Integrating Subjective Beliefs Data into Dynamic Structural Models of Firm Behavior

Conference on Expectations & Learning in Dynamic Structural Models

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Context & Motivation

- Firms' decisions critically depend on their beliefs about future demand, costs, and competitors' behavior.
- These **beliefs can vary across firms** due to:
  a. Differences in firms' capacity to gather and process information.
  b. Heterogeneity in the stochastic processes governing state variables.
- The role of heterogeneity in firms' beiefs as a source of variation in their decisions remains an important but underexplored topic in economics—particularly in structural econometrics.
- Most structural models of firm behavior impose **restrictions that suppress the role of belief heterogeneity**:
  a. Rational expectations.
  b. Limited role for firm-specific, serially correlated state variables, and strong homogeneity in the stochastic processes of these variables.

Purpose of this Paper

- In this context, our primary goal in this paper:

1. To investigate the extent to which data on firms' subjective beliefs can be used to incorporate **richer persistent unobserved heterogeneity in dynamic structural models**.
    - In our structural model **agents can be fully rational**, but there are serially correlated unobserved state variables in many structural components that we typically assume homogeneous across firms.
2. We apply our model/method to study firms' **employment and capital investment** decisions.
    - The empirical analysis draws on data from the *Survey of Business Uncertainty (SBU)*, conducted by the Federal Reserve of Atlanta.
    - The SBU provides a rare monthly panel dataset with firms' beliefs in the form of probability distributions.

Related Literature

1. Growing literature on using subjective beliefs data to allow for richer heterogeneity in dynamic structural models:
   - Pantano & Zheng (2013): dynamic model of college attendance.
     - We allow for time-variant UH, & not restricted to CCP estimation.
   - Arellano, Attanasio, Crossman, & Sancibrian (2024): focused on heterogeneity in income process.
     - We estimate full dynamic structural model.
   - Keiller, de Paula, & Van Reenen (2024): firms' expectations on future output used to relax identification restrictions in PF estimation.
     - Our model endogenizes beliefs & can be used for CF experiments.

2. Research using Structural Models & *Survey of Business Uncertainty*:
   - Barrero (JFS, 2022): Macro model with managers' "biased beliefs".
     - Our focus is not on "biased beliefs" but more on providing a framework that endogenizes and explains these heterogeneous beliefs.

OUTLINE

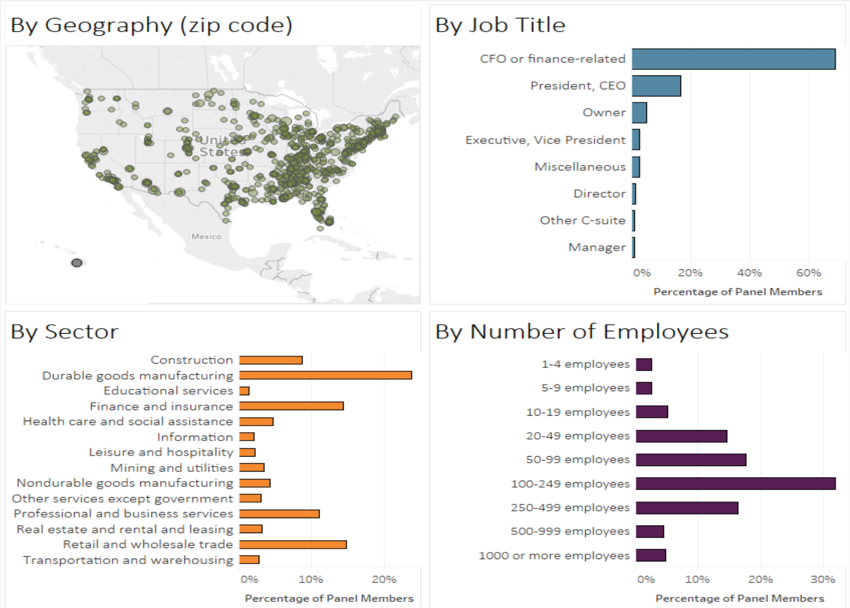
1. The *Survey of Business Uncertainty*
2. Descriptive Evidence on Subjective Beliefs
3. Our Model & Method
4. A Monte Carlo Experiment
5. Summary & Conclusions

1. Survey of Business Uncertainty

Survey of Business Uncertainty – Core Features

- Monthly firm-level panel dataset with information on:
  - Industry, state HQs, size.
  - Monthly revenue, employment, capital investment.
  - Monthly subjective beliefs about probability distributions of revenue, employment, & capital investment 12 months ahead.
- Responding subjects: Senior finance executives (CFOs, CEOs, controllers) in U.S. firms.
- Sample period: October 2014 to July 2023 (106 months).
- Coverage:
  - Panel size: $\approx$ 1,600 responding firms per year.
  - All 50 states. All 4-digit nonfarm industries. Broad firm-size distrib.
  - Response rate: Monthly average of $\approx 45\%$, and $> 60\%$ for large firms.

Survey of Business Uncertainty - Panel Composition



Beliefs Data

- Unlike most existing business surveys, the SBU elicits subjective probability distributions rather than single-point forecasts.
- The SBU elicits five-point subjective distributions over three firm-specific future outcomes: revenue, employment, investment.
    a. Respondents freely select 5 numerical values or Support Points.
    b. Then, they assign a Probability to each point, summing to 100%.
- This flexible design:
    a. Avoids anchoring from pre-specified bins.
    b. Accommodates asymmetry and fat tails in perceived risks.
    c. Captures uncertainty without strong parametric assumptions.

Sales Revenue Questionnaire

SBU Survey of Business Uncertainty

FEDERAL RESERVE BANK of ATLANTA | CHICAGO BOOTH | Stanford University

For the current quarter, what would you estimate the total dollar value of your sales revenue will be?

$ 10,000,000

Looking back, over the last 12 months, what was your approximate percentage sales revenue growth rate?

3 %

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SBU Survey of Business Uncertainty

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Looking ahead, from now to four quarters from now, what approximate percentage sales revenue growth rate would you assign to each of the following scenarios?

The LOWEST percentage sales revenue growth rate would be about:	1	%
A LOW percentage sales revenue growth rate would be about:	2	%
A MIDDLE percentage sales revenue growth rate would be about:	3	%
A HIGH percentage sales revenue growth rate would be about:	4	%
The HIGHEST percentage sales revenue growth rate would be about:	5	%

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SBU Survey of Business Uncertainty

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Please assign a percentage likelihood to the sales revenue growth rates you entered. (Values should sum to 100%)

LOWEST: The likelihood of realizing a **1%** sales revenue growth rate would be:	10	%
LOW: The likelihood of realizing a **2%** sales revenue growth rate would be:	20	%
MIDDLE: The likelihood of realizing a **3%** sales revenue growth rate would be:	40	%
HIGH: The likelihood of realizing a **4%** sales revenue growth rate would be:	20	%
HIGHEST: The likelihood of realizing a **5%** sales revenue growth rate would be:	10	%
Total	100	%

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Employment Questionnaire

SBU Survey of Business Uncertainty

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Currently, what is your number of employees (including part-time)?

100

Looking back, 12 months ago, what was your number of employees (including part-time)?

95

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SBU Survey of Business Uncertainty

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Looking ahead, 12 months from now, what number of employees (including part-time) would you assign to each of the following scenarios?

The LOWEST number of employees would be about:	100
A LOW number of employees would be about:	105
A MIDDLE number of employees would be about:	110
A HIGH number of employees would be about:	120
The HIGHEST number of employees would be about:	125

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SBU Survey of Business Uncertainty

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Please assign a percentage likelihood to the number of employees you entered above. (Values should sum to 100%)

LOWEST CASE: The likelihood of employing about **100** people 12 months from now would be:	0 %
LOW CASE: The likelihood of employing about **105** people 12 months from now would be:	0 %
MIDDLE CASE: The likelihood of employing about **110** people 12 months from now would be:	0 %
HIGH CASE: The likelihood of employing about **120** people 12 months from now would be:	0 %
HIGHEST CASE: The likelihood of employing about **125** people 12 months from now would be:	0 %
Total	0 %

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2. Descriptive Evidence on Subjective Beliefs

Subjective Expectations & Uncertainty

- Let $\{y_{it}^{j} : j = 1, 2, 3, 4, 5\}$ and $\{p_{it}^{j} : j = 1, 2, 3, 4, 5\}$ be the grid points and the probabilities in the elicited beliefs of firm i at month t for variable x.

- **Subjective Expectation:** Mean of the subjective distribution:

$$SE_{y,it} \;=\; \sum_{j=1}^{5} p_{it}^{j}\, y_{it}^{j}$$

- **Subjective Uncertainty:** Standard dev. of subjective distribution:

$$SU_{y,it} \;=\; \sqrt{\sum_{j=1}^{5} p_{it}^{j}\, \left(y_{it}^{j} - SE_{y,it}\right)^{2}}$$

Fact 1: Subjective expectations have high predictive power

- For each variable, we compare Mean Absolute Forecast Error (MAFE) of *subjective expectations* versus a firm-specific VAR(1) model:

$$\ln(y)_{it} = \alpha_i + \beta_{r,i} \ln(r)_{i,t-1} + \beta_{\ell,i} \ln(\ell)_{i,t-1} + \beta_{I,i} \ln(I)_{i,t-1} + \gamma_t + \varepsilon_{it}$$

	MAFE (12 months ahead)	
Variable	*Subjective Beliefs*	*AR1 Model*
log Revenue	0.0597	0.0642
log Employment	0.0775	0.0828
log (1 + Investment)	0.1401	0.1386

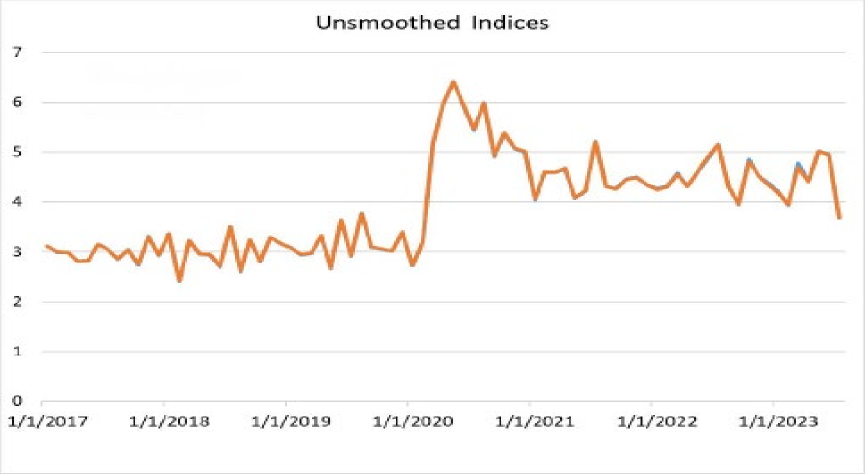
Fact 2: Subjective Uncertainty highly predictive of abs. forecast err

- Regression: Absolute Forecast Error $AFE_{y,i,t+12} \equiv |y_{i,t+12} - SE_{y,it}|$ on Subjective Uncertainty, $SU_{y,it}$:

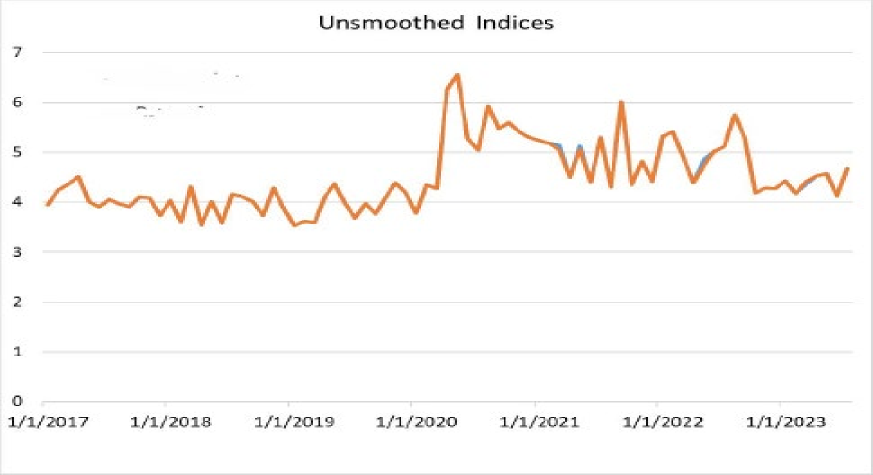
$$AFE_{y,it} = \alpha_i + \beta\, SU_{y,it} + \gamma_{industry,t} + \varepsilon_{it}$$

Variable	$\widehat{\beta}$	*(s.e.)*
log Revenue	0.5023***	(0.1170)
log Employment	0.4937***	(0.0538)
log (1 + Investment)	0.3822***	(0.0873)

Aggregate Uncertainty Index: Sales Growth



Aggregate Uncertainty Index: Employment Growth



3. Model & Method

Basic Framework

- Dynamic model of firm demand for inputs, employment (ℓ_{it}) and capital investment (I_{it}), with Bellman equation:

$$V(s_{it}) = \max_{\ell_{it}, I_{it}} \left\{ \pi\left(\ell_{it}, I_{it}, s_{it}\right) + \beta\, \mathbb{E}_{it}\left[V(s_{i,t+1}) \mid \ell_{it}, I_{it}, s_{it} \right] \right\}$$

- In this paper, we do not challenge the assumption that $\mathbb{E}_{it}$ is based on the **actual probability distribution** $p_{it}(s_{i,t+1} \mid \ell_{it}, I_{it}, s_{it})$.

- Instead, we use the beliefs data from SBU to allow for:

  1. **Richer specification of unobservables** in the state s_{it}.
  2. **Flexible firm-heterogeneity** in the stochastic process of the state variables.

States, Unobservables, Decisions, & Outcomes

1. **Observable state variables**: $\boldsymbol{x}_{it} = (\ \ell_{i,t-1}, k_{it}\ )$
2. **Persistent unobservable state variables**: $\boldsymbol{\omega}_{it} \in \mathbb{R}^d$, with

$$\boldsymbol{\omega}_{i,t+1} \sim f_{\omega,i}(\,\boldsymbol{\omega}_{i,t+1} \mid \boldsymbol{\omega}_{it})$$

   capturing firm-specific productivity, demand, and cost shocks.

3. **i.i.d. unobservable state variables**: $\boldsymbol{\varepsilon}_{it} \sim$ i.i.d. $T1 - EV$
4. **Outcome variable: Revenue**: r_{it}
5. **Outcome variables: Firm's Beliefs**: $\left\{B_{it}^{(r)}(u),\, B_{it}^{(\ell)}(u),\, B_{it}^{(I)}(u)\right\}$:

$$B_{it}^{(y)}(u) \;=\; \Pr\left(y_{i,t+12} \leq u \mid \boldsymbol{\omega}_{it},\, \boldsymbol{x}_{it},\, \ell_{it},\, I_{it}\right)$$

Flexible "Reduced-Form" Model for Beliefs

- **Factor Quantile model with Endogenous regressors**.
- For each var. y and support point $u \in \{1, 2, 3, 4\}$ **(12 equations)**:

$$\ln\left(\frac{B_{it}^{(y)}(u)}{1 - B_{it}^{(y)}(u)}\right) = \left(\ell_{it},\, I_{it},\, \boldsymbol{x}'_{it},\, \boldsymbol{\omega}'_{it}\right)\, \boldsymbol{\beta}_i^{(y)}(u)$$

- To deal with **Endogeneity of the regressors** $(\ell_{it}, I_{it}, \ell_{i,t-1}, k_{it})$, we exploit the Markov structure of $\boldsymbol{\omega}_{it}$.
- Firm-specific VAR(1) model:

$$\boldsymbol{\omega}_{i,t+1} = \Omega_i\, \boldsymbol{\omega}_{it} + \boldsymbol{\xi}_{i,t+1}$$

  with $\mathbb{E}(\boldsymbol{\xi}_{i,t+1} \mid \ell_{it}, I_{it}, \boldsymbol{x}_{it}) = 0$

- With large $T = 106$, we can estimate $\boldsymbol{\beta}_i^{(y)}(u)$ at firm level. We can identify up to **12 factors / unobservable variables** in $\boldsymbol{\omega}_{it}$.

Estimation Procedure

- **Step 1: Estimation of factor model.** Consistent estimation of ω_{it}.
- **Step 2: Estimation of the dynamic structural model** treating $\widehat{\omega}_{it}$ a observable state variables.

  a. Flexible specification of structural functions in terms of ω_{it}:
     - Revenue function: $r_i(\ell_{it}, I_{it}, x_{it}, \omega_{it})$.
     - Profit function: $\pi_i(\ell_{it}, I_{it}, x_{it}, \omega_{it})$.
     - Transition probabilities: $f_{\omega,i}(\omega_{i,t+1} \mid \omega_{it})$.

  b. Remaining unobservables (ε_{it}) are i.i.d., no endogeneity.

  c. Accounting for estimation error in $\widehat{\omega}_{it}$.

Testing for Rationality

- A **key restriction in our model and method** is that beliefs $B_{it}^{(y)}(u)$ and the model predictions about $\Pr(y_{i,t+12} \leq u \mid \ell_{it}, I_{it}, \boldsymbol{x}_{it}, \boldsymbol{\omega}_{it})$ are **deterministic functions of the same variables**: $(\ell_{it}, I_{it}, \boldsymbol{x}_{it}, \boldsymbol{\omega}_{it})$.

- While our approach is in the spirit of the rationality restriction

$$B_{it}^{(y)}(u) \;=\; \Pr(y_{i,t+12} \leq u \mid \ell_{it}, I_{it}, \boldsymbol{x}_{it}, \boldsymbol{\omega}_{it}),$$

  our method does not impose this restriction.

- Therefore, even before the estimation of the structural but after the estimation of $\boldsymbol{\omega}_{it}$, this rationality restriction can be tested.

4. Monte Carlo Experiment

Monte Carlo Experiment: DGP

- **Profit function:**

$$\pi_{it} \;=\; r_{it} - AC_{it}^{\ell} - AC_{it}^{k}$$

- **Revenue function:** Cobb-Douglas

$$r_{it} \;=\; \exp\left\{\omega_{it}^{r,0} + \omega_{it}^{r,1}\ \ln(\ell_{it}) + (1 - \omega_{it}^{r,1})\ \ln(k_{it})\right\}$$

- **Adjustment Costs:**

$$\begin{aligned} AC_{it}^{\ell} &= \ell_{it} + \exp\{\omega_{it}^{\ell}\}\ (\ell_{it} - \ell_{i,t-1})^2 \\ AC_{it}^{k} &= I_{it} + \exp\{\omega_{it}^{k}\}\ \left(\frac{I_{it}}{k_{it}}\right)^2 \end{aligned}$$

- **Stochastic Process** for $(\omega_{it}^{r,0},\ \omega_{it}^{r,1},\ \omega_{it}^{\ell},\ \omega_{it}^{k})$:
  - Linear VAR(1) where all the parameters (including variances of shocks) are firm-specific: $6 \times 4 = 24$ parameters per firm.
  - Each firm-specific parameter comes from a Uniform distribution.

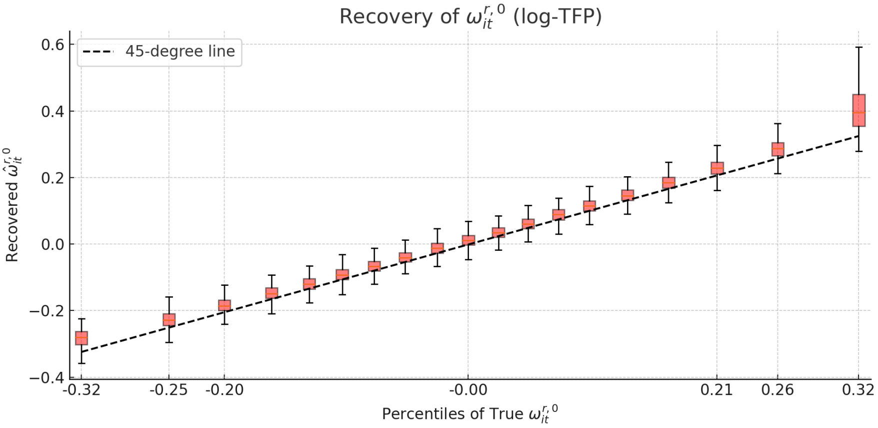
Beliefs Data in the Monte Carlo

1. Data: $N = 1,000$ and $T = 60$ months.
2. For each firm i, we solve its dynamic programming model and simulate state and decision variables.
3. For every simulated observation (i, t) and any variable $y = r, \ell, I$, we generate rational elicited beliefs.

- For values values u corresponding to percentiles 5, 25, 50, 75, and 95 in the firm-specific ergodic distribution of variable y:

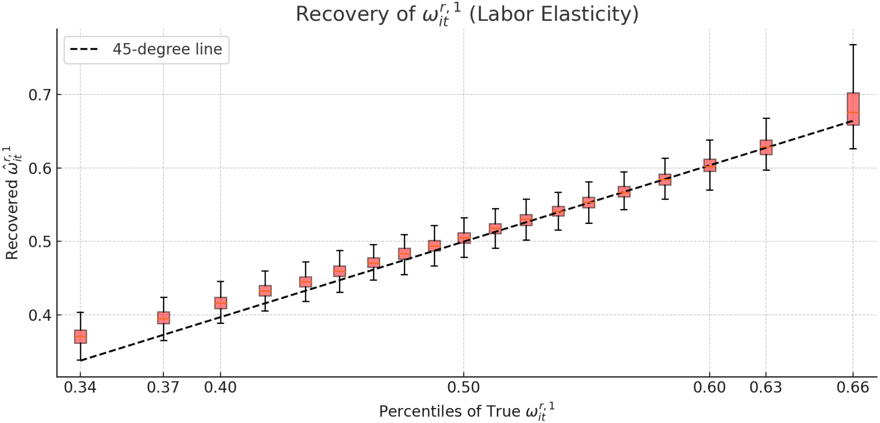
$$B_{it}^{(y)}(u) \;=\; \Pr\left(y_{i,t+12} \leq u \mid \ell_{it},\, I_{it},\, \boldsymbol{x}_{it},\, \boldsymbol{\omega}_{it}\right)$$

Recovery of log-TFP: $\omega_{it}^{r,0}$



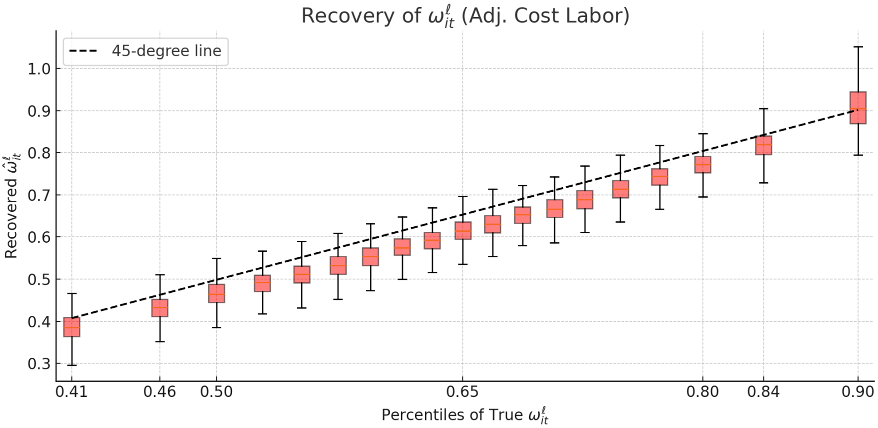
- Box plot of recovered vs. true $\omega_{it}^{r,0}$.
- $R^2 = 0.95$ in simulated sample.

Recovery of labor elasticity $\omega_{it}^{r,1}$



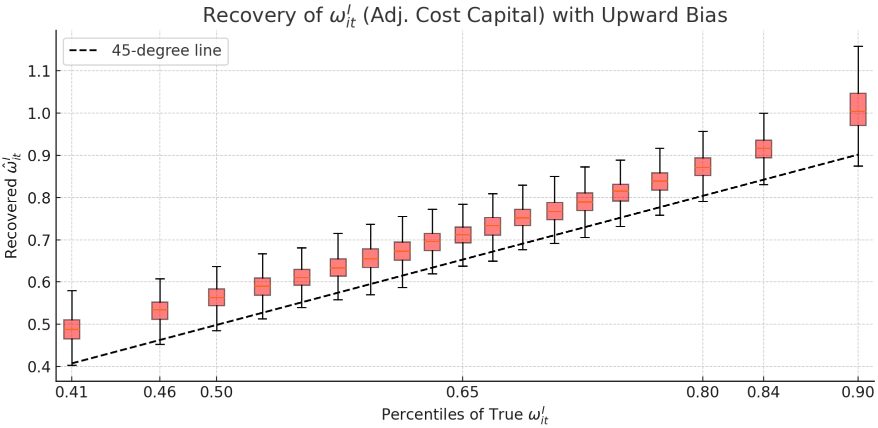
- $R^2 = 0.92$ for $\omega_{it}^{r,1}$ recovery.
- Some shrinkage toward mean.

Recovery of Labor Adjustment Cost Shocks: ω_{it}^{ℓ}



- $R^2 = 0.89$ for ω_{it}^{ℓ}
- Less precise than for $\omega_{it}^{r,0}$ and $\omega_{it}^{r,1}$. Still captures most variation.

Recovery of Capital Adjustment Cost Shocks: ω_{it}^{k}



- $R^2 = 0.87$ for ω_{it}^{k}
- Less precise than for $\omega_{it}^{r,0}$ and $\omega_{it}^{r,1}$. Still captures most variation.

Estimated Transition Processes – VAR

- Mean values of True and Estimated autoregressive parameters.

	True ρ	Estimated $\hat{\rho}$	Std. Err.
$\omega_{it}^{r,0}$ (log-TFP)	0.85	0.84	(0.12)
$\omega_{it}^{r,1}$ (labor elast)	0.60	0.59	(0.13)
ω_{it}^{ℓ} (adj. labor)	0.50	0.48	(0.14)
ω_{it}^{k} (adj. capital)	0.55	0.53	(0.14)

Estimated Structural Parameters

	True	Bias	RMSE
PF Intercept	0.00	0.0002	0.0033
PF slope labor	0.50	0.0006	0.0282
PF slope capital	0.50	-0.010	0.0337
Linear Cost Labor	1.00	0.0166	0.0887
Quadratic AC Labor	0.35	-0.0111	0.0355
Linear Cost Capital	1.00	-0.0325	0.0689
Quadratic AC Capital	0.65	0.0559	0.0870

Summary & Next Steps

- We propose an approach to recover multiple serially correlated unobservable state variables from firms' beliefs about the distribution of future revenue, employment, and investment.
- Monte Carlo experiments show promising results.
  a. Quite Accurate "inversion" /recovery of unobservables.
  b. Estimation of stochastic process & structural parameters show small bias.

Next: Apply method to SBU data.