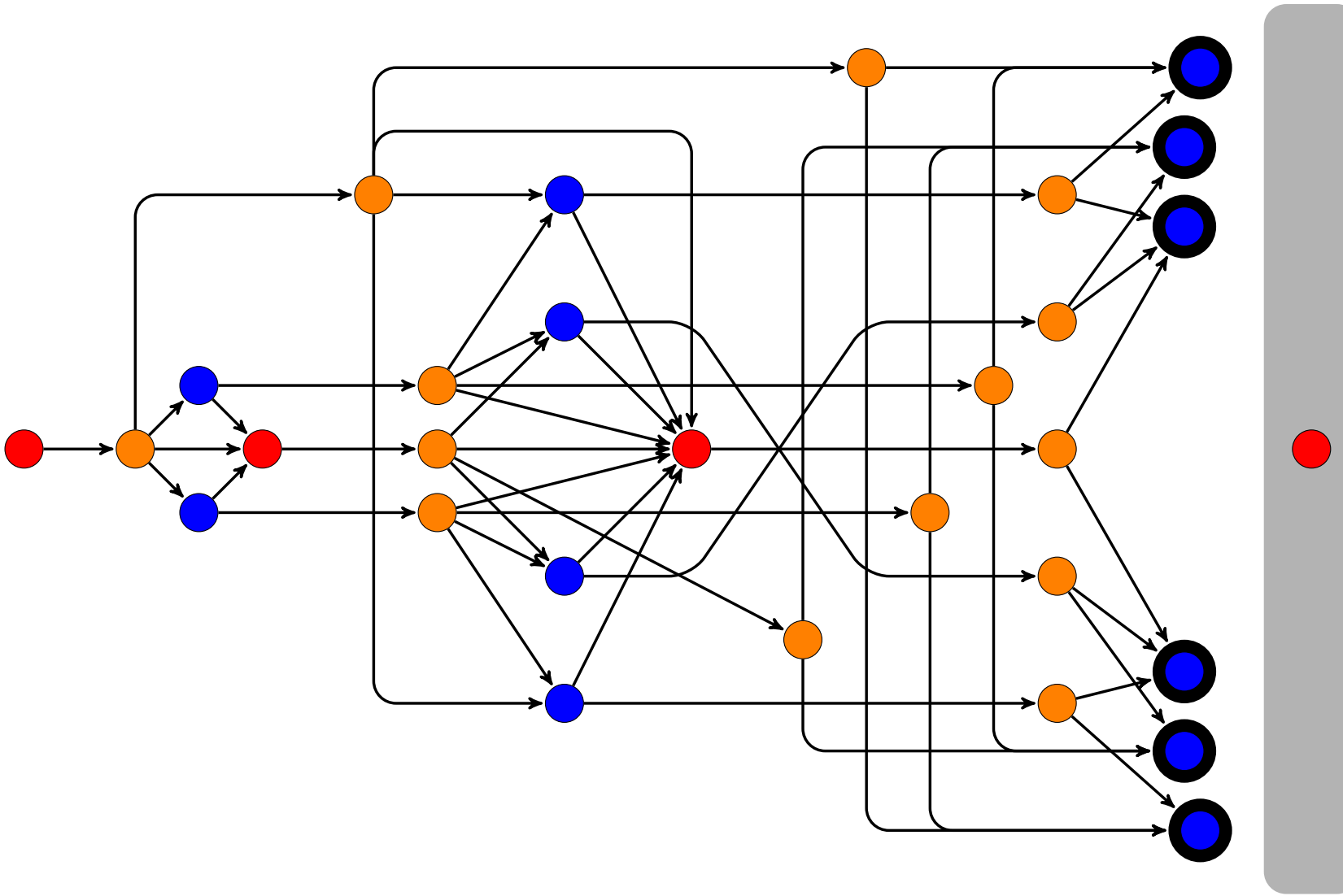

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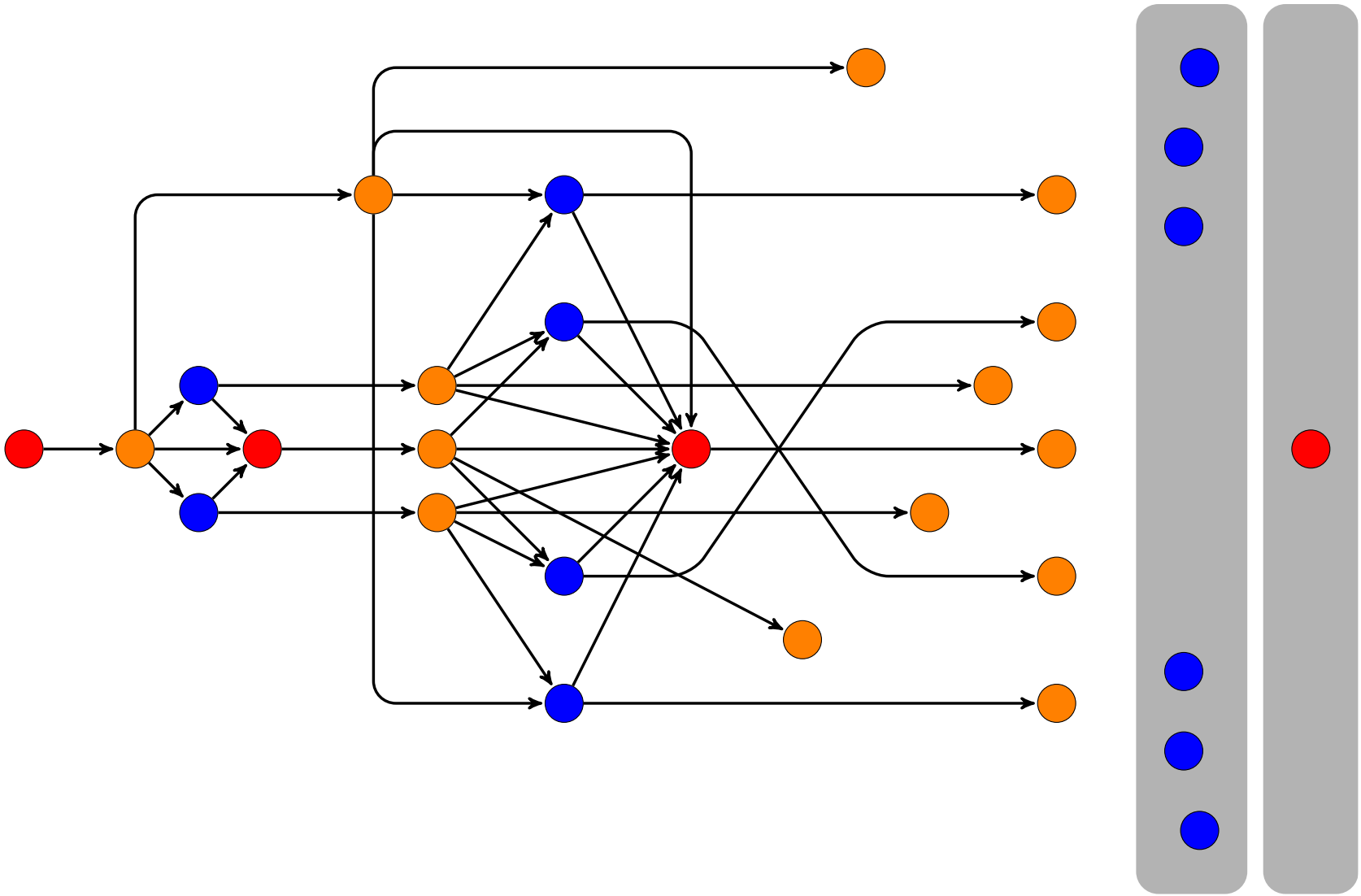

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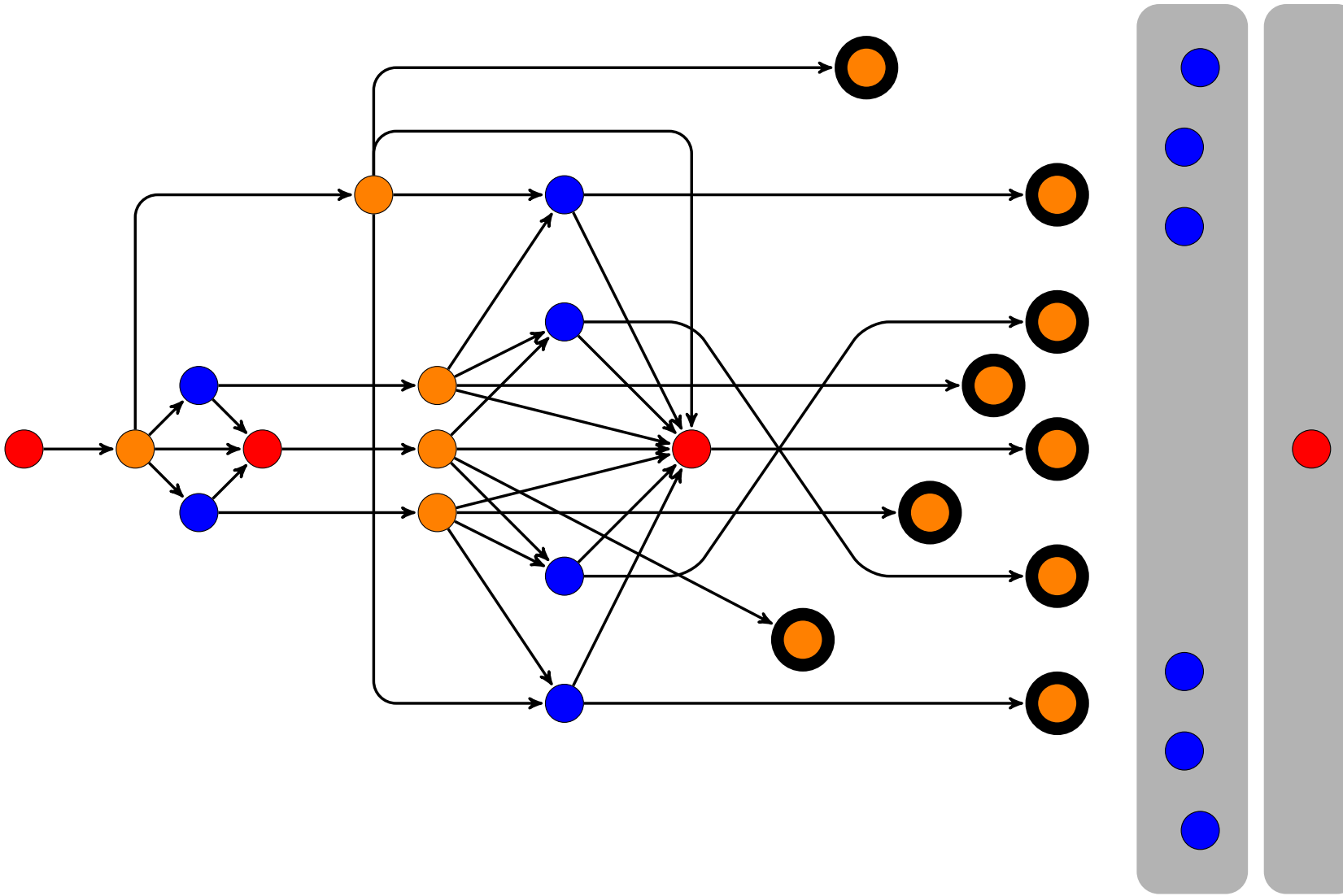

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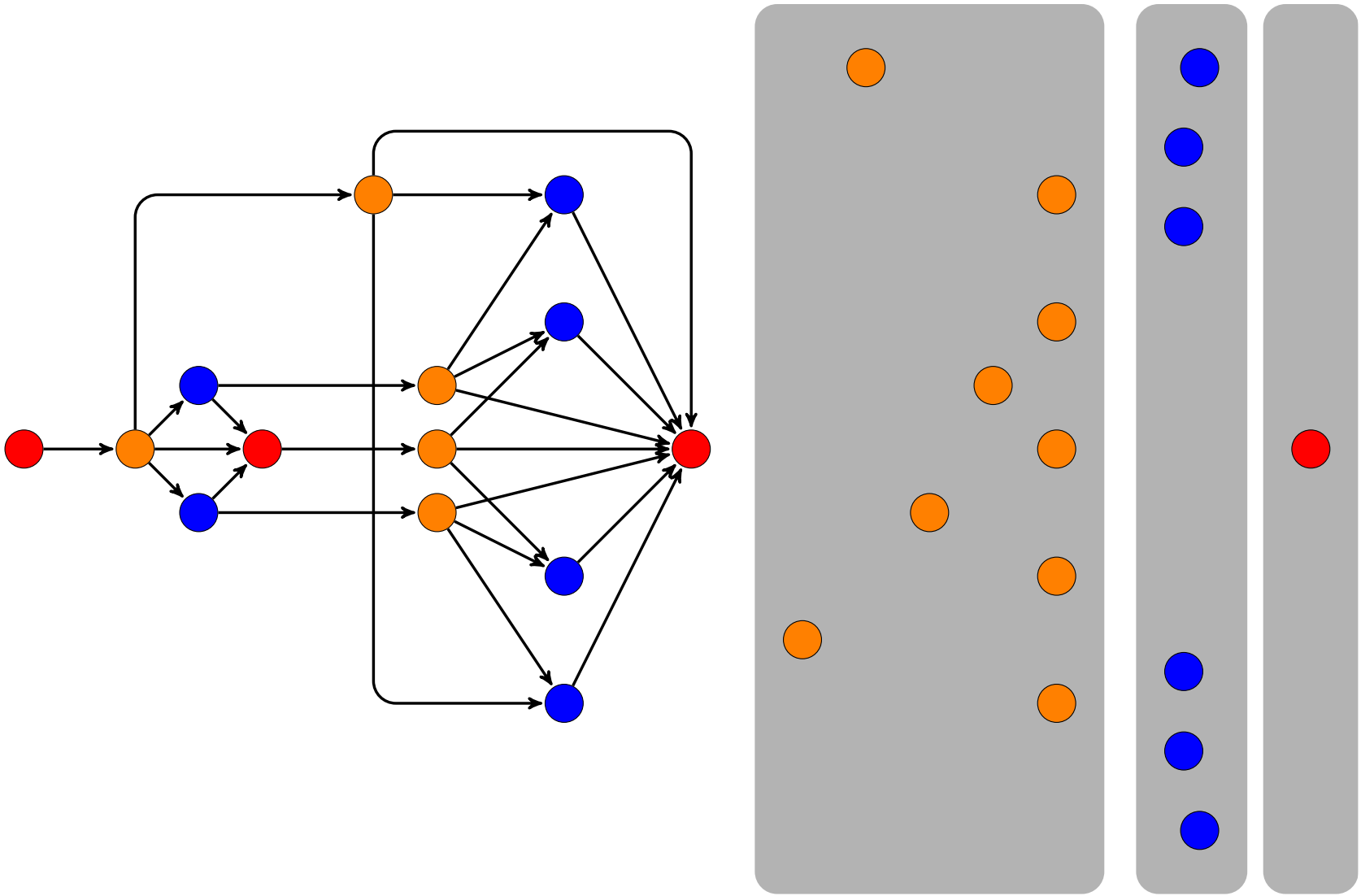

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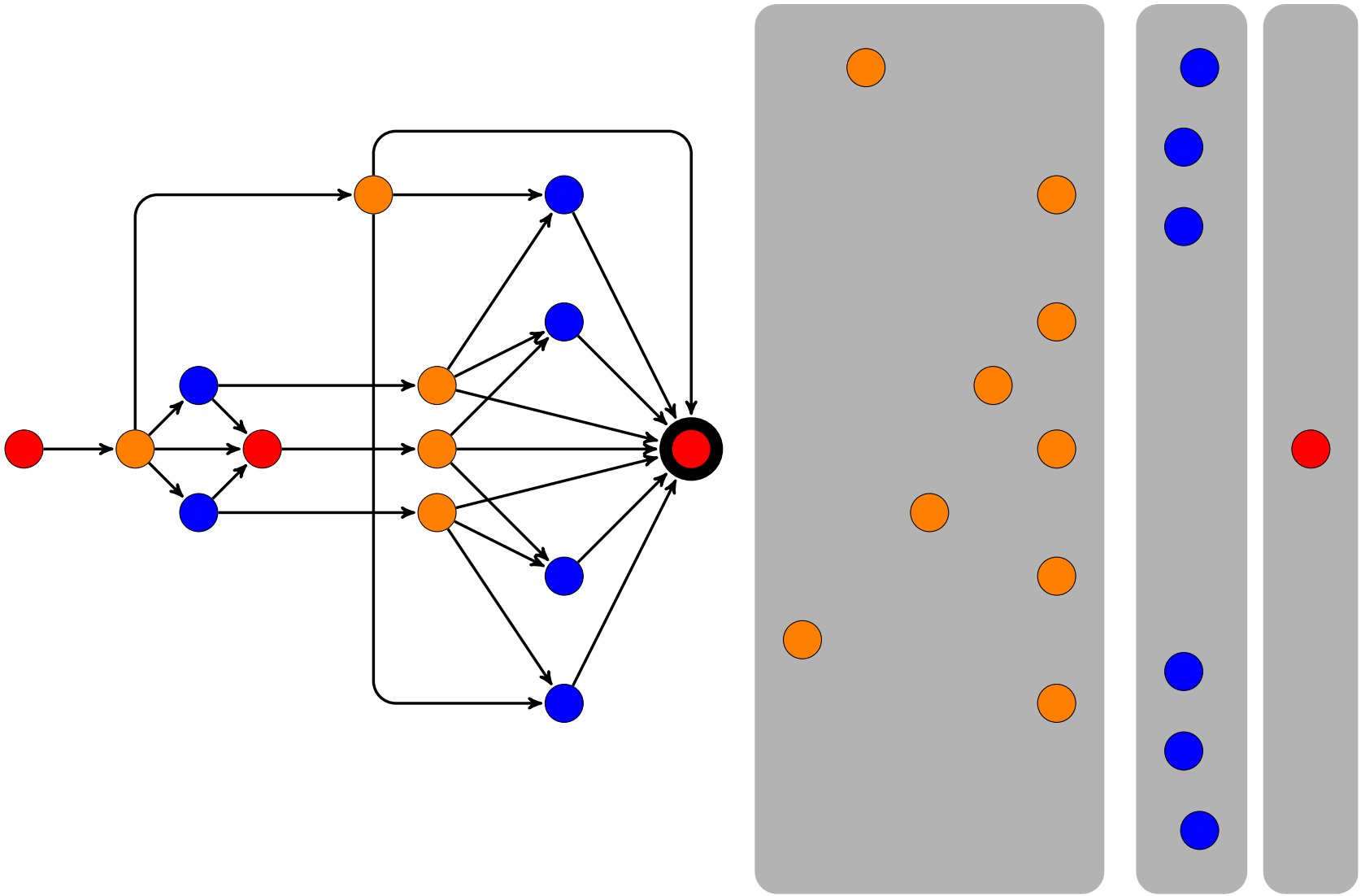

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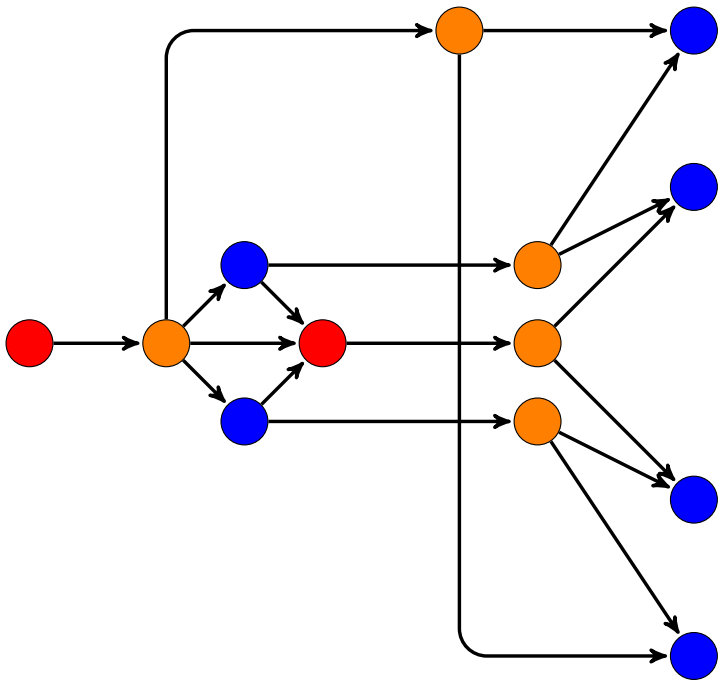
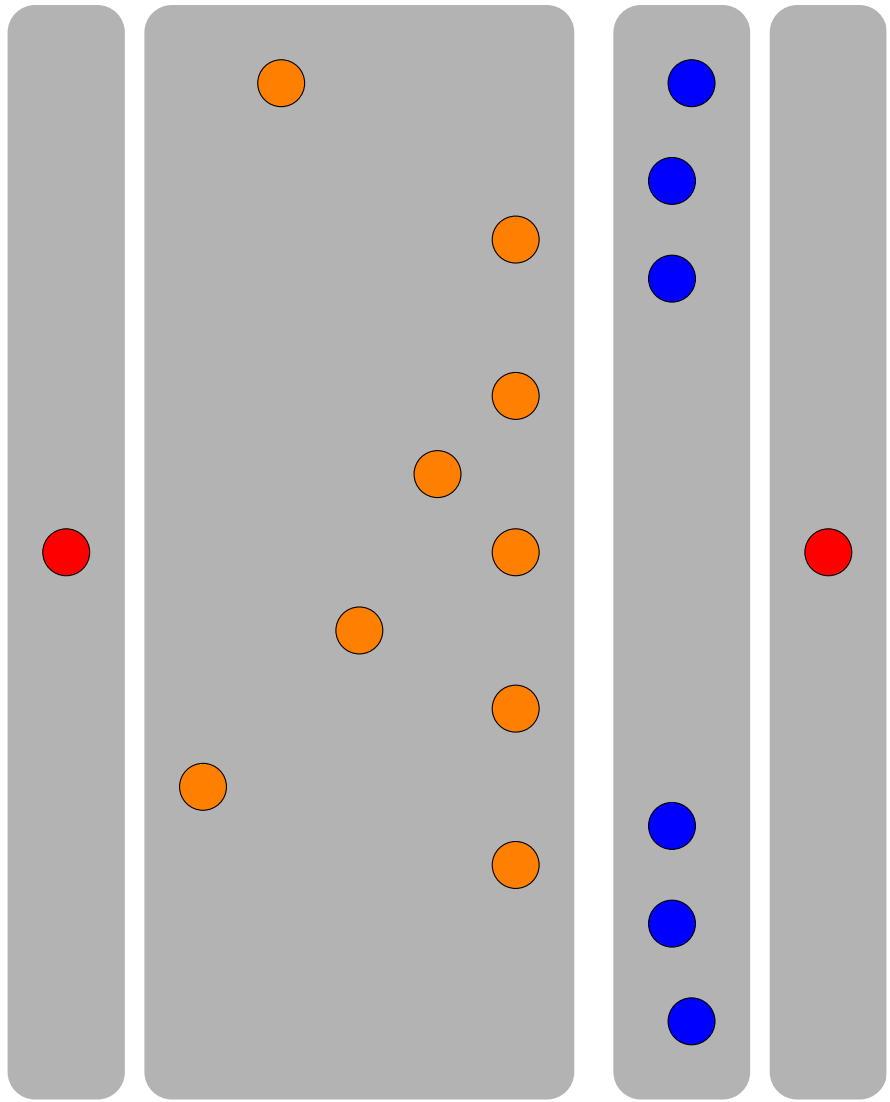
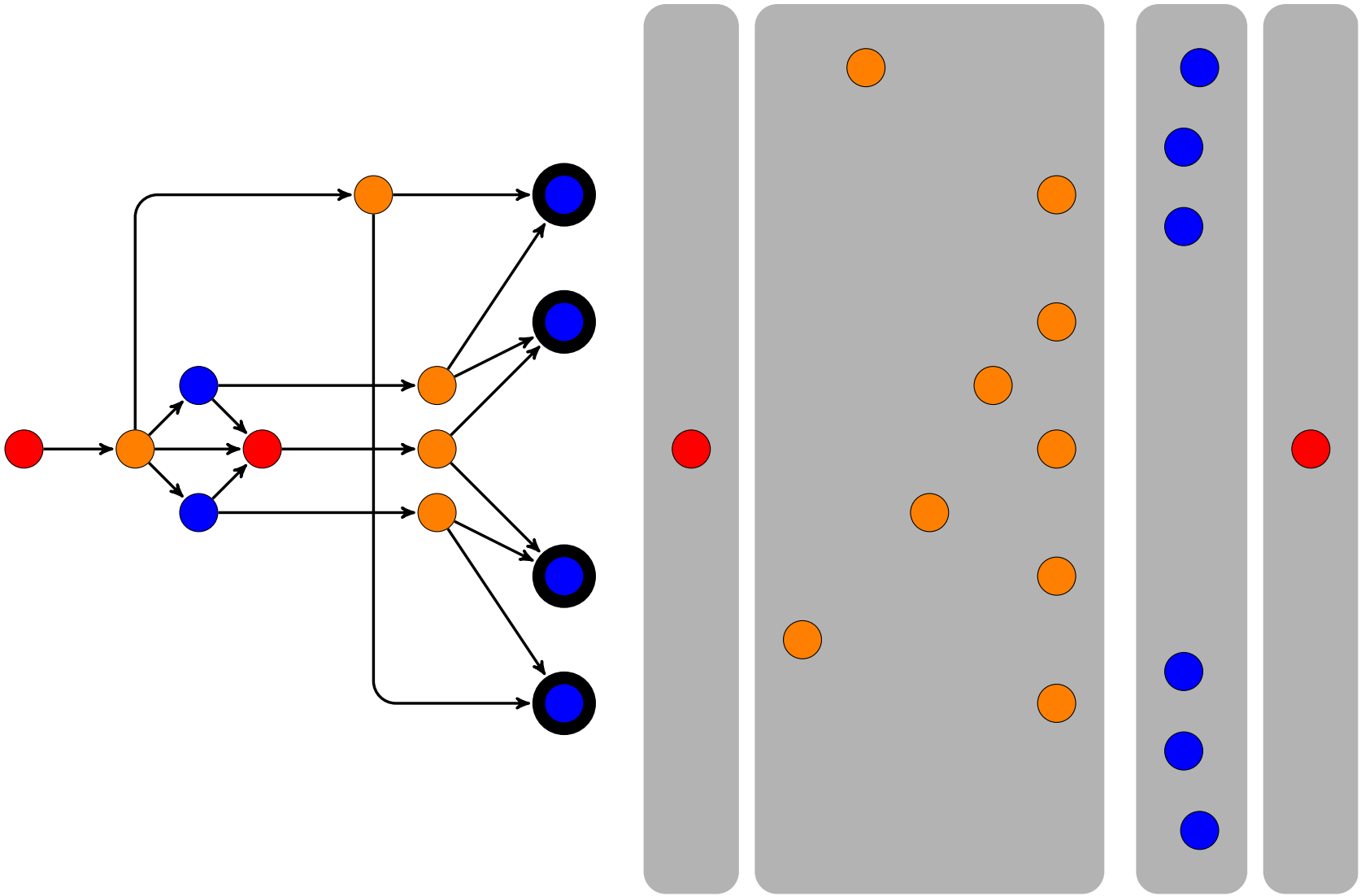




DAG recursion to partition S into stages

Identify and then remove terminal states



DAG recursion to partition S into stages

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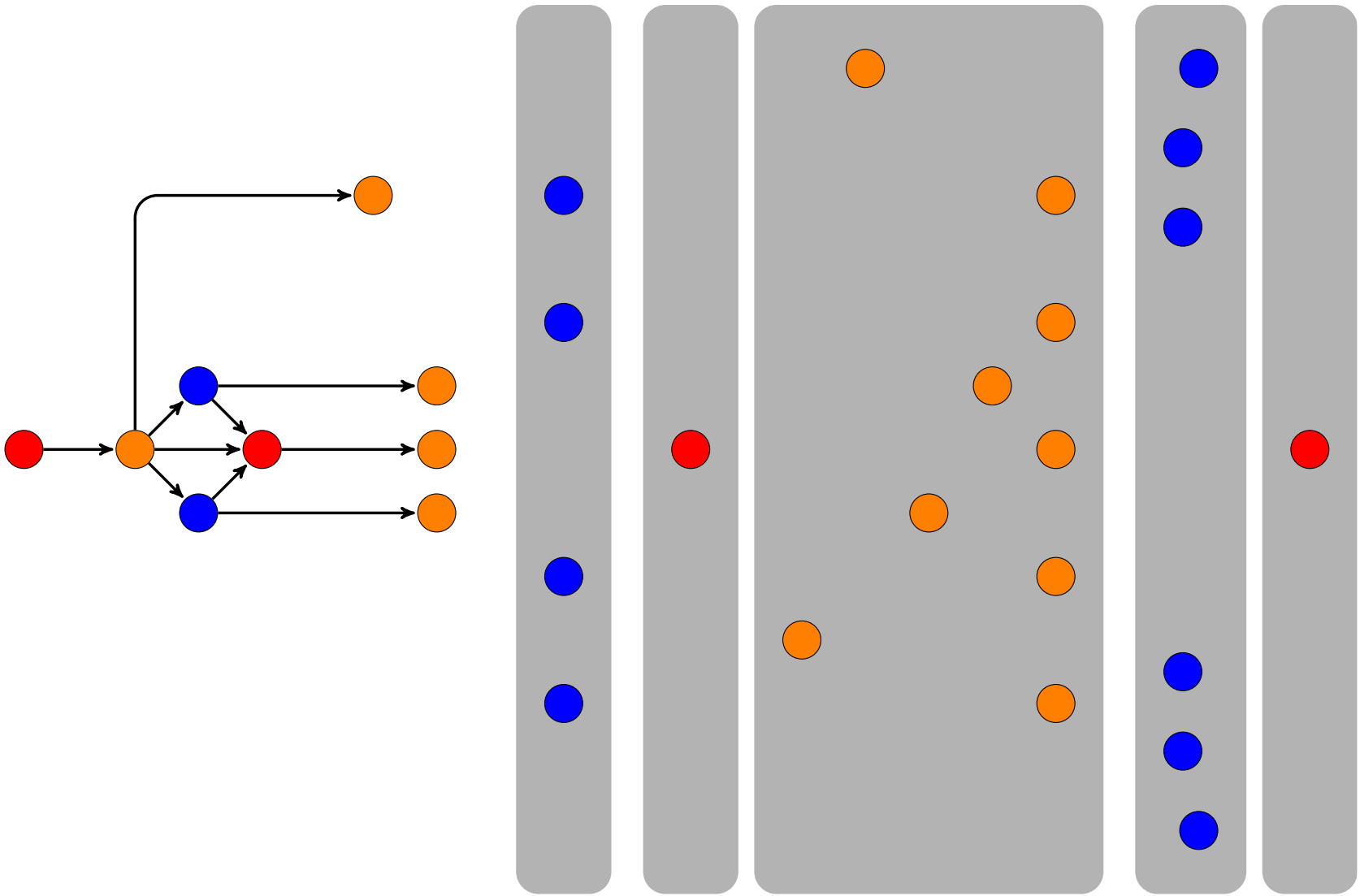

DAG recursion to partition S into stages

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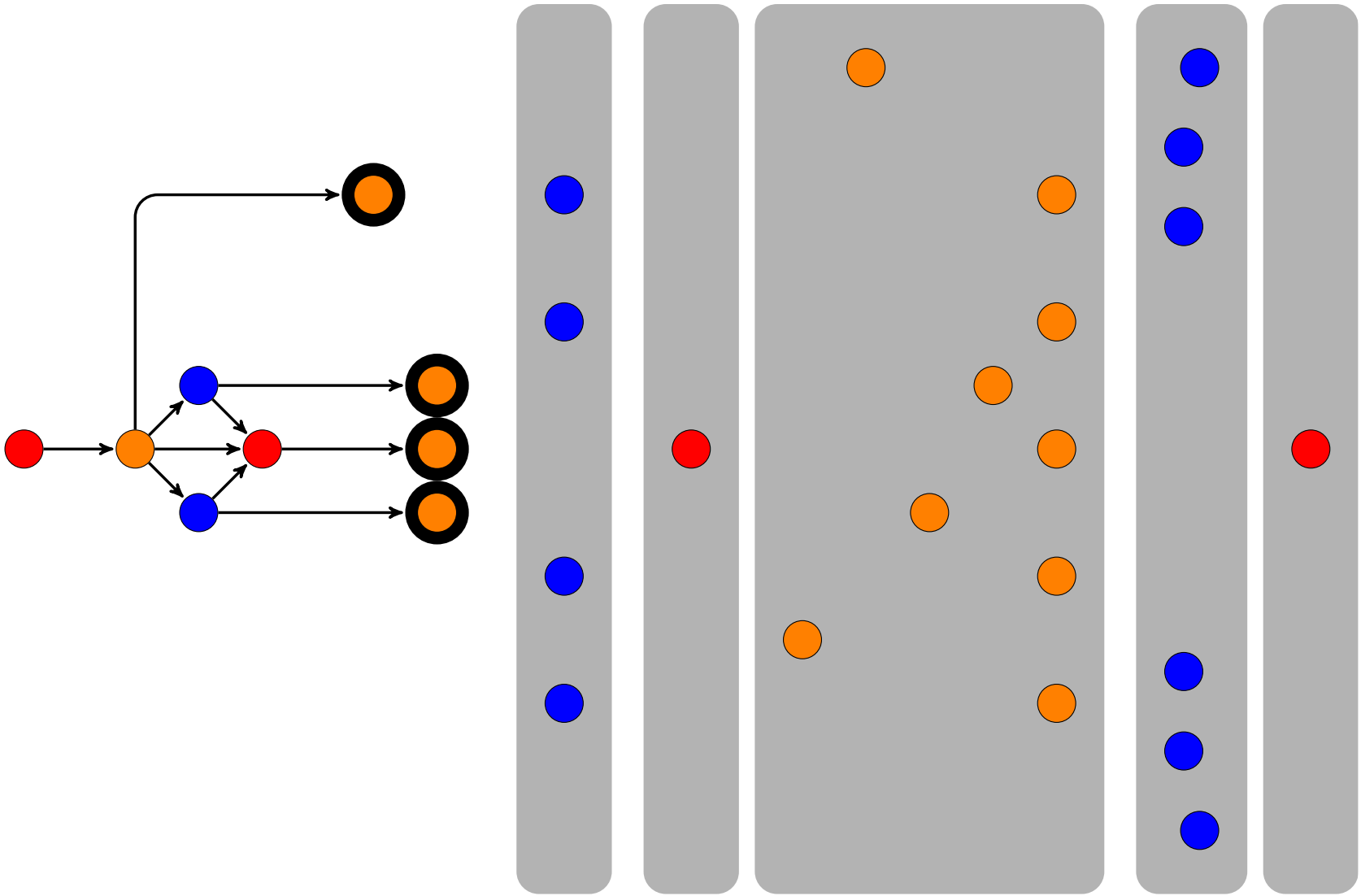

DAG recursion to partition S into stages

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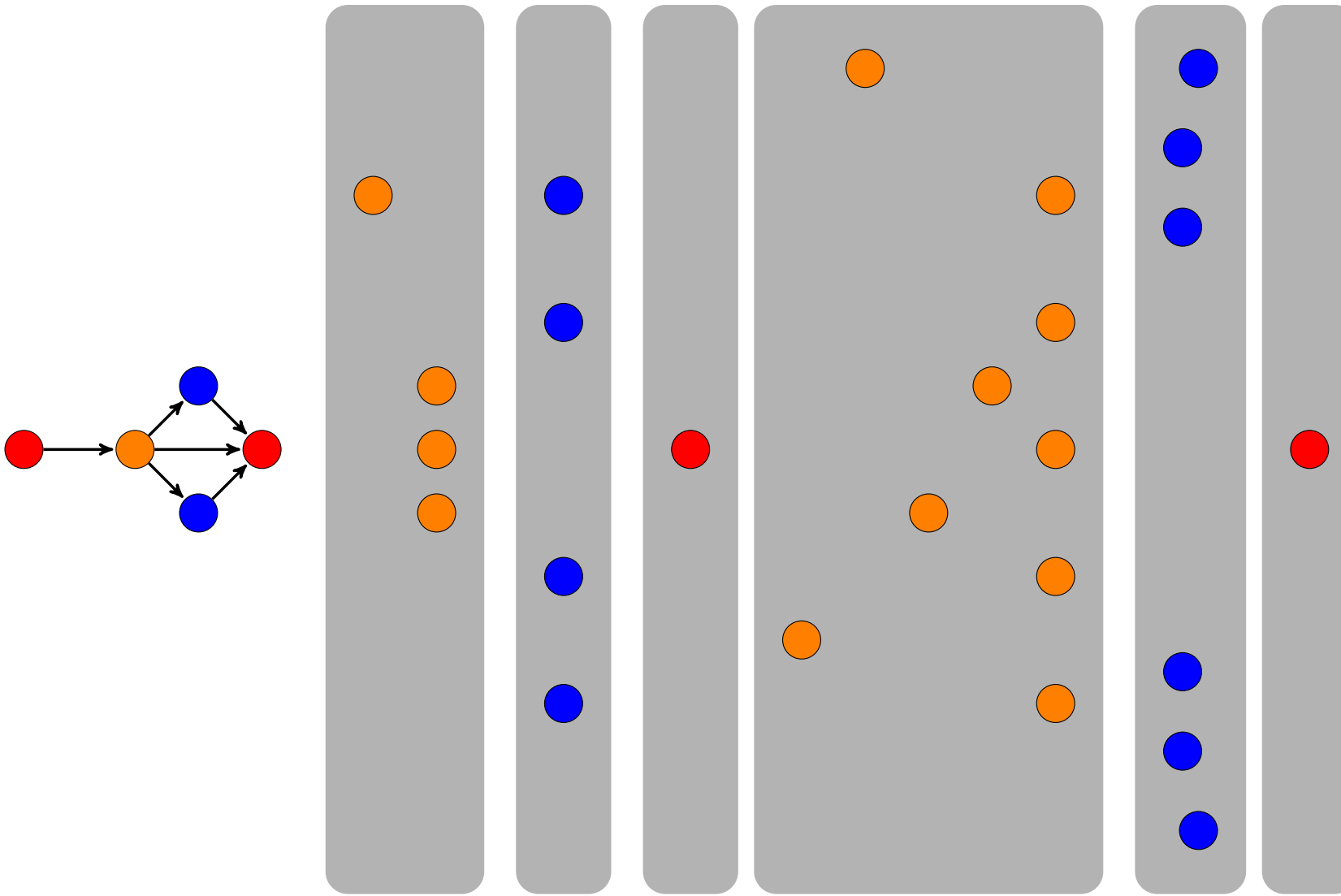
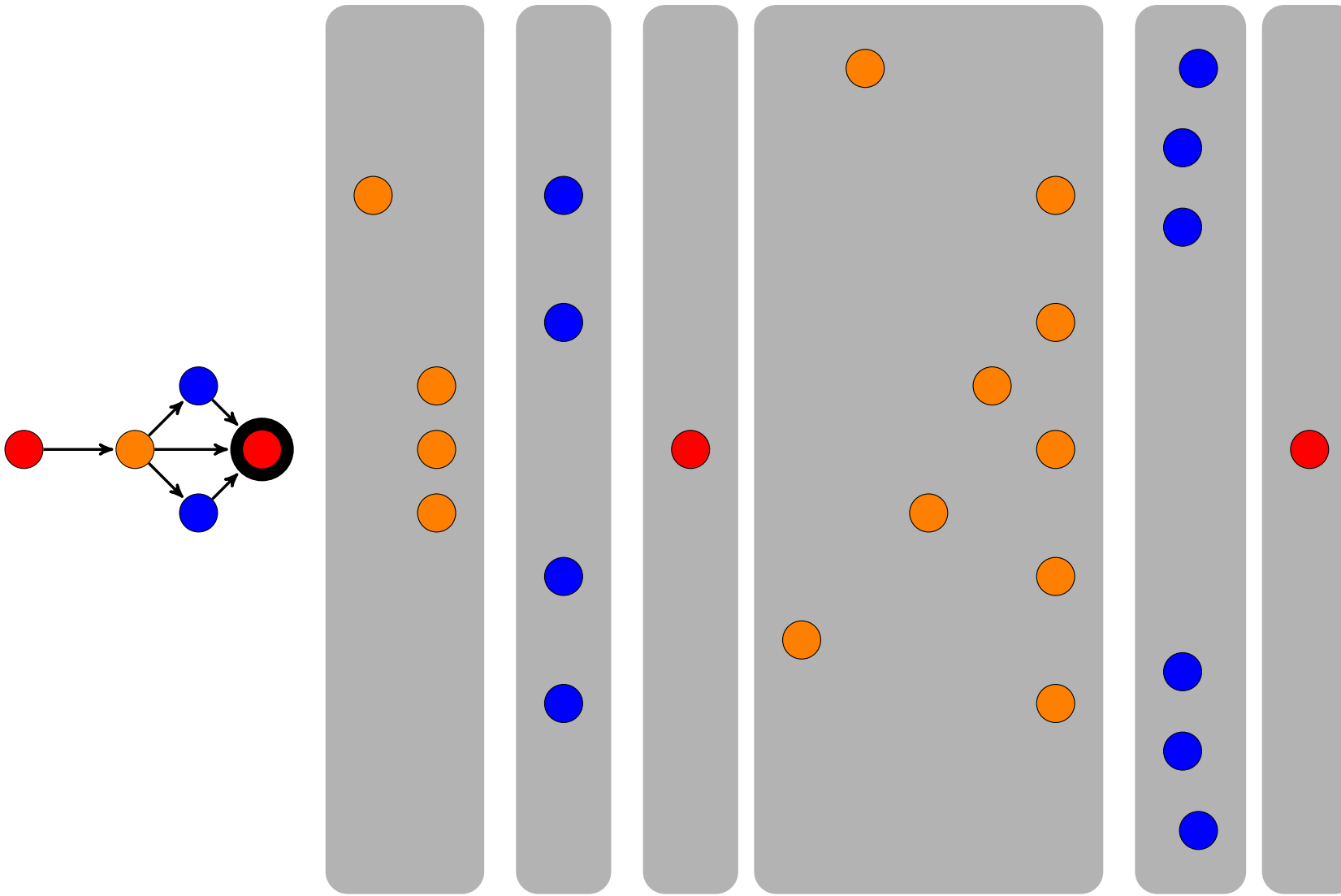


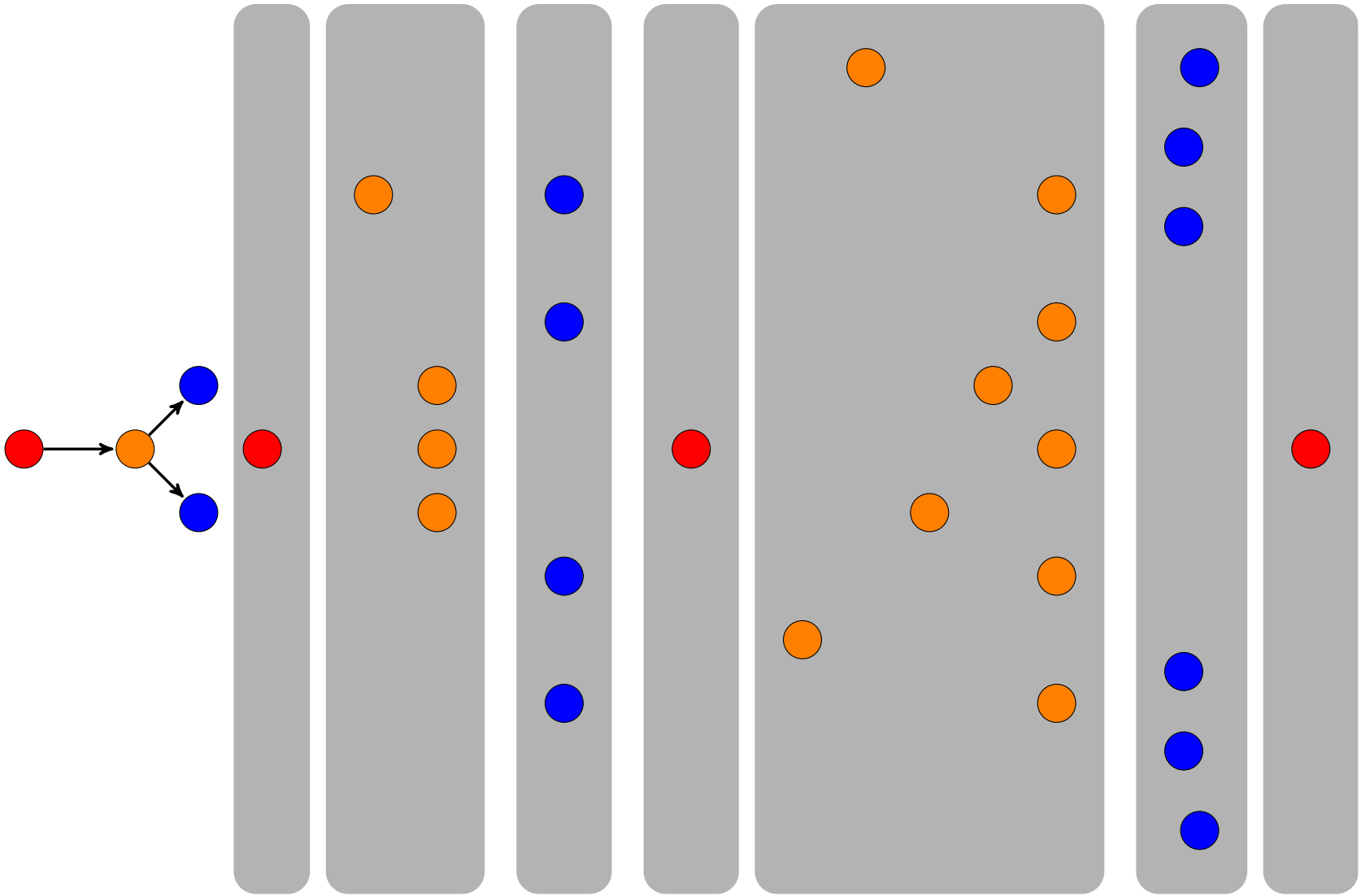
DAG recursion to partition S into stages

Identify and then remove terminal states



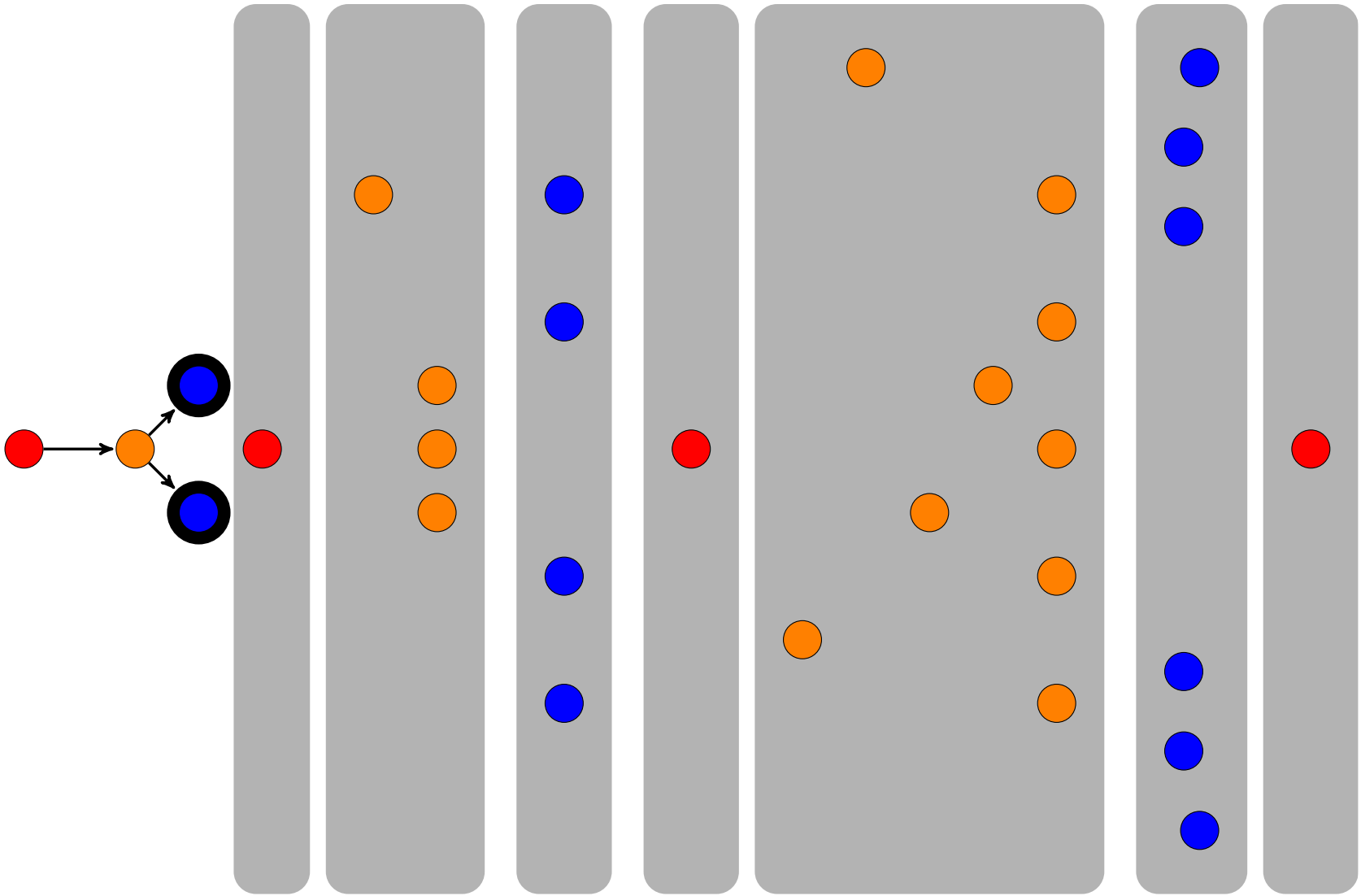
DAG recursion to partition S into stages

Identify and then remove terminal states



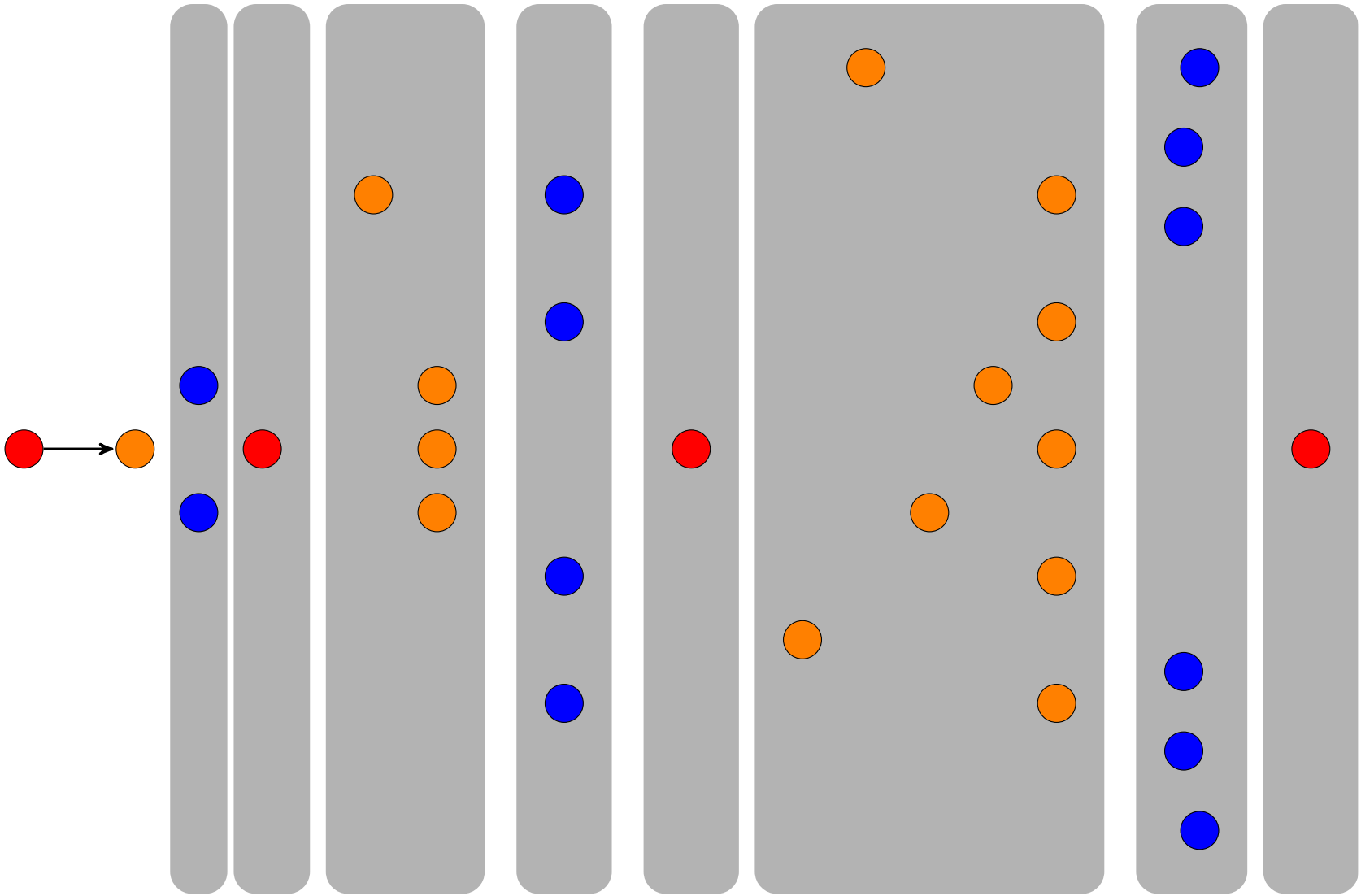
DAG recursion to partition S into stages

Identify and then remove terminal states



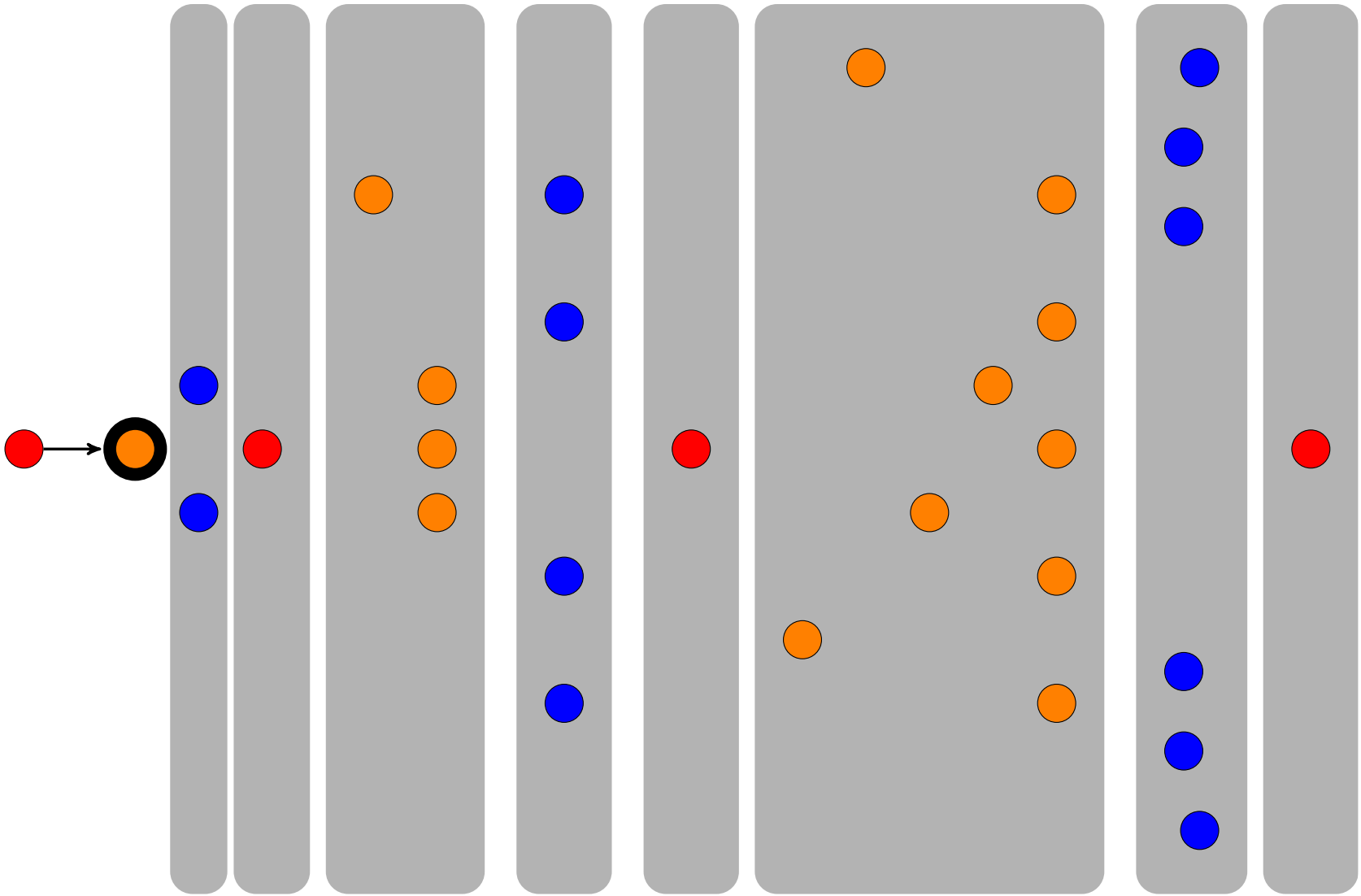
DAG recursion to partition S into stages

Identify and then remove terminal states



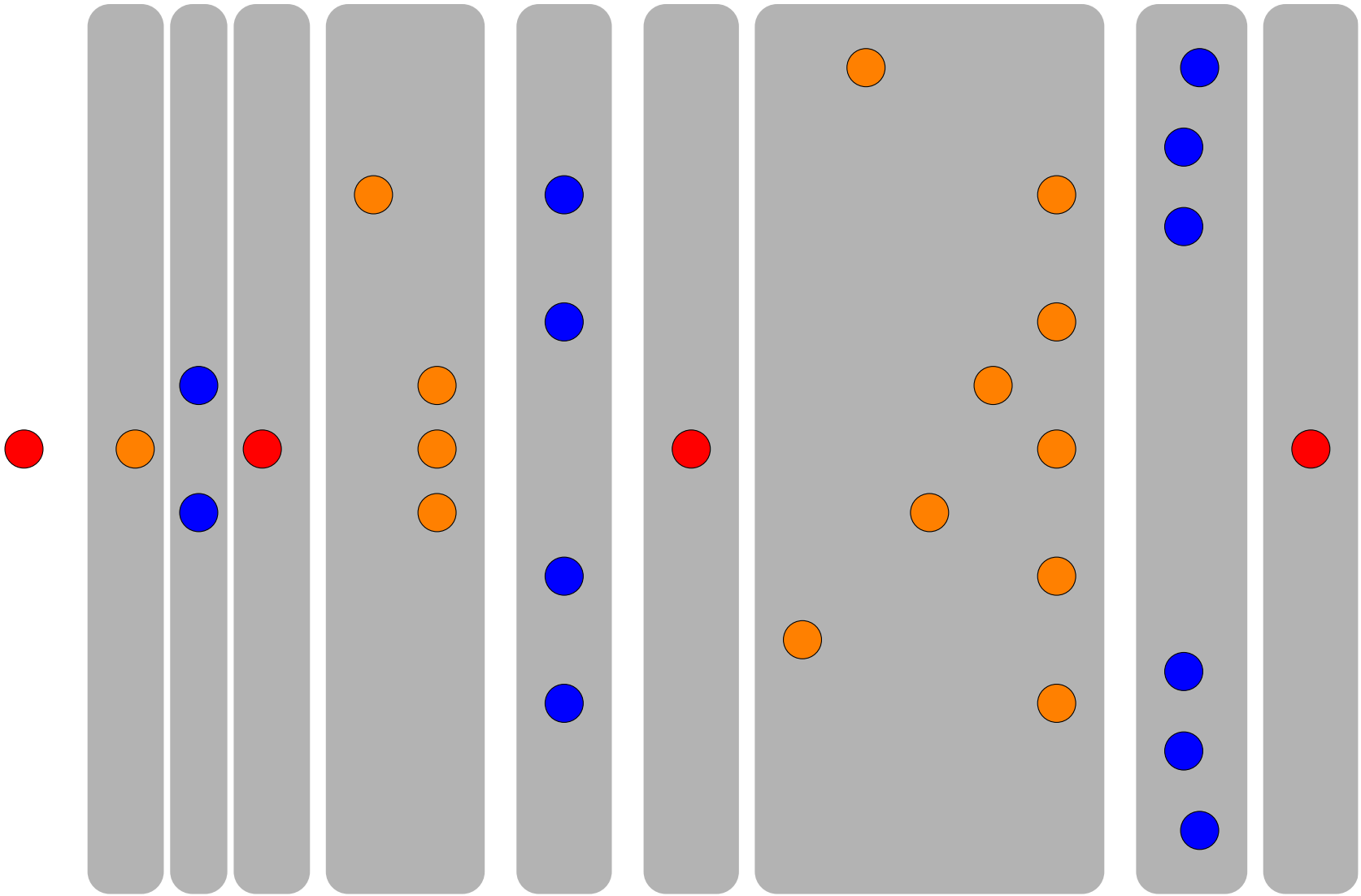
DAG recursion to partition S into stages

Identify and then remove terminal states



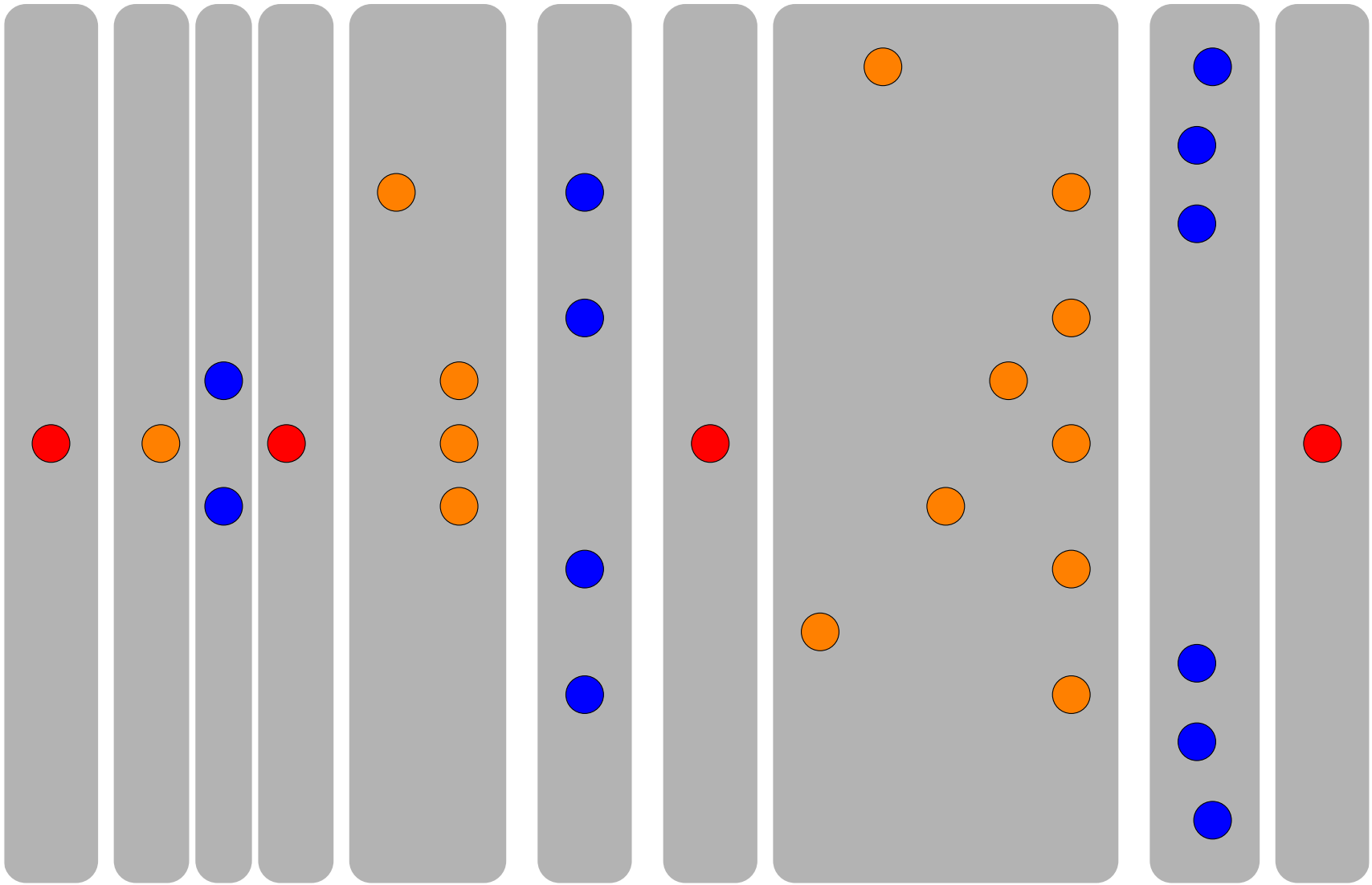
DAG recursion to partition S into stages

Identify and then remove terminal states

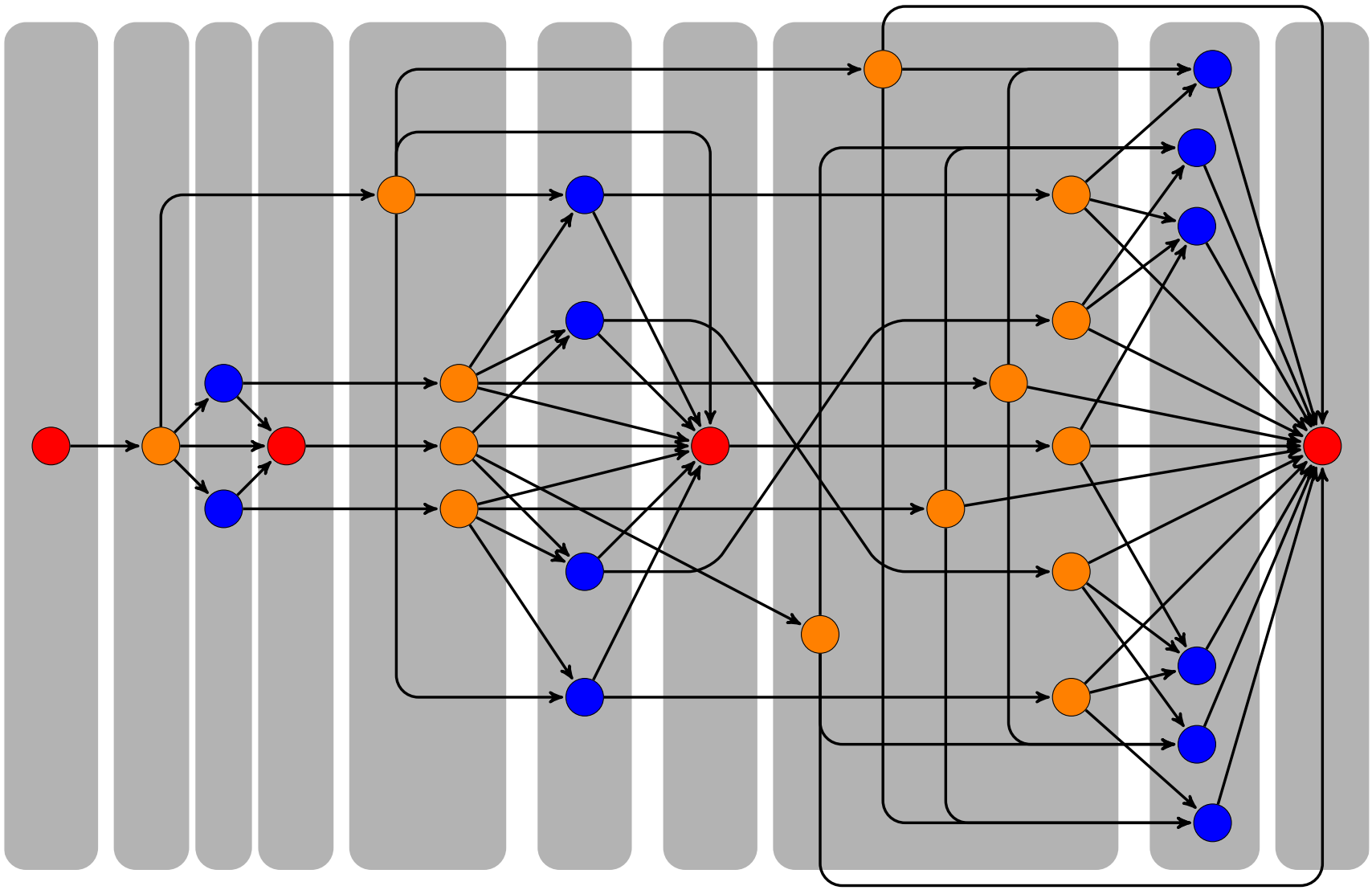


DAG recursion to partition S into stages

Identify and then remove terminal states

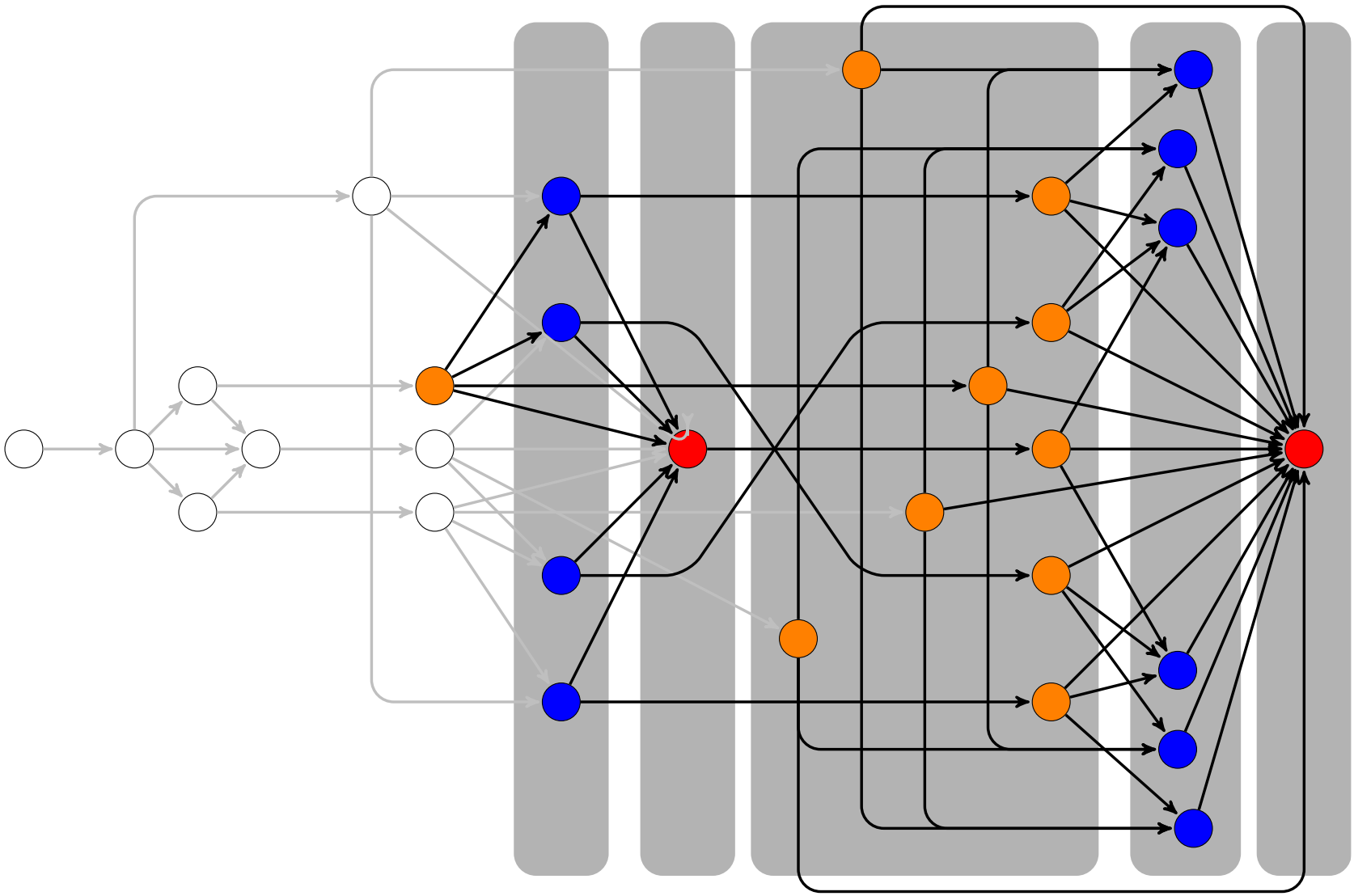


Total order on the set of stages



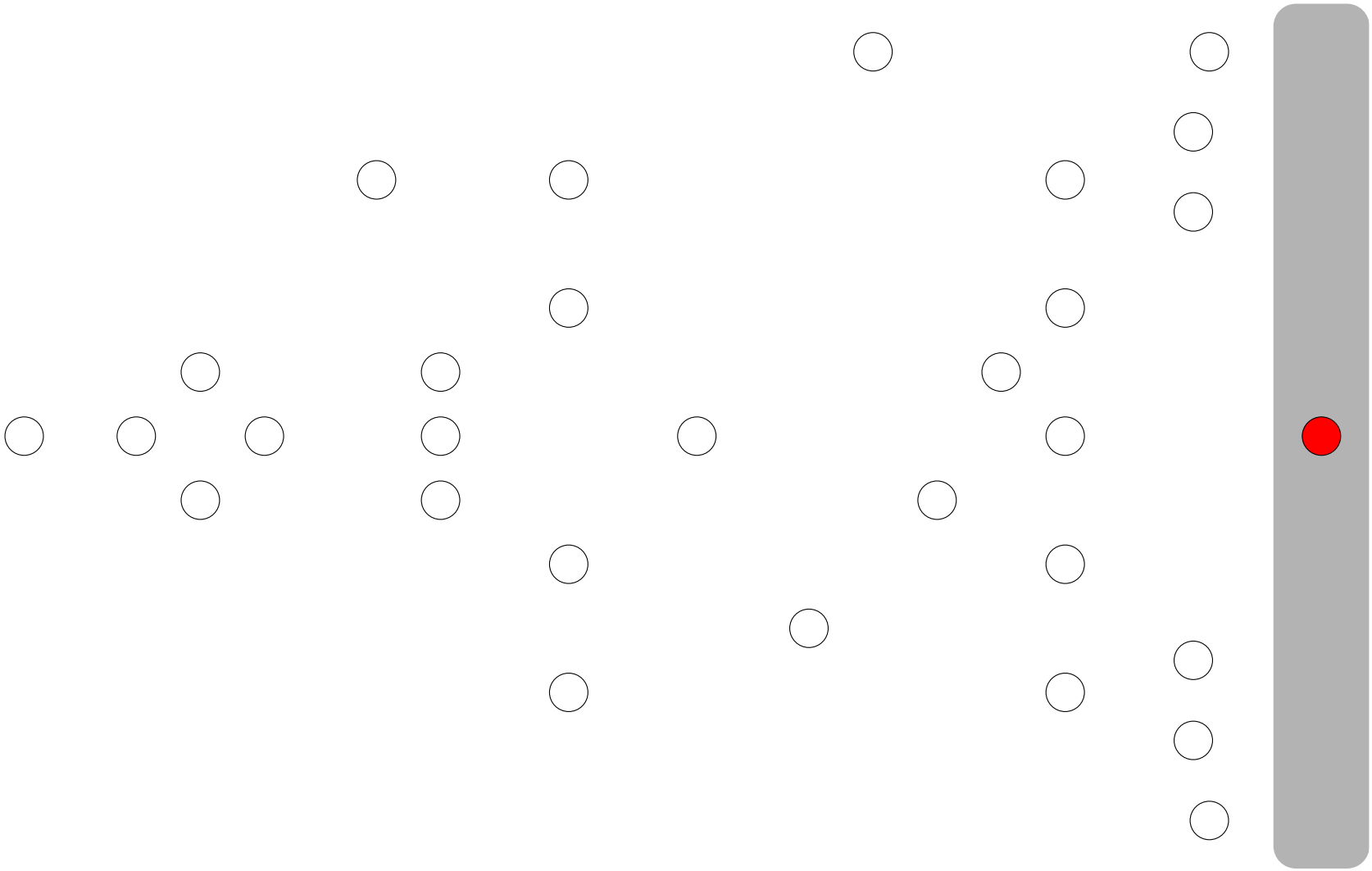
Subgames of DDG and continuation strategies

Subgames and continuation strategies



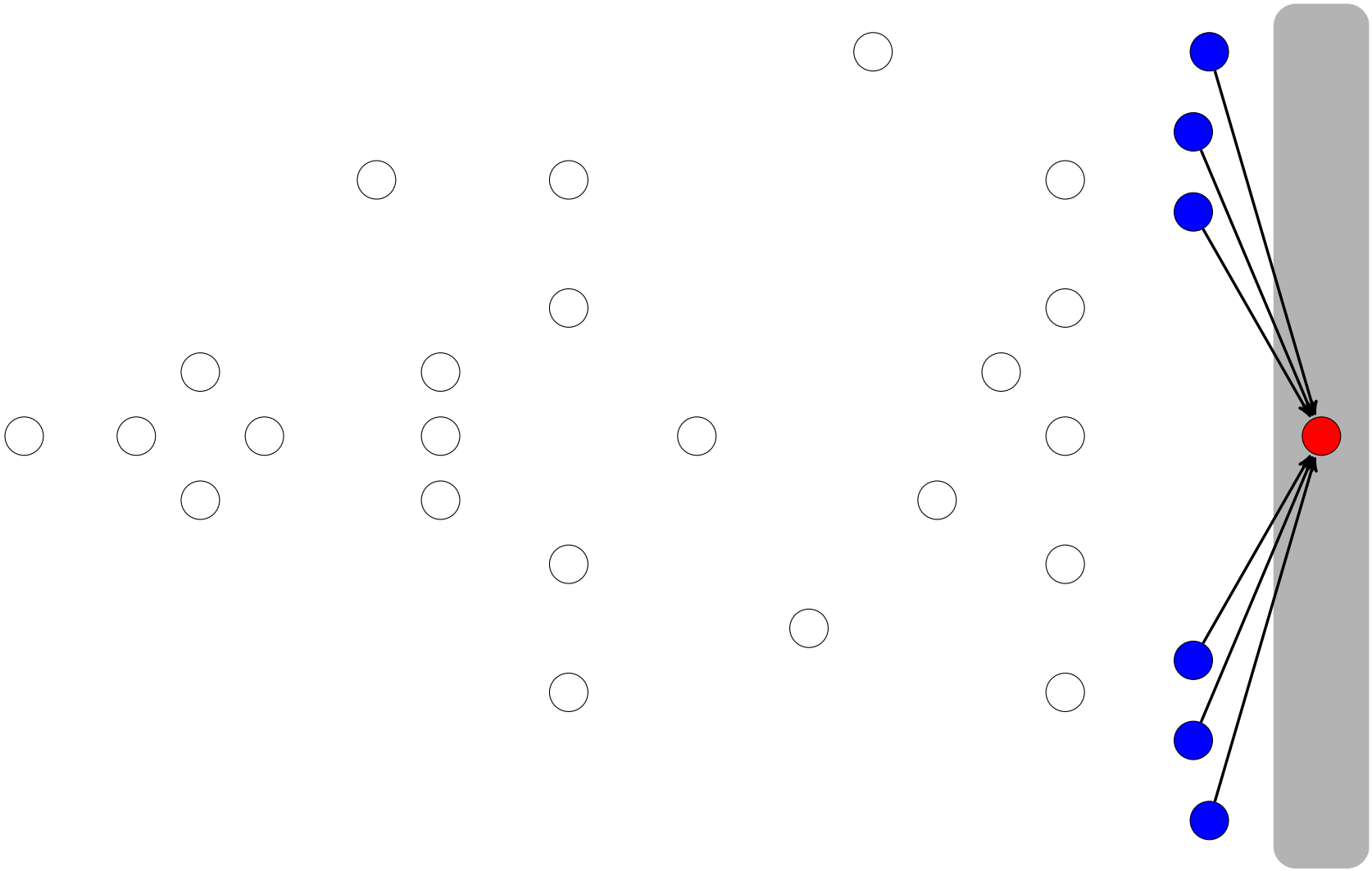
State recursion algorithm

Backward induction on stages of DDG



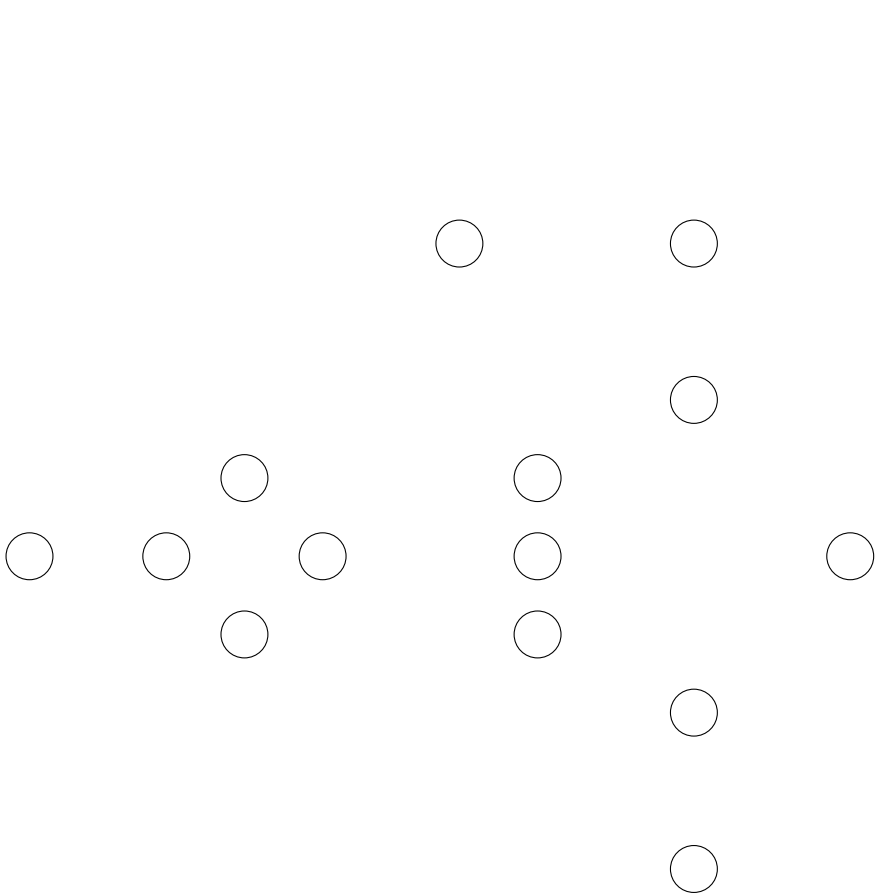
State recursion algorithm

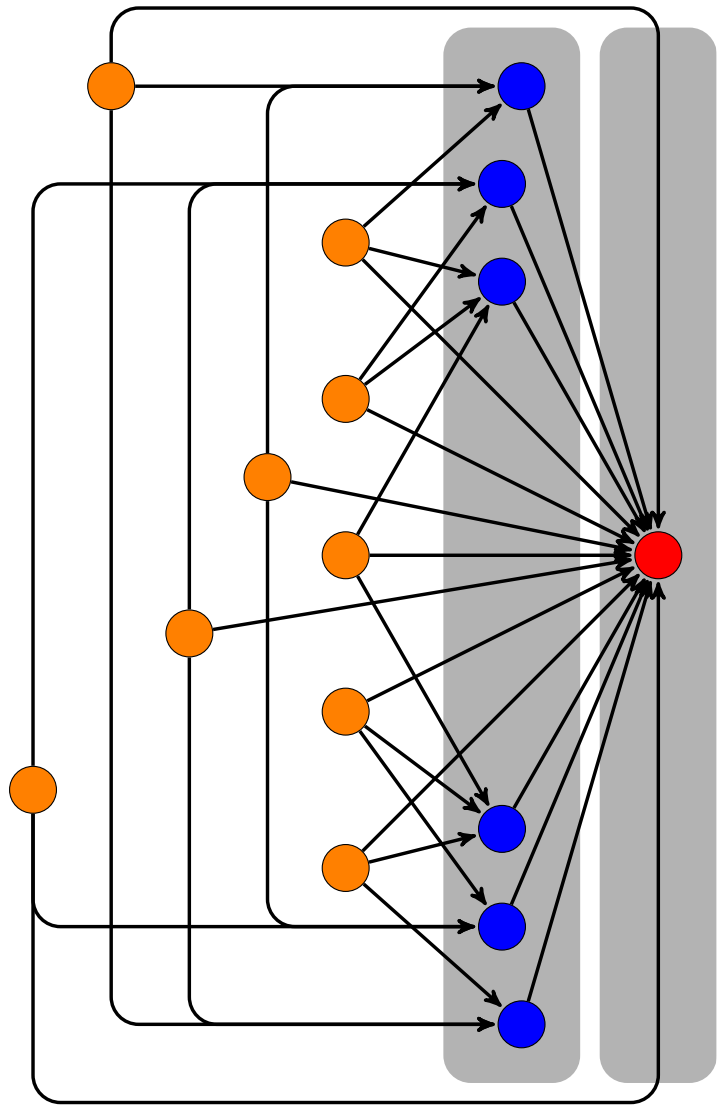
Backward induction on stages of DDG



State recursion algorithm

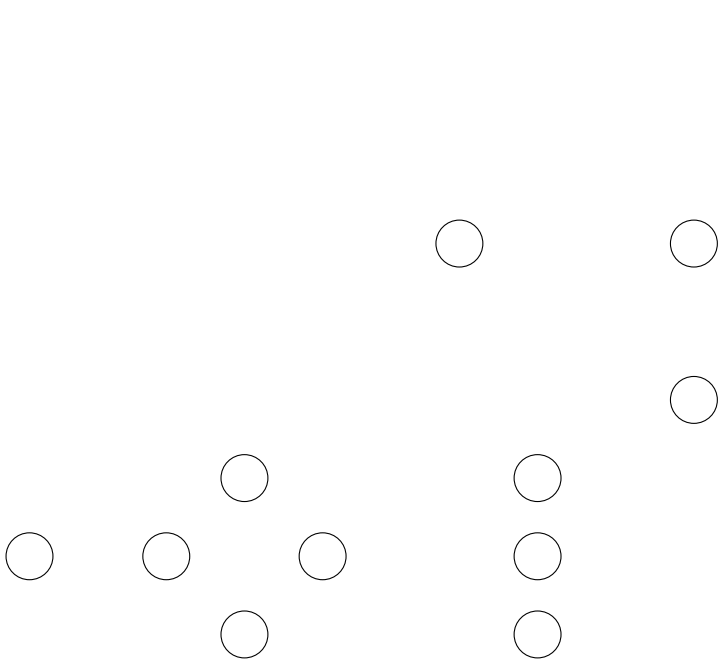
Backward induction on stages of DDG

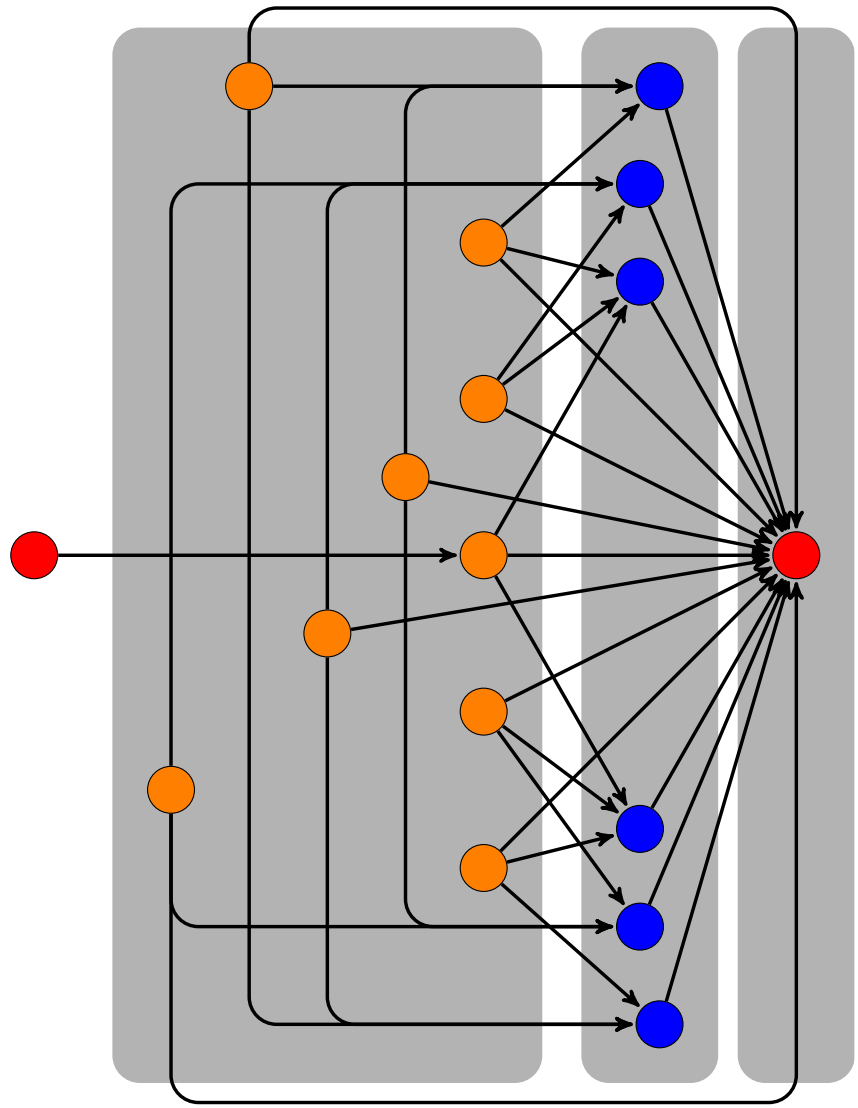




State recursion algorithm

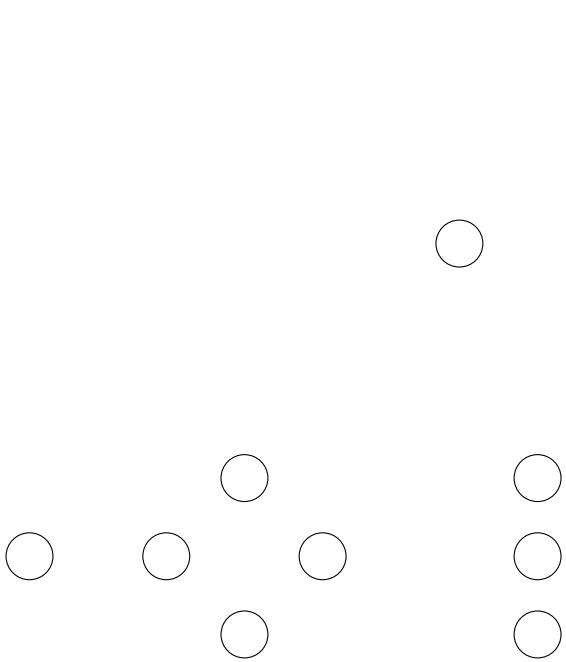
Backward induction on stages of DDG

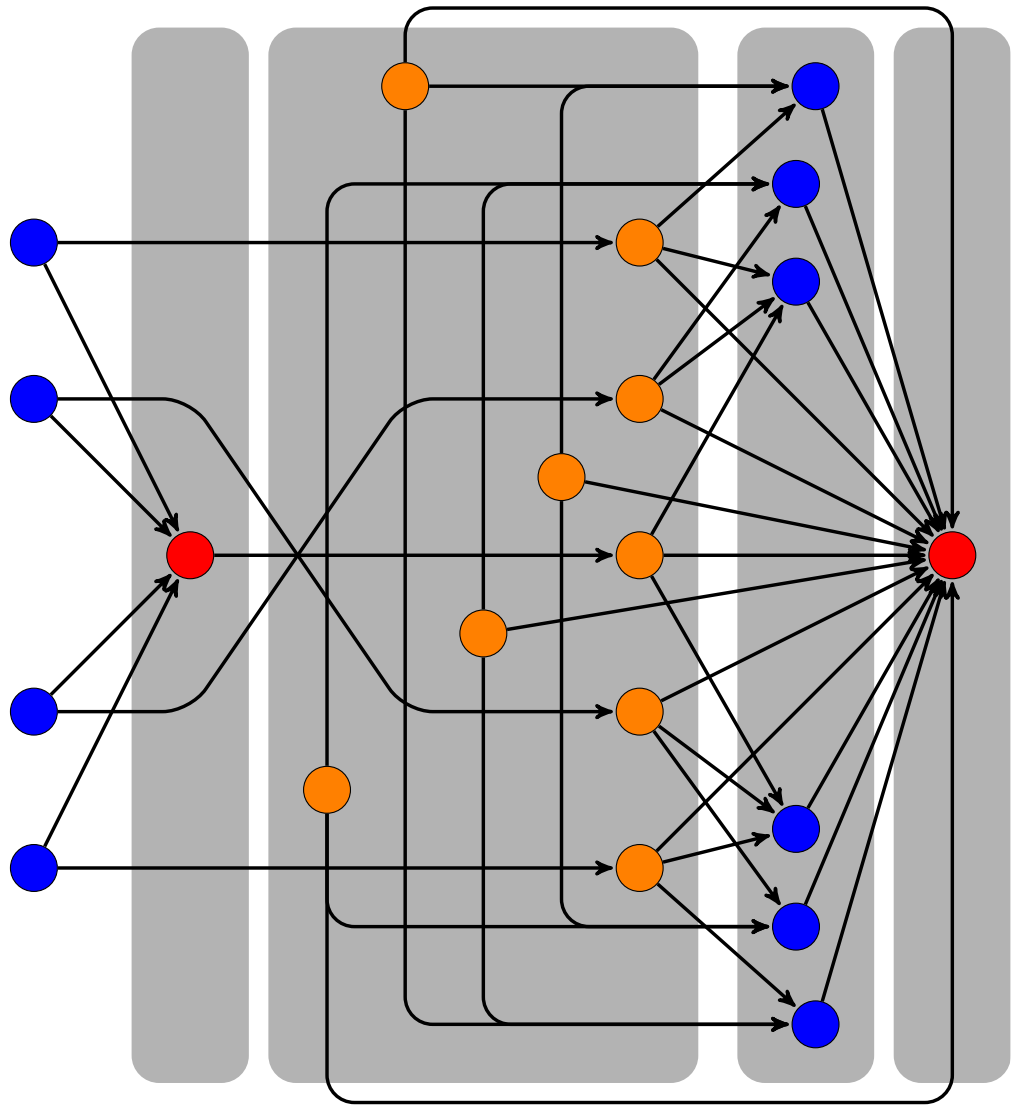




State recursion algorithm

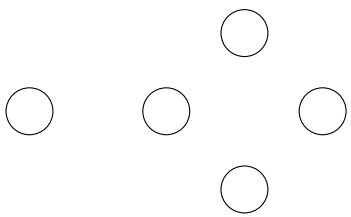
Backward induction on stages of DDG

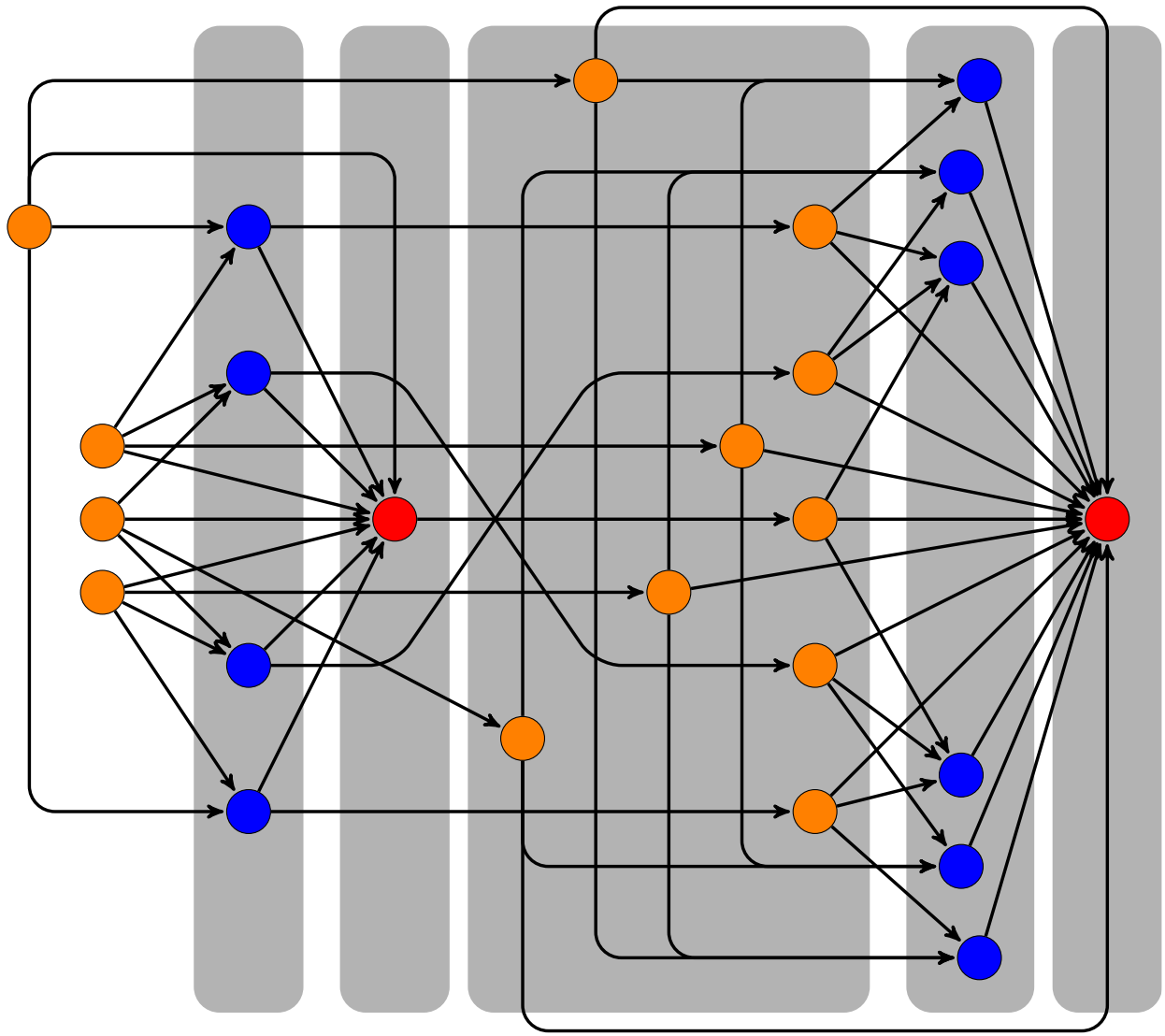




State recursion algorithm

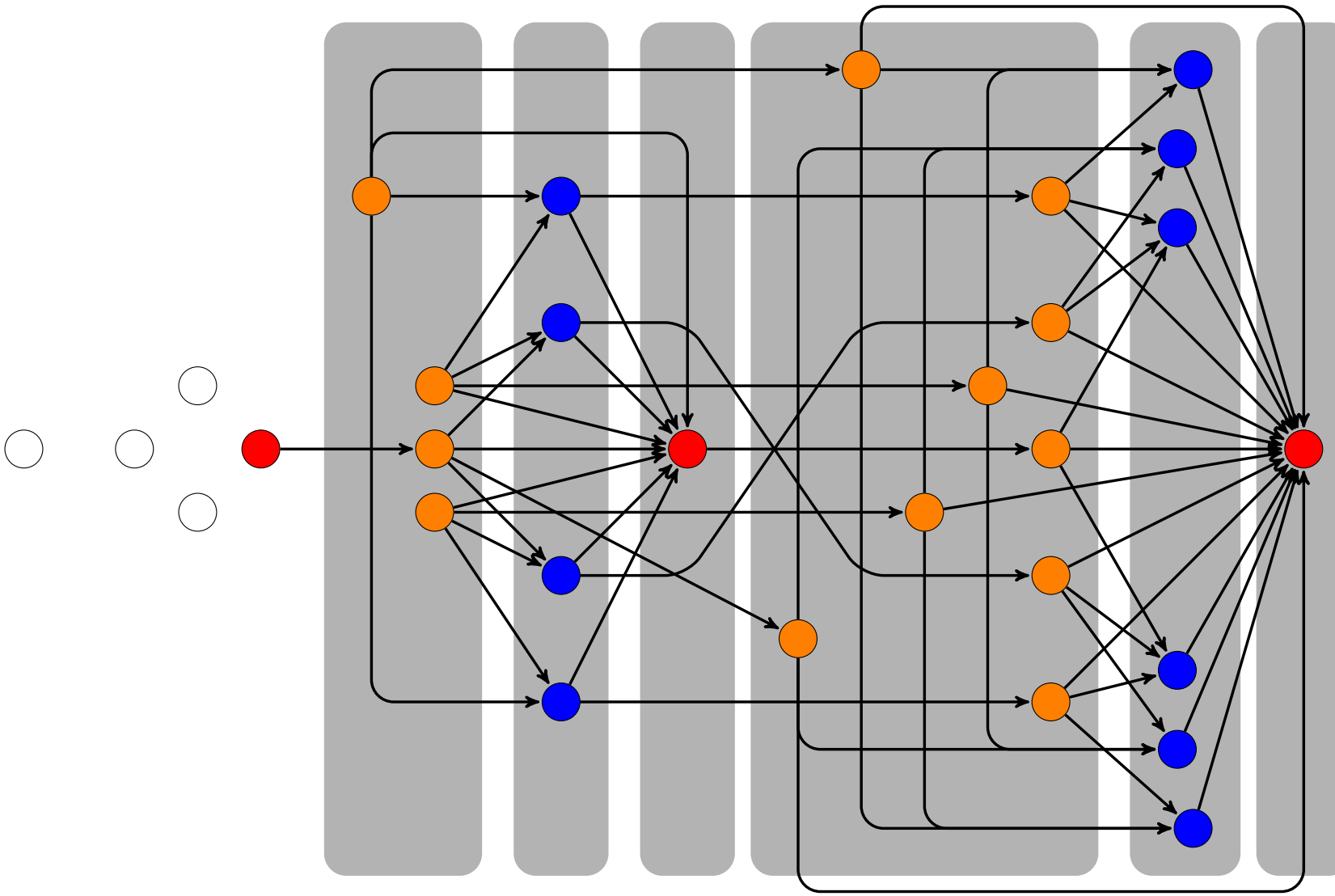
Backward induction on stages of DDG





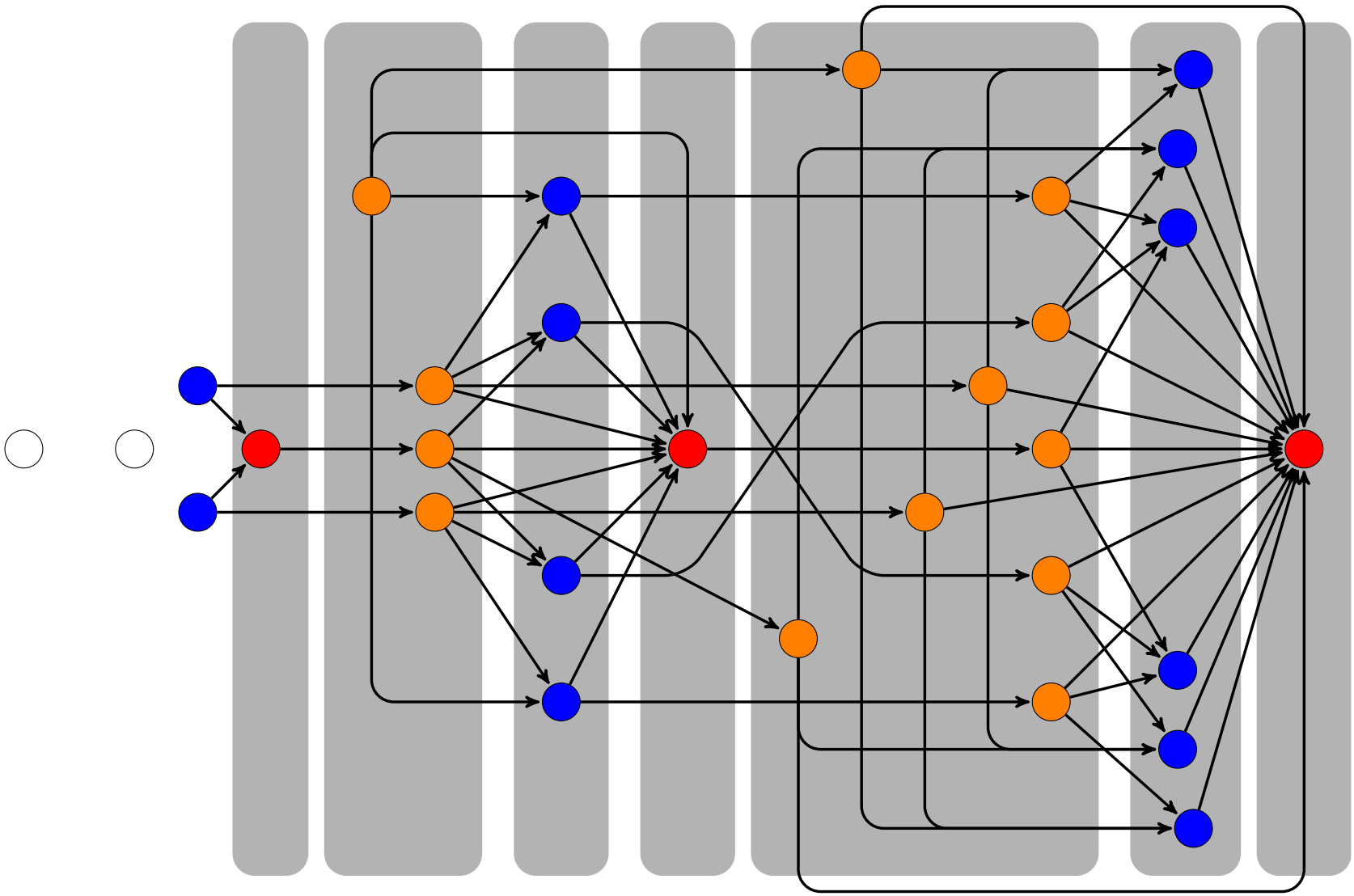
State recursion algorithm

Backward induction on stages of DDG



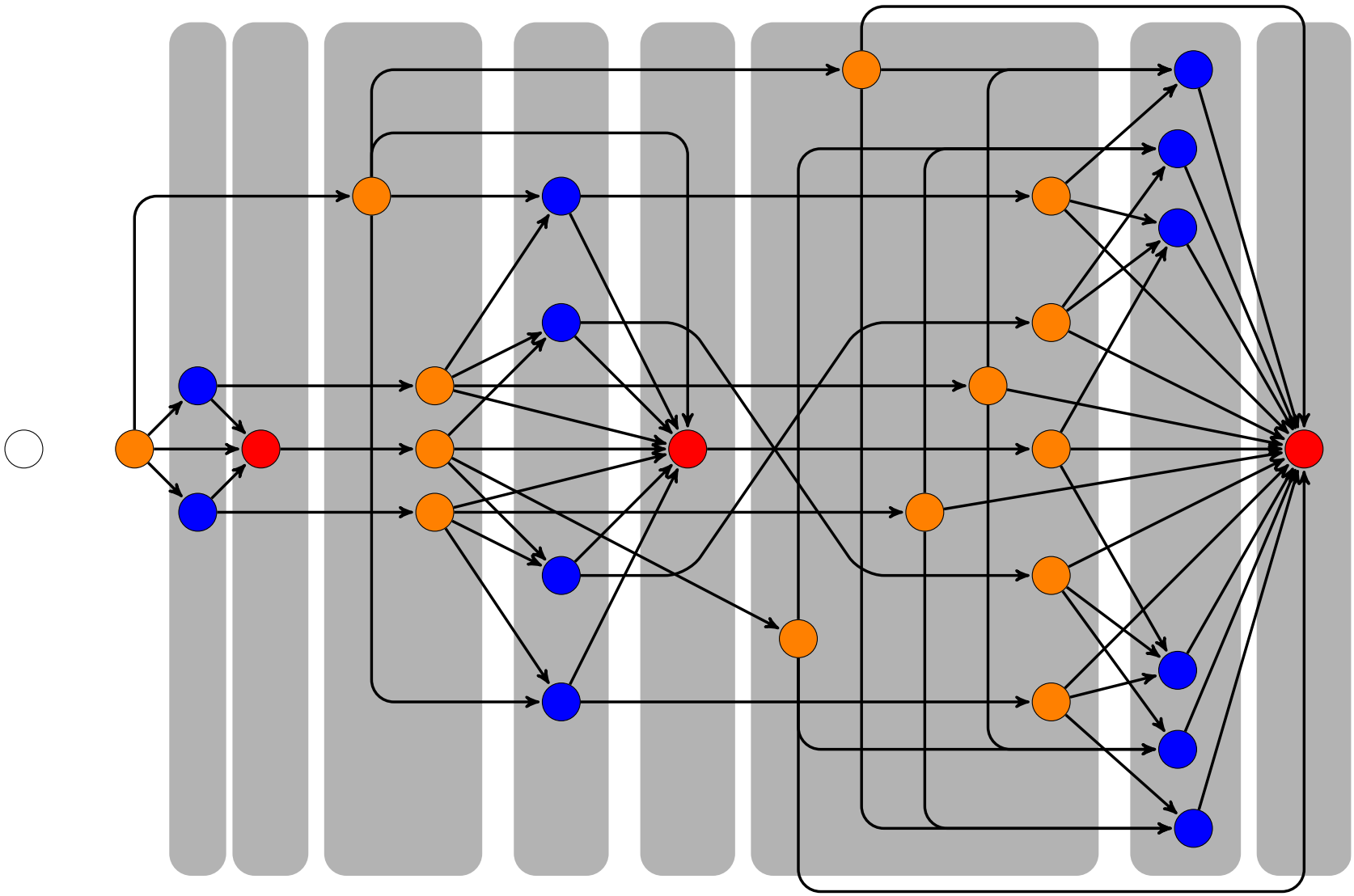
State recursion algorithm

Backward induction on stages of DDG



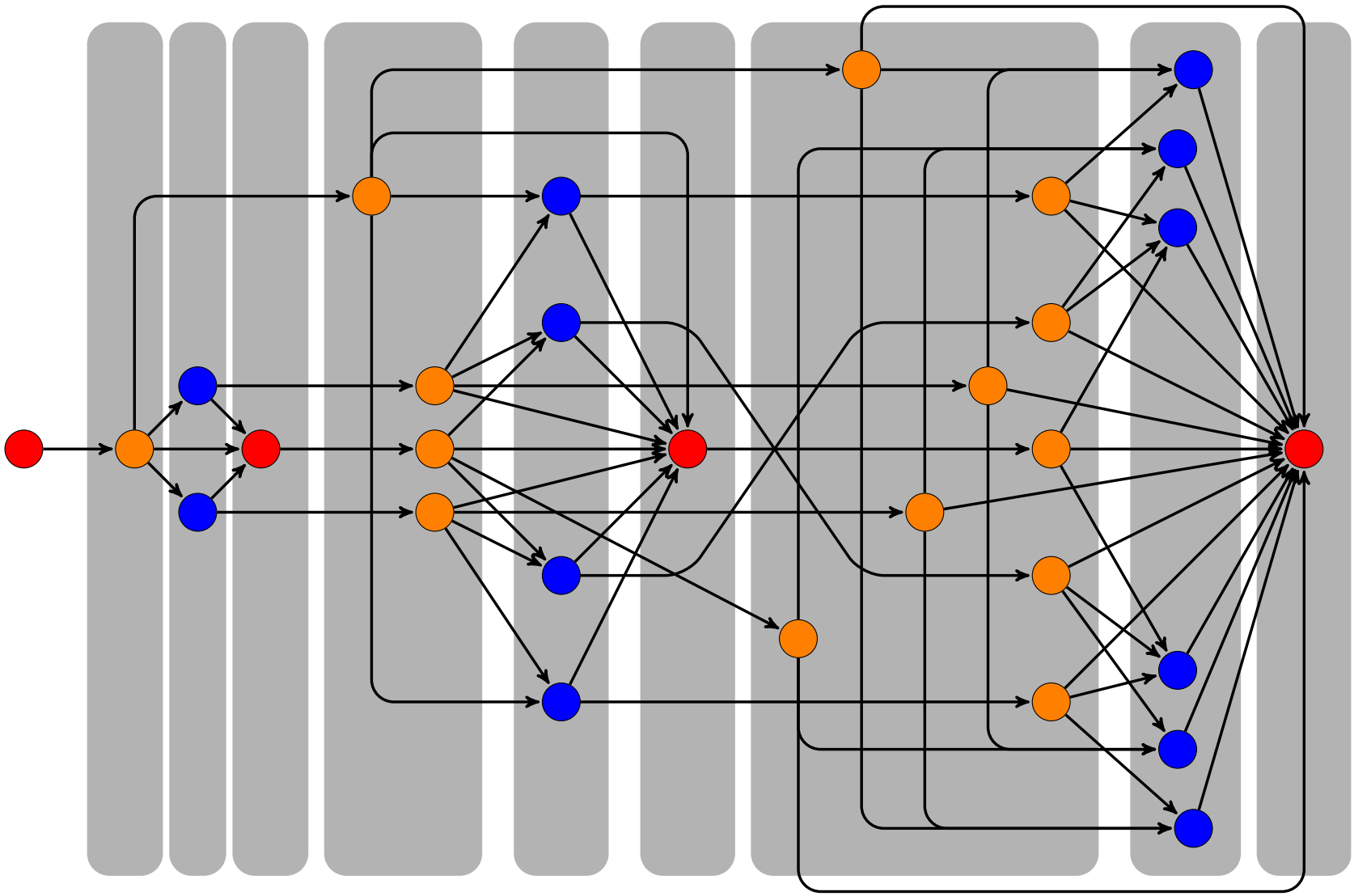
State recursion algorithm

Backward induction on stages of DDG



State recursion algorithm

Backward induction on stages of DDG



State Recursion versus Backward Induction

- State recursion – generalization of backward induction
- Runs on state space instead of time periods
- Time (t) evolves as $t \to t+1$ with probability 1
- For stages of state space (τ) transitions are stochastic and not necessarily sequential
- Yet, probability of going $\tau \to \tau'$ is zero when $\tau' < \tau$
- With multiplicity, state recursion is performed conditional of a particular equilibrium selection rule (ESR)

State Recursion and Model Transitions

- Bellman equation in integrated/expected value function form

$$V_i^\sigma(x) = \int \max_{a_i \in A} \{v_i^\sigma(a_i, x) + \eta \varepsilon_i[a_i]\} g_i(d\varepsilon_i) = \eta \log \sum_{a_i \in A} \exp\left(\frac{v_i^\sigma(a_i, x)}{\eta}\right) + \eta\gamma$$

  - $V_i^\sigma(x)$ – value function integrated over taste shocks ε_i of player i
  - $v_i^\sigma(x)$ – choice-specific value function for action a_i

- Taking expectation over choices with choice probabilities $P_i^\sigma(a_i|x)$

$$V_i^\sigma(x) = \sum_{a_i \in A} P_i^\sigma(a_i|x) \left[\pi_i^\sigma(a_i, x) + e_i^\sigma(a_i, x) + \beta \sum_{x' \in X} V_i^\sigma(x') f_i^\sigma(x'|a_i, x)\right]$$

  - $e_i^\sigma(x)$ – expectation of taste shock ε_i conditional on action a_i chosen
  - $f_i^\sigma(x'|a_i, x)$ – transition probability under σ and action a_i by player i

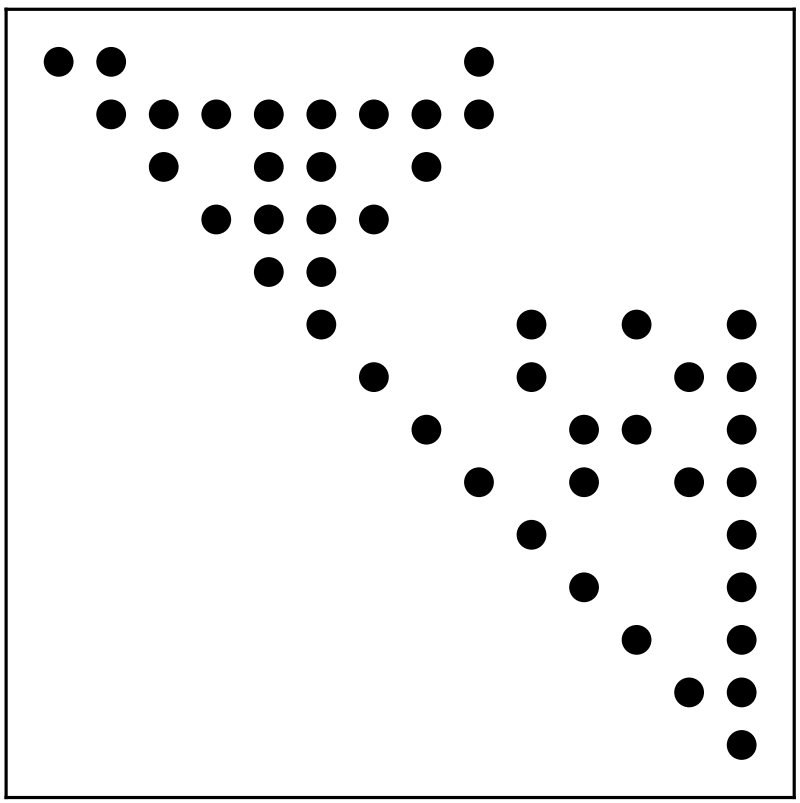
- Stacking over the points of state space, in matrix form

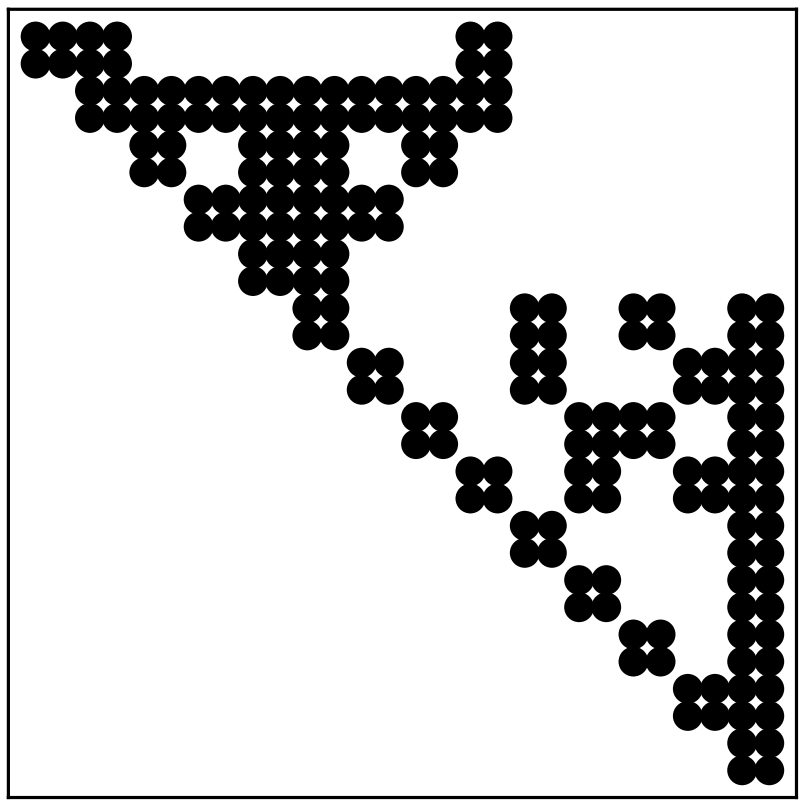
$$V_i^\sigma = (I - \beta F)^{-1} \sum_{a_i \in A} P_i^\sigma(a_i) * [\pi_i^\sigma(a_i) + e_i^\sigma(a_i)],$$

Upper block-triangularity of the transision matrix F is directionality

F in simultaneous and alternating move leapfrogging

- Simultaneous move: 14 points in the state space for (c_1, c_2, c_3)
- Alternating move: 28 points with $m \in \{1, 2\} \to 2 \times 2$ blocks



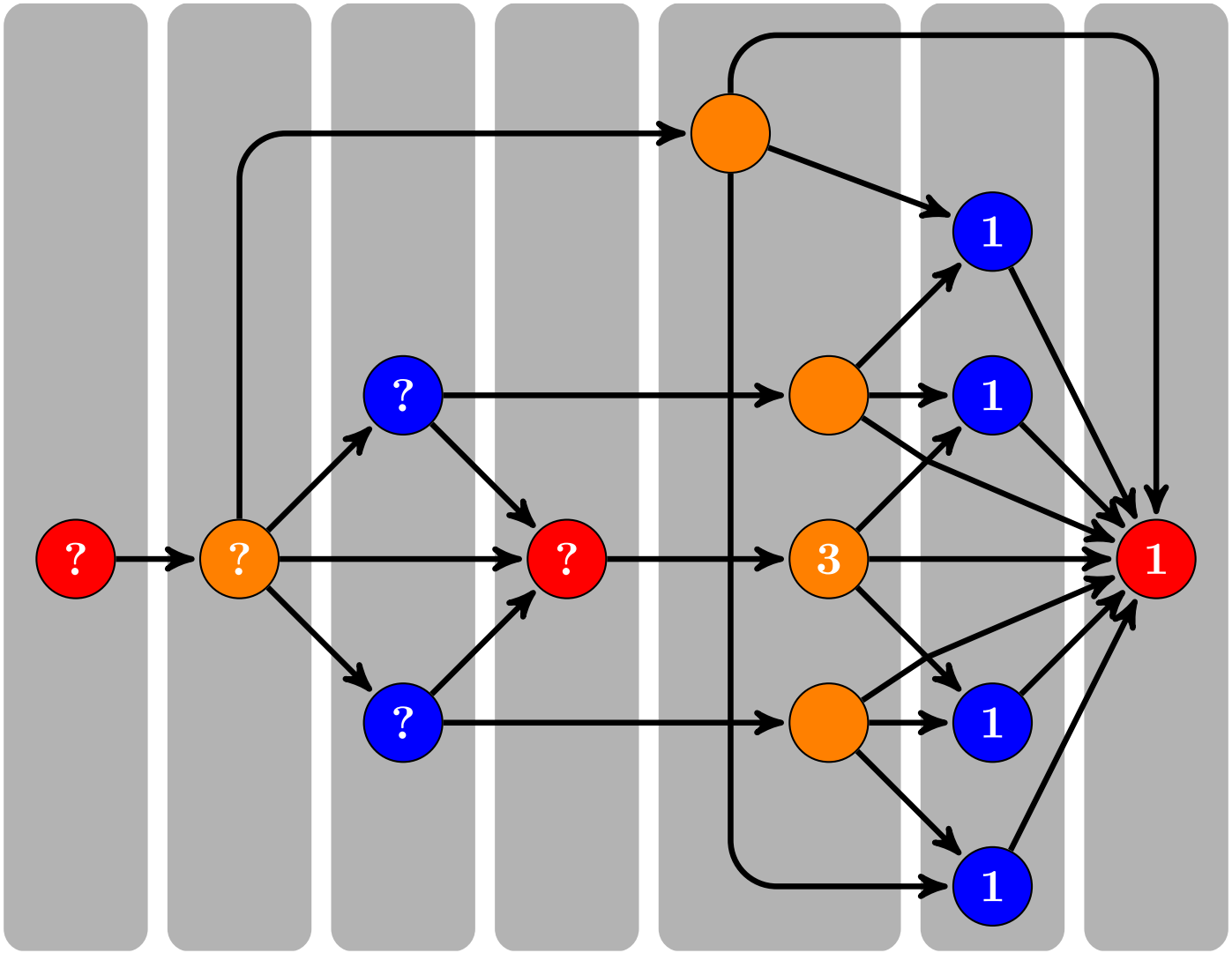


ROAD MAP

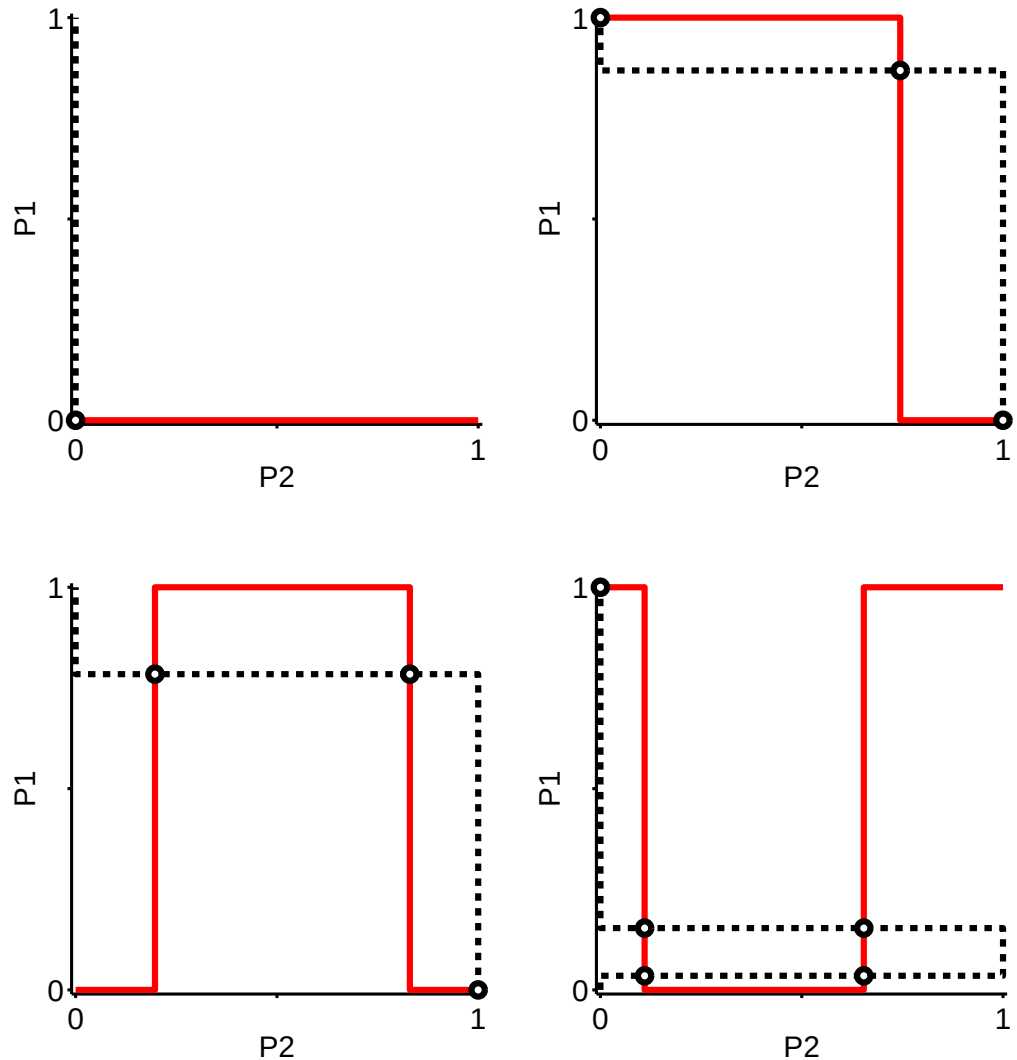
1. Collusion of Australian corrugated fibre packaging (CFP) producers
   - Collusion between Amcor and Visy
   - Bertrand pricing and investment game
   - Solution concept: Markov perfect equilibrium (MPE)
2. Experiment with the model
3. State recursion algorithm
   - Theory of directional dynamic games (DDGs)
4. Recursive lexicographical search (RLS) algorithm
5. Full solution for the leapfrogging game
6. Structural estimation of directional dynamic games with Nested RLS method

Multiplicity of stage equibiria

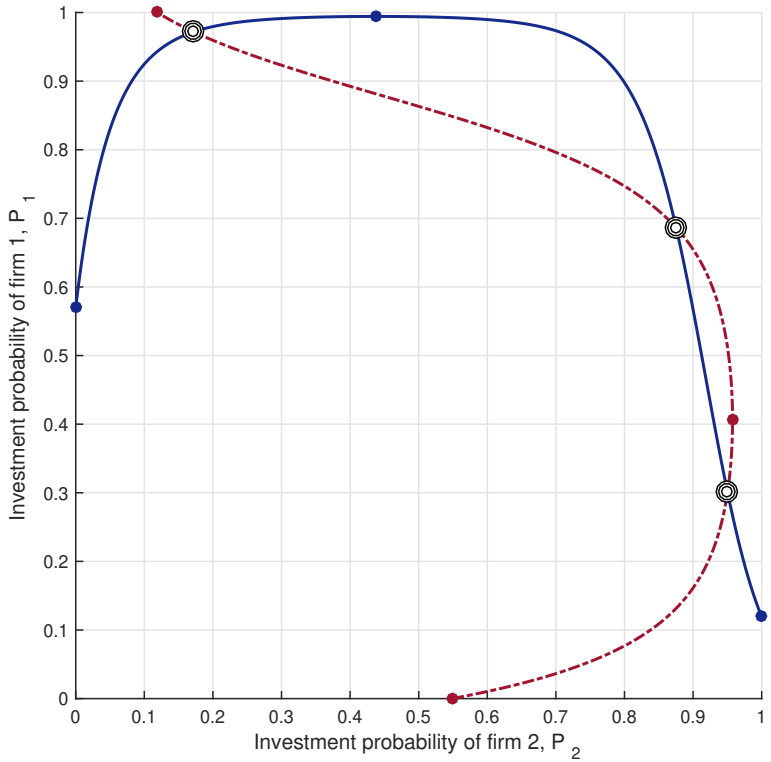
Number of equilibria in the higher stages depends on the selected equilibria

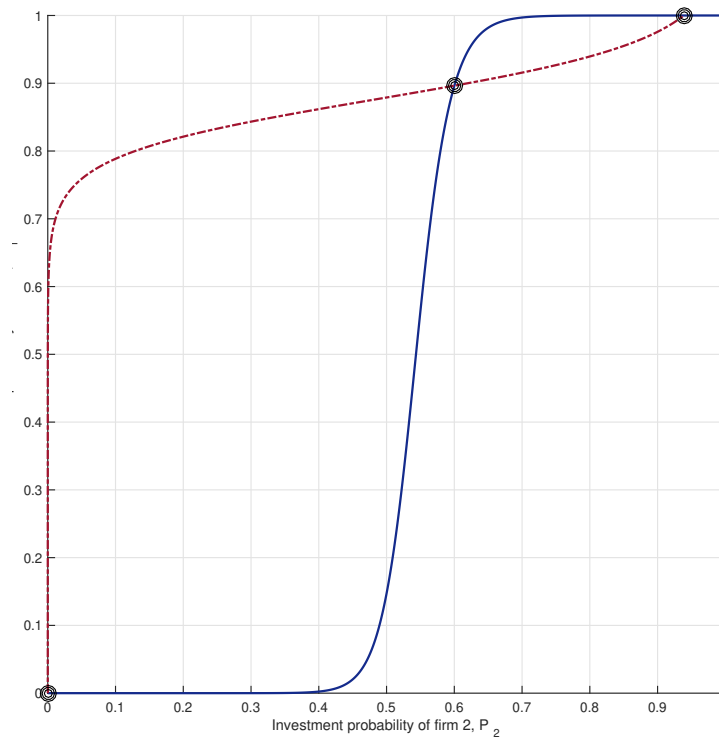


Best response correspondences for $\eta = 0$

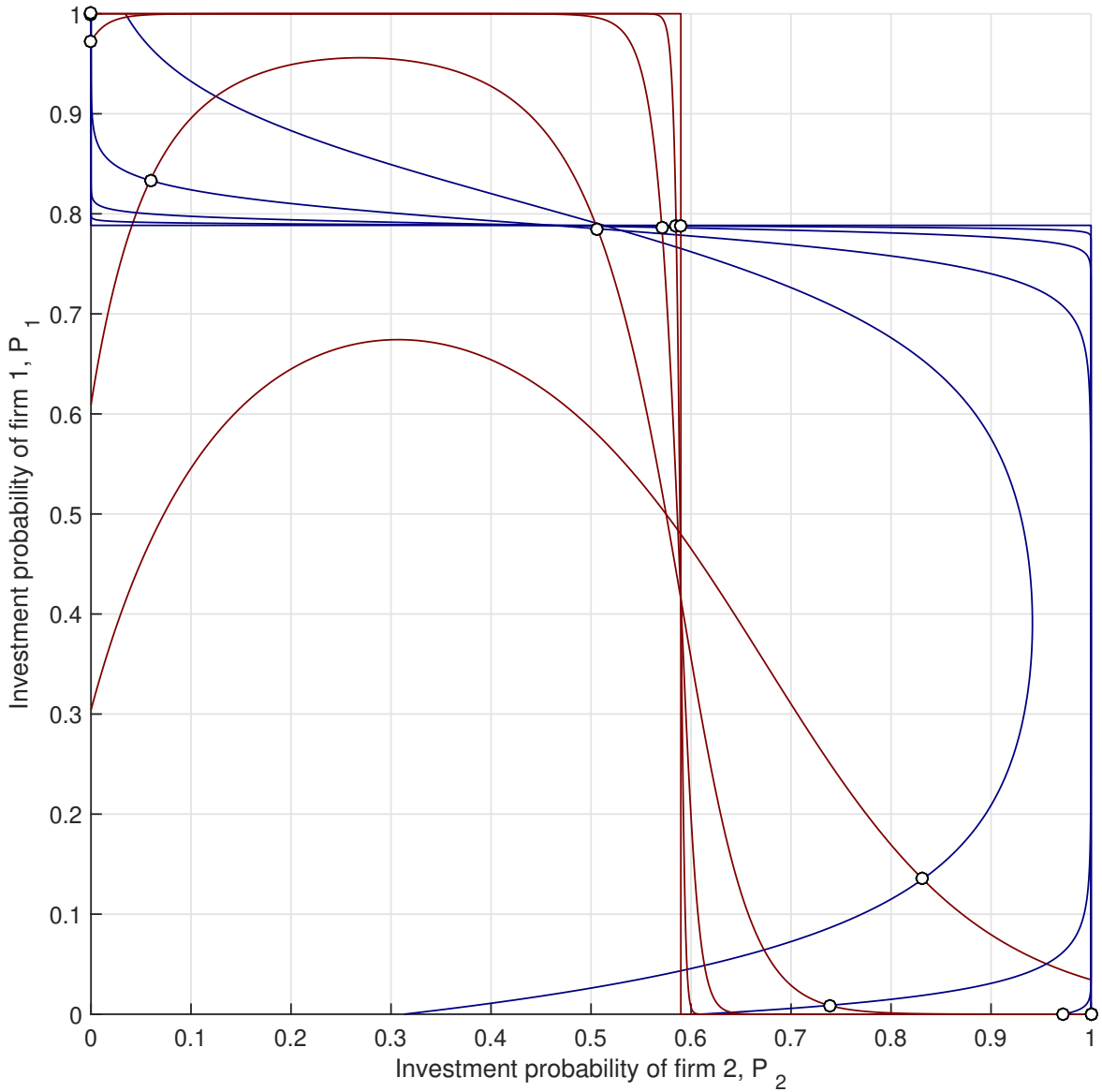


Best response functions for $\eta > 0$





Best response functions as $\eta \to 0$



Recursive Lexicographic Search Algorithm

Building blocks of RLS algorithm:

1. State recursion algorithm solves the game conditional on equilibrium selection rule (ESR)
2. RLS algorithm efficiently cycles through all feasible ESRs

Challenge:

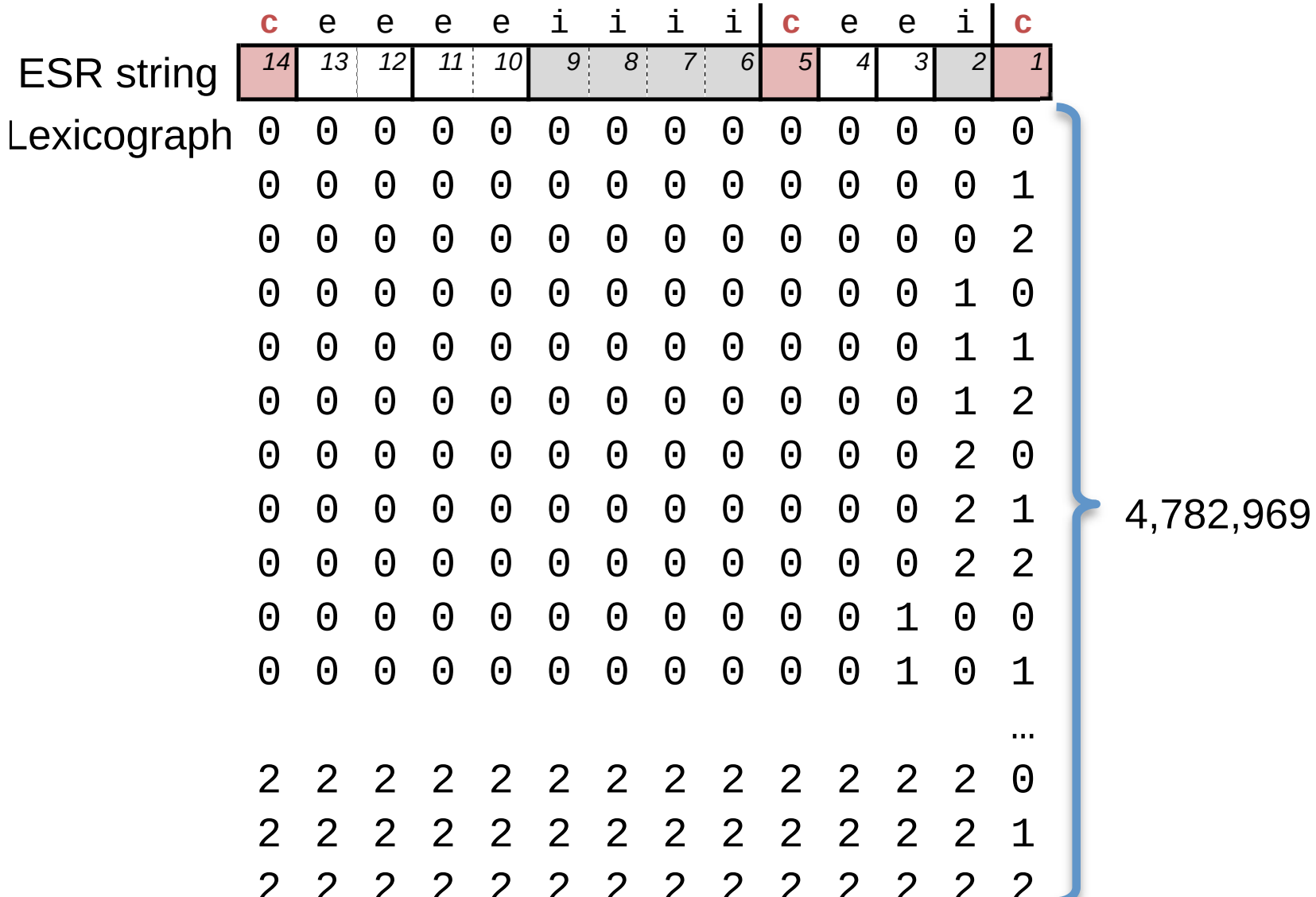
- Choice of a particular MPE for any stage game at any stage
- may alter the set and even the number of stage equilibria at earlier stages

Need to find feasible ESRs

- ESR = string of digits that index the selected stage equilibrium in each point

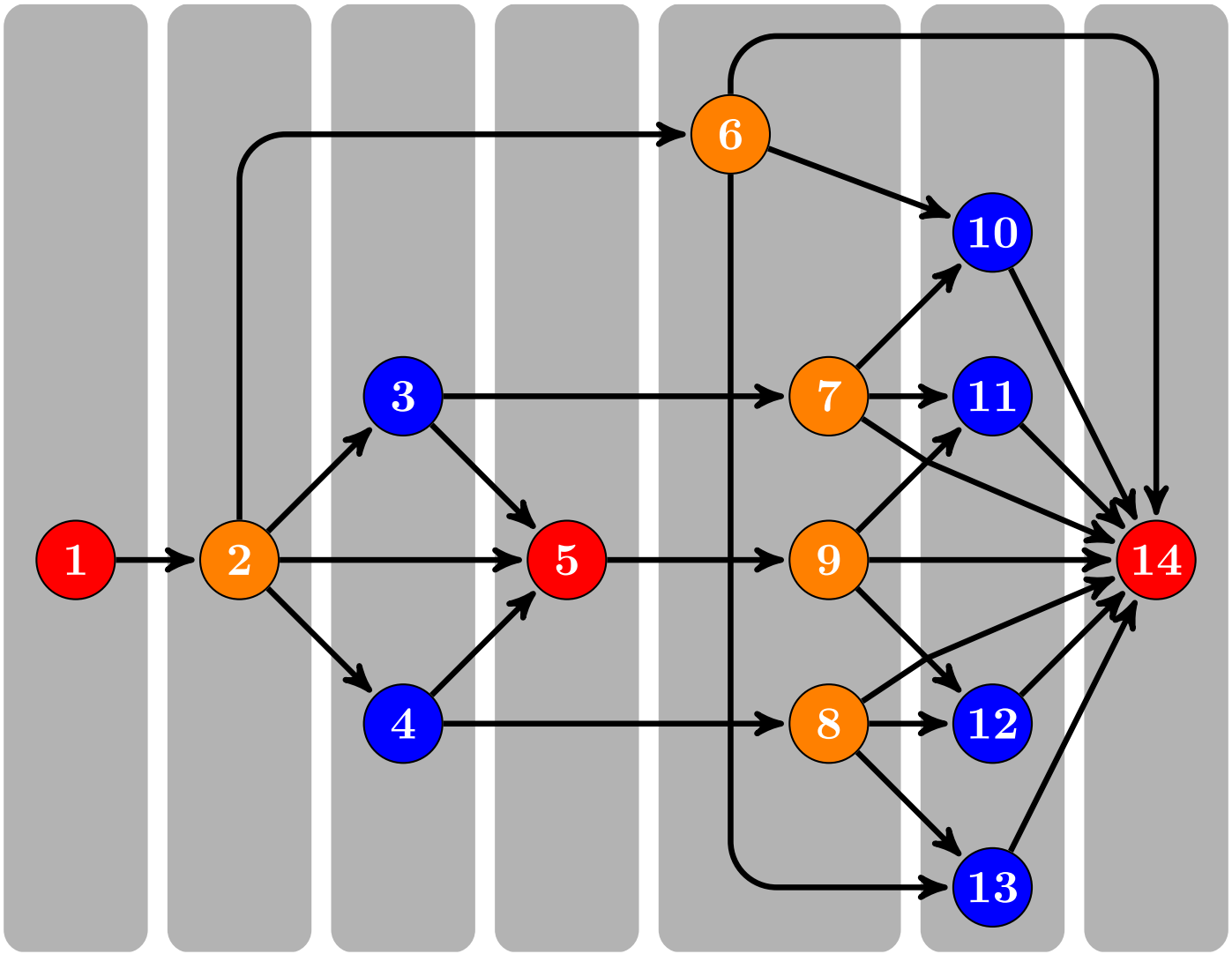
All possible ESR strings in lexicographic order

- Bound the maximum number of equilibria in each stage by $K = 3$



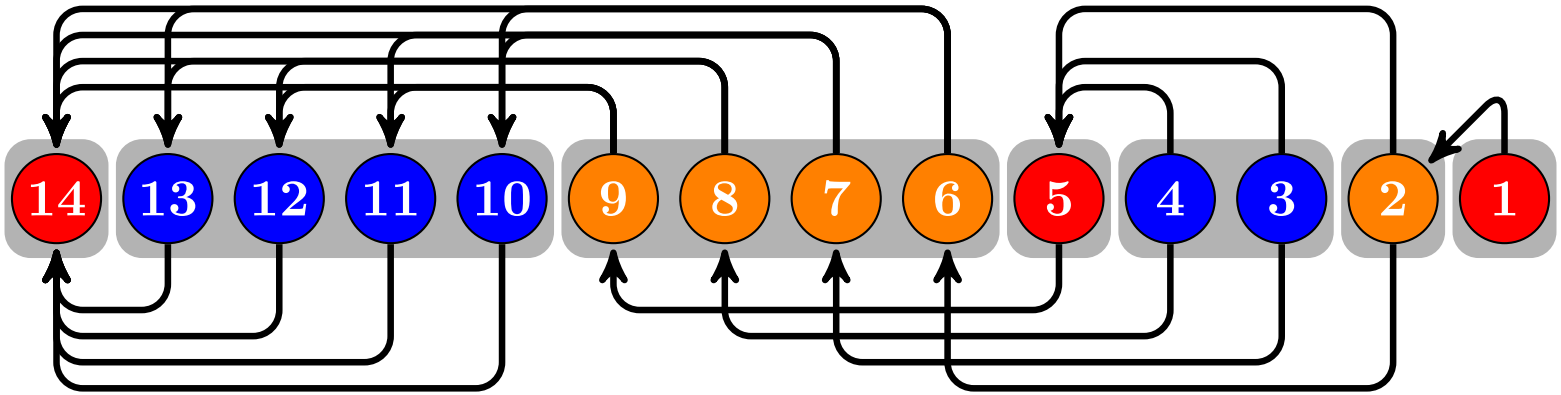
Indexing of points in the state space

Lower index for dependent points, highest for terminal stage

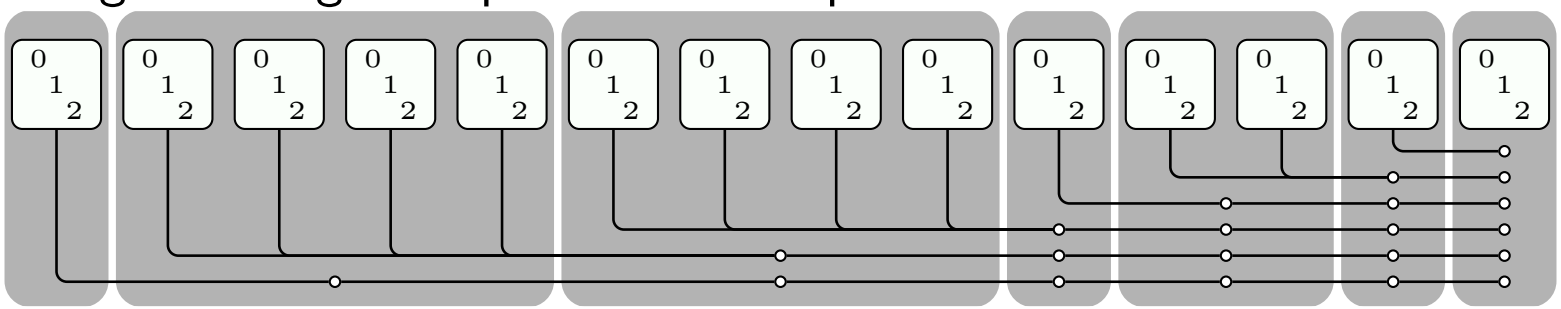


Preserving stage order in ESR strings

Formalization of the ESR as stings of digits

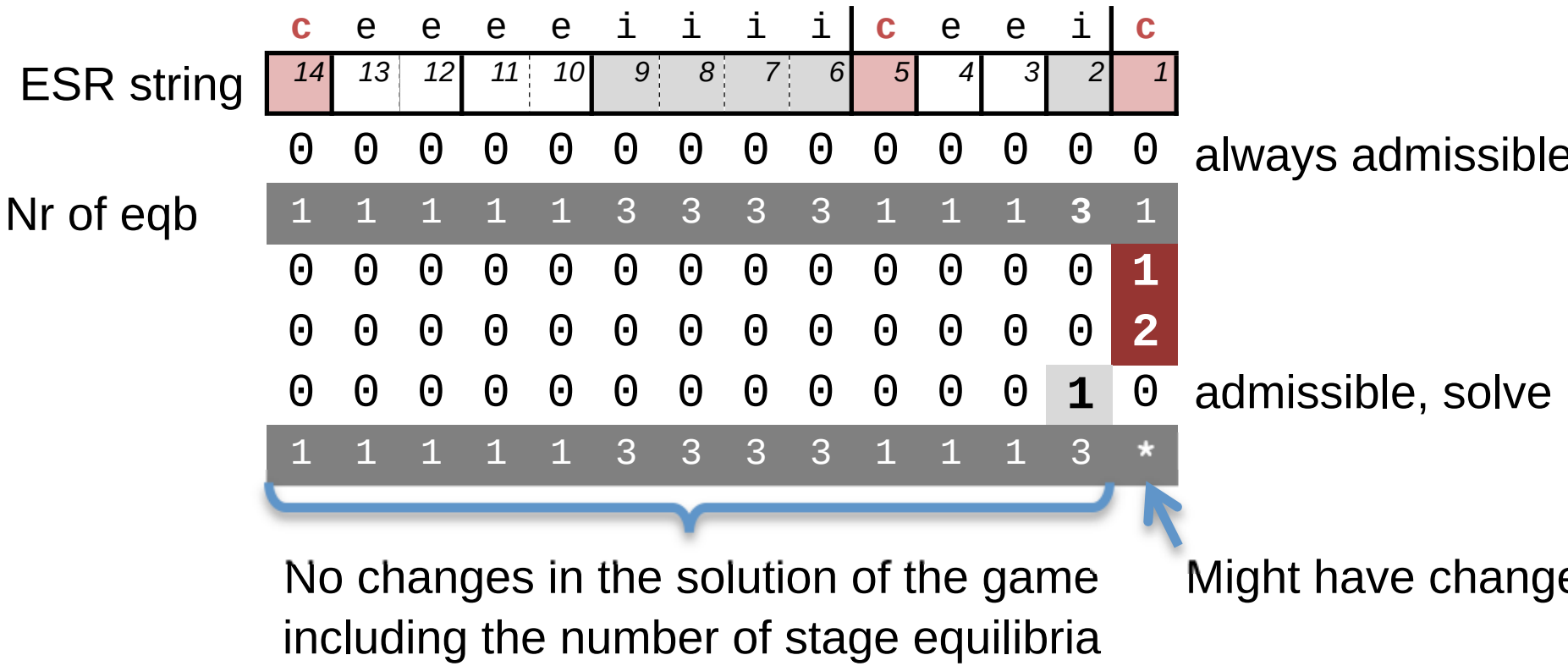


- Digits arranged to preserve the dependence structure

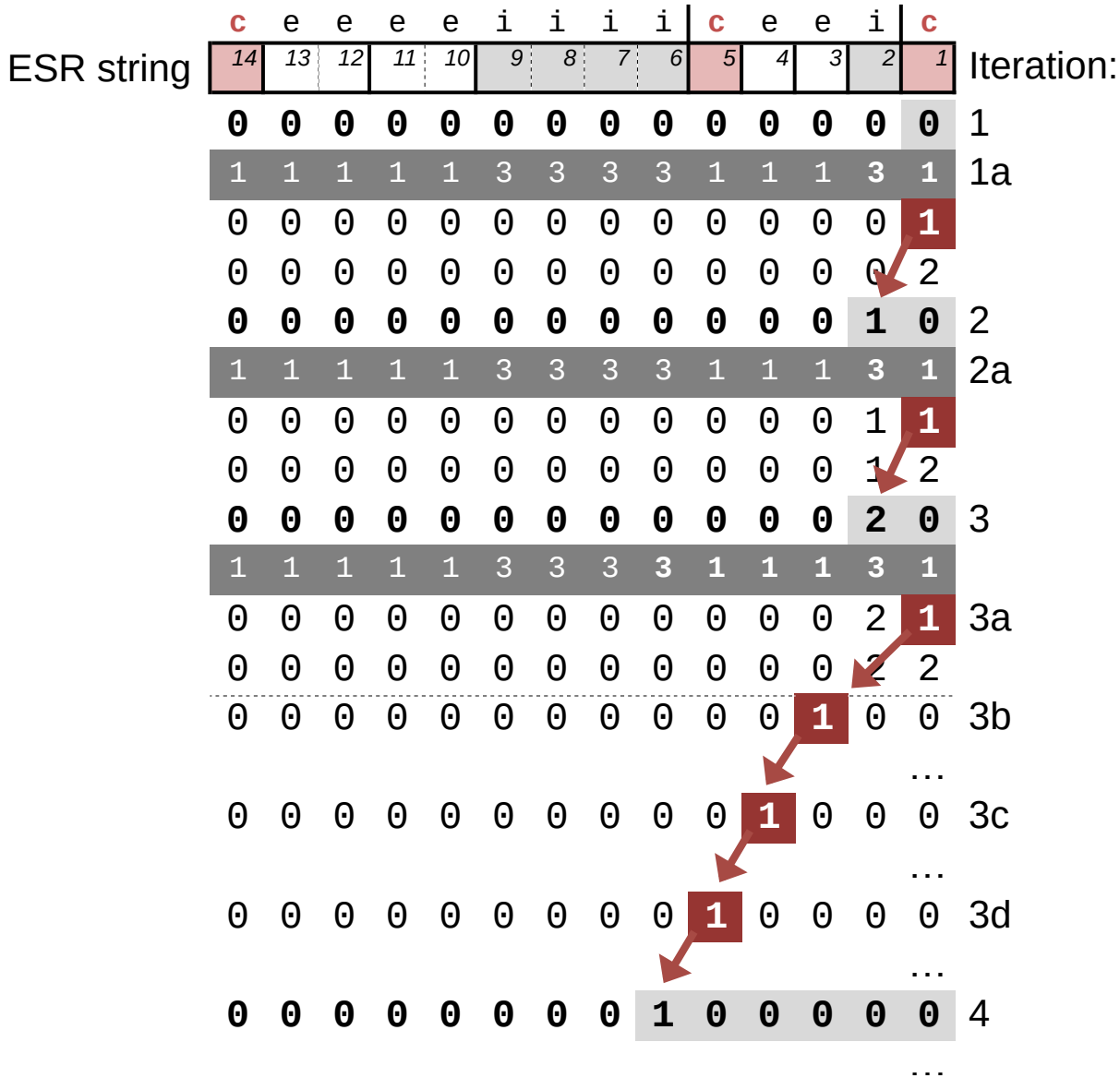


Recalculation of feasibility condition for new ESR

Avoid recalculation of subgames



Jumping over blocks of infeasibles ESRs

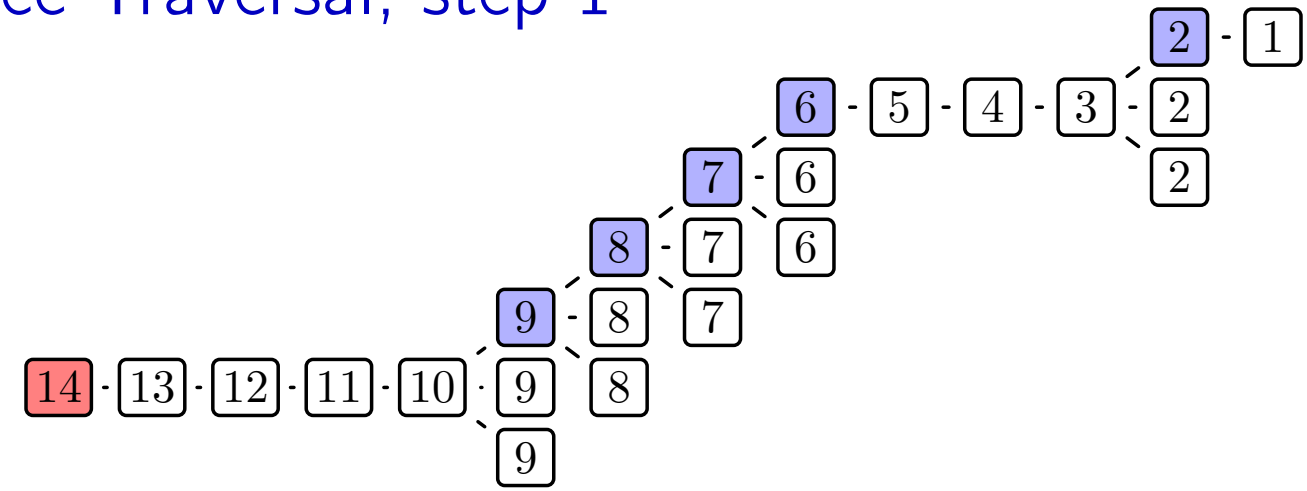


Recursive Lexicographic Search (RLS) Algorithm

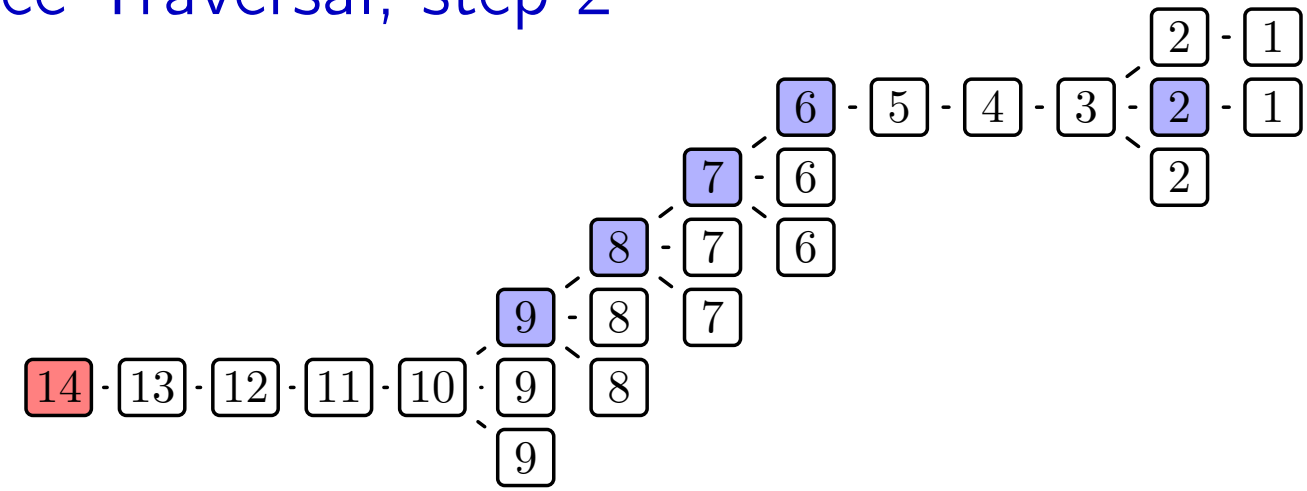
1. Set ESR $= (0, \ldots, 0)$
2. Run State Recursion using the current ESR
3. Save the number of equilibria in every stage game as ne(ESR)
4. Add 1 to the ESR in bases ne(ESR) to obtain new feasible ESR
5. Stopping rule: successor function exceeds the maximum number with given number of digits
6. Return to step 2

RLS = tree traversal!

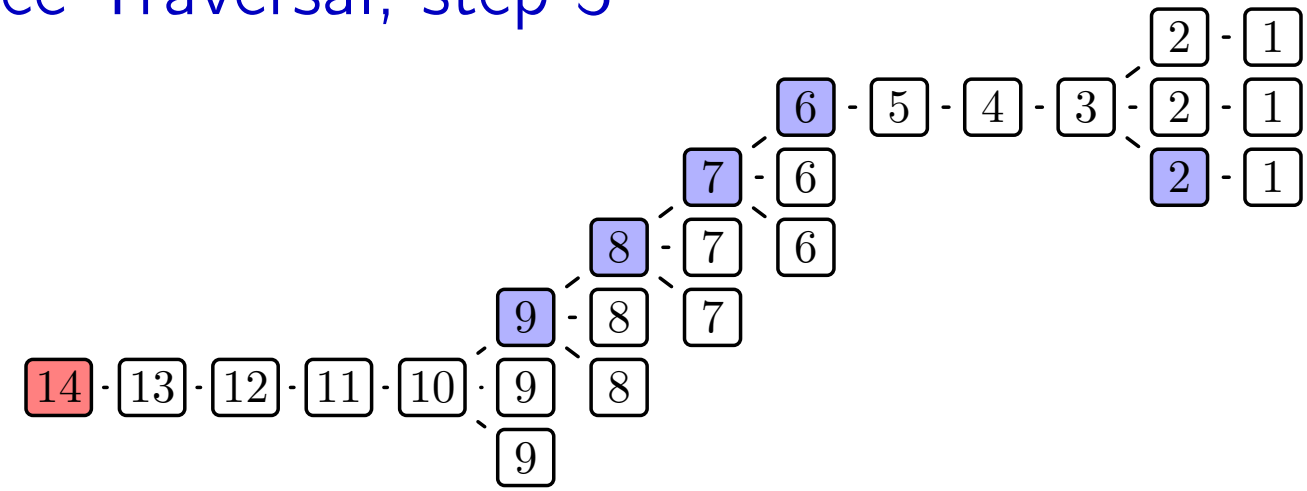
RLS Tree Traversal, step 1



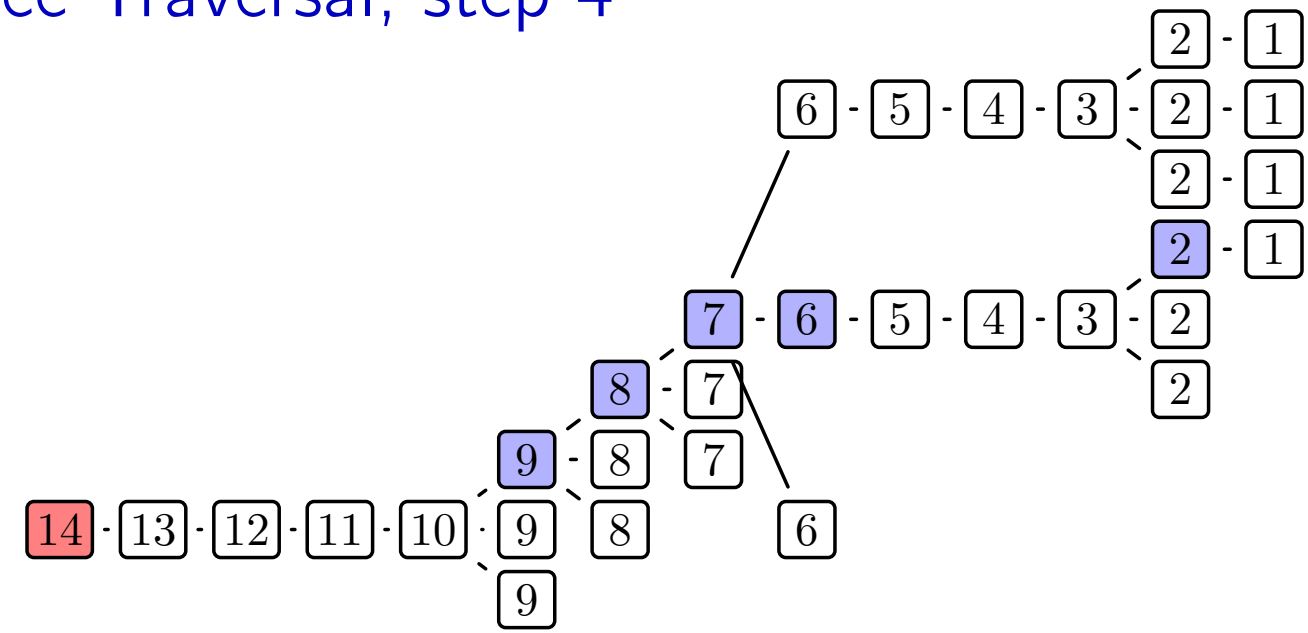
RLS Tree Traversal, step 2



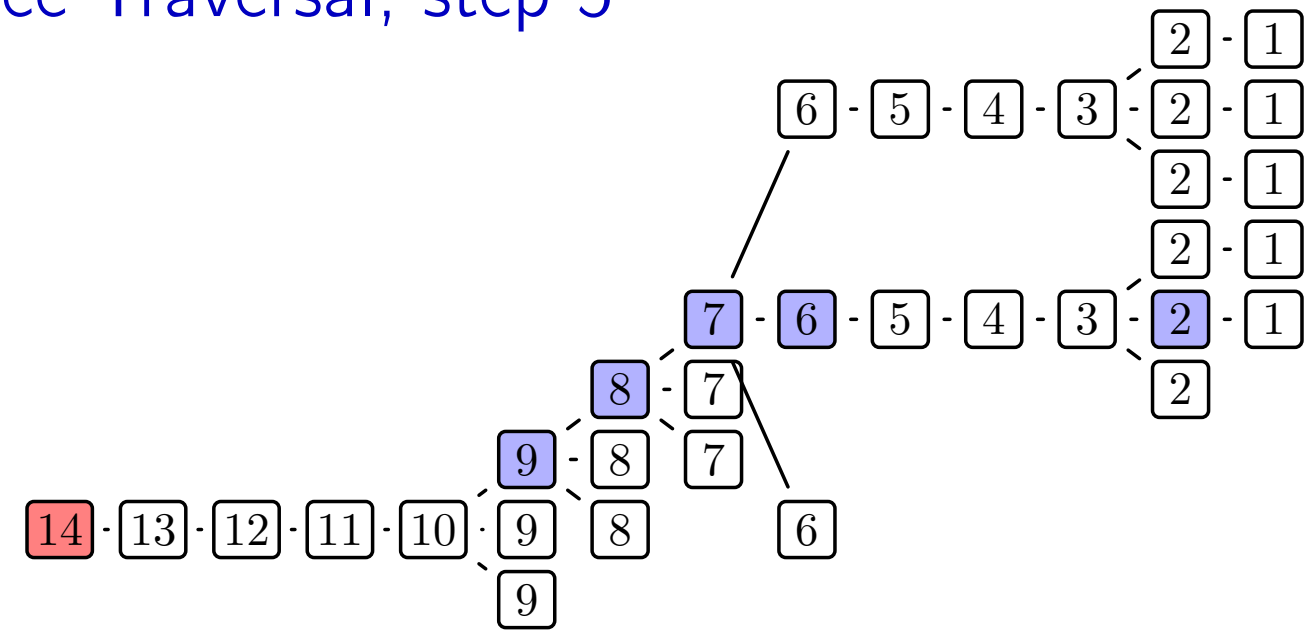
RLS Tree Traversal, step 3



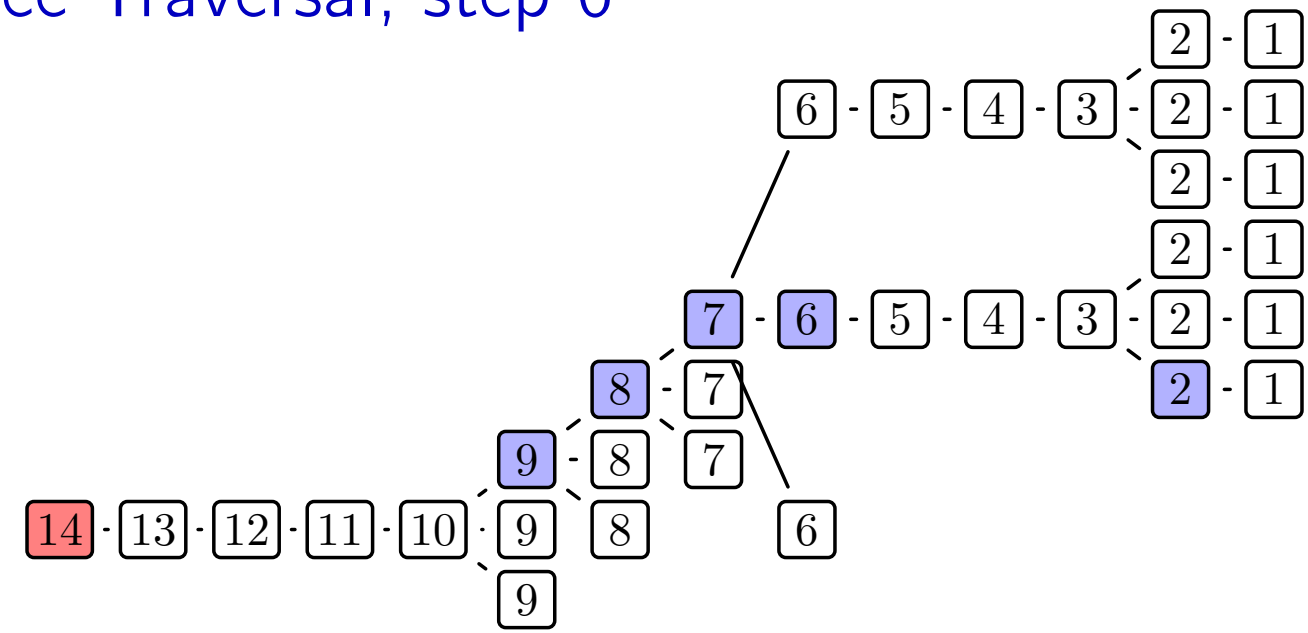
RLS Tree Traversal, step 4



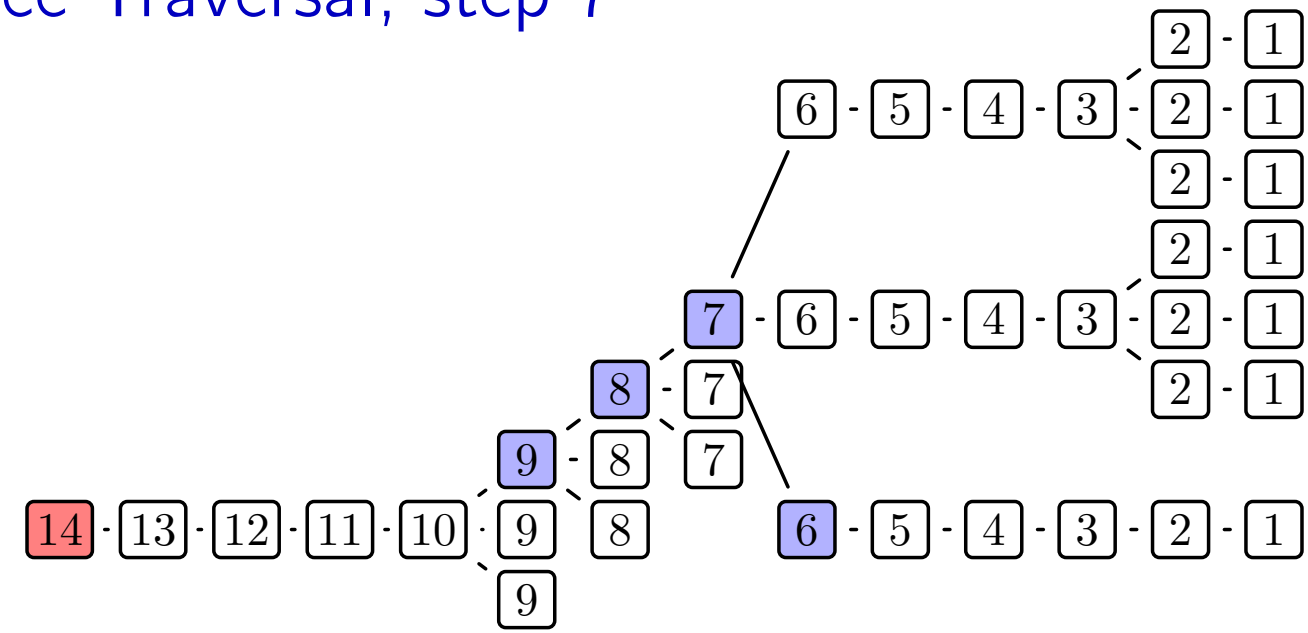
RLS Tree Traversal, step 5



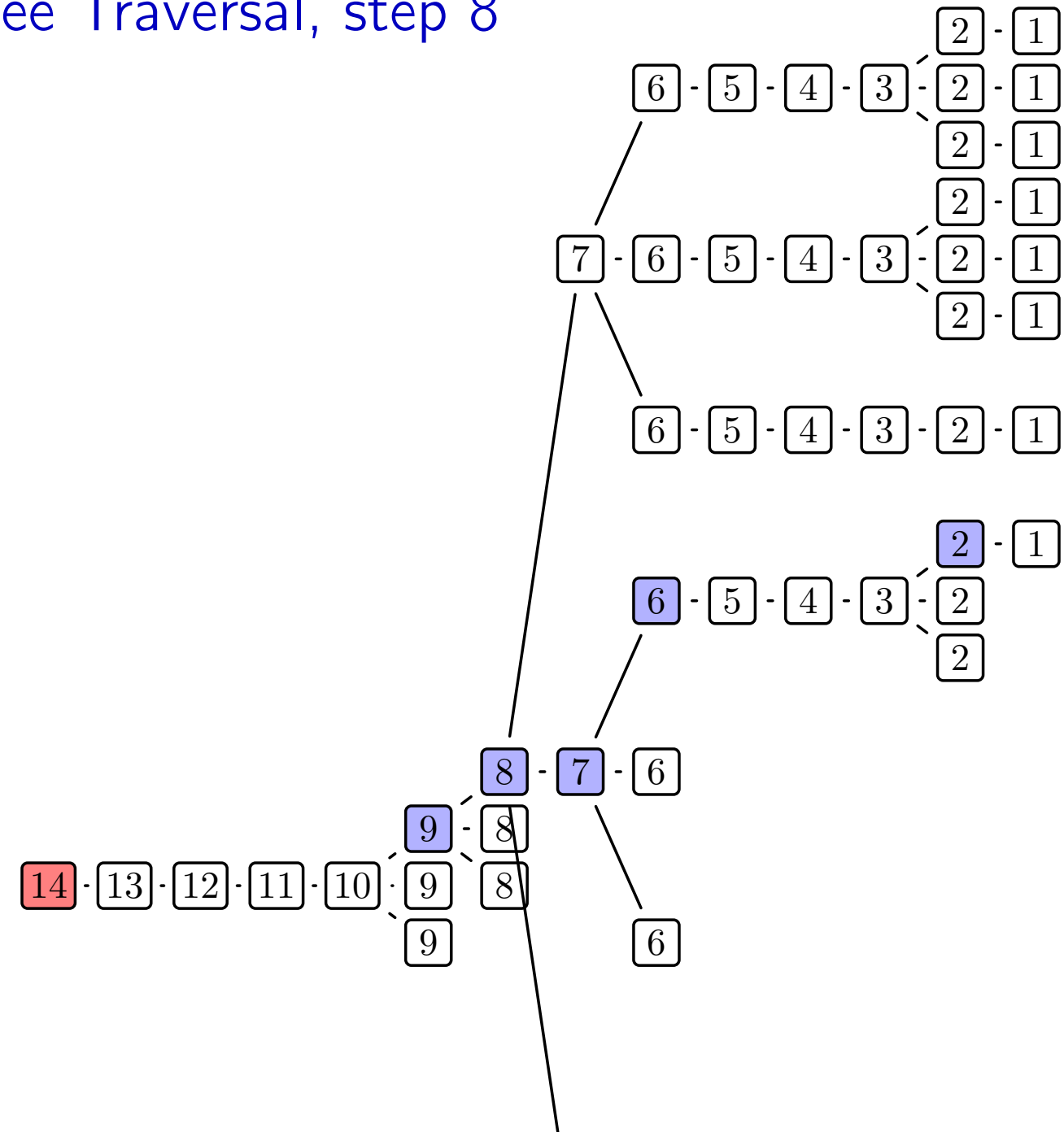
RLS Tree Traversal, step 6



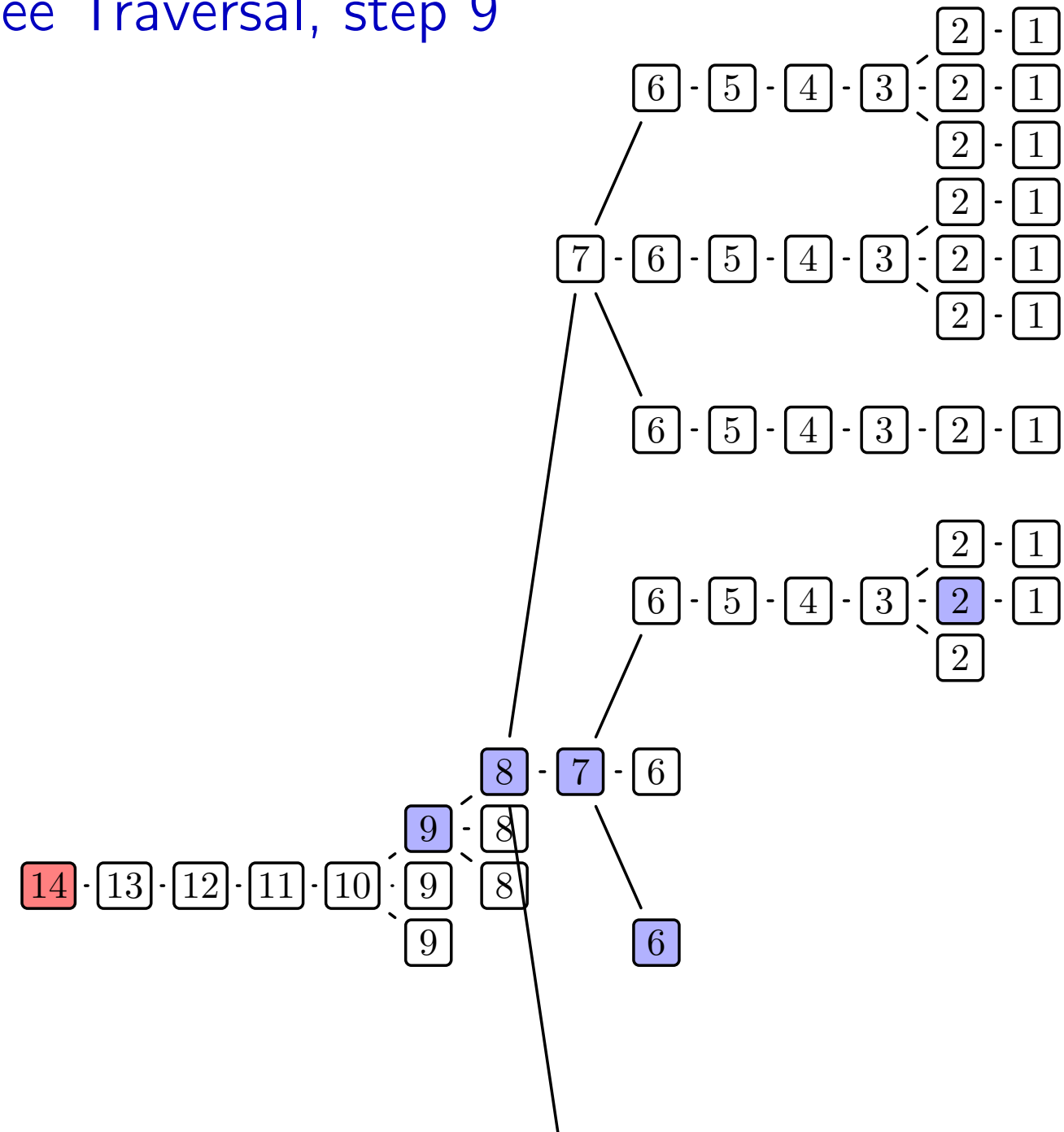
RLS Tree Traversal, step 7



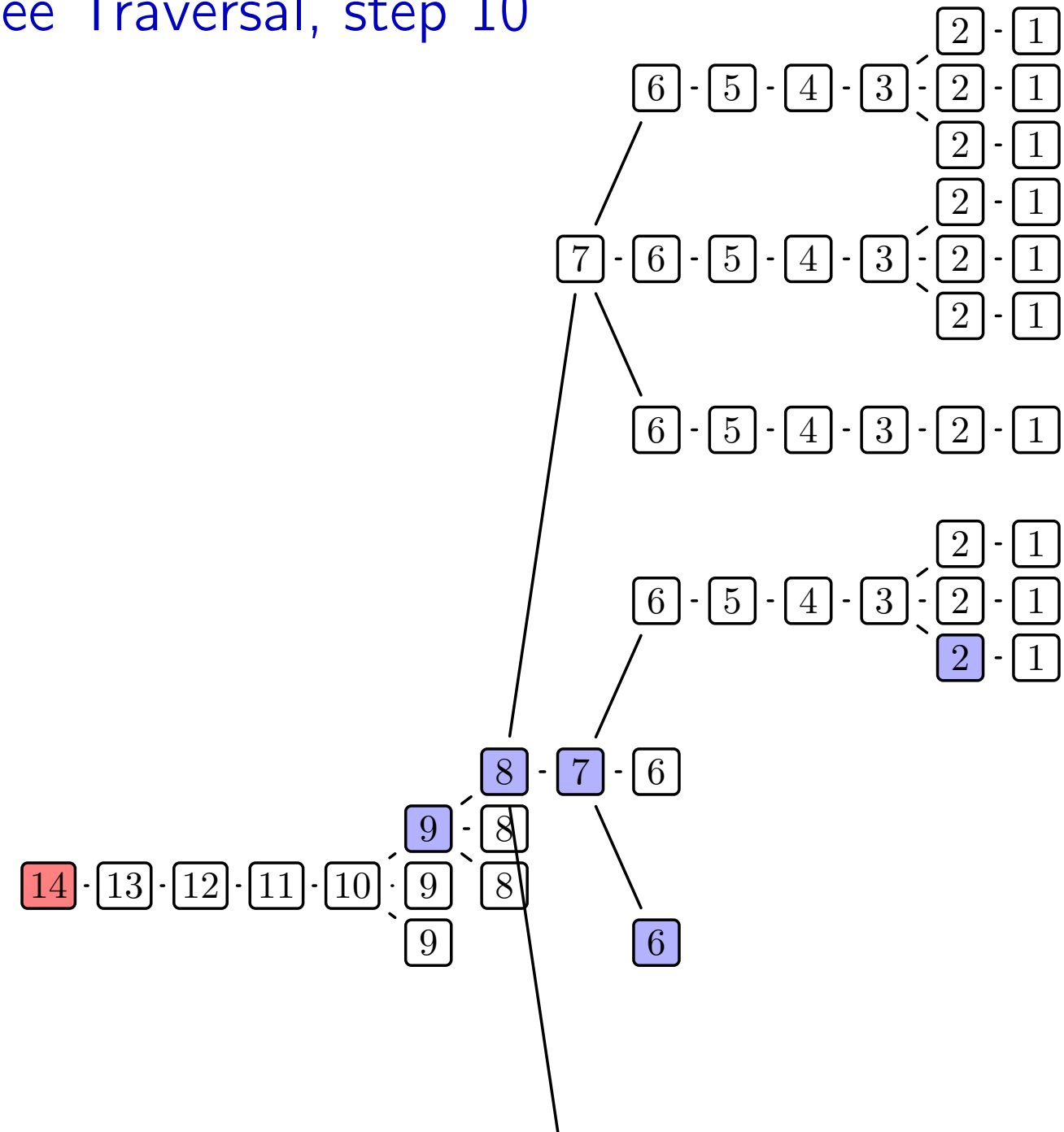
RLS Tree Traversal, step 8



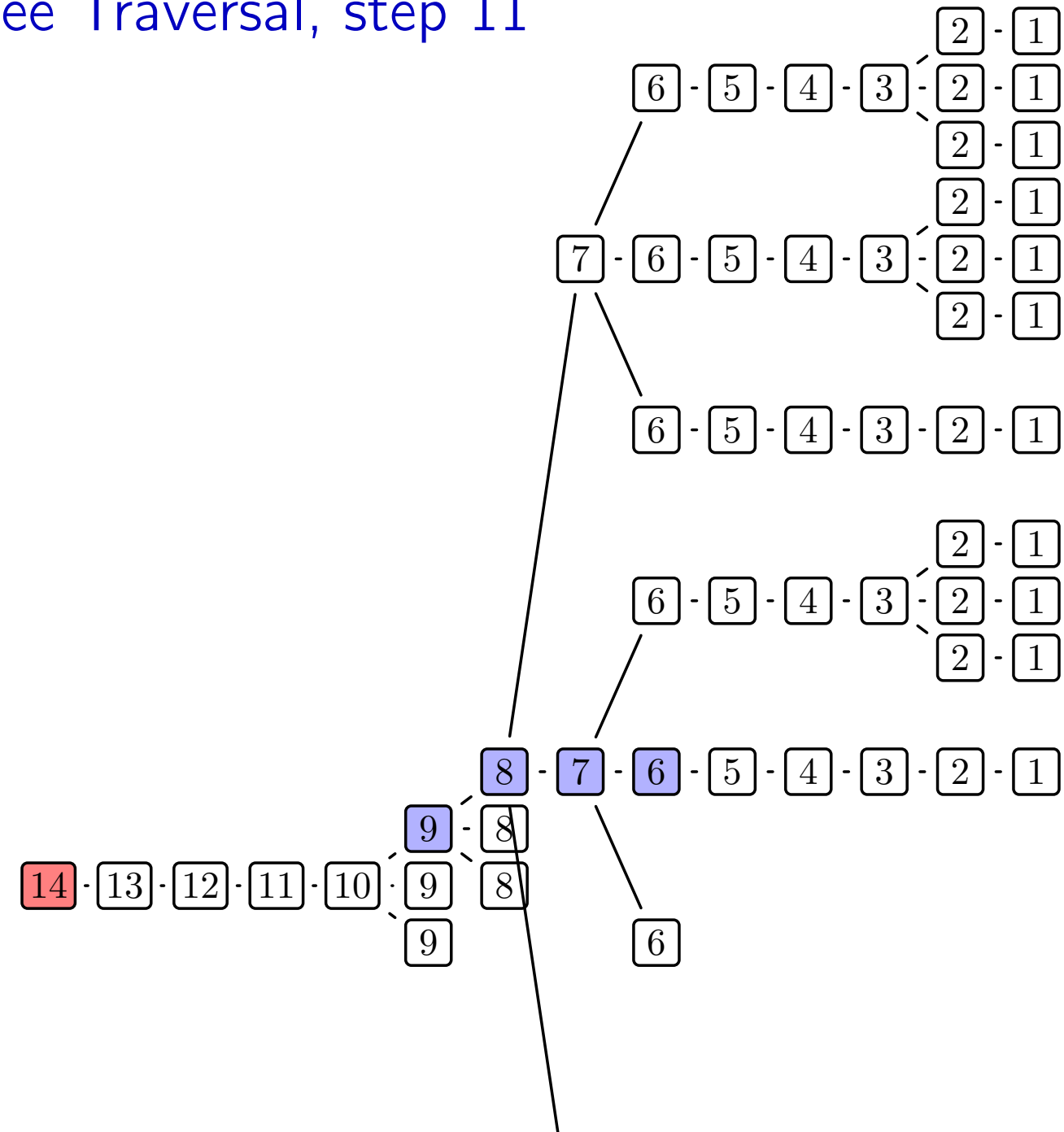
RLS Tree Traversal, step 9



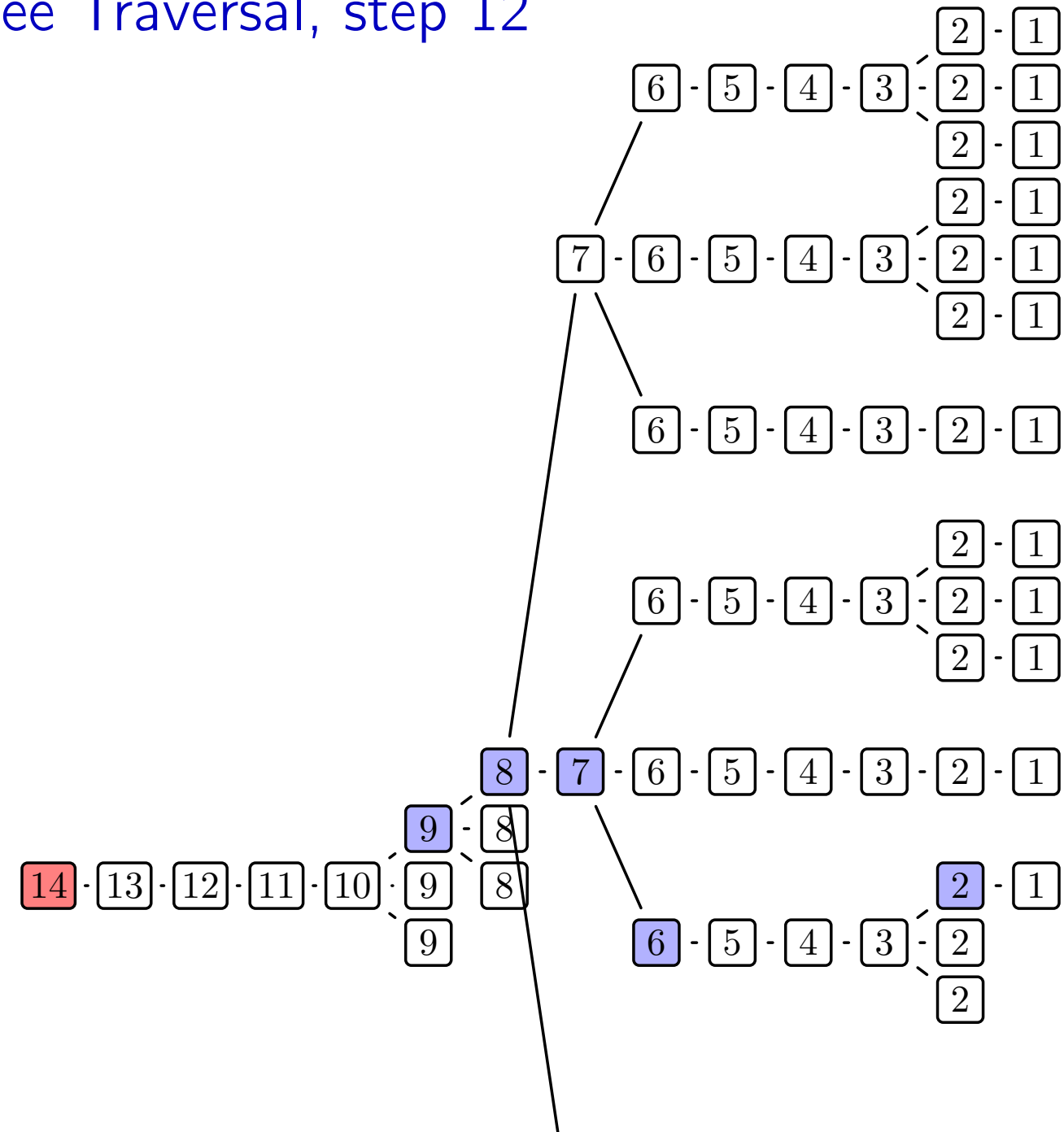
RLS Tree Traversal, step 10



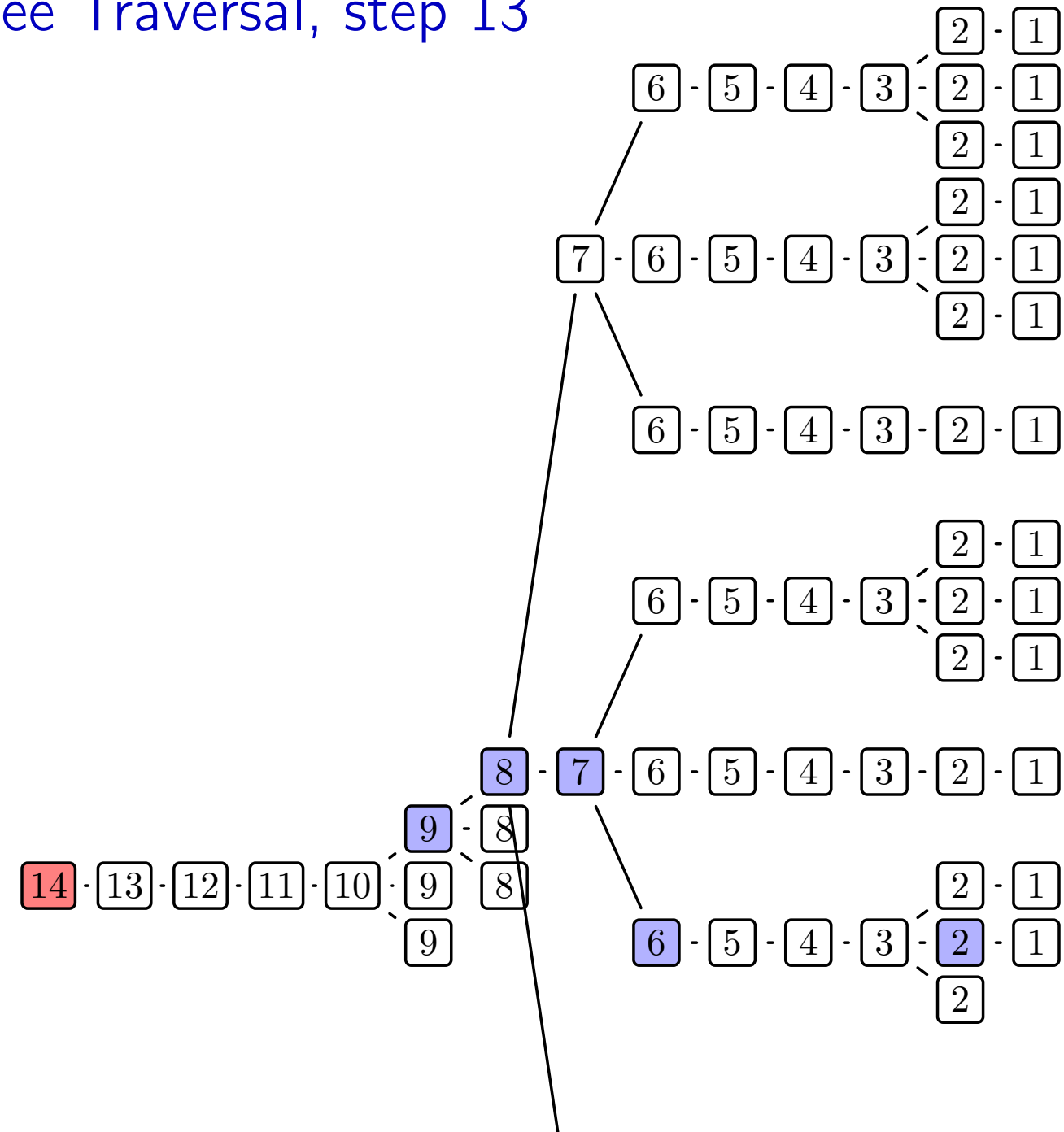
RLS Tree Traversal, step 11



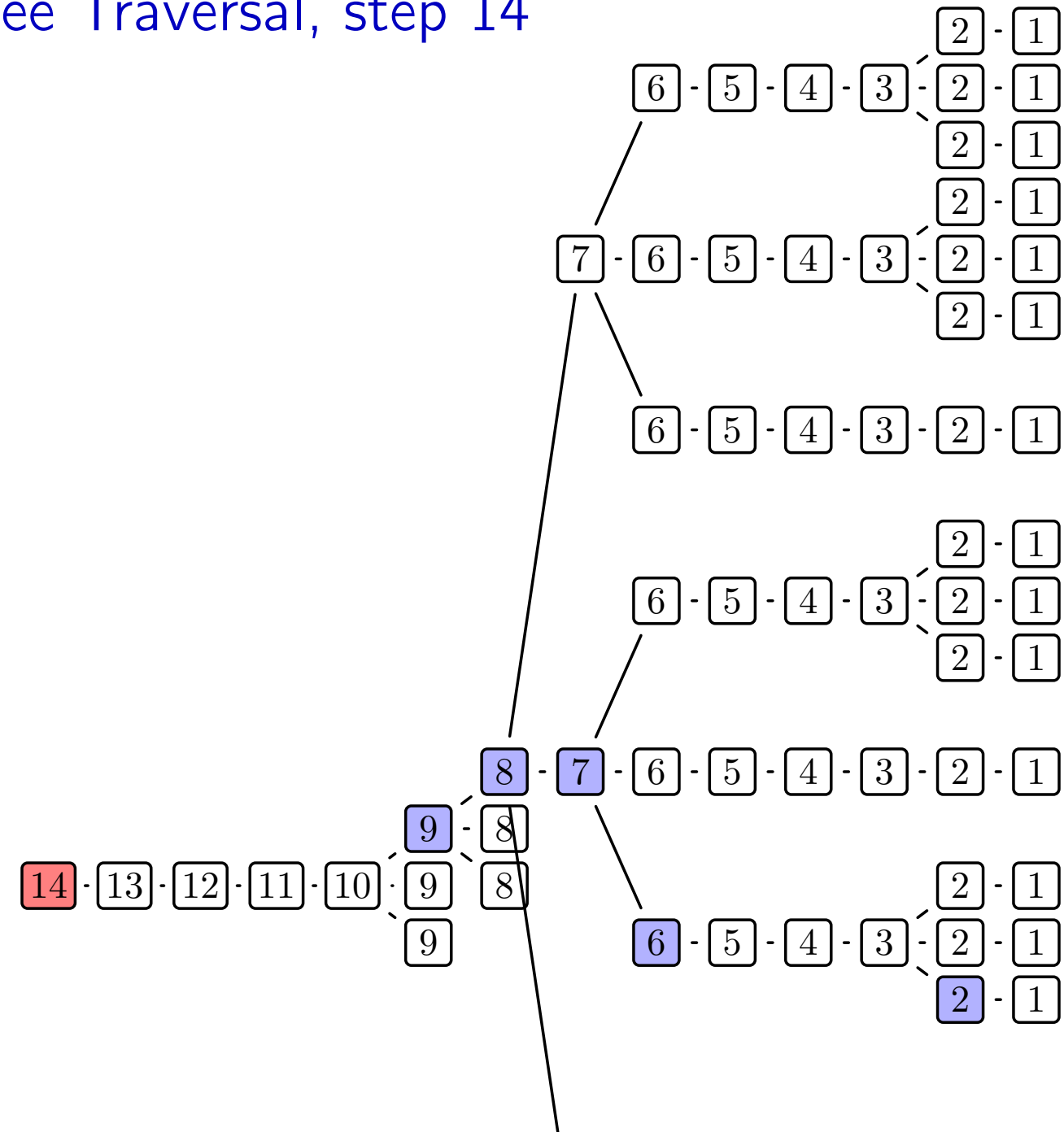
RLS Tree Traversal, step 12



RLS Tree Traversal, step 13



RLS Tree Traversal, step 14



Main result of the RLS Algorithm

Theorem (Decomposition theorem, strong)

Assume there exists an algorithm that can find all MPE of every stage game of the DDG, and that the number of these equilibria is finite in every stage game.
Then the RLS algorithm finds all MPE of the DDG in a finite number of steps, which equals the total number of MPE.

 Iskhakov, Rust and Schjerning, 2016, ReStud
"Recursive lexicographical search: Finding all markov perfect equilibria of finite state directional dynamic games."

Main result of the RLS Algorithm

Theorem (Decomposition theorem, weak)

Assume there exists an algorithm that can find at least one MPE of every stage game of the DDG, and that the number of these equilibria is finite in every stage game.
Then the RLS algorithm finds some (at least one) MPE of the DDG in a finite number of steps, which does not exceed the total number of MPE.

RLS algorithm: running times

$K = 3$

Simultaneous moves	$n = 3$	$n = 4$
Upper bound on number of MPE	4,782,969	3,948,865,611
Actual number of equilibria	127	46,707
Time used	0.008 sec.	0.334 sec.

Simultaneous moves		$n = 5$
Upper bound on number of MPE	174,449,211,009,120,166,087,753,728	
Actual number of equilibria		192,736,405
Time used		45 min.

Alternating moves		$n = 5$
Upper bound on number of MPE	174,449,211,009,120,166,087,753,728	
Actual number of equilibria		1
Time used		0.006 sec.

Resolution to the Bertrand investment paradox

Theorem (Solution to Bertrand investment paradox)

If investment is socially optimal at a state point $(c_1, c_2, c) \in S$*, then*

- *no investment by both firms cannot be an MPE outcome in the subgame starting from* (c_1, c_2, c) *in either the simultaneous or alternating move versions of the dynamic game.*

We show:

1. Many types of endog. coordination is possible in equilibrium
   - Leapfrogging (alternating investments)
   - Preemption (investment by cost leader)
   - Duplicative (simultaneous investments)
2. The equilibria are generally inefficient due to over-investment
   - Duplicative or excessively frequent investments

Multiplicity of equilibria

Theorem (Sufficient conditions for uniqueness)

In the dynamic Bertrand investment and pricing game a sufficient condition for the MPE to be unique is that

1. *firms move in alternating fashion (i.e. $m \neq 0$), and,*
2. *for each $c > 0$ in the support of π we have $\pi(c|c) = 0$.*

1. If firms move simultaneously,
   equilibrium is generally not unique.
2. If technological change is stochastic,
   equilibrium is generally not unique.

 Iskhakov, Rust and Schjerning, 2018, IER

"The dynamics of Bertrand price competition with cost-reducing investments."

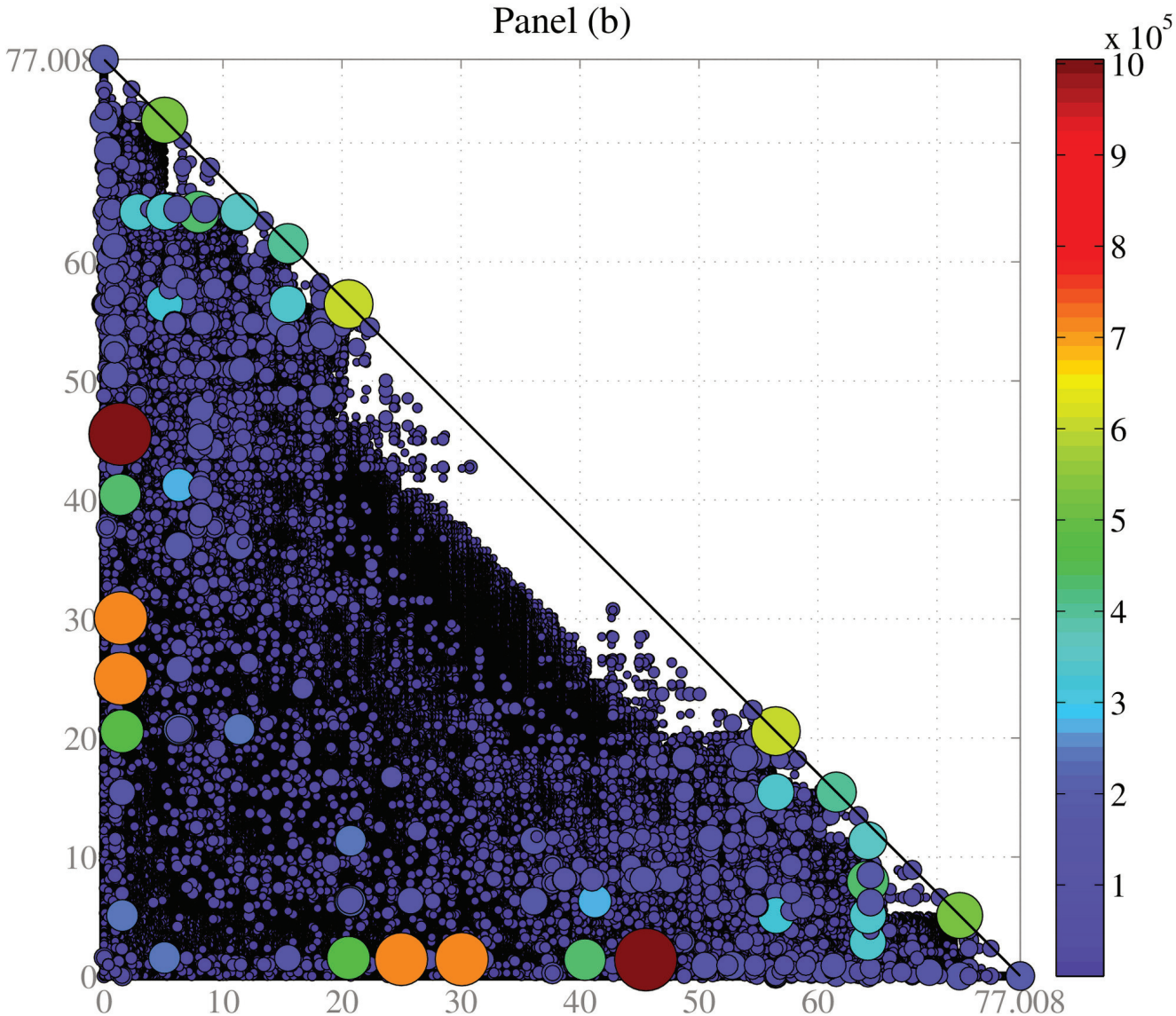
Pay-offs in the simultaneous move game

Theorem (Triangular payoffs in the simultaneous move game)

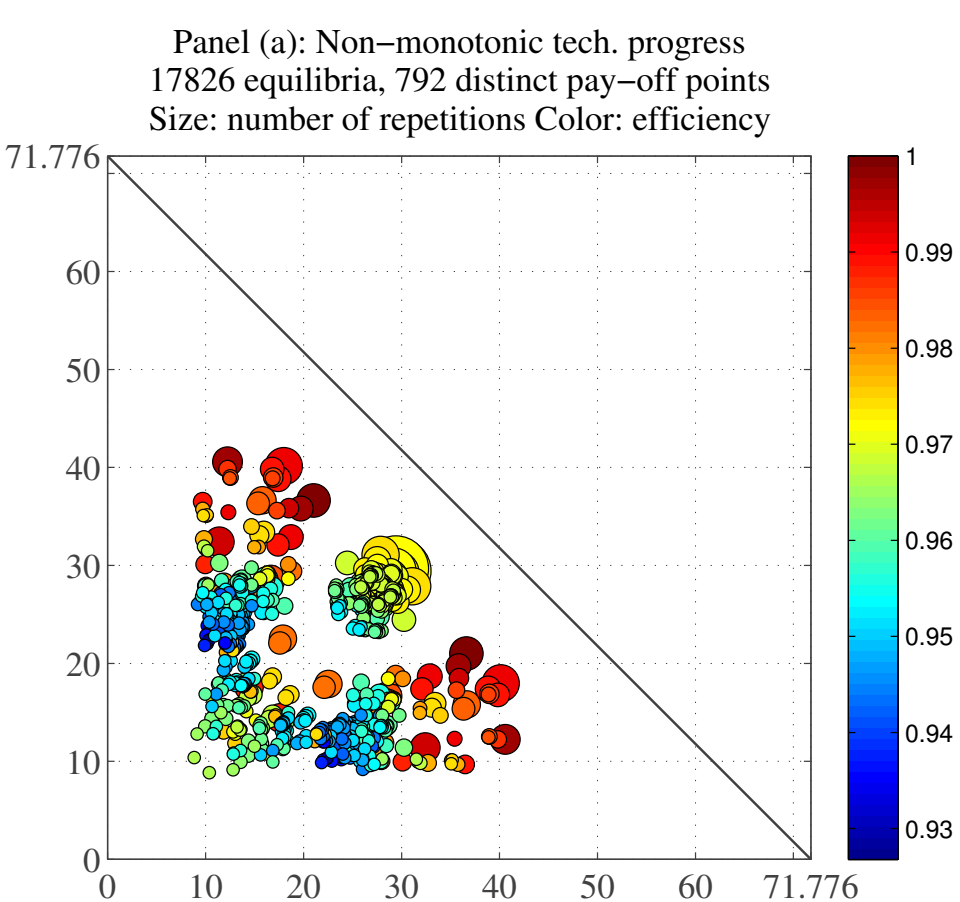
Suppose that the $\{c_t\}$ process has finite support, that there are no idiosyncratic shocks to investment (i.e. $\eta = 0$) and that firms move simultaneously

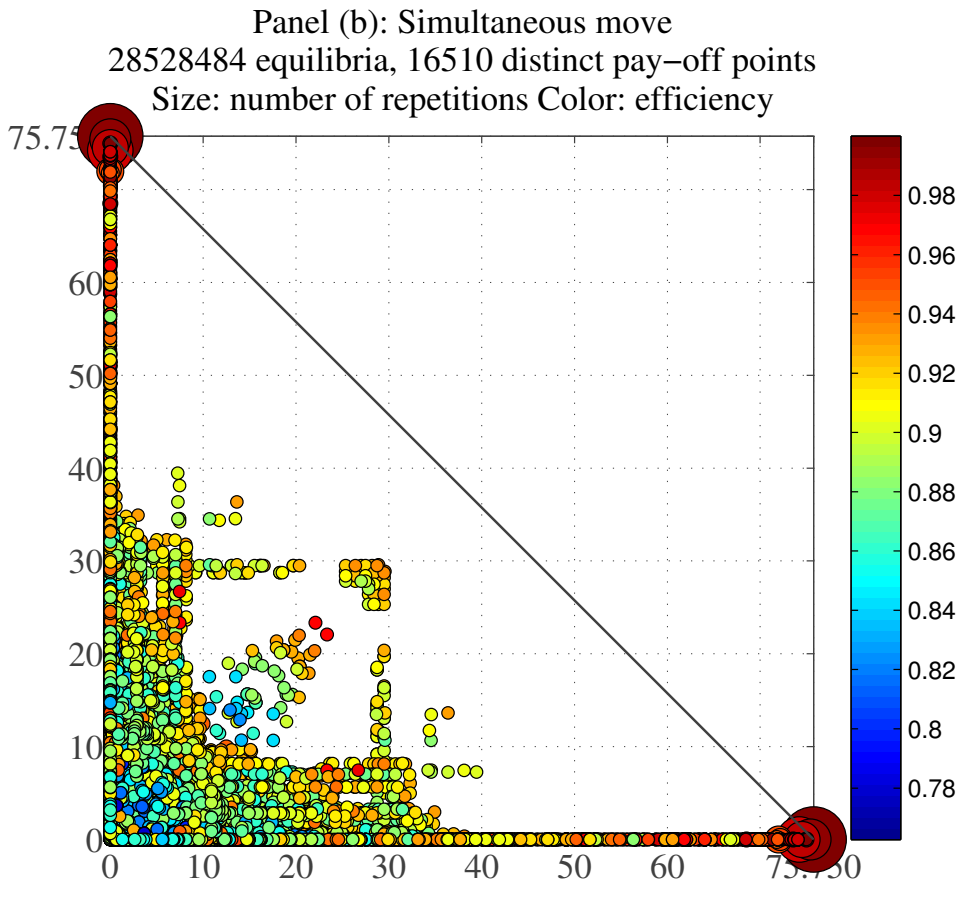
- *The (convex hull of the) set of the expected discounted equilibrium payoffs at the apex state $(c_0, c_0, c_0) \in S$ is a triangle*
- *The vertices of this triangle are at the points $(0, 0)$, $(0, V_M)$ and $(V_M, 0)$ where $V_M = v_{N,i}(c_0, c_0, c_0)$ is the expected discounted payoff to firm i in the monopoly equilibrium where firm i is the monopolist investor.*

Pay-off map


Panel (b)
x 10^5
10
9
8
7
6
5
4
3
2
1
77.008
60
50
40
30
20
10
0
0
10
20
30
40
50
60
77.008

Pay-offs: alternating vs simultaneous move games

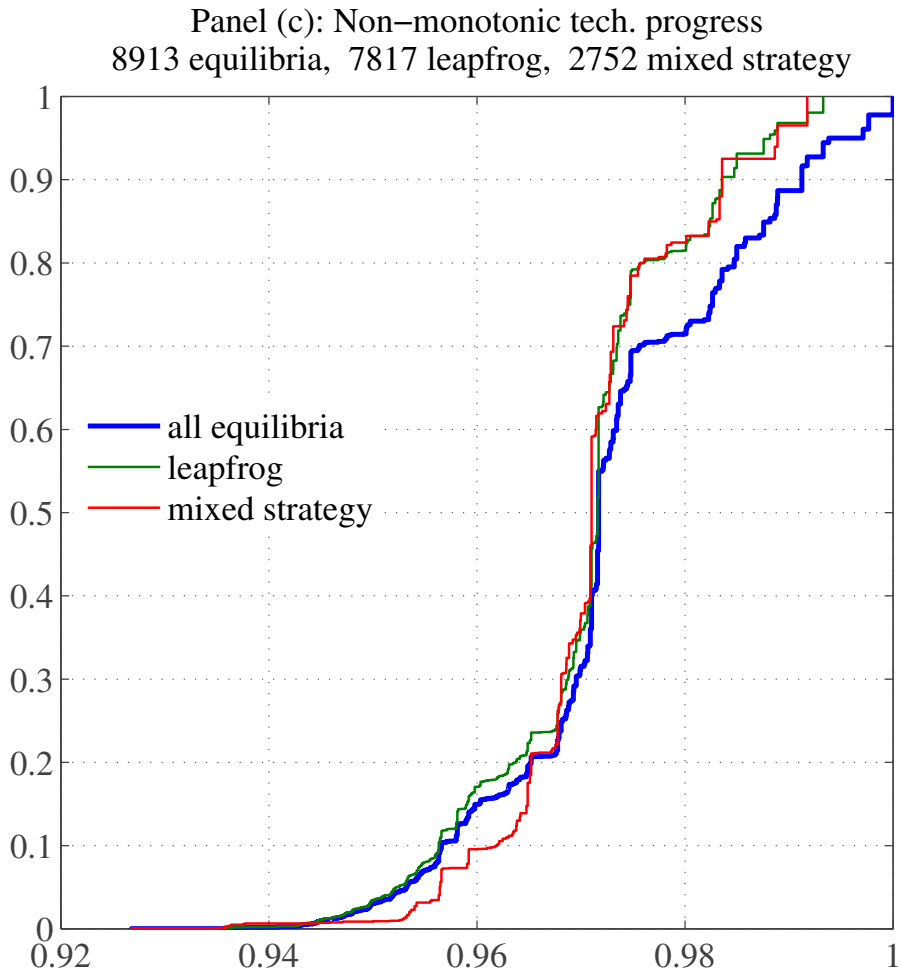


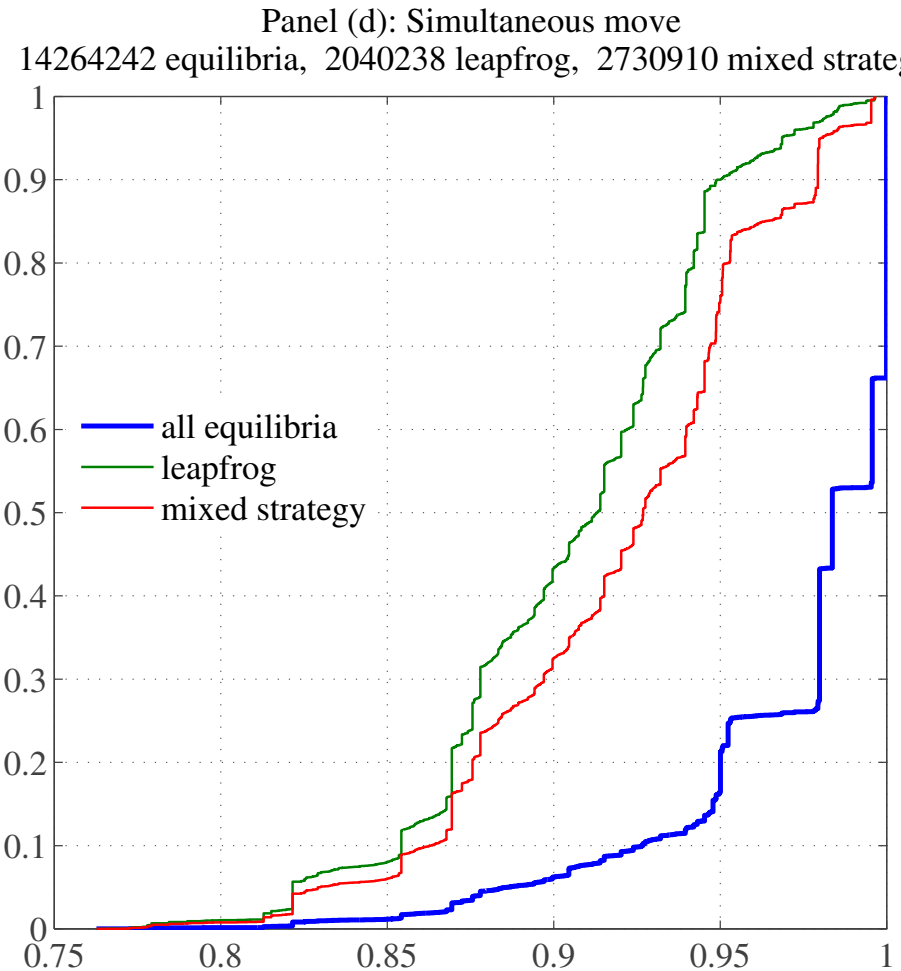


Theorem (Inefficiency of mixed strategy equilibria)

A necessary condition for efficiency in the dynamic Bertrand investment and pricing game is that along MPE path only pure strategy stage equilibria are played.

Efficiency: alternating vs simultaneous move games





Riordan and Salant: Full Preemption

Theorem (**Riordan and Salant, 1994**)

The continuous time investment game where

1. *right to move alternates deterministically.*
2. $K(c) = K$ *and is not prohibitively high.*
3. *technological progress is deterministic:* $c(t)$ *is a continuous, decreasing function*

has a unique MPE with

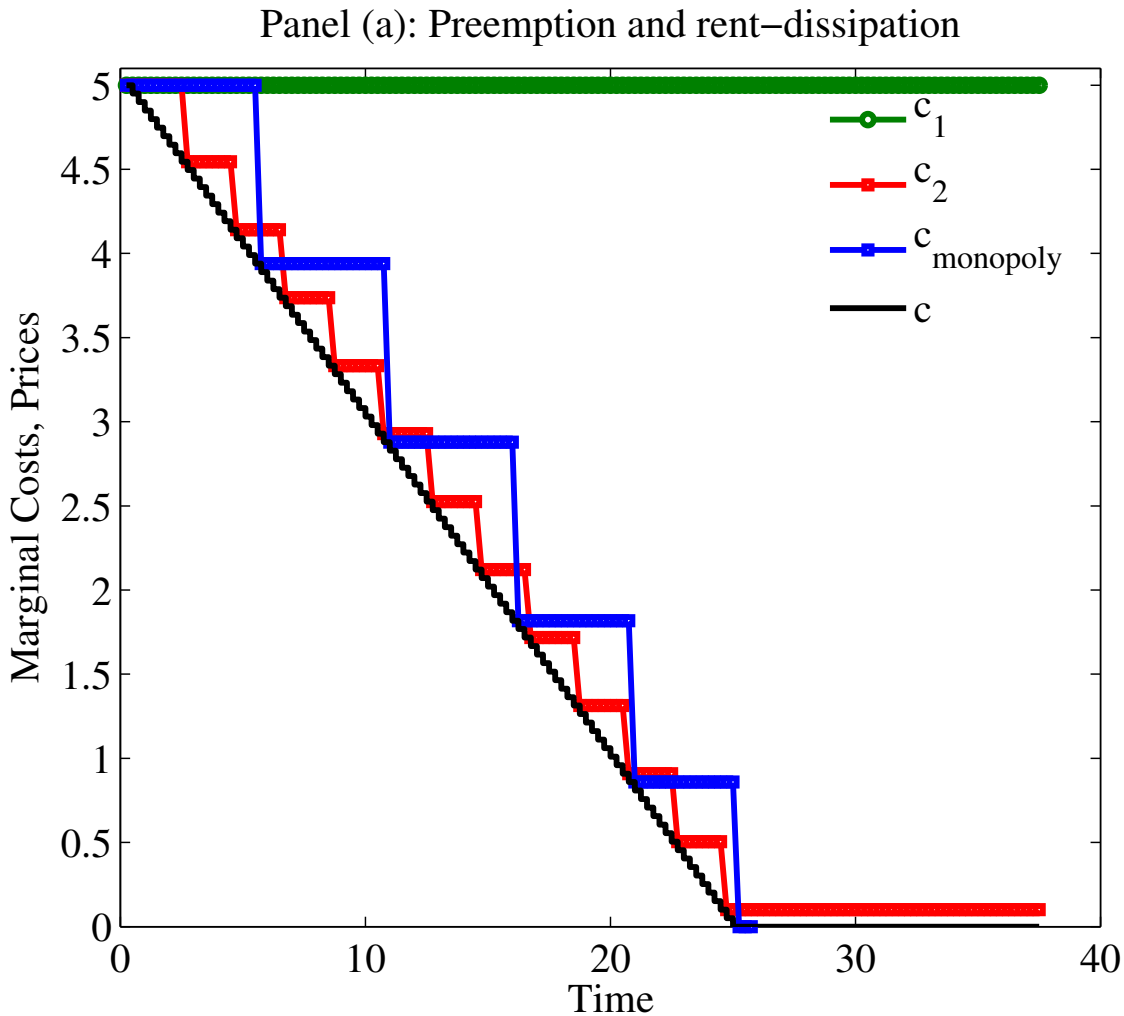
- *preemptive investments: by only one firm and no investment in equilibrium by its opponent.*
- *rent dissipation: discounted payoffs of both firms in equilibrium is* 0*, so the entire surplus is wasted on excessively frequent investments by the preempting firm.*

We show by computing examples and counterexamples

1. Confirm R&S the result with high $K$and small dt
2. Underinvestment: Rent dissapation is not a general outcome - disappears when K is low relative dt
3. Leapfrogging: Preemption is not the general outcome - disappears when K is even lower
4. Random move alternation $\rightarrow$ Leapfrogging
5. Random onestep technology $\rightarrow$ Leapfrogging
6. Random multistep technology $\rightarrow$ Leapfrogging
7. Simultaneous moves $\rightarrow$ Leapfrogging

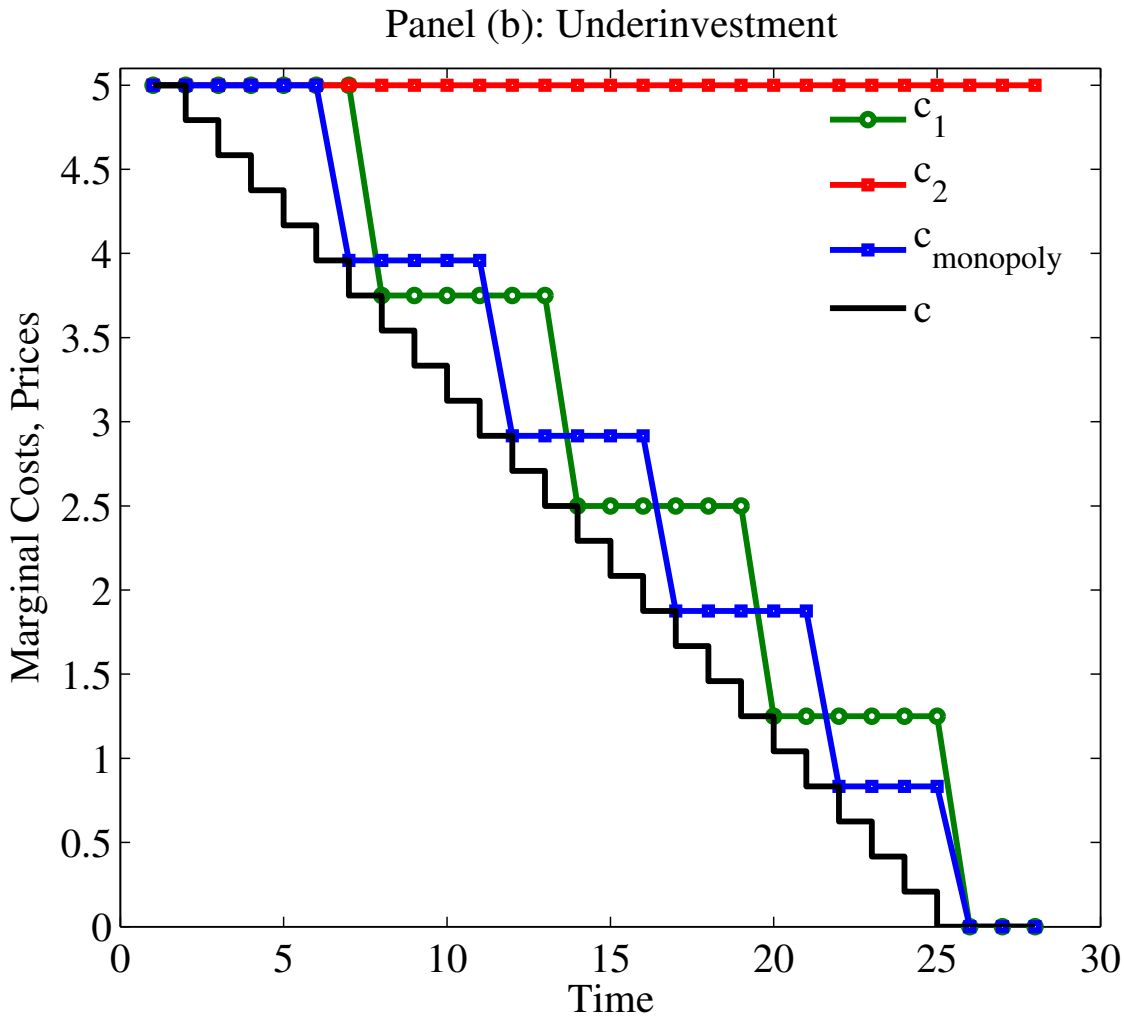
Full preemption and rent dissipation

Confirm R&S the result with high K and small dt



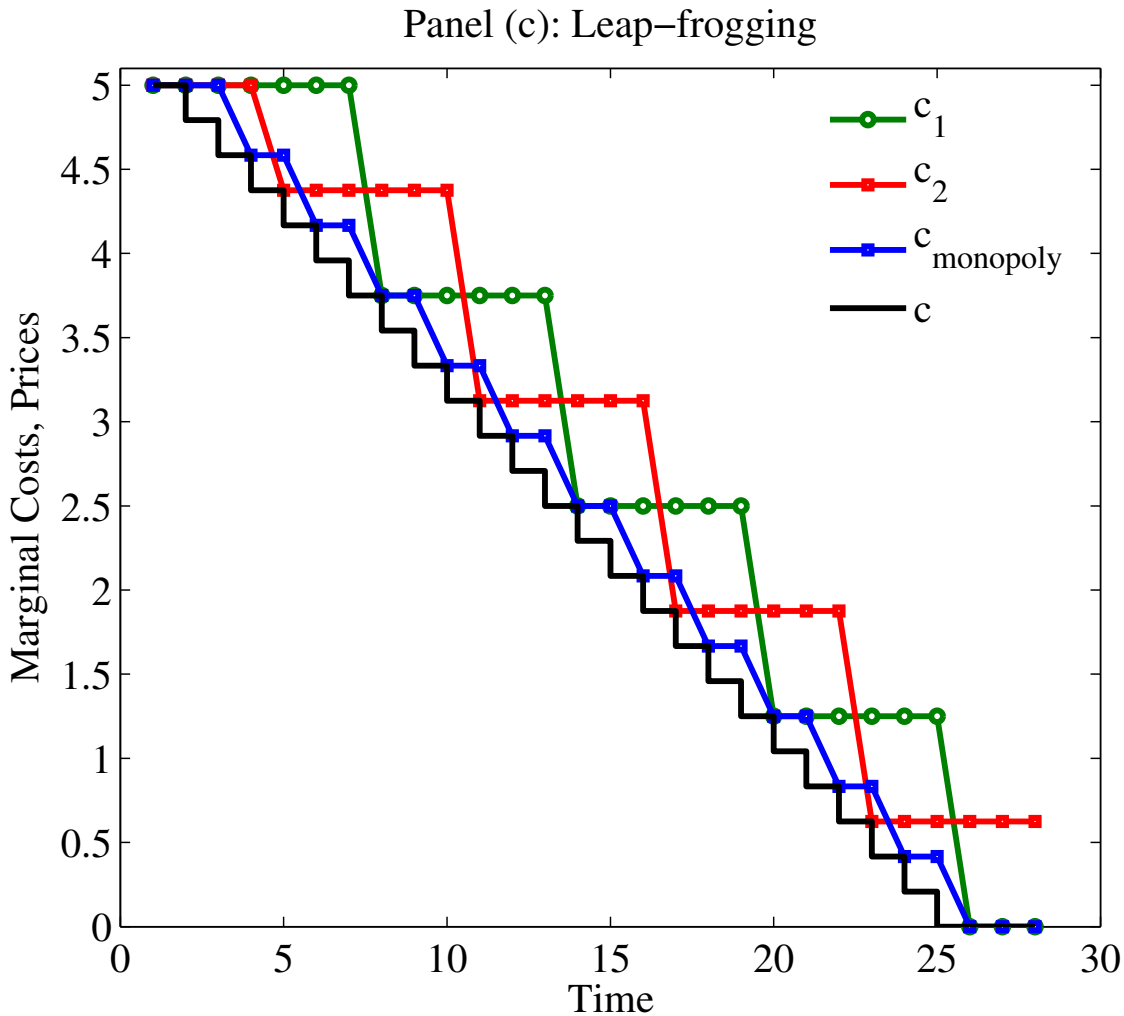
Underinvestment

Rent-dissapation is not a general outcome - disappears when K is low relative dt



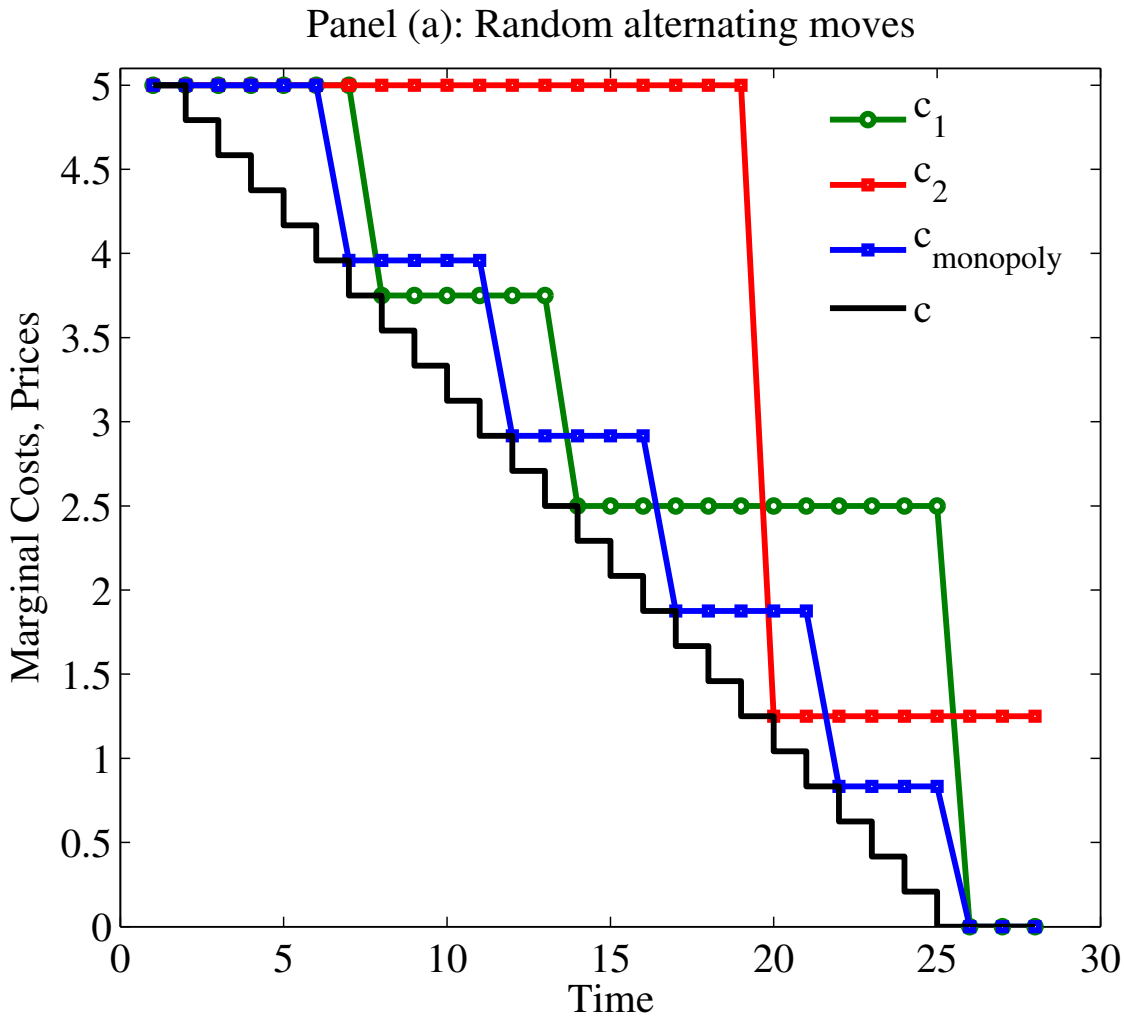
Leap-frogging

Preemption is not the general outcome - disappears when K is even lower



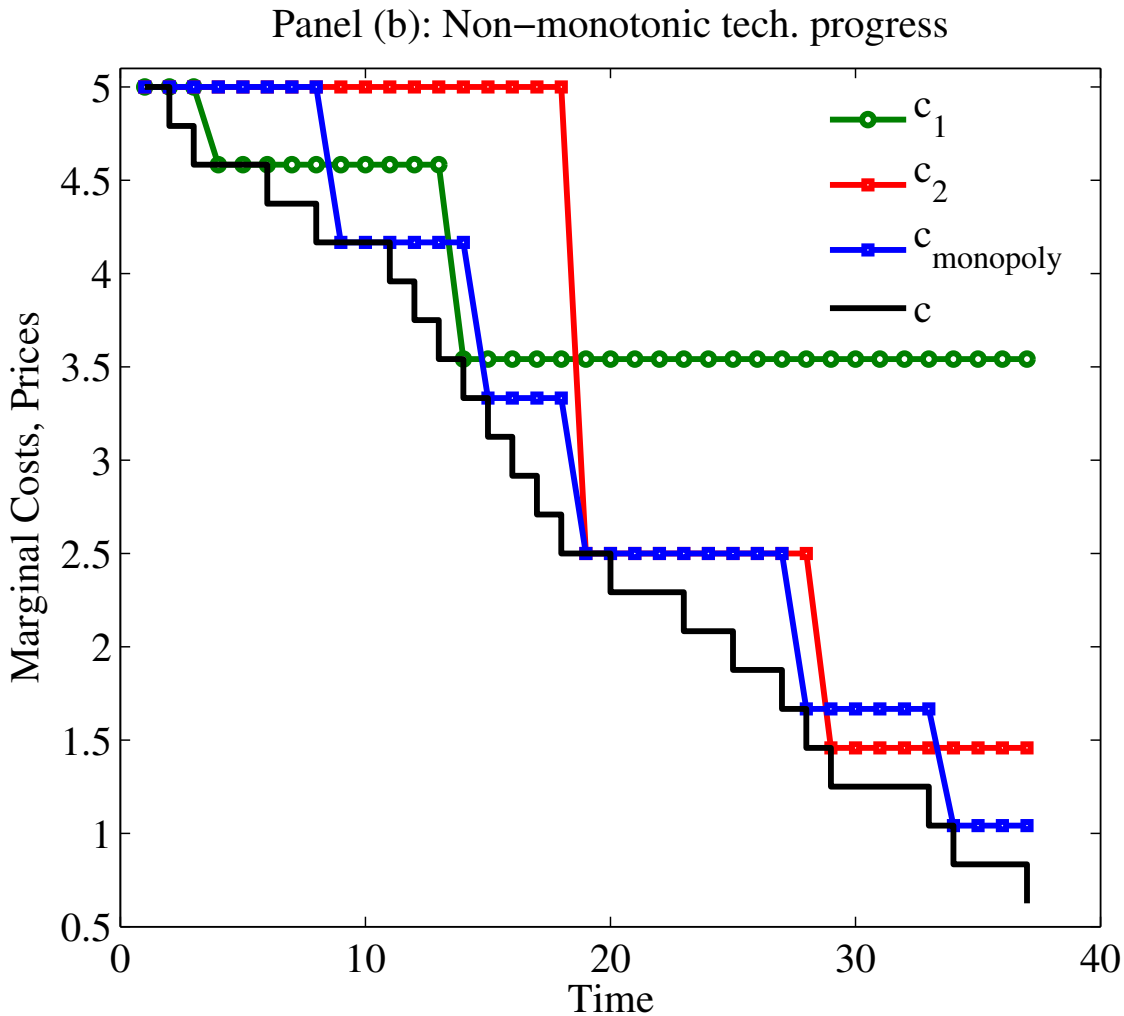
Random alternation → Leapfrogging

Riordan and Salant's result is not robust



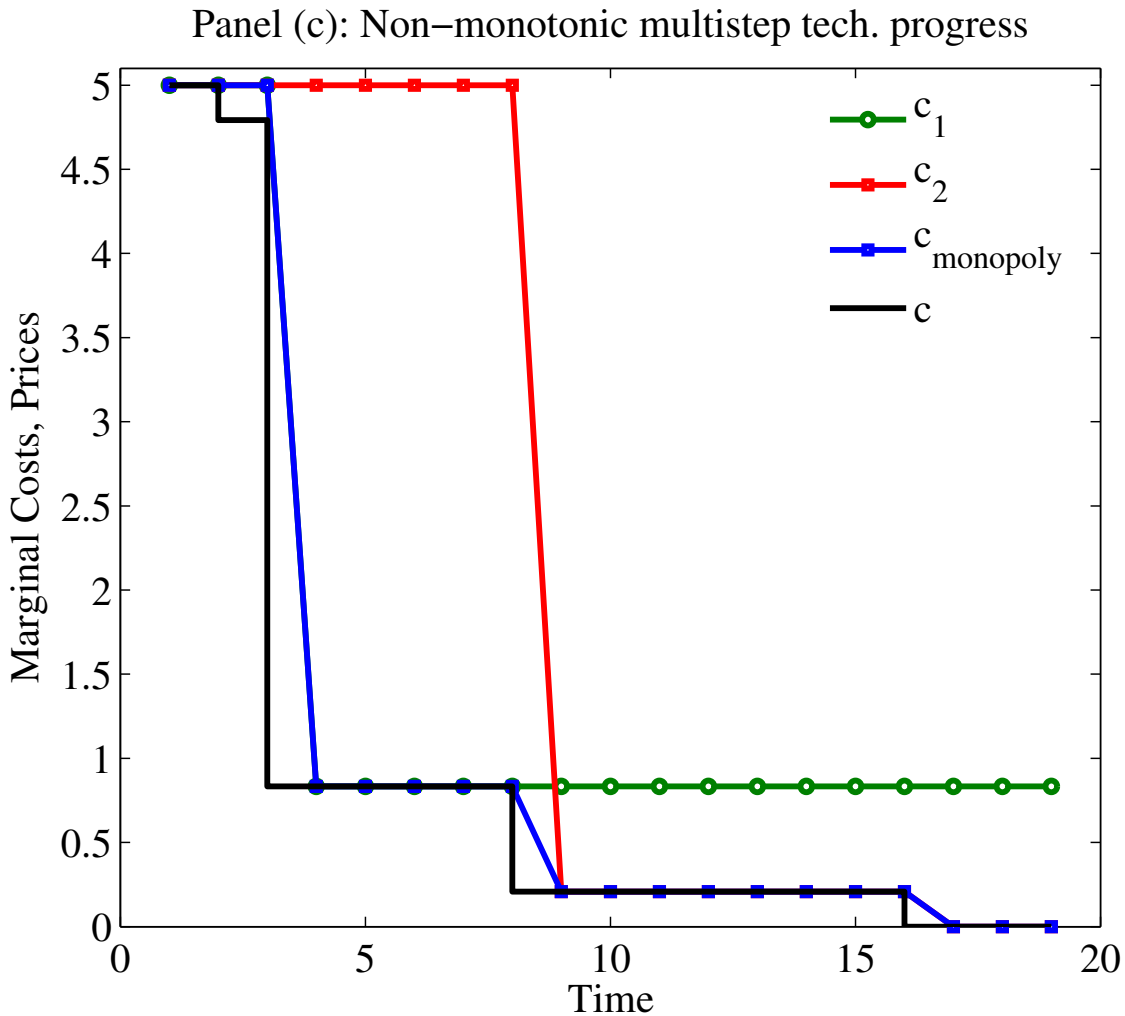
Random onestep technology → Leapfrogging

Riordan and Salant's result is not robust

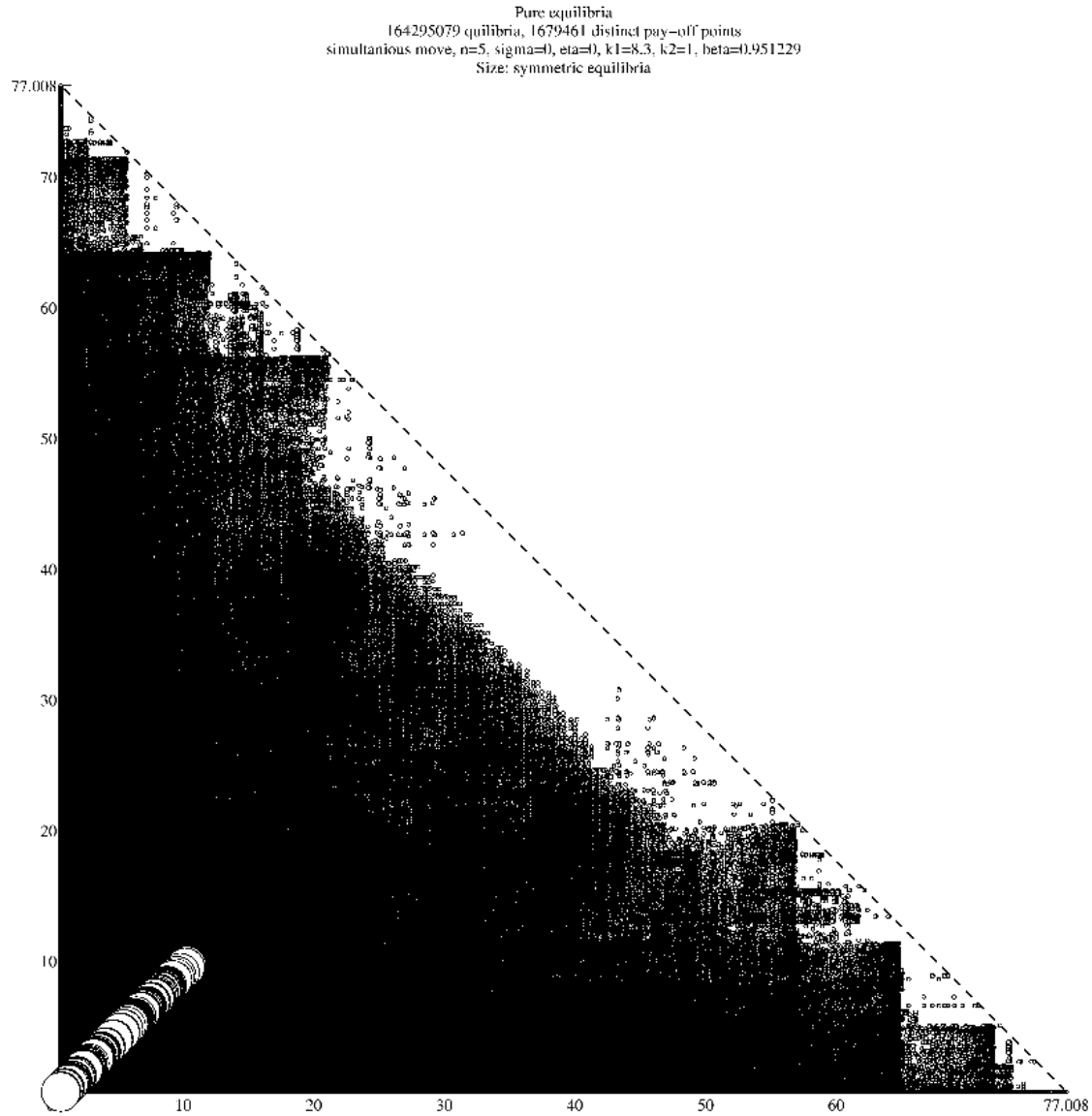


Random multistep technology → Leapfrogging

Riordan and Salant's result is not robust



Symmetric equilibria: $V_1(c_1, c_2, c) = V_2(c_2, c_1, c)$



Failure of homotopy approach in leapfrogging game

Homotopy parameter: η

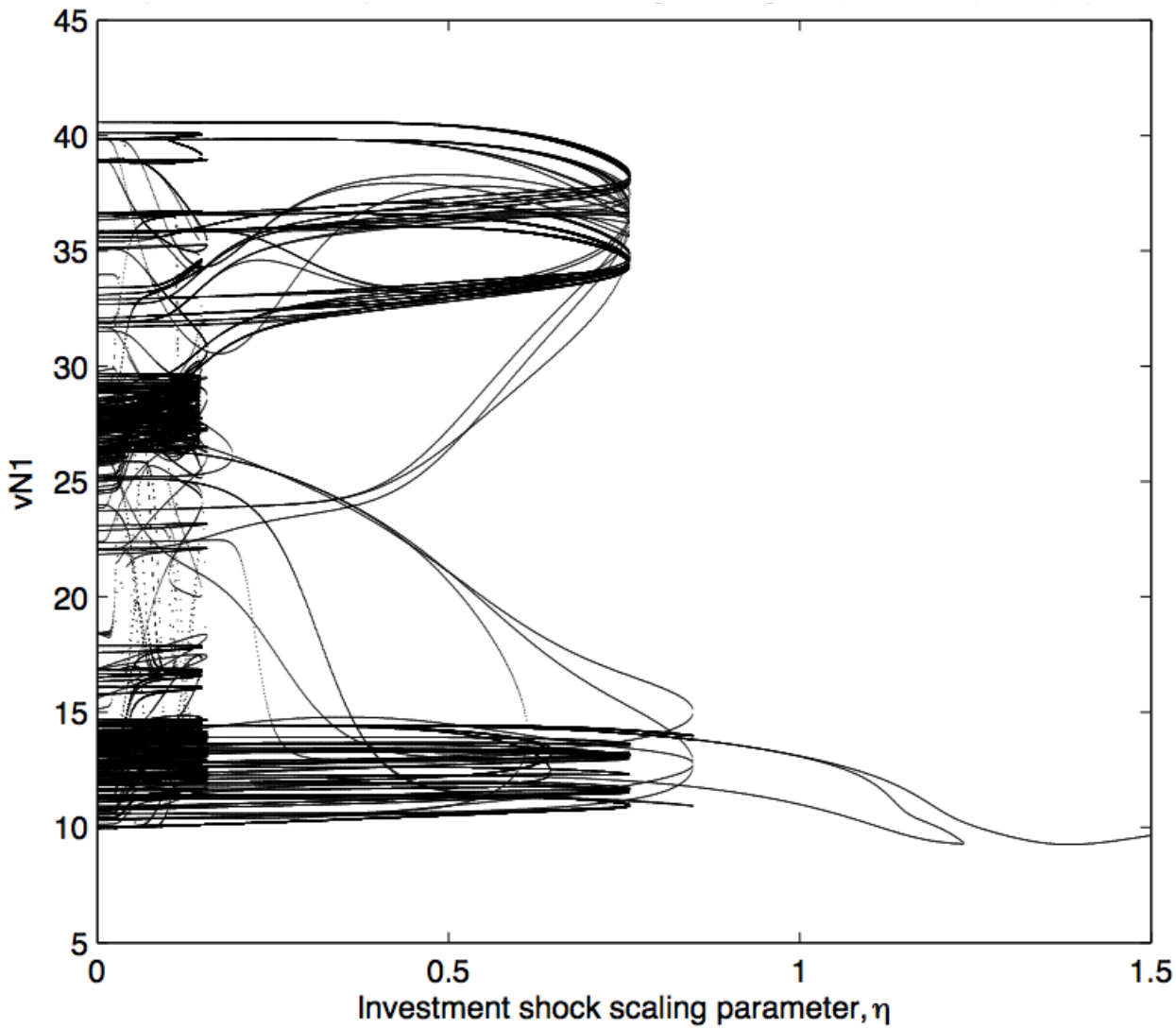
- In each period each firm incurs additive random costs/benefit from not investing and investing
- η is a scaling parameter that index variance of idiosyncratic shocks to investment
- High $\eta \rightarrow$ unique equilibrium $\eta \rightarrow 0 \rightarrow$ multiple equilibria

Problems:

- Multiplicity of equilibria $\rightarrow$ too many bifurcations along the path
- Equilibrium correspondence is not lower hemi-continuous

Failure of homotopy approach in leapfrogging game

Equilibrium correspondance, alternating move game: $V_{N,1}(c_0, c_0, c_0)$ vs. η



Failure of homotopy approach in leapfrogging game

Video: Set of equilibrium outcomes as variance of shocks decreases to zero