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Welfare Comparisons for Biased Learning†

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We study robust welfare comparisons of learning biases (misspecified Bayesian and some forms of non-Bayesian updating). Given a true signal distribution, we deem one bias more harmful than another if it yields lower objective expected payoffs in all decision problems. We characterize this ranking in static and dynamic settings. While the static characterization compares posteriors signal by signal, the dynamic characterization employs an "efficiency index" measuring how fast beliefs converge. We quantify and compare the severity of several well-documented biases. We also highlight disagreements between the static and dynamic rankings, and that some "large" biases dynamically outperform other "vanishingly small" biases. (*JEL* D60, D82, D83, D91)

A growing literature in behavioral economics studies ways in which individuals' learning departs from correct Bayesian updating, whether due to psychological biases and limitations or due to simplifying assumptions about a complex environment. Experimental work has documented a variety of systematic learning biases, such as under- or overreaction to information, overconfidence, and correlation neglect (for a recent survey, see Benjamin 2019). Each such learning bias can lead to inefficient choices in many important economic problems, from career choices to financial investment decisions and voting. Are some learning biases more harmful than others?

In this paper, we study how to compare the welfare costs of different learning biases. Reflecting the many different economic decisions biased agents might face, we take a *robust* approach: given a true signal structure, we deem one bias more harmful than another if it induces a lower objective expected payoff in *all* decision problems. This exercise can be viewed as a counterpart for learning biases of the classic literature on comparisons of statistical experiments: whereas this literature asks when one true signal structure leads to higher payoffs than another in all decision problems assuming that agents interpret signals correctly, we fix a true signal

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structure and robustly compare welfare across different learning biases (i.e., misperceptions of the signal structure).

Our main results characterize this welfare ranking in both a static (one signal observation) and a dynamic (many signal observations) setting. The static ranking can be seen as an analog for learning biases of Blackwell's (1951) order over experiments and is a conservative partial order. The dynamic ranking is a misspecified counterpart of Moscarini and Smith's (2002) dynamic extension of Blackwell's order and is based on quantifying the speed of belief convergence under each learning bias; complementing an active theoretical literature that derives asymptotic beliefs under various learning biases (see literature review), this allows one to rank the wide range of learning biases that induce the same asymptotic beliefs.

Our results provide a welfare-founded approach to quantify and compare the severity of many well-documented learning biases—both for different degrees of the same bias (e.g., varying degrees of under-/overreaction or overconfidence) and across qualitatively different biases (e.g., correlation neglect versus variable neglect). We also highlight several general implications: First, the static and dynamic welfare rankings can disagree; that is, some biases are robustly less harmful than others when agents have access to little data but robustly more harmful in data-rich settings. This is a notable contrast with comparisons of correctly interpreted signal structures: there, signal structures that are more valuable based on one signal observation remain more valuable after any number of signal observations. Second, according to the dynamic ranking, some seemingly "vanishingly small" biases are worse than other "large" biases. Finally, when agents are uncertain about both payoff-relevant states and the signal structure, some biases dynamically outperform correctly specified learning.

Section I sets up the model. A fixed and unknown state of the world θ is drawn from some finite state space, and an agent learns about θ by observing T conditionally i.i.d. signal draws from some true signal structure μ. In modeling how the agent forms her posterior p_T about θ, we allow for any learning bias that can be represented as Bayesian updating under some possibly incorrect perception $\hat{\mu}$ of signal likelihoods. This setting encompasses misspecified Bayesian learning (e.g., overconfidence or correlation neglect) as the leading special case but also several forms of non-Bayesian learning, such as the illustrative under-/overreaction example below; Remark 1 elaborates on the scope and limitations. Upon observing T signals, the agent faces a decision problem, where her utility to each action depends on θ and she maximizes her subjective expected utility given her posterior p_T. We seek to characterize when the agent's welfare under bias $\hat{\mu}^1$ exceeds that under bias $\hat{\mu}^2$ robustly, i.e., regardless of her action set and utility (e.g., risk preferences). Here, taking the perspective of an outside observer, we define welfare as the agent's ex ante expected utility, where expectations are based on the true signal structure μ.

Section II characterizes our welfare rankings in the binary-state setting (Section IVC extends the main results to arbitrary finite states). As a benchmark, we first consider the static welfare ranking, where the agent observes a single signal draw ($T = 1$). Proposition 1 shows that in this case bias $\hat{\mu}^1$ is robustly less harmful than bias $\hat{\mu}^2$ if (and, under a monotonicity condition, only if) each signal is interpreted more accurately under $\hat{\mu}^1$ than under $\hat{\mu}^2$, as formalized by a nested likelihood ratio condition.

Our main focus is on the dynamic welfare ranking, where the number T of signal draws is large. To this end, we introduce a simple *learning efficiency index*: for any true signal structure, this measures how atypical it is for the agent to encounter "confusing" signal sequences, i.e., signal sequences that make the two states indistinguishable based on her perceived signal structure. The key observation, based on arguments from large deviation theory, is that this index quantifies how fast the agent's beliefs converge; that is, for the rate of belief convergence, confusing signal sequences, rather than "extreme" signal sequences that are strongly indicative of either state, are all that matters. Using this observation, Theorems 1 and 3 show that for any biases that give rise to the same asymptotic beliefs, the dynamic welfare ranking is characterized by the learning efficiency index and hence, unlike the static ranking, is generically complete: with correct (respectively, incorrect) asymptotic beliefs, higher (respectively, lower) learning efficiency is better, as this reduces the medium-run likelihood of suboptimal choices in all decision problems. For biases that do not share the same asymptotic beliefs, the difference in asymptotic beliefs can be used to conduct a dynamic welfare comparison.[1]

Based on these characterizations, Section IIC discusses when the static and dynamic rankings disagree, as we illustrate in Example 1. We also identify a unique class of biases—Phillips and Edwards's (1966) model of symmetric under- or overreaction—that attain the same maximal learning efficiency as the correctly specified case. Thus, in dynamic settings, any Phillips-Edwards bias, no matter how significant, robustly outperforms any other bias, even if the latter is vanishingly small. Section IID applies our welfare rankings to several other widely studied learning biases.

Section III considers an agent who, instead of dogmatically perceiving a single signal structure $\hat{\mu}$, entertains a set of possible signal structures and jointly learns about both payoff-relevant states and signal structures. This can be interpreted as capturing some cautiousness against misspecification, and we show that such cautiousness may backfire: even if the set of signal structures the agent entertains is correctly specified (i.e., contains the true signal structure), she may be dynamically worse off than under some forms of misspecification. This is because, while the agent learns both the true state and signal distribution asymptotically, away from the limit, her uncertainty about the signal structure leads to more mistakes, as it increases the likelihood of observing signals that do not allow her to distinguish different states.

EXAMPLE 1 (Illustrative Example: Asymmetric Under-/Overreaction): *Consider the following widely studied learning bias* (*e.g.*, *Möbius et al.* 2022). *An agent learns about some fixed and unknown state θ (e.g., her ability) that is either low, $\underline{\theta}$, or high, $\bar{\theta}$, with prior probabilities $p_0(\underline{\theta}) = p_0(\bar{\theta}) = 1/2$. She observes a sequence $(x_1, \ldots, x_T)$ of T signals, drawn conditionally i.i.d. from the binary set $\{\underline{x}, \bar{x}\}$ with probabilities $\mu_{\bar{\theta}}(\bar{x}) = 0.8$ in state $\bar{\theta}$ and $\mu_{\underline{\theta}}(\underline{x}) = 0.8$ in state $\underline{\theta}$. Thus, signal $\bar{x}$*

[1] While biases that induce correct asymptotic beliefs robustly outperform all biases that lead to incorrect learning, the welfare comparison between two biases that give rise to different incorrect asymptotic beliefs in general depends on the decision problem.

(*respectively,* $\underline{x}$) *is "good news"* (*respectively, "bad news"*) *about* θ*. After each signal observation* x_t*, the agent updates her belief* p_t *using a distorted likelihood ratio,*

$$(1) \qquad \frac{p_t(\overline{\theta})}{p_t(\underline{\theta})} = \frac{p_{t-1}(\overline{\theta})}{p_{t-1}(\underline{\theta})}\left[\frac{\mu_{\overline{\theta}}(x_t)}{\mu_{\underline{\theta}}(x_t)}\right]^{c(x_t)},$$

where $c(x) > 0$ *for each x. The case* $c(\bar{x}) = c(\underline{x}) = 1$ *corresponds to correct Bayesian updating. By contrast,* (1) *can accommodate under- or overreaction to some signals and allows for these departures to vary across different signals. For example,* $c(\bar{x}) > c(\underline{x})$ *captures a form of "ego-biased" updating, where the agent reacts more strongly to good news about her ability than to bad news.*

The experimental literature has estimated distortion functions $c(\cdot)$ *for various subjects.*[2] *One difficulty in interpreting these findings is that, based on casual inspection, it may not be obvious how to evaluate the severity of a given distortion function. Applying the results in this paper, we can provide a welfare-founded answer by characterizing when the agent's welfare under distortion* $c^1(\cdot)$ *exceeds that under* $c^2(\cdot)$ *robustly, i.e., in any decision problem she might face after forming posterior* p_T*.*

Static Ranking.—As a benchmark, suppose the agent observes a single signal draw ($T = 1$)*. Then our results yield an incomplete ranking over distortion functions:* $c^1(\cdot)$ *is less harmful than* $c^2(\cdot)$ *in every decision problem if and only if, for each signal x,*

$$(2) \qquad c^2(x) \leq c^1(x) \leq 1 \quad \text{or} \quad 1 \leq c^1(x) \leq c^2(x),$$

i.e., the interpretation of both signals $\underline{x}$ *and* $\bar{x}$ *is more accurate under* $c^1(\cdot)$ *than* $c^2(\cdot)$*.*

Dynamic Ranking.—Suppose instead that the agent has access to many signal draws. Now, our analysis implies that in every decision problem and for any large enough number T of signal draws, the ranking depends only on the ratio $c^i(\bar{x})/c^i(\underline{x})$*, where welfare is higher the closer this ratio is to* 1*.*

To see the idea, note that for any $c^i(\cdot)$*, the agent's belief* p_T *after T signals satisfies*

$$\frac{1}{T}log\frac{p_T(\overline{\theta})}{p_T(\underline{\theta})} = \left[\nu_T(\bar{x})c^i(\bar{x}) - \nu_T(\underline{x})c^i(\underline{x})\right]log4,$$

where ν_T *is the empirical distribution of signals after T draws. Figure 1 plots beliefs as a function of* ν_T *under distortions* $c^1(\bar{x}) = 0.9, c^1(\underline{x}) = 0.7$ *and* $c^2(\bar{x}) = c^2(\underline{x}) = 0.2$*.*

[2] The estimated parameters in Möbius et al. (2022) are $c(\bar{x}) = 0.27$, $c(\underline{x}) = 0.17$. Subsequent work has estimated how distortion parameters vary with gender and other demographic characteristics (see the survey by Benjamin 2019).

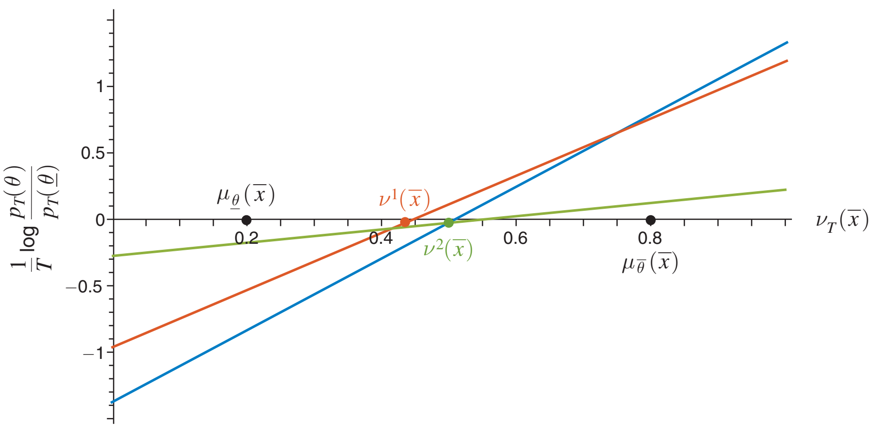

FIGURE 1

Note: Beliefs as a function of the empirical frequency of high signals under correct Bayesian updating (blue) and distortions $c^1(\overline{x}) = 0.9, c^1(\underline{x}) = 0.7$ (orange) and $c^2(\overline{x}) = c^2(\underline{x}) = 0.2$ (green).

At large T, beliefs concentrate on $\overline{\theta}$ if $\nu_T(\overline{x}) > \nu^i(\overline{x})$ and concentrate on $\underline{\theta}$ if $\nu_T(\overline{x}) < \nu^i(\overline{x})$; here, the indistinguishable distribution $\nu^i(\overline{x}) := 1/\left[1 + c^i(\overline{x})/c^i(\underline{x})\right]$ is the empirical frequency that does not allow the agent to distinguish the two states (i.e., p_T remains equal to the prior). Moreover, in each state θ, ν_T converges almost surely to the true signal distribution μ_θ as $T \to \infty$ (by the law of large numbers), so asymptotic beliefs in state θ are determined by whether $\mu_\theta(\overline{x})$ is greater or less than $\nu^i(\overline{x})$. When asymptotic beliefs under two distortions differ, these differences determine the dynamic welfare ranking. However, this leaves a large class of distortions $c^i(\cdot)$ that give rise to the same (correct or incorrect) asymptotic beliefs.

Crucially, our analysis also yields a complete ranking across any such $c^i(\cdot)$. To illustrate, suppose asymptotic beliefs are correct, as is the case for the two distortions in Figure 1. To determine which distortions perform better at large but finite T, we need to take into account that, away from the limit, empirical signal distributions ν_T other than μ_θ can realize. Distortion $c^i(\cdot)$ affects which such ν_T lead to mistakes: beliefs in state $\underline{\theta}$ concentrate on the wrong state $\overline{\theta}$ if $\nu_T(\overline{x}) > \nu^i(\overline{x})$, and beliefs in state $\overline{\theta}$ incorrectly concentrate on $\underline{\theta}$ if $\nu_T(\overline{x}) < \nu^i(\overline{x})$.

Our key insight is that the ex ante probability of observing such mistake-inducing distributions ν_T at large T is quantified by a simple learning efficiency index: this index considers the distance from the indistinguishable distribution ν^i to the true distribution $\mu_{\overline{\theta}}$ or $\mu_{\underline{\theta}}$ in state $\overline{\theta}$ or $\underline{\theta}$, whichever is closer;[3] the smaller this index, the more likely are mistakes at large T. Indeed, as we will see, among all mistake-inducing distributions ν_T, the agent is much more likely at large T to observe "confusing" distributions $\nu_T \approx \nu^i$ that make it difficult to distinguish the two states rather than "extreme"

[3] Our formal analysis quantifies the distance of ν^i to μ_θ using Kullback-Leibler divergence, but for the purpose of comparing the biases in Figure 1, one can consider $\left|\nu^i(\overline{x}) - \mu_\theta(\overline{x})\right|$.

distributions ν_T *that are strongly indicative of the wrong state* $(e.g., \nu_T(\bar{x}) \gg \nu^i(\bar{x})$ *in state* $\underline{\theta})$. *Moreover, the probability of observing such confusing* $\nu_T \approx \nu^i$ *in state* θ *decays more slowly with T the closer* ν^i *is to the true signal distribution* μ_θ. *Finally, the ex ante probability of mistakes is driven by the state* θ *where* ν^i *is closer to* μ_θ, *as mistakes are much more likely to occur in this state.*

In the present example, learning efficiency is maximized when $\nu^i(\bar{x}) = 1/2$, *which is the indistinguishable distribution under correct Bayesian updating, and learning efficiency is higher the smaller* $\left|\nu^i(\bar{x}) - 1/2\right| = \left|1/\left[1 + c^i(\bar{x})/c^i(\underline{x})\right] - 1/2\right|$, *i.e., the closer* $c^i(\bar{x})/c^i(\underline{x})$ *is to 1.*[4] *In Figure 1,* $\nu^2(\bar{x}) = 1/2$, *but* $\nu^1(\bar{x}) = 7/16$, *so the symmetric but significant distortion* $c^2(\cdot)$ *outperforms the asymmetric but more moderate distortion* $c^1(\cdot)$ *at all large T.*

A key implication is that the static and dynamic rankings can disagree. Indeed, since $c^1(\cdot)$ *distorts each individual signal much less than* $c^2(\cdot)$, (2) *implies that* $c^1(\cdot)$ *robustly outperforms* $c^2(\cdot)$ *at* $T = 1$, *reversing the dynamic ranking.*

To understand why such reversals occur, consider Figure 1. *Observe that posteriors after all empirical distributions* ν_T *that are sufficiently skewed toward either* $\underline{x}$ *or* $\bar{x}$ *are closer to correct Bayesian updating under* $c^1(\cdot)$ *than under* $c^2(\cdot)$. *This includes the extreme frequencies* $\nu_T(\bar{x}) \in \{0, 1\}$, *which are the only possible signal frequencies at* $T = 1$. *Importantly, however, the opposite is true after balanced empirical frequencies* $\nu_T(\bar{x}) \approx 1/2$*: here,* $c^2(\cdot)$ *distorts posteriors less than* $c^1(\cdot)$. *Intuitively, under* $c^2(\cdot)$, *the agent updates too little in the direction of* $\underline{\theta}$ *following* $\underline{x}$ *but symmetrically underreacts in the direction of* $\bar{\theta}$ *following* $\bar{x}$, *and these two errors "cancel out" when interpreting balanced signal sequences. As a result, the indistinguishable distribution* $\nu^2(\bar{x}) = 1/2$ *under* $c^2(\cdot)$ *coincides with that under correct Bayesian updating, whereas the indistinguishable distribution* $\nu^1(\bar{x}) = 7/16$ *under* $c^1(\cdot)$ *is distorted relative to correct Bayesian updating. As we saw, how* $c^i(\cdot)$ *affects the indistinguishable distribution is all that matters for welfare at large T. Thus,* $c^2(\cdot)$ *dynamically outperforms* $c^1(\cdot)$, *despite interpreting many signal frequencies* ν_T *much less accurately than* $c^1(\cdot)$.

Finally, we note that measures sometimes used to quantify the severity of bias (1) *in applications, e.g., the difference* $c^i(\bar{x}) - c^i(\underline{x})$ *(Coutts 2019), do not reflect the welfare costs of this bias, as they can disagree with our welfare-founded rankings.*

Related Literature.—We contribute to the theoretical literature on misspecified Bayesian and non-Bayesian learning. Much work studies how learning biases affect asymptotic beliefs, in both single-agent (e.g., Berk 1966; Nyarko 1991; Fudenberg, Romanyuk, and Strack 2017; Heidhues, Koszegi, and Strack 2018, 2021; He 2022; Bushong and Gagnon-Bartsch 2022) and social learning settings (e.g., Eyster and Rabin 2010; Bohren 2016; Gagnon-Bartsch 2017; Frick, Iijima, and Ishii 2020). Several recent papers provide more general criteria to determine convergence to asymptotic beliefs (Bohren and Hauser 2021; Esponda, Pouzo, and Yamamoto 2021; Frick, Iijima, and Ishii 2023; Fudenberg, Lanzani, and Strack 2021).[5] In contrast,

[4] Indeed, in the current setting, the smaller $\left|\nu^i(\bar{x}) - 1/2\right|$, the smaller is $\min_{\theta \in \{\underline{\theta}, \bar{\theta}\}} \left|\nu^i(\bar{x}) - \mu_\theta(\bar{x})\right|$.

[5] Esponda and Pouzo (2016, 2021) formalize Berk-Nash equilibrium, which captures steady states of general misspecified learning dynamics. Spiegler (2016) formalizes a steady-state notion for subjective causal models that

our focus is on the welfare implications of learning biases, which requires us to analyze short-/medium-run beliefs and the speed of belief convergence. We restrict attention to single-agent learning with exogenous i.i.d. signals, but Section IVD briefly discusses some extensions.

Our exercise is broadly related to a number of papers that examine whether and how specific misspecifications can "survive" based on a variety of selection criteria, including performance in competitive markets (e.g., Sandroni 2000; Blume and Easley 2006; Massari 2020), goodness-of-fit tests (e.g., Cho and Kasa 2015; Gagnon-Bartsch, Rabin, and Schwartzstein 2018; Schwartzstein and Sunderam 2021; Ba 2022), voting (e.g., Levy, Razin, and Young 2022), and subjective welfare (e.g., Montiel Olea et al. 2022; Eliaz and Spiegler 2020). More closely related, Fudenberg and Lanzani (2022) and He and Libgober (2021) study selection based on objective welfare. These papers take evolutionary approaches by characterizing which forms of misspecification are stable against mutations. While they conduct their analysis in fixed environments based on long-run outcomes, we compare welfare across all decision problems and consider short-/medium-run beliefs. He and Libgober (2021) show that misspecified agents can be better-off than correctly specified agents under strategic externalities. Section III highlights an alternative mechanism in a single-agent setting: under uncertainty about the signal structure, some forms of misspecification outperform correctly specified learning by speeding up belief convergence.[6]

Finally, as noted in the introduction, our exercise of robustly comparing learning biases is in the same spirit as the classic literature on robustly comparing statistical experiments (e.g., Blackwell 1951; Moscarini and Smith 2002; Azrieli 2014; Mu et al. 2021). In the static setting, Morris (1991) studies a generalization of Blackwell (1951) that compares pairs of true and perceived signal structures. While his ranking is characterized by a system of linear equalities generalizing Blackwell's garbling condition, we obtain a simpler sufficient condition for our static ranking that is also necessary under a comonotonic likelihood ratio property. Our main focus is on the dynamic ranking, which is a misspecified analog of Moscarini and Smith (2002). Indeed, as Remark 2 discusses, when agents are correctly specified (i.e., $\mu = \hat{\mu}$), our efficiency index reduces to the one in Moscarini and Smith (2002).

I. Model

A state θ is drawn once and for all from a finite set Θ according to a full-support distribution $p_0 \in \Delta(\Theta)$. An agent does not observe the realized state θ but learns about θ from signal observations. There is a finite set of signals X, and the agent observes a sequence of T signal draws, $x^T = (x_1, x_2, \ldots, x_T) \in X^T$.[7] Each signal x_t is drawn conditionally i.i.d. according to a true signal structure $\mu := (\mu_\theta)_{\theta \in \Theta}$, where

are captured by directed acyclic graphs (DAGs); he also asks when one DAG is robustly better than another based on objective steady-state payoffs and finds that no two DAGs can be ranked in this way (except if one DAG is fully connected, i.e., correctly specified).

[6] Steiner and Stewart (2016) and Gossner and Steiner (2018) consider static single-agent settings and show that misspecification can be beneficial when agents cannot perfectly implement their subjectively optimal strategies due to the presence of noise.

[7] All results can be generalized to infinite signals, allowing for unbounded likelihood ratios.

$\mu_\theta \in \Delta(X)$ denotes the true signal distribution conditional on state θ. Assume that each μ_θ has full support over X and that $\mu_\theta \neq \mu_{\theta'}$ for all $\theta \neq \theta'$.

The agent's learning from signals is potentially biased. Her perceived signal structure is $\hat{\mu} := (\hat{\mu}_\theta)_{\theta \in \Theta}$, where $\hat{\mu}_\theta \in \mathbb{R}^X_{++}$ captures the agent's perceived likelihood of each signal conditional on state θ and $\hat{\mu}_\theta \neq \hat{\mu}_{\theta'}$ for all $\theta \neq \theta'$. While we assume that $\hat{\mu}_\theta(x) > 0$ for each $x \in X$, we do not require that $\sum_{x \in X} \hat{\mu}_\theta(x) = 1$. As we discuss in Remark 1, this makes it possible to accommodate both misspecified Bayesian learning as well as some common classes of non-Bayesian learning. We call the agent correctly specified if $\hat{\mu} = \mu$. We assume for simplicity that the agent has the correct prior p_0 over states, but this is not important for our analysis.[8]

Upon observing the signal sequence $x^T = (x_1, x_2, \ldots, x_T)$, the agent forms a posterior belief $p_T(\cdot \mid x^T) \in \Delta(\Theta)$ by applying Bayes' rule according to her perceived signal structure. That is, for all $\theta' \in \Theta$,

$$p_T(\theta' \mid x^T) = \frac{p_0(\theta') \prod_{t=1}^T \hat{\mu}_{\theta'}(x_t)}{\sum_{\theta'' \in \Theta} p_0(\theta'') \prod_{t=1}^T \hat{\mu}_{\theta''}(x_t)}. \tag{3}$$

After forming the posterior belief, the agent faces a decision problem, which is a nonempty finite set $A \subseteq \mathbb{R}^\Theta$.[9] Vectors $a \in A$ are (utility) acts, where a_θ denotes the agent's utility from a conditional on state θ. Note that each decision problem jointly summarizes both the agent's action set and her state-dependent utility to each action. At any decision problem A and realized signal sequence x^T, the agent chooses an act $a^*(x^T, \hat{\mu}) \in A$ to maximize her subjective expected utility under her posterior belief $p_T(\cdot \mid x^T)$:

$$a^*(x^T, \hat{\mu}) \in \operatorname*{arg\,max}_{a \in A} \sum_{\theta \in \Theta} p_T(\theta \mid x^T) a_\theta. \tag{4}$$

For ease of exposition, we assume throughout the main text that $\hat{\mu}$ and A are such that (4) admits a unique solution. All results extend to the case with ties, and our proofs in the Appendix allow for ties.[10] We also assume decision problems are nontrivial, i.e., contain no dominant act $a \in A$ with $a_\theta \geq b_\theta$ for all $\theta \in \Theta$ and $b \in A$.

The agent's welfare is her ex ante expected utility to choosing her $\hat{\mu}$-subjectively optimal act $a^*(x^T, \hat{\mu})$ at each x^T. Here, taking the perspective of an outside observer, expectations over signal realizations are based on the *true* signal structure μ. That is, letting μ^T_θ denote the true distribution over signal sequences in X^T conditional on state θ, the agent's welfare is given by

$$W_T(\mu, \hat{\mu}, A) = \sum_{\theta \in \Theta} p_0(\theta) \sum_{x^T \in X^T} \mu^T_\theta(x^T) a^*_\theta(x^T, \hat{\mu}). \tag{5}$$

[8] It is straightforward to allow for incorrect full-support priors (that may vary across biases $\hat{\mu}$), while defining ex ante expected payoffs in (5) based on the true prior p_0. The dynamic welfare characterizations (Theorems 1–4) remain true unchanged, as the prior has a negligible effect at large T. For the static characterization (Proposition 1), we generalize condition (6) to the requirement that after each signal, the posterior under $\hat{\mu}^1$ is a convex combination of those under μ and $\hat{\mu}^2$.

[9] Finiteness can be relaxed when $T = 1$, but it is important for the dynamic case.

[10] The static welfare characterization (Proposition 1) remains valid as long as a fixed tiebreaking rule (i.e., strict total order over acts) is used to select among multiple solutions to (4). The dynamic characterizations (Theorems 1–2) remain valid even when tiebreaking rules vary across biases $\hat{\mu}$.

Given any true signal structure μ, we consider two agents $i = 1,2$ who differ only in their perceived signal structures $\hat{\mu}^i$. We seek to characterize when agent 1's bias is robustly less harmful than agent 2's, in the sense that agent 1's welfare exceeds agent 2's welfare at all decision problems. As a benchmark, we first consider a static welfare ranking: this assumes that agents observe a single signal draw $(T=1)$ and requires that for all A, $W_1(\mu,\hat{\mu}^1,A)$ exceeds $W_1(\mu,\hat{\mu}^2,A)$. Our main focus is on a dynamic welfare ranking: this assumes that agents have access to many signal draws and requires that for all A, $W_T(\mu,\hat{\mu}^1,A)$ exceeds $W_T(\mu,\hat{\mu}^2,A)$ whenever T is large enough.

REMARK 1 (Scope and Limitations): *The leading special case of our setting is misspecified Bayesian learning, where the agent's perceived signal distribution in each state θ is a probability measure $\hat{\mu}_\theta \in \Delta(X)$ but differs from the true signal distribution μ_θ. As is well-known, this formulation can capture many important learning biases. The following are some widely studied examples to which we will apply our results in Section IID.*[11]

- *Under overconfidence, signals (e.g., output levels) are linearly ordered, and the perceived signal distribution $\hat{\mu}_\theta$ in each state first-order stochastically dominates the true output distribution μ_θ, and vice versa for underconfidence.*
- *Under overprecision, the perceived signal structure $\hat{\mu}$ is (Blackwell) more informative than the true μ, and vice versa for underprecision.*
- *Misperceptions about multiple information sources: Suppose signal space $X = \prod_{k=1}^n Z_k$ takes a product form, where each Z_k corresponds to a different information source. Under correlation misperception, $\hat{\mu}_\theta$ misperceives (e.g., neglects) the correlation across different sources relative to the true distribution μ_θ. Neglecting variables (or sparseness) corresponds to the agent perceiving a subset $U \subset \{1,\ldots,n\}$ of sources to be uninformative about the state (given the other sources).*[12] *Under coarse reasoning, sources $\{1,\ldots,n\}$ are partitioned into analogy classes $\mathcal{K}$ with $Z_k = Z_{k'}$ for all k,k' in the same analogy class $K \in \mathcal{K}$, and the agent pools signals within each analogy class.*[13]

By allowing for nonprobabilistic perceived signal likelihoods, $\sum_{x\in X}\hat{\mu}_\theta(x) \neq 1$, our setting can also accommodate some well-documented forms of non-Bayesian learning. For instance, under asymmetric under-/overreaction (Example 1), $\hat{\mu}_{\bar{\theta}}(x)/\hat{\mu}_{\underline{\theta}}(x) = [\mu_{\bar{\theta}}(x)/\mu_{\underline{\theta}}(x)]^{c(x)}$ for each x, where $c(x) > 0$. Similarly, under

[11] The literature has considered versions of these biases in a range of settings (sometimes differing from our specific learning environment). See, e.g., the following papers and references therein: Heidhues, Koszegi, and Strack (2018, 2019) for overconfidence; Daniel and Hirshleifer (2015) for overprecision; Levy and Razin (2015) and Ortoleva and Snowberg (2015) for correlation misperception; Montiel Olea et al. (2022) and Levy, Razin, and Young (2022) for neglecting variables; and Guarino and Jehiel (2013) and Jehiel (2020) for coarse reasoning.

[12] Formally, for all θ, θ', and $y_U \in \prod_{k\in U} Z_k$, $y_{-U} \in \prod_{k\notin U} Z_k$, we have $\hat{\mu}_\theta(y_U,y_{-U})/\hat{\mu}_{\theta'}(y_U,y_{-U}) = \hat{\mu}_\theta(y_{-U})/\hat{\mu}_{\theta'}(y_{-U})$, where $\hat{\mu}_\theta(y_{-U})$ denotes the marginal of $\hat{\mu}_\theta$ over $\prod_{k\notin U} Z_k$. Thus, the agent's posterior following any profile $(z_1,\ldots,z_n)$ does not depend on the signal realizations $(z_k)_{k\in U}$ from sources in U.

[13] Formally, if $x = (z_1,\ldots,z_n)$ and $x' = (z_1',\ldots,z_n')$ are such that $(z_k)_{k\in K}$ is a permutation of $(z_k')_{k\in K}$ for each analogy class $K \in \mathcal{K}$, then $\hat{\mu}_\theta(x) = \hat{\mu}_\theta(x')$ for each θ. Thus, the agent's posterior depends only on the pooled empirical frequency of signals within each analogy class K.

partisan bias, signal likelihood ratios are distorted multiplicatively in the direction of one state, i.e., for each x, $\hat{\mu}_{\bar{\theta}}(x)/\hat{\mu}_{\underline{\theta}}(x) = \eta(x)[\mu_{\bar{\theta}}(x)/\mu_{\underline{\theta}}(x)]$ *for some* $\eta(x) > 1$.[14]

At the same time, by focusing on the simple benchmark of learning from exogenous i.i.d. signals, we rule out that true/perceived signal distributions display intertemporal correlation or vary endogenously with the current belief. We consider both these settings in a companion note (Frick, Iijima, Ishii 2022); see Section IVD for a brief discussion. Moreover, even in i.i.d. settings, the belief-updating formula (3) rules out many important non-Bayesian learning rules that (i) are sensitive to the order of signals (e.g., due to primacy/recency effects) or (ii) distort Bayes' rule in ways that cannot be replicated by Bayesian updating under nonprobabilistic perceived likelihoods $\hat{\mu}$. *As Section IVD discusses, our analysis extends straightforwardly to some cases of (ii); in contrast, (i) would require different techniques.*

II. Welfare Rankings

We now proceed to characterize the welfare rankings. For ease of exposition, we focus throughout this section on a binary state environment, $\Theta = \{\underline{\theta}, \bar{\theta}\}$. Section IVC extends the main results to general finite state spaces.

A. *Static Ranking*

As a benchmark, we first characterize the static welfare ranking. Denote by $\ell_{\mu}(x) := \mu_{\bar{\theta}}(x)/\mu_{\underline{\theta}}(x)$ the true likelihood ratio at signal x and by $\ell_{\hat{\mu}^i}(x) := \hat{\mu}^i_{\bar{\theta}}(x)/\hat{\mu}^i_{\underline{\theta}}(x)$ agent i's perceived likelihood ratio. The following result provides a sufficient condition for $\hat{\mu}^1$ to be less harmful than $\hat{\mu}^2$ based on the static welfare ranking: condition (6) requires agent 1's interpretation of each signal x to be more accurate than agent 2's, in the sense that agent 1's perceived signal likelihood ratio $\ell_{\hat{\mu}^1}(x)$ is in between the true likelihood ratio $\ell_{\mu}(x)$ and agent 2's perception $\ell_{\hat{\mu}^2}(x)$. This yields a fairly demanding partial order over biases, which guarantees that agent 1's posterior following each signal realization is a convex combination of the true posterior and agent 2's posterior and hence that agent 1's objective function is more aligned with the true objective function than agent 2's.[15] Condition (6) is also necessary under a monotonicity property.

PROPOSITION 1: *If, for each* $x \in X$,

$$\ell_{\mu}(x) \geq \ell_{\hat{\mu}^1}(x) \geq \ell_{\hat{\mu}^2}(x) \quad \text{or} \quad \ell_{\mu}(x) \leq \ell_{\hat{\mu}^1}(x) \leq \ell_{\hat{\mu}^2}(x), \tag{6}$$

then $W_1(\mu, \hat{\mu}^1, A) \geq W_1(\mu, \hat{\mu}^2, A)$ *for all decision problems A. The converse holds if* $\mu, \hat{\mu}^1, \hat{\mu}^2$ *satisfy the comonotonic likelihood ratio property; i.e., there is a linear order* $>$ *on signals such that*

$$\min\{\ell_{\mu}(x), \ell_{\hat{\mu}^1}(x), \ell_{\hat{\mu}^2}(x)\} \geq \max\{\ell_{\mu}(x'), \ell_{\hat{\mu}^1}(x'), \ell_{\hat{\mu}^2}(x')\}, \ \forall x > x'. \tag{7}$$

[14] See, e.g., Bohren and Hauser (2021) and Thaler (forthcoming).

[15] The logic is similar to results on time inconsistency, where welfare is monotonic in the degree of preference alignment between ex ante and ex post preferences (e.g., Gul and Pesendorfer 2001).

For the sufficiency direction, we note that condition (6) also implies the strict inequality $W_1(\mu, \hat{\mu}^1, A) > W_1(\mu, \hat{\mu}^2, A)$ in any decision problem in which the agents' chosen acts differ at some x. For the necessity direction, property (7) requires that signals can be ordered in such a way that, under both the true and perceived signal distributions, higher signals are more indicative of $\overline{\theta}$ than lower signals (and moreover, that for any $x > x'$, the true and perceived likelihood ratios at x versus x' can all be separated by a common constant); this rules out biases $\hat{\mu}^i$ that completely reverse the interpretation of some signals x and x' relative to the truth.[16]

EXAMPLE 1 (Continued): *In the illustrative example*, $X = \{\underline{x}, \overline{x}\}$, $\mu_{\overline{\theta}}(\overline{x}) > \mu_{\underline{\theta}}(\overline{x})$, *and* $\ell_{\hat{\mu}^i}(x) = [\ell_\mu(x)]^{c^i(x)}$, *where* $c^i(x) > 0$ *for each agent i and signal x. Thus, the sufficient condition* (6) *for the static welfare ranking is equivalent to the requirement* (2) *that for each signal x, either* $c^2(x) \leq c^1(x) \leq 1$ *or* $1 \leq c^1(x) \leq c^2(x)$. *Moreover, the comonotonic likelihood ratio property* (7) *holds with* $\overline{x} > \underline{x}$. *Thus, by Proposition* 1, *condition* (2) *is both necessary and sufficient for the static welfare ranking*.

B. *Dynamic Ranking: Correct Asymptotic Beliefs*

Our main focus is on characterizing the dynamic welfare ranking, where for any decision problem A, agent 1's welfare $W_T(\mu, \hat{\mu}^1, A)$ exceeds agent 2's welfare $W_T(\mu, \hat{\mu}^2, A)$ whenever T is large enough. This ranking is relevant when agents have access to many signal draws, as is natural in learning settings.

We first note that standard arguments (as in Berk 1966) yield agents' asymptotic beliefs $\lim_{T\to\infty} p_T$: for any $\mu, \nu \in \mathbb{R}^X_+$, define the Kullback-Leibler (KL) divergence of ν relative to μ by $KL(\nu, \mu) := \sum_x \nu(x) \log[\nu(x)/\mu(x)]$.[17] For probability distributions $\mu, \nu \in \Delta(X)$, $KL(\nu, \mu)$ is a standard statistical measure of how "atypical" an observed signal distribution ν is relative to a reference distribution μ; the preceding definition extends this measure to arbitrary nonnegative vectors. In state θ, agent i's belief p_T converges almost surely to a point-mass on state θ' if and only if

$$KL(\mu_\theta, \hat{\mu}^i_{\theta'}) < KL(\mu_\theta, \hat{\mu}^i_{\theta''}), \ \forall \theta'' \in \Theta \backslash \{\theta'\}. \tag{8}$$

That is, i's asymptotic belief concentrates on the state θ' in which, relative to i's perceived signal distribution $\hat{\mu}^i_{\theta'}$, the true long-run signal distribution μ_θ is least atypical.

If asymptotic beliefs under $\hat{\mu}^1$ and $\hat{\mu}^2$ differ, it is straightforward to determine the dynamic welfare ranking based on these different beliefs: if agent 1 learns the true state in both states but agent 2's asymptotic belief is incorrect in at least one state, then for any decision problem A, $W_T(\mu, \hat{\mu}^1, A) > W_T(\mu, \hat{\mu}^2, A)$ for all sufficiently large T. The same is true if agent 1's asymptotic belief is incorrect in only one state but agent 2's is incorrect in both states. Finally, if both agents' asymptotic beliefs are

[16] When $X = \{\underline{x}, \overline{x}\}$, (7) holds if true and perceived signal distributions both interpret $\overline{x}$ (resp. $\underline{x}$) as more indicative of state $\overline{\theta}$ (resp. $\underline{\theta}$), as is the case under most commonly studied biases, including Examples 1–3. With nonbinary signals, (7) is more demanding but is also satisfied in important examples from the literature (e.g., the correlation/variable neglect example in online Appendix E.3).

[17] We impose the standard convention that $0 \log 0 = 0/0 = 0$ and $\log(1/0) = \infty$.

incorrect in only one state and these states differ across agents, then which agent's welfare is higher depends on the decision problem and prior.[18]

Thus, the main challenge is how to compare dynamic welfare when $\hat{\mu}^1$ and $\hat{\mu}^2$ lead to the same asymptotic beliefs. This is the case for many quantitatively and qualitatively different biases when the payoff-relevant state space is coarse (e.g., binary).[19] To address this challenge, we derive a learning efficiency index that characterizes the speed of belief convergence under each bias. To illustrate this approach, in this section, we assume that both agents' biases are small enough that their asymptotic beliefs are correct; by (8), this holds under the following consistency condition. Section IVB shows how this approach extends to the case of incorrect asymptotic beliefs.

ASSUMPTION 1 (Consistency): *For any distinct* $\theta, \theta' \in \Theta$, $KL(\mu_\theta, \hat{\mu}_\theta) < KL(\mu_\theta, \hat{\mu}_{\theta'})$.

DEFINITION 1: *For any true and perceived signal structures* μ *and* $\hat{\mu}$, *define the learning efficiency index by* $w(\mu, \hat{\mu}) := \min_\theta w_\theta(\mu, \hat{\mu})$, *where*

$$(9) \qquad w_\theta(\mu, \hat{\mu}) := \min_{\nu \in \Delta(X)} KL(\nu, \mu_\theta) \quad \text{subject to} \quad KL(\nu, \hat{\mu}_{\underline{\theta}}) = KL(\nu, \hat{\mu}_{\bar{\theta}}).$$

We refer to the constraint $KL(\nu, \hat{\mu}_{\underline{\theta}}) = KL(\nu, \hat{\mu}_{\bar{\theta}})$ *as the* ***indistinguishability condition*** *and to the set* $\mathcal{I}(\hat{\mu}) := \{\nu \in \Delta(X) : KL(\nu, \hat{\mu}_{\underline{\theta}}) = KL(\nu, \hat{\mu}_{\bar{\theta}})\}$ *of distributions satisfying this constraint as the* ***indistinguishable distributions***.[20]

To interpret, suppose $\nu \in \Delta(X)$ is the agent's empirical signal distribution after T draws; i.e., $\nu(x) := (1/T)\sum_{t=1}^{T} 1\{x_t = x\}$ is the fraction of observed signals that are x. If ν satisfies the indistinguishability condition, then observing ν does not help the agent to statistically distinguish states $\underline{\theta}$ and $\bar{\theta}$, as $KL(\nu, \hat{\mu}_{\underline{\theta}}) = KL(\nu, \hat{\mu}_{\bar{\theta}})$ means that ν is equally atypical relative to the agent's perceived signal distributions $\hat{\mu}_{\underline{\theta}}$ and $\hat{\mu}_{\bar{\theta}}$ in the two states.[21] The index $w_\theta(\mu, \hat{\mu})$ quantifies how atypical it is for the agent to observe such "confusing" distributions ν in state θ. Specifically, $w_\theta(\mu, \hat{\mu})$ measures the KL-divergence from the set of indistinguishable distributions to the true signal distribution μ_θ in state θ; hence, the smaller is $w_\theta(\mu, \hat{\mu})$, the more typical it is to observe such distributions ν in state θ. Finally, the learning efficiency index $w(\mu, \hat{\mu})$ considers the minimum across all states of $w_\theta(\mu, \hat{\mu})$; i.e., it focuses on the state θ in which indistinguishable distributions are most typical.

[18] In the nongeneric case where $KL(\mu_\theta, \hat{\mu}_\theta) = KL(\mu_\theta, \hat{\mu}_{\theta'})$ for some $\theta \neq \theta'$, beliefs in state θ almost surely cycle indefinitely. This leads to lower dynamic welfare than correct learning but higher welfare than when beliefs concentrate on an incorrect state.

[19] In continuous state spaces, generic pairs of biases give rise to different asymptotic beliefs. However, several of our main insights extend to that setting (e.g., the static and dynamic rankings can disagree, and certain "large" biases can dynamically outperform other "vanishingly small" biases).

[20] Under Assumption 1, the set of indistinguishable distributions is nonempty.

[21] Equivalently, the agent's posterior p_T after observing ν coincides with the prior p_0.

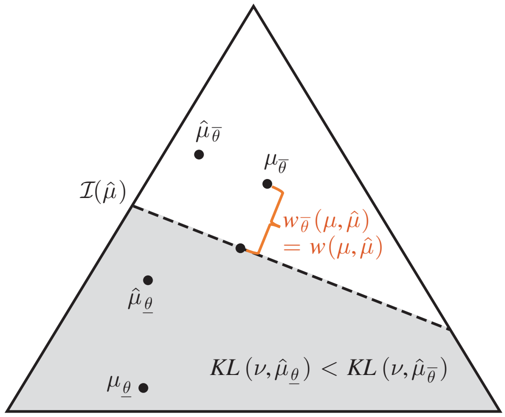

Figure 2. Illustration of Definition 1 and Theorem 1

Notably, an agent's perceived signal structure $\hat{\mu}$ affects the efficiency index $w(\mu, \hat{\mu})$ only through the indistinguishability condition. Geometrically, the set of indistinguishable distributions corresponds to the hyperplane

$$\mathcal{I}(\hat{\mu}) = \left\{\nu \in \Delta(X) : KL(\nu, \hat{\mu}_{\underline{\theta}}) = KL(\nu, \hat{\mu}_{\bar{\theta}})\right\} \tag{10}$$
$$= \left\{\nu \in \Delta(X) : \nu \cdot \log \ell_{\hat{\mu}} = 0\right\},$$

whose normal vector $\log \ell_{\hat{\mu}} := (\log \ell_{\hat{\mu}}(x))_{x \in X}$ is the vector of perceived log-likelihood ratios under $\hat{\mu}$. The learning efficiency index $w(\mu, \hat{\mu})$ is the KL-distance from this hyperplane to whichever true signal distribution $\mu_{\bar{\theta}}$ or $\mu_{\underline{\theta}}$ is closer. Figure 2 depicts this in a setting with $|X| = 3$, in which case $\mathcal{I}(\hat{\mu})$ is a straight line.

The learning efficiency index is defined without reference to a decision problem. Our key result is that this index characterizes the dynamic welfare ranking, i.e., yields a robust comparison across biases that applies at all decision problems and large enough T.

THEOREM 1: *Fix any true signal structure μ and perceived signal structures $\hat{\mu}^1$ and $\hat{\mu}^2$ satisfying Assumption 1. Suppose $w(\mu, \hat{\mu}^1) > w(\mu, \hat{\mu}^2)$. Then for any decision problem A, there exists T^* such that for all $T \geq T^*$, $W_T(\mu, \hat{\mu}^1, A) > W_T(\mu, \hat{\mu}^2, A)$.*

Theorem 1 implies that the dynamic welfare ranking is generically complete for all biases satisfying Assumption 1: given a true signal structure μ, any two such biases $\hat{\mu}^1$ and $\hat{\mu}^2$ can be ranked, except when their efficiency indices $w(\mu, \hat{\mu}^1) = w(\mu, \hat{\mu}^2)$ are exactly tied. Note that while the ranking applies in all decision problems, the number T^* of signal draws beyond which the more efficient bias $\hat{\mu}^1$ leads to higher welfare may depend on the decision problem A. In Section IVA, we show how to use the learning efficiency indices $w(\mu, \hat{\mu}^i)$ to bound T^* for a class of decision problems.

We prove Theorem 1 in Appendix B. The basic idea is as follows. By Assumption 1, both agents' asymptotic beliefs are correct. As a result, comparing their welfare in any decision problem A amounts to comparing how fast $W_T(\mu, \hat{\mu}^i, A)$ converges to the first-best welfare $W^*(A) := \sum_{\theta \in \Theta} p_0(\theta) \max_{a \in A} a_\theta$. We show that this convergence is exponential, with rate given by the learning efficiency index, i.e.,

$$(11) \qquad W^*(A) - W_T(\mu, \hat{\mu}^i, A) = \exp\left[-Tw(\mu, \hat{\mu}^i) + o(T)\right].^{22}$$

Thus, if $w(\mu, \hat{\mu}^1) > w(\mu, \hat{\mu}^2)$, then $W_T(\mu, \hat{\mu}^1, A) > W_T(\mu, \hat{\mu}^2, A)$ for all sufficiently large T. Indeed, (11) implies that the ratio of agent 2 versus agent 1's efficiency loss, $\left[W^*(A) - W_T(\mu, \hat{\mu}^2)\right] / \left[W^*(A) - W_T(\mu, \hat{\mu}^1)\right]$, explodes to infinity as T grows large.

To establish (11), it suffices to show that in each state θ, the probability that the agent's choice under bias $\hat{\mu}$ is suboptimal, i.e., $a^*(x^T, \hat{\mu}) \notin \arg\max_{a \in A} a_\theta$, vanishes exponentially at rate $w_\theta(\mu, \hat{\mu})$. The convergence in (11) is then determined by the slowest rate $w(\mu, \hat{\mu}) = \min_\theta w_\theta(\mu, \hat{\mu})$ at which mistakes decay across states.

The key observation is the following tight connection between suboptimal choices and indistinguishability. Suppose the true state is $\bar{\theta}$ and the observed signal sequence is x^T, with corresponding empirical signal distribution ν_T. The agent chooses suboptimally at large T if $KL(\nu_T, \hat{\mu}_{\underline{\theta}}) < KL(\nu_T, \hat{\mu}_{\bar{\theta}})$, as in this case her posterior concentrates on the incorrect state $\underline{\theta}$ (by analogous arguments to (8)). These mistake-inducing distributions ν_T are shown in the gray region in Figure 2. Crucially, it turns out that

$$(12) \qquad \lim_{T \to \infty} \mu_{\bar{\theta}}^T \left(KL(\nu_T, \hat{\mu}_{\underline{\theta}}) \approx KL(\nu_T, \hat{\mu}_{\bar{\theta}}) \,\middle|\, a^*(x^T, \hat{\mu}) \notin \arg\max_{a \in A} a_{\bar{\theta}} \right) = 1,$$

where the approximation "≈" becomes arbitrarily precise as T grows large. That is, in the unlikely event that the agent chooses suboptimally at large T, this is almost surely because her observed signals make it difficult to distinguish the two states (i.e., $KL(\nu_T, \hat{\mu}_{\underline{\theta}}) \approx KL(\nu_T, \hat{\mu}_{\bar{\theta}})$) rather than because they are strongly indicative of the incorrect state $\underline{\theta}$ (i.e., $KL(\nu_T, \hat{\mu}_{\underline{\theta}}) \ll KL(\nu_T, \hat{\mu}_{\bar{\theta}})$).

Property (12) reflects the *large deviation principle* formalized by Sanov's theorem: when an unlikely event occurs, it is overwhelmingly likely to occur in the least atypical way.[23] Figure 2 illustrates this in the current setting. Among all the mistake-inducing distributions ν_T in the gray region, the ones that are least atypical in state $\bar{\theta}$ (i.e., KL-closest to the true signal distribution $\mu_{\bar{\theta}}$) are the ones nearest to the dashed line representing the indistinguishable distributions $\mathcal{I}(\hat{\mu})$. This is because, by Assumption 1, $\mu_{\bar{\theta}}$ lies outside the gray region. Finally, Sanov's theorem also implies that the probability of observing empirical distributions with $KL(\nu_T, \hat{\mu}_{\underline{\theta}}) \approx KL(\nu_T, \hat{\mu}_{\bar{\theta}})$ decays exponentially at a rate given by the KL-divergence between the set of indistinguishable distributions and the true signal distribution $\mu_{\bar{\theta}}$. As Figure 2 illustrates, this is precisely the learning efficiency index $w_{\bar{\theta}}(\mu, \hat{\mu})$.

[22] By definition, $\lim_{T \to \infty} o(T)/T = 0$, so term $o(T)$ has a negligible effect on the convergence in (11).

[23] Formally, Sanov's theorem states that, for any set of empirical distributions $D \subseteq \Delta(X)$ that is the closure of its interior, $\mu_{\bar{\theta}}^T(\nu_T \in D) = \exp\left[-\inf_{\nu \in D} KL(\nu, \mu_{\bar{\theta}}) T + o(T)\right]$.

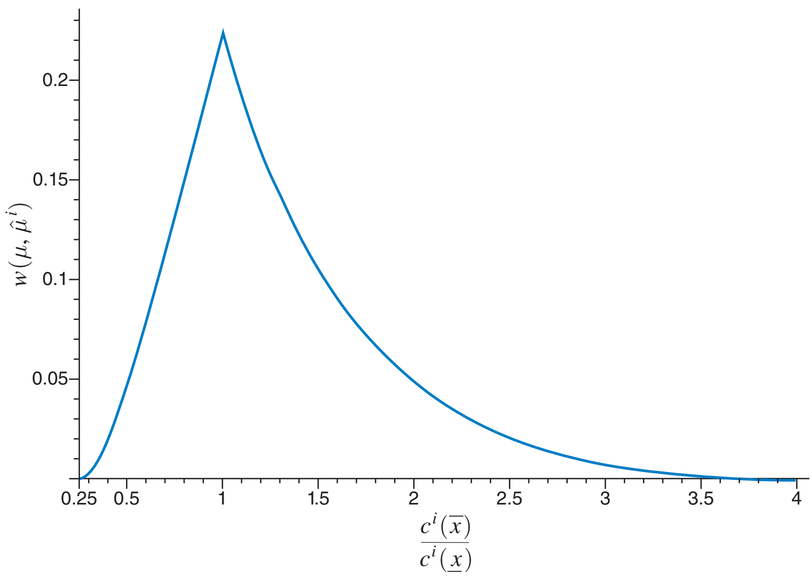

FIGURE 3

Notes: Efficiency index in Example 1 as a function of $c^i(\bar{x})/c^i(\underline{x})$ when $\mu_{\bar{\theta}}(\bar{x}) = 0.8 = \mu_{\underline{\theta}}(\underline{x})$. Assumption 1 holds if and only if $1/4 < c^i(\bar{x})/c^i(\underline{x}) < 4$. When $c^i(\bar{x})/c^i(\underline{x}) \leq 1$, we have $w(\mu, \hat{\mu}^i) = w_{\underline{\theta}}(\mu, \hat{\mu}^i) = KL(\nu^i, \mu_{\underline{\theta}})$. When $c^i(\bar{x})/c^i(\underline{x}) \geq 1$, we have $w(\mu, \hat{\mu}^i) = w_{\bar{\theta}}(\mu, \hat{\mu}^i) = KL(\nu^i, \mu_{\bar{\theta}})$. Here, $\nu^i(\bar{x}) = 1/[1 + c^i(\bar{x})/c^i(\underline{x})]$.

EXAMPLE 1 (Continued): *In the illustrative example, $\log \ell_{\hat{\mu}^i}(x) = c^i(x) \log \ell(x)$ for each agent i and signal $x \in \{\bar{x}, \underline{x}\}$. Under Assumption 1,*[24] *agent i has a unique indistinguishable distribution ν^i, where $\nu^i(\bar{x}) = -\log \ell_\mu(\underline{x})/\{-\log \ell_\mu(\underline{x}) + [c^i(\bar{x})/c^i(\underline{x})] \log \ell_\mu(\bar{x})\}$. Thus, as Figure 3 shows, the efficiency index $w(\mu, \hat{\mu}^i)$ depends on the distortion function $c^i(\cdot)$ only through the ratio $c^i(\bar{x})/c^i(\underline{x})$, and dynamic welfare is higher the closer this ratio is to 1.*

REMARK 2: *Moscarini and Smith (2002)—henceforth, MS—study robust comparisons of true signal structures for correctly specified agents. MS consider an index over signal structures μ given by $w^{\mathrm{MS}}(\mu) := -\min_{\lambda \in [0,1]} \log \sum_x \mu_{\underline{\theta}}(x)^\lambda \mu_{\bar{\theta}}(x)^{1-\lambda}$ and show that if $w^{\mathrm{MS}}(\mu^1) > w^{\mathrm{MS}}(\mu^2)$, then for any nontrivial decision problem, a correctly specified agent's expected payoff under μ^1 is higher than under μ^2 for all sufficiently large T.*

Using the variational formula (e.g., Dupuis and Ellis 2011, Lemma 6.2.3(f)), one can show that $w(\mu, \mu) = w^{MS}(\mu)$; i.e., our efficiency index reduces to MS's index when agents are correctly specified. Moreover, the same argument as in Theorem 1 implies that if $w(\mu^1, \hat{\mu}^1) > w(\mu^2, \hat{\mu}^2)$, then $W_T(\mu^1, \hat{\mu}^1, A) > W_T(\mu^2, \hat{\mu}^2, A)$ for all A and sufficiently large T. This result nests both Theorem 1 and MS's characterization

[24] Assumption 1 holds if and only if $[-\log \ell_\mu(\underline{x})/\log \ell_\mu(\bar{x})][\mu_{\bar{\theta}}(\underline{x})/\mu_{\bar{\theta}}(\bar{x})] < c^i(\bar{x})/c^i(\underline{x}) < [-\log \ell_\mu(\underline{x})/\log \ell_\mu(\bar{x})][\mu_{\underline{\theta}}(\underline{x})/\mu_{\underline{\theta}}(\bar{x})]$. Section IVB revisits the example when Assumption 1 is violated.

and allows for welfare comparisons across agents who differ both in terms of the true signal structures μ^i they face and in their misperceptions $\hat{\mu}^i$.[25]

C. *Dynamic versus Static Welfare Rankings*

A key implication of our analysis is that one learning bias may be robustly less harmful than another when agents have access to little data but robustly more harmful under richer data. This is a notable contrast with the fact that, in the context of comparing true signal structures, Moscarini and Smith's (2002) dynamic ranking extends Blackwell's (1951) static order, as the Blackwell order is preserved under T-fold repetition of signal draws. Figure 4 illustrates the disagreement between our static and dynamic rankings in a specific decision problem: here, the asymmetric distortion function $c^1(\cdot)$ yields higher welfare than the symmetric distortion function $c^2(\cdot)$ at small T, but this pattern reverses at $T^* = 8$, with a welfare gap in the opposite direction that remains significant at all moderate T.

Proposition 1 and Theorem 1 shed light on the general source of disagreements between the static and dynamic rankings. As formalized by the nested likelihood ratio condition (6), $\hat{\mu}^1$ dominates $\hat{\mu}^2$ according to the static ranking if the interpretation of each signal is more accurate under $\hat{\mu}^1$ than under $\hat{\mu}^2$. However, (6) allows that $\ell_\mu(x) \geq \ell_{\hat{\mu}^1}(x) \geq \ell_{\hat{\mu}^2}(x)$ for some signals x, but $\ell_\mu(y) \leq \ell_{\hat{\mu}^1}(y) \leq \ell_{\hat{\mu}^2}(y)$ for other signals y, so misinferences from x and y go in opposite directions. In this case, agent 2's inferences from signal sequences containing both x and y can be more accurate than agent 1's because agent 2's errors cancel out more, as we saw in Example 1. Theorem 1 formalizes how such canceling out of opposite errors influences the dynamic welfare ranking. All that matters is how each bias affects the set of indistinguishable distributions $\mathcal{I}(\hat{\mu}^i)$, and by (10), this set depends only on the *relative* interpretations of different signals, as captured by its normal vector $(\log \ell_{\hat{\mu}^i}(x))_x$.

The above discussion has two implications.

Ranking at All T.—First, if (6) is strengthened to require misinferences from all signals to go in the same direction, then there is no scope for opposite errors to cancel out. In this case, agent 1 is better-off than agent 2 not only robustly across all decision problems but also after any number of signal draws.

PROPOSITION 2: *Suppose that*

$$\ell_\mu(x) \geq \ell_{\hat{\mu}^1}(x) \geq \ell_{\hat{\mu}^2}(x),\ \forall x \in X \tag{13}$$

$$\text{or} \quad \ell_\mu(x) \leq \ell_{\hat{\mu}^1}(x) \leq \ell_{\hat{\mu}^2}(x),\ \forall x \in X.$$

Then for all decision problems A and all T, $W_T(\mu, \hat{\mu}^1, A) \geq W_T(\mu, \hat{\mu}^2, A)$.

[25] MS's and our proofs employ related large deviation arguments. One difference is that there is no direct counterpart in MS of the observation that welfare at large T is determined by the set of indistinguishable distributions, which is key for our comparison of learning biases. Moreover, with nonbinary states (Section IVC), MS's argument relies explicitly on correctly specified learning.

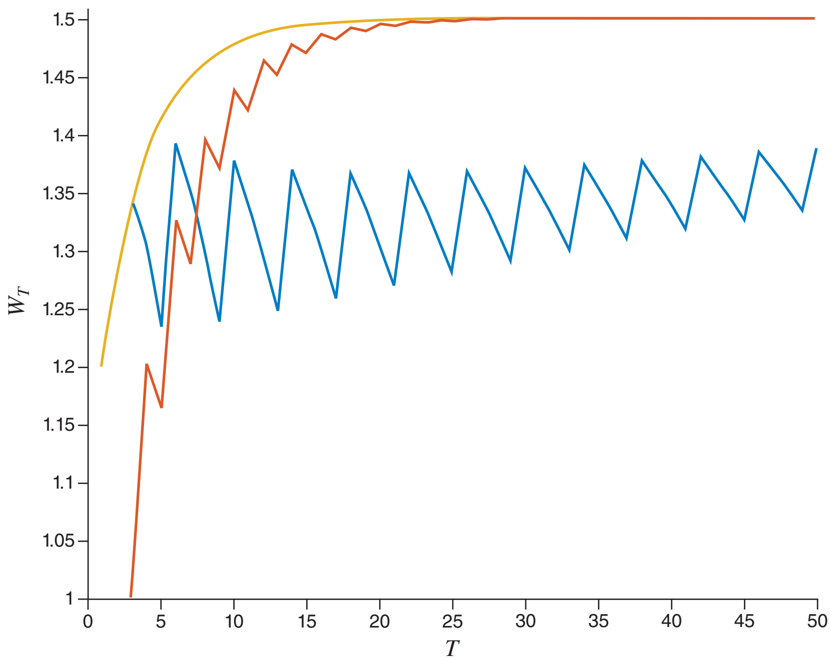

Figure 4

Notes: Welfare as a function of T at decision problem $A = \{(1,0),(0,2)\}$ when $\mu_{\bar{\theta}}(\bar{x}) = 0.8 = \mu_{\underline{\theta}}(\underline{x})$, $p_0 = (1/2, 1/2)$. Yellow: welfare under correct specification. Blue: welfare under Example 1 with $c^1(\bar{x}) = 0.56$, $c_1(\underline{x}) = 0.18$. Orange: welfare under $c^2(\bar{x}) = c_2(\underline{x}) = 0.14$. Welfare in all three cases approximates the first-best $W^*(A) = 1.5$ as $T \to \infty$.

In Example 1, (13) holds if and only if either $c^2(\underline{x}) \geq c^1(\underline{x}) \geq 1 \geq c^1(\bar{x}) \geq c^2(\bar{x})$ or $c^2(\underline{x}) \leq c^1(\underline{x}) \leq 1 \leq c^1(\bar{x}) \leq c^2(\bar{x})$.

Maximally Efficient Biases.—Second, there is a unique class of biases $\hat{\mu}$ that achieve maximal learning efficiency and hence, based on the dynamic ranking, robustly outperform any other bias, no matter how small. Generalizing Example 1, this class corresponds precisely to Phillips and Edwards's (1966) model of symmetric under-/overreaction, where all likelihood ratios are distorted by a constant power c.[26]

PROPOSITION 3: *For any true and perceived signal structures μ and $\hat{\mu}$ satisfying Assumption 1, we have $w(\mu,\hat{\mu}) = w(\mu,\mu)$ if and only if $\hat{\mu}$ is a Philipps-Edwards bias, i.e., there exists $c > 0$ such that $\ell_{\hat{\mu}}(x) = (\ell_{\mu}(x))^c$ for all $x \in X$.*

Trivially, correctly specified agents perform at least as well as biased agents in all decision problems and at all T, as welfare $W_T(\mu,\hat{\mu},A)$ is evaluated based on

[26] We note that all such biases satisfy Assumption 1. Related to the special role of constant under-/overreaction in our setting, Fudenberg, Lanzani, and Strack (2023) show that, in an active learning environment with limited recall, the limit strategies of agents who forget past experiences at a uniform rate (which can be recast as constant underreaction) are self-confirming equilibria.

the true signal structure μ. Proposition 3 shows that Philipps-Edwards biases are the unique class of $\hat{\mu}$ to achieve the same maximal learning efficiency $w(\mu,\hat{\mu}) = w(\mu,\mu)$ as correctly specified agents.[27] The maximal efficiency of such biases reflects the canceling out logic above. Distorting all likelihood ratios (both for signals that are indicative of $\bar{\theta}$ and ones that are indicative of $\underline{\theta}$) by the same power is the only way to preserve the set of correctly specified indistinguishable distributions $\mathcal{I}(\mu)$, as it amounts to multiplying its normal vector $\left(\log \ell_\mu(x)\right)_x$ by a constant.

Proposition 3 implies the following. Consider any Philipps-Edwards bias $\hat{\mu}^1$, no matter how significant its distortion factor c, and any non-Philipps-Edwards bias $\hat{\mu}^2$, no matter how close $\hat{\mu}^2$ is to the true signal structure μ (based on any metric). Then $\hat{\mu}^2$ leads to strictly lower learning efficiency than $\hat{\mu}^1$. Hence, by (11), in any decision problem, the efficiency loss under $\hat{\mu}^2$ explodes relative to that under $\hat{\mu}^1$ after sufficiently many signal draws. Thus, in data-rich settings, some seemingly "vanishingly small" biases perform robustly worse than other "large" biases.

D. *Additional Examples*

To further illustrate our results, we consider their implications for some widely studied examples of misspecified Bayesian learning mentioned in Remark 1.

EXAMPLE 2 (Overconfidence): *Suppose agents $i = 1,2$ are learning about the quality of a project, $\theta \in \{\underline{\theta},\bar{\theta}\} \subseteq \mathbb{R}$ with $\underline{\theta} < \bar{\theta}$. Signals are binary, $\underline{x}$ ("failure") or $\bar{x}$ ("success"). The true and perceived signal distributions in each state take the form $\mu_\theta(\bar{x}) = g(\theta,\beta^*)$ and $\hat{\mu}^i_\theta(\bar{x}) = g(\theta,\beta^i)$. Here, β^* denotes agents' true ability, the output function g is strictly increasing in both the project quality and ability, and β^i denotes agent i's perceived ability. Assume $\beta^2 > \beta^1 > \beta^*$, so both agents are overconfident about their ability but agent 2's overconfidence is more severe.*

COROLLARY 1: *If the output function g is additively separable, the less overconfident $\hat{\mu}^1$ outperforms the more overconfident $\hat{\mu}^2$ at all decision problems A and all T. For general g, $\hat{\mu}^1$ and $\hat{\mu}^2$ can be incomparable based on the static welfare ranking, but based on the dynamic welfare ranking, $\hat{\mu}^1$ always strictly outperforms $\hat{\mu}^2$.*

For additively separable g, the likelihood ratios $\ell_\beta(\bar{x}) := g(\bar{\theta},\beta)/g(\underline{\theta},\beta)$ and $\ell_\beta(\underline{x}) := \left[1 - g(\bar{\theta},\beta)\right]/\left[1 - g(\underline{\theta},\beta)\right]$ are decreasing in β, so the result follows from Proposition 2. However, for general g (e.g., if project quality θ and ability β have a complementary effect on success), $\ell_\beta(\bar{x})$ or $\ell_\beta(\underline{x})$ is in general nonmonotonic in β (see online Appendix E.1), so the less overconfident agent may misinterpret some signals more severely than the more overconfident agent. In this case, Proposition 1 implies that if only one signal is observed, the less overconfident agent performs strictly worse in some decision problems.

[27] By (11), $w(\mu,\hat{\mu}) = w(\mu,\mu)$ means that in all decision problems, the efficiency loss under Philipps-Edwards biases $\hat{\mu}$ vanishes at the same exponential rate as under correct specification. However, this does not imply that $\hat{\mu}$ leads to the same welfare as under correct specification for all large T. Indeed, iterated application of Proposition 1 implies that for any Philipps-Edwards biases $\hat{\mu}^1,\hat{\mu}^2$ whose distortion factors satisfy $1 \leq c^1 < c^2$ or $c^2 < c^1 \leq 1$, we have $W_T(\mu,\hat{\mu}^1,A) \geq W_T(\mu,\hat{\mu}^2,A)$ for all A and T, with strict inequality if the chosen acts under $\hat{\mu}^1$ and $\hat{\mu}^2$ differ at some signal sequences.

In contrast, for the dynamic welfare ranking, all that matters are the signal sequences that make the two states indistinguishable. Online Appendix E.1 shows that overconfident agents' indistinguishable distributions feature a larger fraction of high signals relative to the correctly specified case, and more so the greater their overconfidence. Thus, dynamically, the less overconfident agent 1 is robustly better-off.

If agent 2's overconfidence is severe but agent 1's is moderate, agent 2 mislearns (grows confident in $\underline{\theta}$ in both states), while agent 1 learns the true state. In this case, the result is driven by a difference in asymptotic beliefs, which are the focus of much existing work (e.g., Heidhues, Koszegi, and Strack 2018). Corollary 1 additionally sheds light on the medium-run effect of overconfidence: agent 1 is strictly better-off even when agents' asymptotic beliefs are the same (both correct or both incorrect).[28]

EXAMPLE 3 (Over- versus Underprecision): *Assume signals $X = \{\bar{x}, \underline{x}\}$ are binary and true and perceived signal structures belong to $M = \{\mu' : \mu'_{\bar{\theta}}(\bar{x}), \mu'_{\underline{\theta}}(\underline{x}) > 1/2\}$. Suppose agent 1 (respectively, agent 2) suffers from overprecision (respectively, underprecision), i.e., overestimates (respectively, underestimates) the Blackwell informativeness of μ. Thus, their perceived signal structures $\hat{\mu}^{\mathrm{OP}}$ (respectively, $\hat{\mu}^{\mathrm{UP}}$) satisfy $\hat{\mu}^{\mathrm{OP}}_{\bar{\theta}}(\bar{x}) > \mu_{\bar{\theta}}(\bar{x}) > \hat{\mu}^{\mathrm{UP}}_{\bar{\theta}}(\bar{x}) > \hat{\mu}^{\mathrm{UP}}_{\underline{\theta}}(\bar{x}) > \mu_{\underline{\theta}}(\bar{x}) > \hat{\mu}^{\mathrm{OP}}_{\underline{\theta}}(\bar{x})$.*

COROLLARY 2: *Fix any true signal structure μ. According to the static welfare ranking, no overprecision bias $\hat{\mu}^{\mathrm{OP}}$ and underprecision bias $\hat{\mu}^{\mathrm{UP}}$ can be compared. However, according to the dynamic welfare ranking, there exist a range of overprecision biases $\hat{\mu}^{\mathrm{OP}}$ satisfying Assumption 1 that are outperformed by every underprecision bias $\hat{\mu}^{\mathrm{UP}}$.*

To see why the static ranking is degenerate, note that $\ell_{\hat{\mu}^{OP}}(\bar{x}) > \ell_{\mu}(\bar{x}) > \ell_{\hat{\mu}^{UP}}(\bar{x})$. Thus, condition (6) is violated, so Proposition 1 implies that $\hat{\mu}^{OP}$ and $\hat{\mu}^{UP}$ are incomparable. In contrast, when T is sufficiently large, then by Theorem 1, the learning efficiency index allows for a robust comparison of generically any $\hat{\mu}^{OP}$ and $\hat{\mu}^{UP}$ satisfying Assumption 1.[29] *Notably, as online Appendix E.2 shows, there is a strictly positive lower bound $\underline{w}^{UP} > 0$ on the learning efficiency index of all underprecision biases $\hat{\mu}^{UP}$, whereas the learning efficiency of overprecision biases $\hat{\mu}^{OP}$ can be arbitrarily close to zero. Hence, any $\hat{\mu}^{OP}$ with $w(\mu, \hat{\mu}^{OP}) < \underline{w}^{UP}$ is outperformed by every underprecision bias. The basic reason for this difference is that strong overprecision can cause extreme distortions to the indistinguishability condition $KL(\nu, \hat{\mu}^{OP}_{\underline{\theta}}) = KL(\nu, \hat{\mu}^{OP}_{\bar{\theta}})$ because $KL(\nu, \hat{\mu}_{\theta})$ explodes as $\hat{\mu}_{\theta}$ approaches a point-mass on signal $\underline{x}$ or $\bar{x}$. In contrast, underprecision does not have this effect because here, $\hat{\mu}^{UP}_{\theta}$ is bounded by the true distributions $\mu_{\underline{\theta}}$ and $\mu_{\bar{\theta}}$. Thus, even though both agents learn the correct state, the rate of learning under overprecision can be arbitrarily slow, whereas the impact of underprecision on the speed of learning is limited.*[30]

[28] In the case of incorrect asymptotic beliefs, the result follows from Theorem 3.

[29] Assumption 1 is satisfied by every underprecision bias $\hat{\mu}^{UP}$ but can be violated by overprecision biases for which $\hat{\mu}^{OP}_{\bar{\theta}}(\bar{x})$ or $\hat{\mu}^{OP}_{\underline{\theta}}(\underline{x})$ is very close to 1. In the latter case, $\hat{\mu}^{OP}$ leads to mislearning, so is also outperformed by every underprecision bias according to the dynamic ranking.

[30] If the assumption that signal distributions belong to M is relaxed to $\mu'_{\bar{\theta}}(\bar{x}) > \mu'_{\underline{\theta}}(\bar{x})$, then learning efficiency under underprecision can also be arbitrarily close to zero. However, every $\hat{\mu}^{\mathrm{UP}}$ still leads to correct learning,

EXAMPLE 4 (Neglecting Correlation versus Neglecting Variables): *Consider signals from two sources, $X = Z_1 \times Z_2$. Assume that $Z_1 = Z_2 = Z$ and the true signal distribution μ_θ in each state θ has the same marginal $\mathrm{marg}_Z \mu_\theta$ over Z_1 and Z_2, where $\mathrm{marg}_Z \mu_{\underline{\theta}} \neq \mathrm{marg}_Z \mu_{\bar{\theta}}$. Suppose agent 1 neglects correlation. Her perceived signal distribution $\hat{\mu}_\theta^{CN} = marg_Z \mu_\theta \times marg_Z \mu_\theta$ at each θ features the correct marginals over the two sources but misperceives them to be independent. Agent 2 neglects variables. Her perceived likelihood ratios satisfy $\hat{\mu}_{\bar{\theta}}^{VN}(z_1, z_2)/\hat{\mu}_{\underline{\theta}}^{VN}(z_1, z_2) = \mathrm{marg}_Z \mu_{\bar{\theta}}(z_1)/\mathrm{marg}_Z \mu_{\underline{\theta}}(z_1)$ for all (z_1, z_2), so she updates beliefs using the correct marginal signal distribution for source 1 but ignores signals from source 2, which she misperceives to be uninformative given source 1.*

Proposition 1 implies that the welfare comparison at $T = 1$ again in general depends on the decision problem (see online Appendix E.3 for details). However, by Theorem 1, the dynamic welfare ranking yields a robust comparison.[31]

COROLLARY 3: *For any true signal structure μ, $w(\mu, \hat{\mu}^{CN}) > w(\mu, \hat{\mu}^{VN})$. Thus, according to the dynamic welfare ranking, neglecting correlation is strictly less harmful than neglecting variables.*

If the true signal distributions of the two sources are close to independent, it is intuitive that neglecting their correlation is less harmful than ignoring one source. Perhaps more surprisingly, Corollary 3 shows that correlation neglect is less harmful than variable neglect even when the two sources are highly correlated.[32] *In the latter case, one might expect the opposite ranking: Source 2 is indeed fairly uninformative given source 1, so ignoring it may seem fairly innocuous; in contrast, as the literature highlights (e.g., Levy and Razin 2015; Ortoleva and Snowberg 2015), correlation neglect leads to significant overreaction to some signal sequences due to the mistake of "double-counting" matching signal realizations from the two sources. However, online Appendix E.3 shows that, as far as the set of indistinguishable distributions is concerned, such double-counting leads to less distortion relative to the correctly specified case than neglecting one source, and as a result, $w(\mu, \hat{\mu}^{CN}) > w(\mu, \hat{\mu}^{VN})$.*

III. Extension: Cautiousness against Misspecification

Suppose that, instead of dogmatically perceiving a particular signal structure $\hat{\mu}$, an agent entertains multiple possible signal structures and jointly learns about both payoff-relevant states and signal structures. Such uncertainty about the signal structure can be interpreted as capturing some cautiousness against misspecification. For instance, in the context of Example 2, an agent may seek to guard against over- or underconfidence about her ability by considering a range of possible ability levels

whereas severe overprecision leads to mislearning (compare footnote 29), so there remains a range of overprecision biases that are dynamically outperformed by every underprecision bias.

[31] Both $\hat{\mu}^{CN}$ and $\hat{\mu}^{VN}$ satisfy Assumption 1. This is clear for $\hat{\mu}^{VN}$. For $\hat{\mu}^{CN}$, note that for all $\theta \neq \theta'$, $KL(\mu_\theta, \hat{\mu}_\theta^{CN}) = \sum_k KL(\mathrm{marg}_{Z_k} \mu_\theta, \mathrm{marg}_{Z_k} \mu_\theta) = 0 < \sum_k KL(\mathrm{marg}_{Z_k} \mu_\theta, \mathrm{marg}_{Z_k} \mu_{\theta'}) = KL(\mu_\theta, \hat{\mu}_{\theta'}^{CN})$.

[32] The full-support assumption on μ_θ rules out perfect correlation between the two sources. However, $w(\mu, \hat{\mu}^{CN}) > w(\mu, \hat{\mu}^{VN})$ holds for all full-support μ displaying arbitrarily strong correlation, with the gap in learning efficiency vanishing as μ approaches perfect correlation.

and jointly updating her beliefs about the project quality and her ability. We extend the dynamic welfare ranking to this setting. A key implication is that cautiousness against misspecification can hurt. Even if the set of signal structures the agent entertains is correctly specified, i.e., contains the true signal structure, we show that she may be worse off than a misspecified agent.

As in Section I, consider a finite set Θ of payoff-relevant states from which the true state is drawn according to a full-support distribution p_0, and a fixed true signal structure $\mu := (\mu_\theta)_{\theta\in\Theta}$. The agent learns jointly about payoff-relevant states and signal structures, under a possibly misspecified model that may assign zero probability to the true signal structure. Let $\hat{M}$ denote the compact set of signal structures the agent deems possible, where $\hat{\mu} := (\hat{\mu}_\theta)_{\theta\in\Theta} \in (\Delta(X))^\Theta$ for each $\hat{\mu} \in \hat{M}$. The agent's prior belief is some full-support $q_0 \in \Delta(\hat{M} \times \Theta)$. The agent is correctly specified if her subjective model $\hat{M}$ contains the true signal structure μ.[33]

Upon observing a signal sequence $x^T = (x_1, \ldots, x_T)$, generated i.i.d. according to the true signal structure, the agent Bayesian-updates her belief to $q_T(\cdot|x^T) \in \Delta(\hat{M} \times \Theta)$; i.e., for all measurable $E \subseteq \hat{M} \times \Theta$, $q_T(E|x^T) = [\int_E \hat{\mu}_\theta^T(x^T) dq_0(\hat{\mu},\theta)]/[\int_{\hat{M}\times\Theta} \hat{\mu}'^T_{\theta'}(x^T) dq_0(\hat{\mu}',\theta')]$.

At any decision problem $A \subseteq \mathbb{R}^\Theta$, the agent chooses a subjectively optimal act according to her posterior $\mathrm{marg}_\Theta q_T$ over payoff-relevant states, where as before, we assume away indifferences. As in (5), define the agent's welfare $W_T(\mu, q_0, A)$ as her objective expected utility based on the true signal structure μ and prior p_0 over Θ.

For simplicity, we continue to focus on binary payoff-relevant states, $\Theta = \{\underline{\theta}, \overline{\theta}\}$. Moreover, we restrict attention to agents who correctly learn the payoff-relevant state θ, as is ensured by the following generalization of Assumption 1.[34]

ASSUMPTION 2: *For any distinct* $\theta, \theta' \in \Theta$, $\min_{\hat{\mu}\in M} KL(\mu_\theta, \hat{\mu}_\theta) < \min_{\hat{\mu}\in M} KL(\mu_\theta, \hat{\mu}_{\theta'})$.

We obtain the following generalization of Theorem 1. Define the learning efficiency index by $w(\mu, \hat{M}) := \min_\theta w_\theta(\mu, \hat{M})$, where

$$w_\theta(\mu, \hat{M}) := \min_{\nu\in\Delta(X)} KL(\nu, \mu_\theta) \quad \text{subject to} \quad \min_{\hat{\mu}\in\hat{M}} KL(\nu, \hat{\mu}_{\underline{\theta}}) = \min_{\hat{\mu}\in\hat{M}} KL(\nu, \hat{\mu}_{\overline{\theta}}).$$

The indistinguishability condition $\min_{\hat{\mu}\in\hat{M}} KL(\nu, \hat{\mu}_{\underline{\theta}}) = \min_{\hat{\mu}\in\hat{M}} KL(\nu, \hat{\mu}_{\overline{\theta}})$ again captures the empirical signal distributions ν that do not allow the agent to tell apart states $\underline{\theta}$ and $\overline{\theta}$. Here, in each state θ, the agent uses the signal distribution $\hat{\mu}_\theta$ that comes closest to ν among all the distributions she deems possible. When $\hat{M}$ is a singleton, the index reduces to the one in Definition 1.

[33] We assume that each $\hat{\mu}_\theta \in \Delta(X)$, but the same analysis extends to the case where $\hat{\mu}_\theta \in \mathbb{R}^X_{++}$. The agent's prior belief $\mathrm{marg}_\Theta q_0$ need not match the true prior p_0 (recall footnote 8).

[34] Thus, we isolate the effect of uncertainty about the signal structure on the speed of convergence of beliefs about θ rather than on asymptotic beliefs. Absent Assumption 2, (even correctly specified) agents may fail to learn the true θ asymptotically (due to identification problems that lead to incomplete learning), making them dynamically worse off than any agent satisfying Assumption 2.

THEOREM 2: *Fix any* μ *and* $q_0^i \in \Delta(\hat{M}^i \times \Theta)$ $(i = 1,2)$ *with* $\hat{M}^i$ *satisfying Assumption 2. Suppose* $w(\mu,\hat{M}^1) > w(\mu,\hat{M}^2)$. *Then for any decision problem A, there exists* T^* *such that for all* $T \geq T^*$, $W_T(\mu,q_0^1,A) > W_T(\mu,q_0^2,A)$.

Clearly, based on objective welfare, an agent who is certain of the true signal structure (i.e., whose model is $\hat{M} = \{\mu\}$) weakly outperforms any other agent at all decision problems and all T. Thus, the learning efficiency index $w(\mu,\{\mu\})$ is maximal.

However, a key implication of Theorem 2 is that correctly specified but uncertain agents can be robustly worse off than some misspecified agents at all large T. As the following example illustrates, there are subjective models $\hat{M}^1,\hat{M}^2$ such that $\mu \in \hat{M}^2 \backslash \hat{M}^1$ but $w(\mu,\{\mu\}) > w(\mu,\hat{M}^1) > w(\mu,\hat{M}^2)$. Here, by Assumption 2, the correctly specified but uncertain agent 2 asymptotically learns the true signal distribution μ_θ, so her long-run inferences are the same as those of a correctly specified and certain agent. However, away from the limit, at any large but finite T, agent 2's uncertainty about μ increases the probability of medium-run mistakes, as it distorts her set of indistinguishable distributions. This leads to lower learning efficiency, and hence dynamic welfare, than for a range of misspecified models $\hat{M}^1$, suggesting a sense in which cautiousness against misspecification can backfire.[35]

EXAMPLE 5 (Overconfidence versus Sophistication): *As in Example 2, suppose* $X = \{\underline{x},\bar{x}\}$ *and true and perceived signal distributions take the form* $\mu_\theta^\beta(\bar{x}) = g(\theta,\beta)$, *where g is increasing in both the project quality* θ *and ability* β *and* β^* *denotes the agent's true ability. The maximal learning efficiency is achieved by* $\hat{M} = \{\mu^{\beta^*}\}$, *i.e., under certainty about the true ability. Here, the unique indistinguishable distribution* ν^* *is the KL-midpoint of* $\mu_{\underline{\theta}}^{\beta^*}$ *and* $\mu_{\bar{\theta}}^{\beta^*}$ *(i.e.,* $KL(\nu^*,\mu_{\underline{\theta}}^{\beta^*}) = KL(\nu^*,\mu_{\bar{\theta}}^{\beta^*})$*).*

Next, consider a sophisticated agent 2 who guards against over- or under-confidence by entertaining a range of ability levels that includes the truth: $\hat{M}^2 = \{\mu^\beta : \beta \in [\underline{\beta},\bar{\beta}]\}$ *with* $\beta^* \in [\underline{\beta},\bar{\beta}]$. *Agent 2 learns the true ability* β^* *in the long run. However, crucially, her indistinguishable distribution* ν^2 *generically differs from the indistinguishable distribution* ν^* *under certainty about* β^*. *Indeed, as Figure 5 (right) illustrates,* ν^2 *is the KL-midpoint of* $\mu_{\underline{\theta}}^{\bar{\beta}}$ *and* $\mu_{\bar{\theta}}^{\underline{\beta}}$; *thus, at all large but finite T, agent 2's mistakes are driven by signal sequences that do not allow her to distinguish low project quality but high ability* $(\underline{\theta},\bar{\beta})$ *versus high quality but low ability* $(\bar{\theta},\underline{\beta})$. *This is true even though agent 2 assigns vanishing probability to any abilities other than* β^* *as T grows large.*

Thus, ν^2 *can feature an even larger fraction of high signals than* ν^1, *as Figure 5 illustrates.*[36] *When this is the case,* $w(\mu^{\beta^*},\hat{M}^1) > w(\mu^{\beta^*},\hat{M}^2)$, *so by Theorem 2, the overconfident agent robustly outperforms the sophisticated agent at all large T.*

[35] Notably, since the learning efficiency indices $w(\mu,\hat{M}^i)$ do not depend on the priors q_0^i, agent 2 is worse off than agent 1 even if her prior assigns high probability to the true signal structure μ.

[36] For fixed $\mu_{\bar{\theta}}^{\underline{\beta}}(\bar{x}) < \mu_{\bar{\theta}}^{\beta^1}(\bar{x})$, this happens if $\mu_{\underline{\theta}}^{\bar{\beta}}(\bar{x})$ is sufficiently larger than $\mu_{\underline{\theta}}^{\beta^1}(\bar{x})$. It is unimportant that $\hat{M}^1$ is a singleton: the sophisticated agent is also outperformed by some misspecified agents who entertain a range of abilities that does not include β^*. Thus, the result is not driven by the sophisticated agent forming beliefs over higher-dimensional variables (a force studied by Blume and Easley 2006 and Montiel Olea et al. 2022 in different settings).

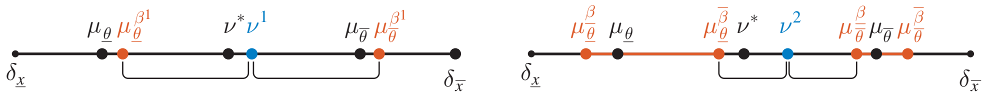

FIGURE 5

Note: Indistinguishable distributions under overconfidence (left) versus sophistication (right).

IV. Discussion

A. *How Many Signals Are Needed for the Dynamic Ranking?*

By Theorem 1, the learning efficiency index provides a robust welfare ranking of biases when sufficiently many signals, $T \geq T^*$, are observed, and we have seen that this can disagree with the static ranking based on one signal observation. Thus, it is important to understand how many signals T^* are needed in order for the dynamic ranking to apply to a given decision problem. We now show that the learning efficiency index can also be used to obtain an upper bound on this number T^*.

We continue to assume binary states $\Theta = \{\overline{\theta}, \underline{\theta}\}$. For simplicity, we also focus on binary signals $X = \{\overline{x}, \underline{x}\}$ and binary decision problems $A = \{\overline{a}, \underline{a}\}$. Assume without loss that $\mu_{\overline{\theta}}(\overline{x}) > \mu_{\underline{\theta}}(\overline{x})$. Observe that under correct specification, there is a unique indistinguishable distribution, $\mathcal{I}(\mu) = \{\nu^*\}$. The following result considers environments with prior indifference, where agents are indifferent between the two acts based on the prior (i.e., $\sum_\theta p_0(\theta)\overline{a}_\theta = \sum_\theta p_0(\theta)\underline{a}_\theta$); online Appendix F extends to general (binary) environments.

PROPOSITION 4: *Take any true signal structure μ and perceived signal structures $\hat{\mu}^1$ and $\hat{\mu}^2$ satisfying Assumption 1. Suppose $\Delta w := w(\mu, \hat{\mu}^1) - w(\mu, \hat{\mu}^2) > 0$ and denote by ν^* the correctly specified indistinguishable distribution. Let $\underline{T}$ be the smallest integer such that the following two inequalities hold:*

$$\frac{1}{\sqrt{2T}} \exp(T \Delta w) \geq 2 \frac{\ell_\mu(\overline{x})}{\ell_\mu(\underline{x})}, \ \forall T \geq \underline{T}, \tag{14}$$

$$\mu_{\underline{\theta}}(\overline{x}) + 1/\underline{T} < \nu^*(\overline{x}) < \mu_{\overline{\theta}}(\overline{x}) - 1/\underline{T}. \tag{15}$$

Then, in all environments with prior indifference, $W_T(\mu, \hat{\mu}^1, A) > W_T(\mu, \hat{\mu}^2, A)$ for all $T \geq \underline{T}$.

Proposition 4 shows how to compute an upper bound $\underline{T}$ on the number of signal draws beyond which the more efficient bias $\hat{\mu}^1$ leads to higher welfare than the less efficient bias $\hat{\mu}^2$ in all environments with prior indifference. The bound $\underline{T}$ depends on the biases only through the difference Δw in the learning efficiency of $\hat{\mu}^1$ versus $\hat{\mu}^2$, as captured by the key inequality (14). Inequality (15) is a technical

bias $\hat{\mu}^2$	$\underline{T}$	ΔW_6	ΔW_{12}	ΔW_{18}	ΔW_{24}
$c^2(\bar{x}) = 0.3, c^2(\underline{x}) = 0.06$	12	0.285	0.276	0.26	0.248
$c^2(\bar{x}) = 0.3, c^2(\underline{x}) = 0.1$	14	0.085	0.129	0.06	0.07
$c^2(\bar{x}) = 0.3, c^2(\underline{x}) = 0.15$	19	0.085	0.043	0.021	0.01

FIGURE 6

Notes: Fix $\mu_{\bar{\theta}}(\bar{x}) = 0.85 = \mu_{\underline{\theta}}(\underline{x})$ and the Philipps-Edwards bias $\hat{\mu}^1$ with distortion $c^1(\bar{x}) = c^1(\underline{x}) = 0.1$. For three asymmetric underreaction biases $\hat{\mu}^2$ with distortions $c^2(\cdot)$, the table shows the bound $\underline{T}$ from Proposition 4, as well as the welfare gap $\Delta W_T := W_T(\mu, \hat{\mu}^1, A) - W_T(\mu, \hat{\mu}^2, A)$ under decision problem $A = \{(1,0),(0,1)\}$ and prior $p_0 = (1/2, 1/2)$ at $T = 6, 12, 18, 24$.

condition that depends only on the true signal structure μ and can be dropped up to slightly complicating the right-hand side of (14) (see online Appendix F).[37]

Importantly, $\underline{T}$ need not be very large. Indeed, the right-hand side of the key inequality (14) is a constant, and in the left-hand side $\exp(T\Delta w)$ grows much faster than $\sqrt{T}$; moreover, the larger the learning efficiency gap Δw, the faster (14) is met. When $\underline{T}$ is moderate, beliefs under one or both biases can still be far from having concentrated on the correct state at $\underline{T}$. Thus, the difference in mistake probabilities between the two biases and the welfare gap $W_{\underline{T}}(\mu, \hat{\mu}^1, A) - W_{\underline{T}}(\mu, \hat{\mu}^2, A)$ can be significant, as we already saw in Figure 4. To further illustrate, Figure 6 computes $\underline{T}$ for several additional biases and shows nonnegligible welfare gaps (compared to the first-best welfare $W^*(A) = 1$) at the surrounding T. Note that $\underline{T}$ is in general a nontight upper bound on the number T^* of signals beyond which the dynamic ranking applies, so T^* can be even smaller, and the welfare gaps at T close to T^* can be even larger.[38] Online Appendix F derives general bounds on the gap in mistake probabilities and welfare.

B. *Dynamic Ranking with Incorrect Asymptotic Beliefs*

We extend the dynamic welfare ranking to the case where asymptotic beliefs are incorrect in some states. As discussed in Section IIB, if two biases lead to different asymptotic beliefs, these beliefs determine their dynamic welfare ranking. Thus, we focus on comparing biases $\hat{\mu}^1$ and $\hat{\mu}^2$ that give rise to the same incorrect asymptotic beliefs. The main insight is that the ranking over such biases is again generically complete and can be characterized using the learning efficiency index, except that in states in which mislearning occurs, agents with a *lower* efficiency index are better-off.

We continue to assume binary states, $\Theta = \{\underline{\theta}, \bar{\theta}\}$. We drop Assumption 1 but impose the following assumption on both biases, which guarantees that the sets of indistinguishable distributions (10) are nonempty.[39]

ASSUMPTION 3 (Nondegeneracy): *For each $\theta \in \Theta$, there exists $\nu \in \Delta(X)$ such that $KL(\nu, \hat{\mu}_\theta) < KL(\nu, \hat{\mu}_{\theta'})$ for all $\theta' \neq \theta$.*

[37] Note that (15) is satisfied for all large enough T, as the indistinguishability condition $KL(\nu^*, \mu_{\underline{\theta}}) = KL(\nu^*, \mu_{\bar{\theta}})$ implies $\mu_{\underline{\theta}}(\bar{x}) < \nu^*(\bar{x}) < \mu_{\bar{\theta}}(\bar{x})$.

[38] For example, in Figure 6, $T^* = 3$ for the first two examples, and $T^* = 5$ for the third.

[39] Assumption 3 is always satisfied for misspecified Bayesian agents (i.e., if $\hat{\mu}_\theta \in \Delta(X)$ for all θ) but rules out extreme forms of non-Bayesian learning: in particular, under binary states, it rules out biases for which all signal realizations lead an agent to update her belief in the same direction.

The following result shows that the dynamic welfare ranking is again characterized by the state-dependent efficiency indices $w_\theta\left(\mu, \hat{\mu}^i\right)$.

THEOREM 3: *Fix any true signal structure μ and perceived signal structures $\hat{\mu}^1$ and $\hat{\mu}^2$ satisfying Assumption 3. Suppose one of the following is true:*

(*i*) (*Mislearning in one state*)*: Under both $\hat{\mu}^1$ and $\hat{\mu}^2$, asymptotic beliefs concentrate on state θ in both states,*[40] *and* $\min\left\{w_\theta\left(\mu, \hat{\mu}^2\right), w_{\theta'}\left(\mu, \hat{\mu}^1\right)\right\} < \min\left\{w_\theta\left(\mu, \hat{\mu}^1\right), w_{\theta'}\left(\mu, \hat{\mu}^2\right)\right\}$ *for $\theta' \in \Theta \backslash \{\theta\}$.*

(*ii*) (*Mislearning in both states*)*: Under both $\hat{\mu}^1$ and $\hat{\mu}^2$, asymptotic beliefs concentrate on state $\bar{\theta}$ in state $\underline{\theta}$ and on state $\underline{\theta}$ in state $\bar{\theta}$,*[41] *and* $w\left(\mu, \hat{\mu}^1\right) < w\left(\mu, \hat{\mu}^2\right)$.

Then, for any decision problem A, there exists T^* such that for all $T \geq T^*$, $W_T\left(\mu, \hat{\mu}^1, A\right) > W_T\left(\mu, \hat{\mu}^2, A\right)$.

To understand the result, consider the first case, where both agents' asymptotic beliefs are always a point-mass on θ. Then, in state θ, both agents choose optimally in the long run, so a faster rate of convergence, i.e., a higher efficiency index $w_\theta(\mu, \hat{\mu})$, is better for welfare. However, in state θ', both agents choose the same suboptimal act in the long run, so now slower convergence, i.e., a lower $w_{\theta'}(\mu, \hat{\mu})$, is better. Thus, if both (a) $w_\theta\left(\mu, \hat{\mu}^2\right) < w_\theta\left(\mu, \hat{\mu}^1\right)$ and (b) $w_{\theta'}\left(\mu, \hat{\mu}^1\right) < w_{\theta'}\left(\mu, \hat{\mu}^2\right)$, then agent 1 is better off. Notably, Theorem 3 shows that the same conclusion obtains under a weaker condition that only compares the minima of both sides of inequalities (a) and (b), allowing for a complete ranking across all such biases.

EXAMPLE 1 (Continued): *Returning to the illustrative example, Theorem 3 implies that among distortion functions $c^i(\cdot)$ that induce the same asymptotic beliefs, higher dynamic welfare is again characterized by the ratio $c^i(\bar{x})/c^i(\underline{x})$ being closer to* 1.[42] *Figure 7 illustrates this. Beyond correct learning, there are two possible cases: asymptotic beliefs concentrate on $\bar{\theta}$* (*respectively, $\underline{\theta}$*) *in both states if $c^i(\bar{x})/c^i(\underline{x})$ is sufficiently greater* (*respectively, smaller*) *than* 1.[43] *In the former case, $c^1(\cdot)$ dynamically outperforms $c^2(\cdot)$ if $1 < c^1(\bar{x})/c^1(\underline{x}) < c^2(\bar{x})/c^2(\underline{x})$, and in the latter case if $1 > c^1(\bar{x})/c^1(\underline{x}) > c^2(\bar{x})/c^2(\underline{x})$.*

C. *Dynamic Ranking with General States*

We extend the dynamic welfare ranking to general finite state spaces Θ.[44] While under binary states, the learning efficiency index yields a generically complete

[40] This occurs if and only if $KL\left(\mu_{\bar{\theta}}, \hat{\mu}^i_\theta\right) < KL\left(\mu_{\bar{\theta}}, \hat{\mu}^i_{\theta'}\right)$ and $KL\left(\mu_{\underline{\theta}}, \hat{\mu}^i_\theta\right) < KL\left(\mu_{\underline{\theta}}, \hat{\mu}^i_{\theta'}\right)$ for $\theta' \neq \theta$ and $i = 1, 2$.

[41] This occurs if and only if $KL\left(\mu_{\underline{\theta}}, \hat{\mu}^i_{\bar{\theta}}\right) < KL\left(\mu_{\underline{\theta}}, \hat{\mu}^i_{\underline{\theta}}\right)$ and $KL\left(\mu_{\bar{\theta}}, \hat{\mu}^i_{\bar{\theta}}\right) > KL\left(\mu_{\bar{\theta}}, \hat{\mu}^i_{\underline{\theta}}\right)$ for $i = 1, 2$.

[42] Assumption 3 holds, as all distortion factors $c^i(x)$ are positive.

[43] The former happens if $c^i(\bar{x})/c^i(\underline{x}) > \left[-\log \ell_\mu(\underline{x})/\log \ell_\mu(\bar{x})\right]\left[\mu_{\underline{\theta}}(\underline{x})/\mu_{\underline{\theta}}(\bar{x})\right]$, the latter if $c^i(\bar{x})/c^i(\underline{x}) < \left[-\log \ell_\mu(\underline{x})/\log \ell_\mu(\bar{x})\right]\left[\mu_{\bar{\theta}}(\underline{x})/\mu_{\bar{\theta}}(\bar{x})\right]$, and in between these bounds there is correct learning. Case (ii) of Theorem 3 (mislearning in both states) never arises under asymmetric under-/overreaction, as $c^i(x)$ is assumed positive.

[44] For the static ranking, a sufficient condition for $\hat{\mu}^1$ to dominate $\hat{\mu}^2$ is that after each signal, the interim belief under $\hat{\mu}^1$ is a convex combination of the beliefs under μ and $\hat{\mu}^2$, as in Section IIA.

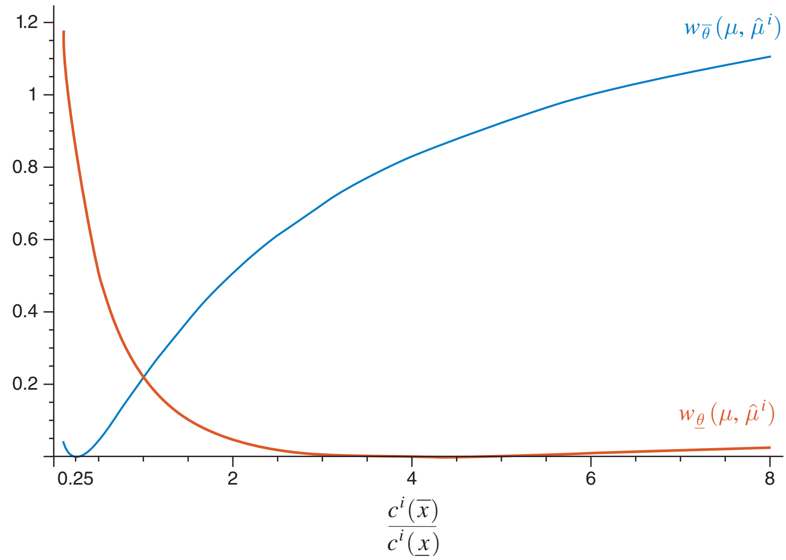

FIGURE 7

Notes: Efficiency indices when $\mu_{\overline{\theta}}(\overline{x}) = 0.8 = \mu_{\underline{\theta}}(\underline{x})$. Figure 3 discussed the case $1/4 < c^i(\overline{x})/c^i(\underline{x}) < 4$ (correct learning). Asymptotic beliefs concentrate on $\overline{\theta}$ in both states if $c^i(\overline{x})/c^i(\underline{x}) > 4$. For any $\hat{\mu}, \hat{\mu}'$ in this region, $w_{\underline{\theta}}(\mu, \hat{\mu}) < w_{\overline{\theta}}(\mu, \hat{\mu}')$. Thus, by Theorem 3, dynamic welfare is determined by $w_{\underline{\theta}}(\mu, \hat{\mu}^i)$ and is higher the lower $w_{\underline{\theta}}(\mu, \hat{\mu}^i)$, i.e., the smaller $c^i(\overline{x})/c^i(\underline{x})$. Analogously, asymptotic beliefs concentrate on $\underline{\theta}$ in both states if $c^i(\overline{x})/c^i(\underline{x}) < 1/4$; now, welfare is higher the lower $w_{\overline{\theta}}(\mu, \hat{\mu}^i)$, i.e., the greater $c^i(\overline{x})/c^i(\underline{x})$.

ranking over biases with the same asymptotic beliefs, a complete ranking is not possible with more than two states. For example, consider any decision problem A in which all acts yield the same utilities in states θ and θ'. Then any bias that only affects inferences between θ and θ' is payoff irrelevant in A but may affect welfare in decision problems in which the utilities at θ and θ' differ. However, we show that, up to controlling for such redundancies of equivalent states, a generalization of the learning efficiency index again yields a generically complete ranking.

Given any decision problem A, consider the partition S_A over Θ whose cells are $S_A(\theta) := \{\theta' \in \Theta : \arg\max_{a \in A} a_{\theta'} = \arg\max_{a \in A} a_\theta\}$ for each θ. That is, S_A divides Θ into equivalence classes of states that share the same ex post optimal act. Under binary Θ, all nontrivial A induce the same partition.

For any nondegenerate partition S over Θ (i.e., with $S(\theta) \neq \Theta$), define the S-learning efficiency index by $w(\mu, \hat{\mu}, S) := \min_\theta w_\theta(\mu, \hat{\mu}, S(\theta))$, where for each θ,

$$w_\theta(\mu, \hat{\mu}, S(\theta)) := \min_{\nu \in \Delta(X)} KL(\nu, \mu_\theta)$$

$$\text{subject to} \quad \min_{\theta' \in S(\theta)} KL(\nu, \hat{\mu}_{\theta'}) = \min_{\theta' \notin S(\theta)} KL(\nu, \hat{\mu}_{\theta'}).$$

Generalizing Definition 1, the indistinguishability condition in state θ now captures the set of empirical distributions ν based on which the agent is unable to distinguish whether the state is in $S(\theta)$ or in $\Theta \backslash S(\theta)$. As before, $w_\theta(\mu, \hat{\mu}, S(\theta))$ measures how atypical such empirical distributions ν are relative to the true signal distribution μ_θ, and the index $w(\mu, \hat{\mu}, S)$ considers the minimum across all states.

To extend Theorem 1, we impose the following generalization of Assumption 1. Given a partition S of Θ, we require that in each state θ, the agent asymptotically assigns probability one to the correct cell $S(\theta)$. The condition does not restrict asymptotic beliefs over states in $S(\theta)$, allowing for some forms of mislearning and for comparisons across biases whose asymptotic beliefs need not coincide in each state (online Appendix G extends Theorem 3, allowing for mislearning of the cell $S(\theta)$).

ASSUMPTION 4 (S-consistency): *For every* $\theta \in \Theta$, $\min_{\theta' \in S(\theta)} KL(\mu_\theta, \hat{\mu}_{\theta'}) < \min_{\theta' \notin S(\theta)} KL(\mu_\theta, \hat{\mu}_{\theta'})$.

We also focus on perceived signal structures that satisfy Assumption 3, and on regular decision problems A, where $a_\theta \neq a'_\theta$ for all θ and distinct $a, a' \in A$.

THEOREM 4: *Let S be a nondegenerate partition over Θ. Fix any true signal structure μ and perceived signal structures $\hat{\mu}^1$ and $\hat{\mu}^2$ satisfying Assumptions 3 and 4. Suppose $w(\mu, \hat{\mu}^1, S) > w(\mu, \hat{\mu}^2, S)$. Then for any regular decision problem A with $S_A = S$, there exists T^* such that for all $T \geq T^*$, $W_T(\mu, \hat{\mu}^1, A) > W_T(\mu, \hat{\mu}^2, A)$.*

Thus, up to restricting to decision problems that feature the same classes S of equivalent states, the dynamic welfare ranking is again generically complete and is characterized by the S-learning efficiency index.

D. *Conclusion*

This paper conducts a robust comparison of objective welfare across a wide range of learning biases. Our core results characterize this welfare comparison in dynamic settings, using a learning efficiency index. Complementing a focus in the literature on asymptotic beliefs, this index determines the speed of belief convergence under each bias by quantifying the likelihood with which agents encounter signal sequences that do not allow them to distinguish different states. We highlight that learning efficiency can be strictly lower for smaller biases or for biases that are less harmful in static settings, and that correctly specified but uncertain agents can be outperformed by misspecified agents. We apply our results to quantify and compare the severity of various commonly studied biases. Our findings may serve as a starting point to robustly evaluate interventions aimed at mitigating the effect of learning biases.[45]

We have focused on learning biases that can be represented as Bayesian updating under some possibly incorrect perception of signal likelihoods. As the proofs in Appendix B clarify, the key properties of this setting are that (i) posterior beliefs depend only on the empirical signal distribution (rather than the exact signal sequence) and (ii) for almost all empirical signal distributions $\nu \in \Delta(X)$, posteriors concentrate on some state as $T \to \infty$. These properties rule out many non-Bayesian

[45] For example, the previous version (Frick, Iijima, and Ishii 2021, Section 5) quantified the extent to which coarsening signals improves welfare under learning biases.

updating rules but are shared by some prominent non-Bayesian models that are not formally nested by our setting.[46] In the latter models, the same large deviation techniques as in this paper can be used to conduct dynamic welfare comparisons.

Our companion note (Frick, Iijima, and Ishii 2022) moves beyond the current i.i.d. setting in two natural ways that lead to violations of (i). First, we extend the dynamic characterization based on learning efficiency to Markovian signals. This extension can accommodate additional learning biases, such as the gambler's/hot-hand fallacies and forms of intertemporal correlation neglect. Second, we extend the dynamic welfare analysis to simple settings where the true/perceived signal distributions can depend endogenously on the agent's current belief, capturing some forms of misspecified active learning or belief-dependent updating (e.g., confirmation bias).

Appendix A. Proof of Proposition 1

We prove a slight generalization of Proposition 1 that allows decision problems to feature ties. For each decision problem A, we assume some arbitrary strict total order $\succ_A$ over A such that whenever (4) admits multiple solutions for an agent, she chooses the $\succ_A$-optimal act among these solutions. Note that both agents are assumed to use the same tiebreaking rule.

Suppose (6) holds for all $x \in X$. Fix any decision problem A. For each realized signal x, let p_x denote the posterior belief under μ, and $\hat{p}_x^i$ denote the posterior belief under $\hat{\mu}^i$ $(i = 1,2)$. Let $a_x^i \in \arg\max_{a \in A} a \cdot \hat{p}_x^i$ denote the action chosen by agent i (with tiebreaking according to $\succ_A$ in case of indifference). By (6) and the fact that Θ is binary, there is $\beta_x \in [0,1]$ such that $\hat{p}_x^1 = \beta_x p_x + (1 - \beta_x)\hat{p}_x^2$. Thus, for all $a \in A$,

$$a \cdot \hat{p}_x^1 = \beta_x a \cdot p_x + (1 - \beta_x) a \cdot \hat{p}_x^2. \tag{A1}$$

We claim that

$$a_x^1 \cdot p_x \geq a_x^2 \cdot p_x. \tag{A2}$$

Indeed, if $\beta_x = 0$, then by (A1), both agents share the same interim payoffs. Thus, $a_x^1 = a_x^2$ (using the assumption that agents follow the same tiebreaking rule), which implies (A2). If $\beta_x > 0$, then (A2) follows from (A1) and the fact that $a_x^1 \cdot \hat{p}_x^1 \geq a_x^2 \cdot \hat{p}_x^1$ and $a_x^1 \cdot \hat{p}_x^2 \leq a_x^2 \cdot \hat{p}_x^2$. Finally, by (5) and (A2), for each x,

$$\begin{aligned} W_1(\mu, \hat{\mu}^1, A) &= \sum_\theta p_0(\theta) \sum_x \mu_\theta(x) a_x^1 \cdot p_x \geq \sum_\theta p_0(\theta) \sum_x \mu_\theta(x) a_x^2 \cdot p_x \\ &= W_1(\mu, \hat{\mu}^2, A). \end{aligned}$$

[46] An example is the model in de Clippel and Zhang (2022) (which nests, e.g., the models of under-/overreaction, divisible updating, and base-rate neglect in Epstein, Noor, and Sandroni 2010; Cripps 2018; and Benjamin, Bodoh-Creed, and Rabin 2019): Applied to our setting, the posterior is $p_T(\cdot \mid x^T) = D(p_T^*(\cdot \mid x^T))$, where $p_T^*(\cdot \mid x^T)$ is the correct Bayesian posterior and $D : \Delta(\Theta) \to \Delta(\Theta)$ is a distortion function. Property (i) always holds; if D is continuous and maps point-mass beliefs to point-mass beliefs, so does property (ii). Here, it is essential that beliefs are updated only once based on the entire signal sequence x^T, rather than sequentially after each signal realization x_t. The latter case in general leads to different posteriors (except in Cripps 2018) and violates property (i).

For the converse direction, assume that $\mu, \hat{\mu}^1, \hat{\mu}^2$ satisfy the comonotonic likelihood ratio property for some linear order $>$ on X. Suppose that (6) is violated at some $x^* \in X$. We will construct a decision problem A such that $W_1(\mu, \hat{\mu}^1, A) < W_1(\mu, \hat{\mu}^2, A)$. Since (6) is violated at x^*, we either have (i) $\ell_{\hat{\mu}^1}(x^*) > \ell_{\hat{\mu}^2}(x^*)$ and $\ell_{\hat{\mu}^1}(x^*) > \ell_{\mu}(x^*)$ or (ii) $\ell_{\hat{\mu}^1}(x^*) < \ell_{\hat{\mu}^2}(x^*)$ and $\ell_{\hat{\mu}^1}(x^*) < \ell_{\mu}(x^*)$. We consider only case (i), as the argument for case (ii) is analogous. Take any $\ell^* \in (\max\{\ell_\mu(x^*), \ell_{\hat{\mu}^2}(x^*)\}, \ell_{\hat{\mu}^1}(x^*))$. The comonotonic likelihood ratio property ensures that for each $i = 1, 2$, we have $\ell_\mu(x)$, $\ell_{\hat{\mu}^i}(x) > \ell^*$ if $x > x^*$, while we have $\ell_\mu(x), \ell_{\hat{\mu}^i}(x) < \ell^*$ if $x < x^*$.

Consider a decision problem $A = \{\bar{a}, \underline{a}\}$ such that $\bar{a}_{\bar{\theta}} - \underline{a}_{\bar{\theta}} > 0 > \bar{a}_{\underline{\theta}} - \underline{a}_{\underline{\theta}}$ and $(\bar{a}_{\bar{\theta}} - \underline{a}_{\bar{\theta}})\ell^*[p_0(\bar{\theta})/p_0(\underline{\theta})] + \bar{a}_{\underline{\theta}} - \underline{a}_{\underline{\theta}} = 0$. Let a_x (respectively, a_x^i) denote the act that maximizes the conditional expected payoff according to μ (respectively, $\hat{\mu}^i$) at signal x. By construction, $a_x = a_x^1 = a_x^2 = \bar{a}$ for $x > x^*$, $a_x = a_x^1 = a_x^2 = \underline{a}$ for $x < x^*$, but $a_x = a_x^2 = \underline{a} \neq \bar{a} = a_x^1$ for $x = x^*$. This implies $W_1(\mu, \mu, A) = W_1(\mu, \hat{\mu}^2, A) > W_1(\mu, \hat{\mu}^1, A)$, as desired. ∎

Appendix B. Proofs of Theorems 1–4

B.1. *Preliminaries*

The following preliminary results will be used in the Proofs of Theorems 1–4. These results analyze the agent's asymptotic behavior under a more general class of strategies than those that were introduced in the main text. Specifically, we focus on strategies that have well-behaved limits, which depend only on empirical signal distributions.

Define a limit strategy at A as an upper-hemicontinuous correspondence $\phi : \Delta(X) \rightrightarrows A$, with $|\phi(\nu)| = 1$ for a dense set of $\nu \in \Delta(X)$. We say a strategy $\sigma : \bigcup_{T \in \mathbb{N}} X^T \to A$ is induced by limit strategy ϕ if for any $a \in A$ and compact $K \subseteq \Delta(X)$ with $\phi(\nu) = \{a\}$ for all $\nu \in K$, there is T^* such that $\sigma(x^T) = a$ whenever $T \geq T^*$ and $\nu_{x^T} \in K$. Thus, at large T, behavior following any signal sequence x^T depends only on the empirical signal distribution ν_{x^T}, as specified by $\phi(\nu_{x^T})$.

Let $\Pr_\theta$ denote the probability measure over signal sequences induced by repeated i.i.d. draws according to the true signal distribution μ_θ in state θ. The following lemma characterizes the asymptotic decay rate of the probability that the agent's choice deviates from the action under the long-run signal distribution μ_θ.

LEMMA 1: *Suppose that strategy σ at a decision problem A is induced by a limit strategy ϕ. Take any state θ and act $a^* \in A$ such that $\phi(\mu_\theta) = \{a^*\}$. Then*

$$\text{(B1)} \quad \lim_{T\to\infty} \frac{1}{T} \log \Pr_\theta\left[\sigma(x^T) \neq a^*\right] = -\inf_{\nu \in \Delta(X)} KL(\nu, \mu_\theta) \quad s.t. \quad \phi(\nu) \neq \{a^*\}.$$

PROOF:

Take any $d > \inf_{\nu \in \Delta(X)} KL(\nu, \mu_\theta)$ s.t. $\phi(\nu) \neq \{a^*\}$. Then, there exists ν with $KL(\nu, \mu_\theta) < d$ and $\phi(\nu) \neq \{a^*\}$. By the definition of ϕ and the continuity of $KL(\cdot, \mu_\theta)$, up to slightly perturbing ν, it is without loss to assume $\phi(\nu) = \{a'\}$ for some $a' \neq a^*$. Since ϕ is upper-hemicontinuous, there is a closed ball $B \ni \nu$ with

$\phi(\nu') = \{a'\}$ for all $\nu' \in B$; we take the ball small enough that $\inf_{\nu' \in B} KL(\nu', \mu_\theta) \leq d$. Since σ is induced by ϕ, there exists T^* such that $\sigma(x^T) = a'$ for all $T \geq T^*$ and x^T with $\nu_{x^T} \in B$. Thus,

$$\begin{aligned}\liminf_{T\to\infty} \frac{1}{T} \log \Pr_\theta\left[\sigma\left(x^T\right) \neq a^*\right] &\geq \liminf_{T\to\infty} \frac{1}{T} \log \Pr_\theta\left[\sigma\left(x^T\right) = a'\right] \\ &\geq \liminf_{T\to\infty} \frac{1}{T} \log \Pr_\theta\left[\nu_{x^T} \in B\right] \geq -d,\end{aligned}$$

where the last inequality uses Sanov's theorem (e.g., Dembo and Zeitouni 2010). Since d can be arbitrarily close to $\inf_{\nu \in \Delta(X)} KL(\nu, \mu_\theta)$ s.t. $\phi(\nu) \neq \{a^*\}$, we have $\liminf_{T\to\infty} (1/T) \log \Pr_\theta\left[\sigma\left(x^T\right) \neq a^*\right] \geq -\inf_{\nu \in \Delta(X)} KL(\nu, \mu_\theta)$ s.t. $\phi(\nu) \neq \{a^*\}$.

Take any $d < \inf_{\nu \in \Delta(X)} KL(\nu, \mu_\theta)$ s.t. $\phi(\nu) \neq \{a^*\}$. Then, $\phi(\nu) = \{a^*\}$ for all $\nu \in K := \left\{\nu \in \Delta(X) : KL(\nu, \mu_\theta) \leq d\right\}$, which is compact. Since σ is induced by ϕ, there exists T^* such that $\sigma\left(x^T\right) = a^*$ for all $T \geq T^*$ and x^T with $\nu_{x^T} \in K$. Thus,

$$\limsup_{T\to\infty} \frac{1}{T} \log \Pr_\theta\left[\sigma\left(x^T\right) \neq a^*\right] \leq \limsup_{T\to\infty} \frac{1}{T} \log \Pr_\theta\left[\nu_{x^T} \notin K\right] = -d,$$

where the equality again uses Sanov's theorem. Since d can be arbitrarily close to $\inf_{\nu \in \Delta(X)} KL(\nu, \mu_\theta)$ *s.t.* $\phi(\nu) \neq \{a^*\}$, we have $\limsup_{T\to\infty} (1/T) \log \Pr_\theta\left[\sigma\left(x^T\right) \neq a^*\right] \leq -\inf_{\nu \in \Delta(X)} KL(\nu, \mu_\theta)$ *s.t.* $\phi(\nu) \neq \{a^*\}$. ∎

The following lemma derives the limit strategy ϕ that induces the agent's strategy considered in our model.

LEMMA 2: *Fix any $\hat{\mu}$ such that Assumption 3 holds and any regular decision problem A. Consider any strategy $\sigma : \bigcup_{T \in \mathbb{N}} X^T \to A$ satisfying* (4) *under the posterior belief p_T induced by* (3) *under $\hat{\mu}$.*[47] *Then, σ is induced by the limit strategy ϕ of the form*

$$\phi(\nu) = \bigcup_{\theta \in \arg\min_{\theta' \in \Theta} KL\left(\nu, \hat{\mu}_{\theta'}\right)} \arg\max_{a \in A} a_\theta. \tag{B2}$$

PROOF:

Note that ϕ as defined by (B2) is upper-hemicontinuous since each $KL(\cdot, \hat{\mu}_\theta)$ is continuous. To verify that $|\phi(\nu)| = 1$ on a dense set of ν, it suffices to show that $\arg\min_{\theta' \in \Theta} KL(\nu, \hat{\mu}_{\theta'})$ is a singleton on a dense set of ν because each $\arg\max_{a \in A} a_\theta$ is a singleton (by the regularity of A). To show this, take any $\nu \in \Delta(X)$ and $\theta \in \arg\min_{\theta' \in \Theta} KL(\nu, \hat{\mu}_{\theta'})$, i.e., $\nu \cdot \log \hat{\mu}_\theta \geq \nu \cdot \log \hat{\mu}_{\theta'}$ for all $\theta' \neq \theta$. By Assumption 3, there exists $\nu' \in \Delta(X)$ with $\nu' \cdot \log \hat{\mu}_\theta > \nu' \cdot \log \hat{\mu}_{\theta'}$ for all $\theta' \neq \theta$. Thus, for all large $n > 0$, $\{\theta\} = \arg\min_{\theta' \in \Theta} KL([\nu' + (n-1)\nu]/n, \hat{\mu}_{\theta'})$, as desired.

To see that σ is induced by ϕ, take any $a \in A$ and compact $K \subseteq \Delta(X)$ such that $\phi(\nu) = \{a\}$ for all $\nu \in K$. Let $\Theta^* := \left\{\theta : \{a\} = \arg\max_{a \in A} a_\theta\right\}$. For each $\nu \in K$,

[47] We allow arbitrary tiebreaking in case (4) admits multiple solutions.

$\phi(\nu) = \{a\}$ and (B2) imply $\max_{\theta\in\Theta^*}\nu\cdot\log\hat{\mu}_\theta > \max_{\theta'\notin\Theta^*}\nu\cdot\log\hat{\mu}_{\theta'}$. Since K is compact, there exists $\kappa > 0$ such that $\max_{\theta\in\Theta^*}\nu\cdot\log\hat{\mu}_\theta > \max_{\theta'\notin\Theta^*}\nu\cdot\log\hat{\mu}_{\theta'} + \kappa$ for all $\nu \in K$. By (4) and the definition of Θ^*, there exists $\gamma > 0$ such that $\sigma(x^T) = a$ holds for all x^T with $\max_{\theta\in\Theta^*}\min_{\theta'\notin\Theta^*}\log\left[p_T(\theta|x^T)/p_T(\theta'|x^T)\right] \geq \gamma$. Fix T^* large enough that $T^* \geq \left\{\gamma + \max_{\theta,\theta'\in\Theta}\log\left[p_0(\theta)/p_0(\theta')\right]\right\}/\kappa$. Then, for all $T \geq T^*$, any x^T with $\nu_{x^T} \in K$, and any $\theta^* \in \arg\max_{\theta\in\Theta^*}\nu_{x^T}\cdot\log\hat{\mu}_\theta$ and $\theta' \notin \Theta^*$, we have

$$\log\frac{p_T(\theta^*|x^T)}{p_T(\theta'|x^T)} = \log\frac{p_0(\theta^*)}{p_0(\theta')} + T\sum_x \nu_{x^T}(x)\log\frac{\hat{\mu}_{\theta^*}(x)}{\hat{\mu}_{\theta'}(x)} \geq \gamma.$$

Thus, $\sigma(x^T) = a$ by the choice of γ. ∎

Henceforth, denote welfare conditional on state θ by $W_{\theta T}(\mu,\hat{\mu},A) := \sum_{x^T}\mu_\theta^T(x^T)a_\theta^*(x^T,\hat{\mu})$.

B.2. *Proof of Theorems 1 and 4*

Below, we prove Theorem 4. Theorem 1 follows immediately from Theorem 4. Indeed, Assumption 1 implies Assumptions 3 and 4; moreover, for binary states, it is without loss of generality to focus on regular decision problems.[48]

The following lemma relates (B1) and the S-learning efficiency index.

LEMMA 3: *Let S be a nondegenerate partition over Θ. Take any μ and $\hat{\mu}$ such that Assumption 4 holds. Then, for any regular decision problem A with $S_A = S$ and $\theta \in \Theta$,*

$$w_\theta(\mu,\hat{\mu},S(\theta)) = \inf_{\nu\in\Delta(X)} KL(\nu,\mu_\theta) \quad s.t. \quad \phi(\nu) \neq \{a^*\},$$

where ϕ is as given by (B2) *and* $\{a^*\} = \arg\max_{a\in A} a_\theta$.

PROOF:

Since ϕ is given by (B2) and $\{a^*\} = \arg\max_{a\in A} a_\theta$, the optimization in (B1) can be written as

$$\inf_{\nu\in\Delta(X)} KL(\nu,\mu_\theta) \quad s.t. \quad \min_{\theta'\in S_A(\theta)} KL(\nu,\hat{\mu}_{\theta'}) \geq \min_{\theta'\notin S_A(\theta)} KL(\nu,\hat{\mu}_{\theta'}).$$

In this problem, the set of ν satisfying the constraint is (i) compact (by continuity of KL) and (ii) nonempty (as it includes $\mu_{\theta'}$ for $\theta' \notin S_A(\theta)$ by Assumption 4). Hence, the infimum is achieved by some ν.

Thus, it suffices to show that the infimum of the above problem is achieved by some ν at which the constraint binds. For this, take any ν with $\min_{\theta'\in S_A(\theta)} KL(\nu,\hat{\mu}_{\theta'}) >$

[48] To see the latter, suppose A is not regular. Then some distinct acts $a,a' \in A$ have the same payoff in some state. When Θ is binary, this implies $a \geq a'$ or $a' \geq a$. Then removing the dominated act does not change the agent's choices since her posterior after every signal history has full support.

$\min_{\theta' \notin S_A(\theta)} KL(\nu, \hat{\mu}_{\theta'})$. Assumption 4 ensures $\nu \neq \mu_\theta$, which implies that the objective $KL(\alpha\nu + (1-\alpha)\mu_\theta, \mu_\theta)$ with $\alpha < 1$ is strictly lower than $KL(\nu, \mu_\theta)$ by the convexity of KL. Moreover, choosing α sufficiently close to 1, $\alpha\nu + (1-\alpha)\mu_\theta$ satisfies the constraint of the problem (by the continuity of KL). Hence, ν is not optimal, so the constraint must bind at the optimum. ∎

To complete the Proof of Theorem 4, fix any nondegenerate partition S of Θ and $\mu, \hat{\mu}^1, \hat{\mu}^2$ satisfying Assumptions 3–4. Suppose $w(\mu, \hat{\mu}^1, S) > w(\mu, \hat{\mu}^2, S)$, and consider any regular decision problem A with $S_A = S$. Let σ^i denote the strategy given by (4) under $\hat{\mu}^i$, whose limit strategy ϕ^i satisfies (B2) by Lemma 2.

Take any θ. By Lemma 2 and Assumption 4, $\phi^i(\mu_\theta) = \arg\max_{a \in A} a_\theta$, which is equal to some unique a^* by the regularity of A. Then

$$
\begin{aligned}
w_\theta(\mu, \hat{\mu}^i, S(\theta)) &= -\lim_{T\to\infty} \frac{1}{T} \log \Pr_\theta\left[\sigma^i(x^T) \neq a^*\right] \\
&= -\lim_{T\to\infty} \frac{1}{T} \log\left[\max_{a\in A} a_\theta - W_{\theta T}(\mu, \hat{\mu}^i, A)\right],
\end{aligned}
$$

where the first equality follows from Lemmas 1 and 3, and the second equality from the fact that $\{a^*\} = \arg\max_{a\in A} a_\theta$. Therefore,

$$
\begin{aligned}
W^*(A) - W_T(\mu, \hat{\mu}^i, A) &= \sum_\theta p_0(\theta)\left[\max_{a\in A} a_\theta - W_{\theta T}(\mu, \hat{\mu}^i, A)\right] \\
&= \sum_\theta p_0(\theta) \exp\left[-T w_\theta(\mu, \hat{\mu}^i, S(\theta)) + o(T)\right] \\
&= \exp\left[-T w(\mu, \hat{\mu}^i, S) + o(T)\right],
\end{aligned}
$$

where the final equality uses $w(\mu, \hat{\mu}^i, S) := \min_\theta w_\theta(\mu, \hat{\mu}^i, S(\theta))$. This implies that there exists T^* such that for all $T \geq T^*$, $W^*(A) - W_T(\mu, \hat{\mu}^2, A) > W^*(A) - W_T(\mu, \hat{\mu}^1, A)$, i.e., $W_T(\mu, \hat{\mu}^1, A) > W_T(\mu, \hat{\mu}^2, A)$. ∎

B.3. *Proof of Theorem 2*

The following two lemmas generalize Lemmas 2 and 3 and admit analogous proofs.

LEMMA 4: *Take any μ and $\hat{M}$ such that Assumption 2 holds. Consider any (full-support) prior $q_0 \in \Delta(\hat{M} \times \Theta)$ and strategy σ at a regular decision problem A given by $\sigma(x^T) \in \arg\max_{a\in A} \sum_{\theta\in\Theta} q_T(\theta \mid x^T) a_\theta$, where $q_T(\cdot)$ is given by Bayesian updating. Then σ is induced by the limit strategy ϕ of the form*

$$
\phi(\nu) = \bigcup_{\theta \in \arg\min_{\theta'\in\Theta} \min_{\hat{\mu}\in M} KL(\nu, \hat{\mu}_{\theta'})} \arg\max_{a\in A} a_\theta. \tag{B3}
$$

LEMMA 5: *Take any μ and $\hat{M}$ such that Assumption 2 holds. Then for any regular decision problem A and $\theta \in \Theta$,*

$$
w_\theta(\mu, \hat{M}) = \inf_{\nu\in\Delta(X)} KL(\nu, \mu_\theta) \quad s.t. \quad \phi(\nu) \neq \{a^*\},
$$

where ϕ *is as given by* (B3) *and* $\{a^*\} = \arg\max_{a \in A} a_\theta$.

Given Lemmas 4–5, the Proof of Theorem 2 is analogous to Theorems 1 and 4. ∎

B.4. *Proof of Theorem 3*

Fix $\hat{\mu}^1$ and $\hat{\mu}^2$ satisfying Assumption 3. Take any decision problem A and let σ^i denote the strategy given by (4) under $\hat{\mu}^i$. We can assume without loss that A contains no dominated acts since agents would never choose such acts given that their posteriors p_T^i have full support at each signal history. As in the Proof of Theorems 1 and 4, we allow for the possibility that (4) features multiple subjectively optimal acts at some posteriors, in which case agents employ (possibly different) tiebreaking rules.

Let $\bar{a}$ (respectively, $\underline{a}$) denote the unique act in A that is ex post optimal at $\bar{\theta}$ (respectively, $\underline{\theta}$). Since A does not contain dominated acts, we have

$$\text{(B4)} \qquad \forall a \in A \backslash \{\bar{a}, \underline{a}\}, \quad [\bar{a}_{\underline{\theta}} < a_{\underline{\theta}} < \underline{a}_{\underline{\theta}}] \quad \text{and} \quad [\underline{a}_{\bar{\theta}} < a_{\bar{\theta}} < \bar{a}_{\bar{\theta}}].$$

Observe that an analog of Lemma 3 holds by the same argument. For each θ,

$$w_\theta(\mu, \hat{\mu}^i) = \inf_{\nu \in \Delta(X)} KL(\nu, \mu_\theta) \quad s.t. \quad \phi^i(\nu) \neq \{a^i\},$$

where ϕ^i is the limit strategy given by (B2) under $\hat{\mu}^i$ and $\{a^i\} = \phi^i(\mu_\theta)$. This value is well defined, as Assumption 3 guarantees that the constraint set is nonempty.

First Part.—We focus on the case $\theta = \bar{\theta}$; the case $\theta = \underline{\theta}$ is analogous. Thus, the probability of agent i choosing $\bar{a}$ goes to 1 as $T \to \infty$ at both states. In particular, by Lemmas 1, 2, and the representation of $w_\theta(\mu, \hat{\mu}^i)$ above, we have

$$w_\theta(\mu, \hat{\mu}^i) = -\lim_{T \to \infty} \frac{1}{T} \log \Pr_\theta[\sigma^i(x^T) \neq \bar{a}]$$

for each i and θ. Moreover, by (B4), $W_{\bar{\theta}T}(\mu, \hat{\mu}^i, A) < \bar{a}_{\bar{\theta}}$ and $W_{\underline{\theta}T}(\mu, \hat{\mu}^i, A) > \bar{a}_{\underline{\theta}}$ hold for all large enough T. Thus,

$$\text{(B5)} \qquad p_0(\bar{\theta})[\bar{a}_{\bar{\theta}} - W_{\bar{\theta}T}(\mu, \hat{\mu}^1, A)] + p_0(\underline{\theta})[W_{\underline{\theta}T}(\mu, \hat{\mu}^2, A) - \bar{a}_{\underline{\theta}}]$$
$$= \exp\left[-T\min\{w_{\bar{\theta}}(\mu, \hat{\mu}^1), w_{\underline{\theta}}(\mu, \hat{\mu}^2)\} + o(T)\right],$$

$$\text{(B6)} \qquad p_0(\bar{\theta})[\bar{a}_{\bar{\theta}} - W_{\bar{\theta}T}(\mu, \hat{\mu}^2, A)] + p_0(\underline{\theta})[W_{\underline{\theta}T}(\mu, \hat{\mu}^1, A) - \bar{a}_{\underline{\theta}}]$$
$$= \exp\left[-T\min\{w_{\bar{\theta}}(\mu, \hat{\mu}^2), w_{\underline{\theta}}(\mu, \hat{\mu}^1)\} + o(T)\right].$$

The assumption that $\min\{w_{\bar{\theta}}(\mu, \hat{\mu}^1), w_{\underline{\theta}}(\mu, \hat{\mu}^2)\} > \min\{w_{\bar{\theta}}(\mu, \hat{\mu}^2), w_{\underline{\theta}}(\mu, \hat{\mu}^1)\}$ implies that for all large enough T, the value of (B5) is smaller than the value of (B6). This is equivalent to $W_T(\mu, \hat{\mu}^1, A) > W_T(\mu, \hat{\mu}^2, A)$.

Second Part.—In this case, the probability of agent i choosing $\bar{a}$ (respectively, $\underline{a}$) in state $\underline{\theta}$ (respectively, $\bar{\theta}$) goes to 1 as $T \to \infty$. In particular, by Lemmas 1, 2, and the representation of $w_\theta(\mu, \hat{\mu}^i)$ above, we have

$$w_{\bar{\theta}}(\mu, \hat{\mu}^i) = -\lim_{T\to\infty} \frac{1}{T} \log \Pr_{\bar{\theta}}\left[\sigma^i(x^T) \neq \underline{a}\right],$$
$$w_{\underline{\theta}}(\mu, \hat{\mu}^i) = -\lim_{T\to\infty} \frac{1}{T} \log \Pr_{\underline{\theta}}\left[\sigma^i(x^T) \neq \bar{a}\right]$$

for each i. Moreover, by (B4), $W_{\bar{\theta}T}(\mu, \hat{\mu}^i, A) > \underline{a}_{\bar{\theta}}$ and $W_{\underline{\theta}T}(\mu, \hat{\mu}^i, A) > \bar{a}_{\underline{\theta}}$ hold for all large enough T. Thus, by (B4), for each i,

$$\text{(B7)} \qquad p_0(\bar{\theta})\left[W_{\bar{\theta}T}(\mu, \hat{\mu}^i, A) - \underline{a}_{\bar{\theta}}\right] + p_0(\underline{\theta})\left[W_{\underline{\theta}T}(\mu, \hat{\mu}^i, A) - \bar{a}_{\underline{\theta}}\right]$$
$$= \exp\left[-Tw(\mu, \hat{\mu}^i) + o(T)\right].$$

The assumption that $w(\mu, \hat{\mu}^1) < w(\mu, \hat{\mu}^2)$ implies that for all large T, the value of (B7) is higher for $i = 1$ than $i = 2$. This is equivalent to $W_T(\mu, \hat{\mu}^1, A) > W_T(\mu, \hat{\mu}^2, A)$. ∎

Appendix C. Proof of Proposition 2

Fix any T. Condition (13) implies that either $\prod_{t=1}^T \ell_\mu(x_t) \geq \prod_{t=1}^T \ell_{\hat{\mu}^1}(x_t) \geq \prod_{t=1}^T \ell_{\hat{\mu}^2}(x_t)$ for all $x^T \in X^T$ or $\prod_{t=1}^T \ell_\mu(x_t) \leq \prod_{t=1}^T \ell_{\hat{\mu}^1}(x_t) \leq \prod_{t=1}^T \ell_{\hat{\mu}^2}(x_t)$ for all $x^T \in X^T$. Thus, analogously to Proposition 1, $W_T(\mu, \hat{\mu}^1, A) \geq W_T(\mu, \hat{\mu}^2, A)$ for all A (where we can again allow for ties provided both agents use the same tiebreaking rule). ∎

Appendix D. Proof of Proposition 3

If $\hat{\mu}$ is a Philipps-Edwards bias, then, as explained in the main text, $\mathcal{I}(\hat{\mu}) = \mathcal{I}(\mu)$, so $w(\mu, \mu) = w(\mu, \hat{\mu})$. To prove the "only if" direction, we will use the following lemma.

LEMMA 6: *Fix any μ. There exists a unique $\nu^* \in \mathcal{I}(\mu)$ such that $w(\mu, \mu) = KL(\nu^*, \mu_{\underline{\theta}}) = KL(\nu^*, \mu_{\bar{\theta}})$. Moreover, $\nu^*(x) > 0$ for all $x \in X$, and there exist $\lambda_{\underline{\theta}}, \lambda_{\bar{\theta}} > 0$ and $C_{\underline{\theta}}, C_{\bar{\theta}} \in \mathbb{R}$ such that*

$$\nabla_\nu KL(\nu^*, \mu_{\underline{\theta}}) = \left(1 + C_{\underline{\theta}} - \lambda_{\underline{\theta}} \log \frac{\mu_{\underline{\theta}}(x)}{\mu_{\bar{\theta}}(x)}\right)_{x\in X},$$
$$\nabla_\nu KL(\nu^*, \mu_{\bar{\theta}}) = \left(1 + C_{\bar{\theta}} - \lambda_{\bar{\theta}} \log \frac{\mu_{\bar{\theta}}(x)}{\mu_{\underline{\theta}}(x)}\right)_{x\in X}.$$

PROOF:

By definition, $\nu \in \mathcal{I}(\mu)$ if and only if $KL(\nu, \mu_{\underline{\theta}}) = KL(\nu, \mu_{\bar{\theta}})$. Thus, $w(\mu, \mu) = KL(\nu^*, \mu_{\underline{\theta}}) = KL(\nu^*, \mu_{\bar{\theta}})$ if and only if $\nu^* \in \arg\min_{\nu \in \mathcal{I}(\mu)} KL(\nu, \mu_{\underline{\theta}}) = \arg\min_{\nu \in \mathcal{I}(\mu)} KL(\nu, \mu_{\bar{\theta}})$. Moreover, $\arg\min_{\nu \in \mathcal{I}(\mu)} KL(\nu, \mu_{\underline{\theta}}) = \{\nu^*\}$ is a singleton since $\mathrm{I}(\mu)$ is nonempty, compact, and convex and $KL(\cdot, \mu_{\underline{\theta}})$ is continuous and (strictly) convex.

To see that $\nu^*(x) > 0$ for all $x \in X$, suppose instead that $\nu^*(\bar{x}) = 0$ for some $\bar{x} \in X$. Consider any other $\hat{\nu} \in \mathcal{I}(\mu)$ with $\hat{\nu}(\bar{x}) > 0$.[49] For each $\varepsilon \in [0,1]$, consider $\nu_\varepsilon := (1-\varepsilon)\nu^* + \varepsilon\hat{\nu} \in \mathcal{I}(\mu)$. Then for each θ, $\lim_{\varepsilon \searrow 0} [KL(\nu_\varepsilon, \mu_\theta) - KL(\nu^*, \mu_\theta)]/\varepsilon = -\infty$, contradicting the fact that $\nu^* \in \arg\min_{\nu \in \mathcal{I}(\mu)} KL(\nu, \mu_\theta)$.

For the final part, consider state $\underline{\theta}$; the argument for state $\bar{\theta}$ is analogous. Since $\nu^*(x) > 0$ for all $x \in X$, $KL(\nu, \mu_{\underline{\theta}})$ is differentiable in ν at ν^*. As shown in the Proof of Lemma 3, $\nu^* \in \Delta(X)$ also solves the following relaxed problem:

$$\min_{\nu \in \Delta(X)} KL(\nu, \mu_{\underline{\theta}}) \quad s.t. \quad KL(\nu, \mu_{\bar{\theta}}) \leq KL(\nu, \mu_{\underline{\theta}})$$
$$= \min_{\nu \in \Delta(X)} KL(\nu, \mu_{\underline{\theta}}) \quad s.t. \quad \sum_{x \in X} \nu(x) \log \frac{\mu_{\underline{\theta}}(x)}{\mu_{\bar{\theta}}(x)} \leq 0.$$

Thus, we have the following first-order conditions at ν^*: there exist Lagrange multipliers $\lambda_{\underline{\theta}} \geq 0$ and $C_\theta \in \mathbb{R}$ such that for all x,

$$\text{(D1)} \qquad \log \frac{\nu^*(x)}{\mu_{\underline{\theta}}(x)} = -\lambda_{\underline{\theta}} \log \frac{\mu_{\underline{\theta}}(x)}{\mu_{\bar{\theta}}(x)} + C_{\underline{\theta}}.$$

If $\lambda_{\underline{\theta}} = 0$, then (D1) implies $\nu^* = \mu_{\underline{\theta}}$, which violates the constraint $\sum_{x \in X} \nu(x) \log[\mu_{\underline{\theta}}(x)/\mu_{\bar{\theta}}(x)] \leq 0$ since $\mu_{\underline{\theta}} \neq \mu_{\bar{\theta}}$. Thus, $\lambda_{\underline{\theta}} > 0$. Since $\nabla_\nu KL(\nu^*, \mu') = (1 + \log[\nu^*(x)/\mu'(x)])_{x \in X}$ for any $\mu' \in \operatorname{int}\Delta(X)$, (D1) becomes $\nabla_\nu KL(\nu^*, \mu_{\underline{\theta}}) = (1 + C_{\underline{\theta}} - \lambda_{\underline{\theta}} \log[\mu_{\underline{\theta}}(x)/\mu_{\bar{\theta}}(x)])_{x \in X}$. ∎

PROOF OF PROPOSITION 3 ("Only If" Direction):

Suppose $\hat{\mu}$ is not a Philipps-Edwards bias. Lemma 6 yields a unique $\nu^* \in \mathcal{I}(\mu)$ such that $\nu^*(x) > 0$ for all $x \in X$ and $w(\mu, \mu) = KL(\nu^*, \mu_{\underline{\theta}}) = KL(\nu^*, \mu_{\bar{\theta}})$. First, suppose $\nu^* \notin \mathcal{I}(\hat{\mu})$. Then $KL(\nu^*, \hat{\mu}_{\underline{\theta}}) < KL(\nu^*, \hat{\mu}_{\bar{\theta}})$ or $KL(\nu^*, \hat{\mu}_{\bar{\theta}}) < KL(\nu^*, \hat{\mu}_{\underline{\theta}})$. We focus on the former case, as the latter is analogous. As in the Proof of Lemma 3,

$$\text{(D2)} \qquad w_{\bar{\theta}}(\mu, \hat{\mu}) = \min_{\nu' \in \Delta(X)} KL(\nu', \mu_{\bar{\theta}}) \quad s.t. \quad KL(\nu', \hat{\mu}_{\underline{\theta}}) \leq KL(\nu', \hat{\mu}_{\bar{\theta}}).$$

[49] Such a $\hat{\nu}$ exists: If $\mu_{\bar{\theta}}(\bar{x}) = \mu_{\underline{\theta}}(\bar{x})$, set $\hat{\nu}(\bar{x}) = 1$. If $\mu_{\bar{\theta}}(\bar{x}) > \mu_{\underline{\theta}}(\bar{x})$, *set* $\hat{\nu}(\bar{x}) + \hat{\nu}(\underline{x}) = 1$ and $\hat{\nu}(\bar{x})/\hat{\nu}(\underline{x}) = \log[\mu_{\underline{\theta}}(\underline{x})/\mu_{\bar{\theta}}(\underline{x})]/\log[\mu_{\bar{\theta}}(\bar{x})/\mu_{\underline{\theta}}(\bar{x})]$ for some $\underline{x}$ such that $\mu_{\underline{\theta}}(\underline{x}) > \mu_{\bar{\theta}}(\underline{x})$. The case $\mu_{\bar{\theta}}(\bar{x}) < \mu_{\underline{\theta}}(\bar{x})$ is analogous.

Since $KL(\cdot, \hat{\mu}_{\underline{\theta}})$ and $KL(\cdot, \hat{\mu}_{\bar{\theta}})$ are continuous, $(1-\varepsilon)\nu^* + \varepsilon \mu_{\bar{\theta}}$ satisfies the constraint in (B9) for $\varepsilon \in (0,1)$ sufficiently small. However,

$$KL\big((1-\varepsilon)\nu^* + \varepsilon \mu_{\bar{\theta}}, \mu_{\bar{\theta}}\big) \leq (1-\varepsilon)KL(\nu^*, \mu_{\bar{\theta}}) + \varepsilon KL(\mu_{\bar{\theta}}, \mu_{\bar{\theta}}) < w(\mu, \mu),$$

using convexity of $KL(\cdot, \mu_{\bar{\theta}})$ and $w(\mu,\mu) > 0$. Thus, $w(\mu, \hat{\mu}) \leq w_{\bar{\theta}}(\mu, \hat{\mu}) < w(\mu,\mu)$.

Next, suppose $\nu^* \in \mathcal{I}(\hat{\mu})$. Since $\hat{\mu}$ is not a Philipps-Edwards bias and $\mathcal{I}(\mu) = \{\nu \in \Delta(X) : \nu \cdot \log(\mu_{\bar{\theta}}/\mu_{\underline{\theta}}) = 0\}, \mathcal{I}(\hat{\mu}) = \{\nu \in \Delta(X) : \nu \cdot \log(\hat{\mu}_{\bar{\theta}}/\hat{\mu}_{\underline{\theta}}) = 0\}$, there exists $\nu \in \mathcal{I}(\hat{\mu}) \backslash \mathcal{I}(\mu)$. Then $\nu \cdot \log(\mu_{\underline{\theta}}/\mu_{\bar{\theta}}) < 0$ or $\nu \cdot \log(\mu_{\bar{\theta}}/\mu_{\underline{\theta}}) < 0$. We focus on the former case, as the latter is analogous. For each $\varepsilon \in (0,1)$, define $\nu_\varepsilon := (1-\varepsilon)\nu^* + \varepsilon\nu$. Since $\nu^*, \nu \in \mathcal{I}(\hat{\mu})$, we have $\nu_\varepsilon \in \mathcal{I}(\hat{\mu})$. Moreover, Lemma 6 yields $\lambda_{\underline{\theta}} > 0$ such that

$$\lim_{\varepsilon \to 0} \frac{KL(\nu_\varepsilon, \mu_{\underline{\theta}}) - KL(\nu^*, \mu_{\underline{\theta}})}{\varepsilon} = (\nu - \nu^*) \cdot \nabla_\nu KL(\nu^*, \mu_{\underline{\theta}})$$

$$= -\lambda_{\underline{\theta}} \nu \cdot \log \frac{\mu_{\underline{\theta}}}{\mu_{\bar{\theta}}} < 0.$$

Thus, for $\varepsilon > 0$ sufficiently small, we have $\nu_\varepsilon \in \mathcal{I}(\hat{\mu})$ and $KL(\nu_\varepsilon, \mu_{\underline{\theta}}) < KL(\nu^*, \mu_{\underline{\theta}}) = w(\mu,\mu)$. Hence, $w(\mu, \hat{\mu}) \leq w(\underline{\theta}, \mu, \hat{\mu}) < w(\mu, \mu)$. ■

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