Dynamic Personalized Offers while Learning Changing Tastes

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Abstract

Firms selling products to consumers realize that it takes time to learn consumers' preferences (say, by tracking their online behavior). What makes it even more challenging is that consumers' preferences may change over time, depreciating the value of acquired information. How should the firm personalize its offers and change them dynamically to learn as well as adapt to changing tastes? How should consumers behave in light of these dynamic offers? I build a continuous-time bargaining model with one-sided incomplete information where a buyer's binary type is publicly revealed through Brownian motion and the binary type changes via a Poisson process. In equilibrium, firms will start with high prices which will only be accepted by high-type consumers with positive probability and as belief drifts below a certain threshold, the firm will offer the lowest price that will be accepted by both types immediately. Changing tastes have two effects: a level effect that leaves low value consumers less likely to accept a given offer and a slope effect so that the firm screens high value consumers faster. Hence type change benefits both types of consumers at a cost to the firm. If the firm is restricted to constant prices and can use the acquired information to select consumers, it is better off than under dynamic prices. The continuation bargaining process gets resolved slower under fixed prices than under flexible prices, which makes consumers more willing to accept a given offer quicker. At last, the firm benefits from knowing when the type changes even if the new type is an independent new draw and the firm does not observe the realization.

1 Introduction

In bilateral bargaining where the uninformed party makes all the offers, the most disadvantaged type of the informed party is usually exploited all the surplus. Say, if seller is the informed party, buyer will start with some low prices and slowly screen out lower type sellers (Fudenberg et al., 1991; Deneckere and Liang, 2006; Fuchs and Skrzypacz, 2010). Along the path, buyer updates her belief through seller's rejection decisions and becomes more confident that seller is of higher types. Such a feature exists also in the setting where the buyer has access to some exogenous news source graduating revealing the seller type (Daley and Green, 2020). In practice, the asset type may not be fixed, but instead evolve over time. Such anticipation of type transition and bargaining with the new type will bring new bargaining dynamics.

Consider a venture capital trying to acquire a start-up pharmaceutical company which produces Covid-19 vaccines. The start-up firm privately knows its own value, while VC can conduct investigations of the firm through which it learns firm's value over time. In the meantime VC may negotiate offers with the firm. Sometime in the future, new vaccines or new biochemical technologies may enter the market and change the firm's value. The venture capital may observe such environment change, but it does not know the new value of the firm and needs to investigate again. Hence, the change in firm's value affects the both the firm's continuation value of waiting and VC's value of the acquired information of the firm.

Another example is that a firm tries to sell a good or service to consumers with unobserved preferences. It can collect consumer information through browser cookies or third party platforms and use such information to decide whether to make an price offer or what offer to make. In the usual setting with fixed consumer type, the firm will set gradually decreasing prices to screen high valuation consumers through intertemporal price discrimination. Consumer's preferences, however, may vary over time. Consumers may wait for a high value shock and delay accepting offers, while the firm needs to reevaluate the value of acquired information and make more tempting ones. Will the firm benefit from advertising differentiated prices to sellers based on the acquired information in comparison with the case that it can only use the information to advertise the same price to selected sellers when the type is persistent?

In this paper, I first extend the model by Daley and Green (2020) by introducing exogenous termination to the bargaining with interdependent values and show the payoff equivalence of the uninformed party with the fixed price benchmark. Then I model the exogenous termination specifically as endogenous type transition where the continuation game will be bargaining with the new type. Specifically, I consider an informed seller holding a asset of binary types and an uninformed buyer making frequent offers to the seller. The buyer can learn about the seller's value via a Brownian diffusion process. The type of the asset is persistent and a Poisson arrival event will redraw the type of the asset. Instead of equivalence, the buyer payoff lies between two fixed price benchmark ones when there is expected delay in the continuation bargaining. I also show how the transition affects the bargaining outcome, trade efficiency and both parties' welfare.

With exogenous termination, the buyer would offer low prices when the belief of high type is low, which will only be accepted by the low type. Low type mixes between accepting and rejecting, while high type always rejects. Over time, the belief will go up with updating from both the news process and low type's acceptance decision. Buyer would offer the upper bound of price offer when the belief is sufficiently high and both type sellers would accept immediately. This price upper bound is determined by high type's continuation payoff, and is equivalent to the fixed price in the fixed price benchmark. In the benchmark, the fixed price is set at the lowest price that the high type seller is willing to accept and buyer simply decides when to make such an offer to the seller. Therefore, I can construct two mappings from termination/continuation payoffs to ex-ante expected payoffs. With the same continuation payoffs, the ex-ante expected payoffs of buyer and high type seller are the same whereas the low type seller is better off under equilibrium than under fixed price benchmark.

When the continuation game is specifically modelled as type transition, the payoff may not be realized immediately upon transition. The delay in bargaining with the new type will affect the continuation payoff and hence high type's acceptance decision. The equilibrium will be a fixed point of above mapping, and the upper bound of price offer will be determined endogenously. The fixed price benchmark has two slightly different interpretations under type transition. The first is that the buyer cannot revoke the price offer upon transition whereas the second one can. These two feature no difference with exogenous termination as the game simply ends upon event arrival. With type transition, however, there may be delay in the continuation bargaining and they describe different scenarios. In equilibrium with delay, the buyer is better off than the first benchmark and is worse off than the second benchmark.

2 Literature Review

This paper contributes to literature with one sided incomplete information (Ausubel et al., 2002; Fuchs and Skrzypacz, 2020). Fuchs and Skrzypacz (2010) considers the arrival of exogenous termination events while Daley and Green (2020) considers the arrival of public news. The reduced form of public transition case in this paper combines these two channels and extends the finding in Daley and Green (2020) that the uninformed firm cannot benefit from adjusting prices freely. The equilibrium structure that the uninformed party screens the informed party gradually using early tempting offers and thus induces delay and inefficiency is commonly observed in the literature (Gul and Sonnenschein, 1988; Fudenberg et al., 1991; Deneckere and Liang, 2006; Gerardi et al., 2020) .

The equilibrium definition builds on Daley and Green (2020) and adds to the literature that models bargaining with one-sided private information directly in continuous time. In comparison, Ortner (2020) adds uninformed party's time varying cost to the bargaining. Daley and Green (2020) studies public learning of the private information itself. Lomys (2020) studies public learning of their outside options. Dilmé (2021) analyzes the role of discounting in shaping the bargaining outcome. This paper contributes to the literature by analyzing the interaction between gradual learning of the current private information and expected arrival of new one. It highlights how the expectation of future learning affects the value of currently acquired information and shapes the bargaining dynamics.

The private transition case in this paper contributes to the literature on information acquisition of a evolving type (Kremer et al., 2020). Ortner (2020) also considers a evolving private information but without learning. In contrast, this paper considers gradual learning of the evolving type and finds that the public observability of the transition time plays a role even if the new draw is independent.

3 Model

3.1 Set Up

A seller tries to trade an durable asset of type $\theta \in \{L,H\}$ to a buyer. The buyer privately knows the asset type. Let β_0 denote the prior probability that the seller assigns to $\theta = H$ at time $t = 0$. The seller's cost of parting with the asset is K_θ, where we normalize $K_L = 0 < K_H$. The buyer's value for the asset is V_θ, with $V_H \geq V_L$. There is common knowledge of gains from trade: $V_\theta > K_\theta$ for each θ.

The game is played in continuous time, starting at $t = 0$ with infinite horizon. At each instant t, the buyer makes a price offer to the seller. If the seller accepts an offer of p at time t, the transaction is executed and the game ends. The payoffs to the seller and the buyer respectively are $e^{-rt}(p-K_\theta)$ and $e^{-rt}(V_\theta - p)$, where $r>0$ is the common discount rate. With Poisson arrival rate $\lambda > 0$, a public event may occur that will also end the game, leaving payoffs K_θ^O to type-θ seller and $V^O \geq 0$ to the buyer, where $K_H^O \geq K_L^O \geq 0$. The payoffs here can be interpreted as termination payoffs where some event ends the game or continuation payoff where some public event changes the continuation of the bargaining process. I will discuss in more details how these payoffs affect the both parties' equilibrium payoffs and how they are determined. Both players are risk-neutral, expected utility maximizers.

Prior to reaching an agreement and to exogenous termination, public news about the asset's type is revealed via a Brownian diffusion process X_t, which satisfies $X_0 = 0$ and evolves according to

$$dX_t = \mu_\theta dt + \sigma dB_t,$$

where $B = \{B_t, \mathcal{F}_t, 0 \leq t \leq \infty\}$ is standard Brownian motion on the canonical probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ and ω denotes an arbitrary element of Ω. At each time t, the entire history of news, $\{X_s, 0 \leq s \leq t\}$, is observable to both players.

Let $\mathcal{H}_t$ denote the σ-algebra generated by $\{X_s, 0 \leq s \leq t\}$, which is therefore the public information at time t. The parameters μ_H, μ_L, and σ are common knowledge and without loss of generality, $\mu_H \geq \mu_L$. Define the signal-to-noise ratio $\gamma \equiv (\mu_H - \mu_L)/\sigma$. When $\gamma = 0$, the news is completely uninformative. Larger values of γ imply more informative news. In what follows, assume that $\gamma > 0$.

The timing of the game is as follows. At each instant, the exogenous event may arrive and game ends. Otherwise game continues and there is arrival of news about the seller, and then the buyer makes an offer, and then the seller decides whether to accept offer or not. If the seller accepts the offer, the game ends; if the seller rejects, the game continues to the next instant.

The equilibrium bargaining dynamics will depend on whether a static adverse selection problem can arise and whether there are positive trade surpluses for both types.

Assumption 1. *The* ***static lemons condition (SLC)*** *holds that $K_H > V_L$.*

Assumption 2. *The* ***high surplus condition (HSC)*** *holds that $V_H - K_H > \frac{\lambda}{\lambda+r}(V^O + K_H^O)$, and the* ***low surplus condition (LSC)*** *holds that $V_L > \frac{\lambda}{\lambda+r}(V^O + K_L^O)$.*

Assumption 1 guarantees the static adverse selection problem arise. Assumption 2 guarantees the trade surplus with both types is higher than the expected discounted trade surplus from waiting for exogenous termination, so it's always efficient to trade immediately with the current type than waiting for exogenous termination. With complete information where buyer extracts all the surplus, this assumption guarantees that the buyer is willing to trade with current type than waiting for the termination. Henceforth, I will denote $\tilde{K}_H \equiv K_H + \frac{\lambda}{\lambda+r} K_H^O$ as the lowest price high type is willing to accept and $\kappa \equiv \frac{\lambda}{\lambda+r}\left(K_H^O + V^O\right)$ as the expected discounted termination surplus with a high type seller.

3.2 Fixed Price Benchmark

To understand the role of news process and type transition in shaping the results, I consider a fixed price version of the model in which the buyer cannot adjust price which is similar to the fixed price benchmark as in Daley and Green (2020). The price is exogenously fixed at the lowest price that the high type seller is willing to accept immediately. The buyer observes the news, so she only decides when (if ever) to complete the transaction.

In Daley and Green (2020), the type is fixed and there is no future event, so with buyer having all the bargaining power, the fixed price is simply set at K_H. In this paper, with Poisson arrival of the new event, the high type always has the option to wait for the event instead of accepting the current offer. Therefore, this lowest price depends on the expected arrival of termination event. With termination payoff given by K_H^O for high type seller, the lowest price is the same as equilibrium offer upper bound $\tilde{K}_H$.

4 Equilibrium

Following the literature, I focus on Markovian equilibria with respect to the uninformed party's belief, or public belief and where the low type's payoff is non-decreasing with respect to this belief. I adopt the equilibrium concept developed by Daley and Green (2020). The formal components of equilibrium are defined in Appendix A.

4.1 Equilibrium Conditions

Here I give a brief overview of conditions. On the equilibrium path, seller is sequentially rational to the offer process and that the buyer updates her belief in consistent with the seller's strategy. Off path refinement is imposed with discrete time analog that the seller optimally responds to any offer taking buyer's future strategy as given. Buyer would make offers maximizing her payoff.

Seller Optimality – For a given offer process $P = \{P_t : 0 \leq t \leq \infty\}$, the seller chooses a set of stopping times $\mathcal{T}$ The type-θ seller's optimal stopping problem is

$$(\text{SP}_\theta) \qquad \sup_{\tau \in \mathcal{T}} E^\theta \left[e^{-(\lambda+r)\tau}(P_\tau - K_\theta) \right] + (1 - e^{-(\lambda+r)\tau}) \frac{\lambda}{\lambda + r} K_\theta^O,$$

A mixed strategy for the seller is a distribution over $\mathcal{T}$, which can be represented by the CDF it endows over the type-θ seller's acceptance time for each sample path of news, denoted S^θ.

Belief Consistency – At any time t, if trade has not yet occurred, the buyer assigns a probability, $\beta_t \in [0,1]$, to $\theta = H$. Transform this into the injective log-likelihood ratio, denoted as $Z_t \equiv \ln(\beta_t/(1-\beta_t)) \in \bar{\mathbb{R}}$.

The buyer's belief at time t is conditioned on the history of news and the fact that the seller has rejected all past offers. The belief "at time t" should be interpreted to mean *before* observing the seller's decision at time t. The buyer's belief is the sum of two components, $Z_t = \hat{Z}_t + Q$, which separate the two sources of information to the buyer:

$$Z_t = \underbrace{\ln\left(\frac{\beta_0}{1-\beta_0}\right) + \frac{\gamma}{\sigma}\left(X_t - \frac{\mu_H + \mu_L}{2} t\right)}_{\hat{Z}_t} + \underbrace{\ln\left(\frac{1 - S_{t^-}^H}{1 - S_{t^-}^L}\right)}_{Q_t}. \tag{1}$$

The term $\hat{Z}_t$ is the belief updating based on news, while the term Q is the stochastic process that keeps track of the information conveyed in equilibrium that the seller has rejected all past offers.

Stationarity. – I focus on stationary equilibria, using the uninformed party's belief as the state variable. Let $\Pi_\theta(z)$ denote the expected payoff for the type-θ seller given state z. For any $\tau \in \mathcal{S}^\theta$,

$$\Pi_\theta(z) \equiv E_z^\theta \left[e^{-(\lambda+r)\tau}(P_\tau - K_\theta) + (1 - e^{-(\lambda+r)\tau}) \frac{\lambda}{\lambda + r} K_\theta^O \right],$$

where E_z^θ is the expectation with respect to the probability law of the process Z starting from $Z_0 = z$ and conditional on θ. The expected payoff depends on the news process, the offer process and the expected arrival of exogenous termination.

Similarly, let $W(z)$ denote the expected payoff to the buyer in any given state z:

$$\begin{aligned} W(z) \equiv (1 - \beta(z)) E_z^L &\left[\int_0^\infty e^{-rt} \left(e^{-\lambda t}(V_L - P(Z_t)) + (1 - e^{-\lambda t}) V^O \right) dS_{t^-}^L \right] \\ + \beta(z) E_z^H &\left[\int_0^\infty e^{-rt} \left(e^{-\lambda t}(V_H - P(Z_t)) + (1 - e^{-\lambda t}) V^O \right) dS_{t^-}^H \right], \end{aligned} \tag{2}$$

where $\beta(z) = e^z/(1+e^z)$ is the belief of being high type corresponding to the log-likelihood ratio z. It depends on the acceptance decision of two type sellers and the expected arrival of type transition.

Response to Any Offer. – Let $\sigma_\theta(z,p)$ be the probability that the type-θ seller accepts some offer p. With exogenous termination, the high type seller will not accept an offer equal to his reservation value K_H and earn zero payoff. Instead the lowest price he is willing to accept depends on his termination payoff. Similar to no termination case, in equilibrium the buyer would never offer higher than this lowest price. The following lemma is derived from limit of the discrete time analog.

LEMMA 1. *The buyer never makes an offer greater than $\tilde{K}_H$.*

With this upper bound of price offer, the following conditions are imposed both on and off path.

(R1) If $p \geq \tilde{K}_H$, then $\sigma_H(z,p) = \sigma_L(z,p) = 1$.

For offers less than $\tilde{K}_H$, the seller responds optimally and the buyer's belief updates consistently. Let $\tilde{z}(z,p)$ be the buyer's updated belief if her offer of p is rejected in state z.

(R2) If $p < \tilde{K}_H$, then $\beta\left(\tilde{z}(z,p)\right) = \frac{\beta(z)(1-\sigma_H(z,p))}{\beta(z)(1-\sigma_H(z,p))+(1-\beta(z))(1-\sigma_L(z,p))}$.

Hence, the seller's choice is whether to accept p or reject and get $\Pi_\theta\left(\tilde{z}(z,p)\right)$.

(R3) If $p < \tilde{K}_H$, then $\sigma_\theta(z,p) \in \arg\max_\sigma \sigma(p-K_\theta) + (1-\sigma)\Pi_\theta\left(\tilde{z}(z,p)\right)$.

If $p = P(z) < \tilde{K}_H$, then (R2) and (R3) are implied by *belief consistency* and *seller optimality*. The refinement is to add the same condition to seller for deviating offers.

Buyer Optimality. – The buyer chooses an optimal offer at each instant given seller's strategy. If the buyer offers p in state z, either it will be accepted, earning her $V_\theta - p$ from type θ seller, or rejected, earning her the continuation value from the post rejection belief $\tilde{z}(z,p)$. The offer $P(z)$ is optimal if

$$\begin{aligned} P(z) \in \arg\max_p\; & \beta(z)\sigma_H(z,p)(V_H-p) + (1-\beta(z))\,\sigma_L(z,p)(V_L-p) \\ & + \left(1-\left(\beta(z)\sigma_H(z,p)+(1-\beta(z))\sigma_L(z,p)\right)\right)W\left(\tilde{z}(z,p)\right), \end{aligned} \tag{W1}$$

The updated belief $\tilde{z}(z,p)$ and continuation payoff take into account of the rejection decision of sellers while the value from news acquisition and exogenous termination is included in the form of W. The next condition for W specifies that the buyer makes optimal use of this option to wait and learn. For all $\tau \in \mathcal{T}$,

$$\text{(W2)} \qquad W(z) \geq E_z\left[e^{-(\lambda+r)\tau}W(\hat{Z}_\tau) + (1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}V^O\right].$$

Given the above the equilibrium definitions and the continuation payoffs $\{V^O, K_H^O, K_L^O\}$, I can construct a mapping $\Phi : \mathbb{R}_+^3 \to \mathbb{R}_+^3$ such that $\{W^0, \Pi_H^0, \Pi_L^0\} = \Phi\left(V^O, K_H^O, K_L^O\right)$, where W^0 is buyer's ex-ante expected payoff, Π_θ^0 is type-θ's ex-ante expected payoff. Φ describes the equilibrium behavior that maps continuation payoffs to ex-ante expected equilibrium payoffs. Similarly, I can construct a mapping for the fixed price benchmark $\Phi^F : \mathbb{R}_+^3 \to \mathbb{R}_+^3$ such that $\{W^{F^0}, \Pi_H^{F^0}, \Pi_L^{F^0}\} = \Phi^F\left(V^O, K_H^O, K_L^O\right)$. Φ_B, Φ_H, Φ_L will refer to the buyer, high type seller and low type seller's payoffs in the image of Φ.

THEOREM 1. *Given a set of continuation payoffs $\{V^O, K_H^O, K_L^O\}$, $\exists \bar{\beta} \in (0,1)$ such that when $\beta_0 \geq \bar{\beta}$, $\Phi = \Phi^F$ and when $\beta_0 < \bar{\beta}$, $\Phi_i = \Phi_i^F$ for $i = B, H$, and $\Phi_L > \Phi_L^F$.*

Under exogenous termination, buyer and high type seller always have the same payoff while low type seller is better off under equilibrium than fixed price benchmark when there is delay in bargaining. When the belief is high enough such that there is no delay, buyer and both type sellers will have the same payoff. The next subsection will characterize the equilibrium components and construct the equilibrium mapping.

4.2 Equilibrium Characterization

Now I will solve the equilibrium and construct the equilibrium payoffs. At each instant, the acceptance payoff is strictly higher for the low type seller with lower cost, whereas the option value of waiting is higher for the high type with better news. Hence, the low type has strictly stronger incentive to accept the same offer at each instant. If, at one instant, low type accepts with probability 1 and high type accepts with probability less than 1, then next instant the buyer knows for sure the remaining seller is high type and would offer a strictly higher price (with a discrete jump). Hence, I guess that the equilibrium features gradual screening of low type as in the literature and verifies it. Then I show that the equilibrium is unique following the methods in Daley and Green (2020).

I characterize the equilibrium using a pair (ζ, q), where ζ is the threshold above which the buyer offers $P(z) = \tilde{K}_H$ and trade is immediate. When $z < \zeta$, the buyer offers some $P(z) < \tilde{K}_H$, which the high type rejects and the low type accepts at a state specific hazard rate, $q(z)$ (i.e., proportional to time). In any state $z < \zeta$, the buyer's offer is such that the low type is indifferent between accepting $P(z)$ or waiting until $\tilde{K}_H$ is offered. The next definition gives a formal description of such a profile (ζ, q). Let $V(z) \equiv \beta(z)V_H + (1-\beta(z))V_L$ be the expected value of the asset when the buyer's log-likelihood belief is z.

DEFINITION 1. *For $\zeta \in \bar{\mathbb{R}}$ and measurable function $q : (-\infty, \zeta) \to \mathbb{R}_+$, let $T(\zeta) \equiv \inf\{t : Z_t \geq \zeta\}$ and $\Sigma(\zeta, q)$ be the strategy profile and belief process:*

$$Z_t = \begin{cases} \hat{Z}_t + \int_0^t q(Z_s)ds & \text{if } t < T(\zeta), \\ \text{arbitrary} & \text{otherwise} \end{cases} \tag{3}$$

$$S_t^H = \begin{cases} 0 & \text{if } t < T(\zeta), \\ 1 & \text{otherwise} \end{cases} \tag{4}$$

$$S_t^L = \begin{cases} 1 - e^{-\int_0^t q(Z_s)ds} & \text{if } t < T(\zeta), \\ 1 & \text{otherwise} \end{cases} \tag{5}$$

$$P(z) = \begin{cases} \tilde{K}_H & \text{if } z \geq \zeta, \\ \frac{\lambda}{\lambda+r}K_L^O + E_z^L\left[e^{-(\lambda+r)T(\zeta)}\right]\left(\tilde{K}_H - \frac{\lambda}{\lambda+r}K_L^O\right) & \text{otherwise} \end{cases} \tag{6}$$

$$\sigma_H(z,p) = \begin{cases} 1 & \text{if } p \geq \tilde{K}_H, \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

$$\sigma_L(z,p) = \begin{cases} 1 & \text{if } p \geq \tilde{K}_H, \\ \frac{\beta(P^{-1}(p)) - \beta(z)}{\beta(P^{-1}(p))(1-\beta(z))} & \text{if } p \in (P(z), \tilde{K}_H), \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

$$\tilde{z}(z,p) = \begin{cases} \text{arbitrary} & \text{if } p \geq \tilde{K}_H, \\ P^{-1}(p) & \text{if } p \in (P(z), \tilde{K}_H), \\ z & \text{otherwise} \end{cases} \tag{9}$$

Based on the definition of such $\Sigma(\zeta, q)$ pair, the equilibrium payoff of both type sellers can be constructed, with high type payoff being constant and low type payoff being non-decreasing.

LEMMA 2. *In any* $\Sigma(\zeta,q)$ *profile, the seller's payoffs are given by*

$$\Pi_H(z) = \frac{\lambda}{\lambda+r}K_H^O$$
$$\Pi_L(z) = P(z) = \frac{\lambda}{\lambda+r}K_L^O + E_z^L\left[e^{-(\lambda+r)T(\zeta)}\right]\left(\tilde{K}_H - \frac{\lambda}{\lambda+r}K_L^O\right).$$

where high type's payoff is constant and low type's payoff is non-decreasing and strictly increasing below ζ.

The next theorem proves that there exists such an equilibrium and that the equilibrium is the unique "reasonable" stationary equilibrium. The imposed refinement is that the seller's value function is non-decreasing in the public belief, which means that good news is never harmful to the seller.

THEOREM 2. *There exists a unique pair* (β,q) *such that the strategies and beliefs above constitute an equilibrium of the bargaining game with event arrival. In addition, this is the unique stationary equilibrium in which the seller's value function is non-decreasing.*

In a $\Sigma(\zeta,q)$ profile, along the path, the high type seller plays a pure strategy $\tau^H = T(\zeta)$ whereas the low type mixes. If $t > T(\zeta)$, trade occurs by time t with probability 1. Hence, in (3), the evolution of Z in this event is off-path. Likewise for $\tilde{z}(z,p)$ if $p \geq \tilde{K}_H$ in (9). For $z < \zeta$, only the low type accepts offers, meaning rejection is a (weakly) positive signal that $\theta = H$. Therefore, along the equilibrium path, the buyer's belief conditional on rejection Z, has additional upward drift compared to the belief that updates solely based on news $\hat{Z}$. For $t < T(\zeta)$, the high type always rejects ($S^H_{t^-} = 0$), therefore $Q_t = -\ln(1-S_{t^-})$ and $dQ_t = dS_{t^-}/(1-S_{t^-}) = q(Z_t)dt$. Hence, the additional upward drift of Z, relative to $\hat{Z}$, is the hazard rate of the low type's acceptance, $q(z) \geq 0$, with $dZ_t = d\hat{Z}_t + q(Z_t)dt$.

First I characterize buyer's value function. For $z < \zeta$, the buyer trades at rate $(1-\beta(z))q(z)$, which leads to a net payoff of $V_L - P(z) - W(z)$. If the low type rejects, the buyer receives the discounted expected continuation payoff, and Z evolves according to

$$dZ_t = \frac{\gamma}{\sigma}\left(dX_t - \frac{\mu_H+\mu_L}{2}dt\right) + q(Z_t)dt,$$

which has drift $\left(\gamma^2/2\right)(2\beta(z)-1) + q(Z_t)$ given the buyer's information and volatility γ.

Therefore the buyer's value function satisfies

$$rW(z) = \lambda(V^O - W(z)) + q(z)(1-\beta(z))(V_L - P(z) - W(z)) + \left(\frac{\gamma^2}{2}(2\beta(z)-1) + q(z)\right)W'(z) + \frac{\gamma^2}{2}W''(z). \tag{10}$$

Collecting the q terms gives

$$rW(z) = \underbrace{\lambda(V^O - W(z))}_{\text{Evolution due to type transition}} + \underbrace{\frac{\gamma^2}{2}(2\beta(z)-1)W'(z) + \frac{\gamma^2}{2}W''(z)}_{\text{Evolution due to news}} + q(z)\underbrace{\left((1-\beta(z))(V_L - P(z) - W(z)) + W'(z)\right)}_{\Gamma(z)\equiv\text{Net benefit of screening at z}}. \tag{11}$$

The first two terms on the right-hand side of (11) is the evolution of the buyer's value arising from type transition and news as in (24) for the fixed price game. The third term is the additional value she derives from trade with the low type, which is the product of the trade rate, $q(z)$, and the net benefit of this screening, denoted $\Gamma(z)$. The next lemma demonstrates that the result of zero screening benefit in equilibrium still holds as in Daley and Green (2020).

LEMMA 3. *If the proposed strategies and beliefs constitute an equilibrium, the net benefit of screening must be zero at all beliefs below the threshold, i.e.* $\Gamma(z) = 0$ *for all* $z < \zeta$.

At all $z < \zeta$, (11) reduces to

$$rW(z) = \lambda(V^O - W(z)) + \frac{\gamma^2}{2}(2\beta(z)-1)W'(z) + \frac{\gamma^2}{2}W''(z), \tag{12}$$

Any solution to the ODE has the form

$$W(z) = C_1^* \frac{e^{u_1 z}}{1+e^z} + C_2^* \frac{e^{u_2 z}}{1+e^z} + \frac{\lambda}{\lambda + r}V^O, \tag{13}$$

where $(u_1, u_2) = \frac{1}{2}\left(1 \pm \sqrt{1 + 8(\lambda + r)/\gamma^2}\right)$ and C_1^*, C_2^* are constants. For any candidate threshold ζ, these constants are pinned down by the observation that (i) the buyer's payoff is uniformly bounded between 0 and $V_H + V^O$ (which implies $C_2^* = 0$), (ii) $W(\zeta) = V(\zeta) - \tilde{K}_H$ (which pins down $C_1 = (1+e^\zeta)(V(\zeta) - K_H - \kappa)e^{-u_1\zeta}$, as a function of ζ). It follows that the buyer's payoff under an arbitrary threshold policy ζ is

$$W(z|\zeta) = (1+e^{\zeta})\,(V(\zeta) - K_H - \kappa)\,e^{-u_1\zeta}\,\frac{e^{u_1 z}}{1+e^z} + \frac{\lambda}{\lambda+r}V^O.$$

Buyer takes the termination payoff V^O as given and chooses the optimal threshold that maximizes $W(z|\zeta)$ over all possible values of ζ. The first order condition $\frac{d}{d\zeta}W(z|\zeta) = 0$ gives

$$e^{\zeta}(u_1 - 1) - u_1\frac{(K_H+\kappa) - V_L}{V_H - (K_H+\kappa)} = 0, \tag{14}$$

and has a unique real solution

$$\hat{\zeta} = \ln\left(\frac{(K_H+\kappa)-V_L}{V_H-(K_H+\kappa)}\right) + \ln\left(\frac{u_1}{u_1-1}\right),$$

which gives $C_1^* = \left(\frac{(K_H+\kappa)-V_L}{u_1-1}\right)^{1-u_1}\left(\frac{V_H-(K_H+\kappa)}{u_1}\right)^{u_1}$.

Given $\hat{\zeta}$ and C_1^*, the buyer's value function is given by

$$W(z) = \begin{cases} V(z) - \tilde{K}_H & \text{if } z \geq \hat{\zeta} \\ C_1^*\frac{e^{u_1 z}}{1+e^z} + \frac{\lambda}{\lambda+r}V^O & \text{if } z < \hat{\zeta}, \end{cases} \tag{15}$$

The offers are pinned down by $\Gamma(z) = 0$, which gives the offer process and type-L seller's value function

$$\begin{aligned} \Pi_L(z) = P(z) &= V_L - W(z) + \frac{W'(z)}{1-\beta(z)} \\ &= \begin{cases} \tilde{K}_H & \text{if } z \geq \hat{\zeta} \\ V_L + C_1^*(u_1-1)e^{u_1 z} - \frac{\lambda}{\lambda+r}V^O & \text{if } z < \hat{\zeta}, \end{cases} \end{aligned} \tag{16}$$

whereas type-H seller's value function is given by

$$\begin{aligned} \Pi_H(z) &= E_z^H\left[e^{-(\lambda+r)T(\zeta)}(\tilde{K}_H - K_H) + (1-e^{-(\lambda+r)T(\zeta)})\frac{\lambda}{\lambda+r}K_H^O\right] \\ &= \frac{\lambda}{\lambda+r}K_H^O + E_z^H\left[e^{-(\lambda+r)T(\zeta)}(\tilde{K}_H - K_H - \frac{\lambda}{\lambda+r}K_H^O)\right] \\ &= \frac{\lambda}{\lambda+r}K_H^O, \end{aligned} \tag{17}$$

which is independent of z. This is captured by the observation that the upper bound of the price offer is the lowest price high type seller would accept and that the upper bound is set to make the high type indifferent between accepting it and waiting for the news. Hence, the expected payoff of the high type seller is equal to the discounted ex-ante seller expected payoff.

We can construct the equilibrium mapping from termination/continuation payoff to ex ante expected payoff of buyer and seller under equilibrium and fixed price benchmark.

PROPOSITION 1. *In equilibrium, the mapping from continuation payoff to ex-ante expected payoff can be described as*

$$\Phi_1\left(V^O, K_H^O, K_L^O\right) = \begin{cases} V(z_0) - K_H - \frac{\lambda}{\lambda+r}K_H^O & \text{if } z_0 \geq \hat{\zeta}, \\ C_1^* \frac{e^{u_1 z_0}}{1+e^{z_0}} + \frac{\lambda}{\lambda+r}V^O & \text{if } z_0 < \hat{\zeta} \end{cases}$$

$$\Phi_2\left(V^O, K_H^O, K_L^O\right) = \frac{\lambda}{\lambda+r}K_H^O,$$

$$\Phi_3\left(V^O, K_H^O, K_L^O\right) = \begin{cases} K_H + \frac{\lambda}{\lambda+r}K_H^O & \text{if } z_0 \geq \hat{\zeta}, \\ V_L + C_1^*(u_1-1)e^{u_1 z_0} - \frac{\lambda}{\lambda+r}V^O & \text{if } z_0 < \hat{\zeta} \end{cases}$$

where

$$C_1^* = \left(\frac{(K_H + \frac{\lambda}{\lambda+r}(V^O + K_H^O)) - V_L}{u_1 - 1}\right)^{1-u_1} \left(\frac{V_H - (K_H + \frac{\lambda}{\lambda+r}(V^O + K_H^O))}{u_1}\right)^{u_1}$$

and

$$\hat{\zeta} = \ln\left(\frac{(K_H + \frac{\lambda}{\lambda+r}(V^O + K_H^O)) - V_L}{V_H - (K_H + \frac{\lambda}{\lambda+r}(V^O + K_H^O))}\right) + \ln\left(\frac{u_1}{u_1 - 1}\right).$$

The high type seller is extracted the largest surplus possible and just earn the expected continuation payoff, which is not affected by the bargaining process and only depends on his own continuation value. When the prior belief is relatively optimistic, the buyer and low type seller's payoffs also just depend on high type's continuation value. The buyer finds it better to make the deal sooner than to wait and learn about seller's type. The offer she makes needs to make the high type willing to accept. When the prior belief is relatively low, however, the buyer would find it worthwhile to learn and screen low types gradually. The continuation value of the buyer would push up her equilibrium payoff at each belief and have a level effect on both buyer and low type seller's payoffs. Both high type and buyer's continuation value also affect the value of waiting for the news and hence the slope of buyer and low type's payoffs.

Now we determine the rate of trade $q(z)$ for all $z<\zeta$. For type-L seller, Z evolves according to

$$dZ_t=\left(q(Z_t)-\frac{\gamma^2}{2}\right)dt+\gamma dB_t,$$

so type-L seller's value function satisfies

$$(\lambda+r)\Pi_L(z)=\lambda K_L^O+\left(q(z)-\frac{\gamma^2}{2}\right)\Pi_L'(z)+\frac{\gamma^2}{2}\Pi_L''(z). \tag{18}$$

Hence we have

$$\begin{aligned} q(z)&=\frac{(\lambda+r)\Pi_L(z)-\lambda K_L^O+\frac{\gamma^2}{2}\Pi_L'(z)-\frac{\gamma^2}{2}\Pi_L''(z)}{\Pi_L'(z)} \\ &=\frac{\gamma^2}{2C_1^*}e^{-u_1 z}\left(V_L-\frac{\lambda}{\lambda+r}(V^O+K_L^O)\right) \end{aligned} \tag{19}$$

The final part of the characterization is a description of the seller's reaction to off-path offers in arbitrary state z. The type-θ seller accepts an offer $\hat{p}\neq P(z)$ in state z with probability $\sigma_\theta(z,\hat{p})$, where

$$\begin{aligned} &\sigma_H(z,\hat{p})=\sigma_L(z,\hat{p})=0, \quad \text{if } \hat{p}<P(z); \\ &\sigma_H(z,\hat{p})=\sigma_L(z,\hat{p})=1, \quad \text{if } \hat{p}\geq \tilde{K}_H. \end{aligned} \tag{20}$$

For $z<\zeta$ and $\hat{p}\in(P(z),\tilde{K}_H)$, then $\sigma_H(z,\hat{p})=0$ and

$$\sigma_L(z,\hat{p})=1-e^z\left[\frac{\hat{p}+\frac{\lambda}{\lambda+r}V^O-V_L}{C_1^*(u_1-1)}\right]^{-\frac{1}{u_1}}. \tag{21}$$

A Bayesian consistent belief starting from z given a rejection of such an offer $\hat{p}$ is therefore (from (R2))

$$\hat{z}=\frac{1}{u_1}\ln\left[\frac{\hat{p}+\frac{\lambda}{\lambda+r}V^O-V_L}{C_1^*(u_1-1)}\right]=\Pi_L^{-1}(\hat{p}). \tag{22}$$

That is, $\hat{z}$ is the unique solution to $\Pi_L(z)=\hat{p}$. Therefore, the low type is indifferent and willing to mix between accepting $\hat{p}$ and rejecting to earn $\Pi_L(\hat{z})$ in the continuation play. The continuation value includes both bargaining with the current type and expected payoff from exogenous termination.

4.3 Fixed Price Benchmark

The fixed price game reduces to a standard real option (or stopping) problem for the buyer. Her belief about the value of the asset evolves over time based on news and she has the right to execute the transaction at the fixed price $\tilde{K}_H$ at any time. Without keeping track of seller's rejection decision, the buyer's belief process can be described by $\hat{Z}_t$ alone. The buyer's payoff from stopping is $V(z) - \tilde{K}_H$. Hence, she chooses a stopping time T to solve

$$\begin{aligned} &\sup_T E[e^{-rT}(V(\hat{Z}_T) - \tilde{K}_H)e^{-\lambda T}] + \int_0^T e^{-rt} V^O \lambda e^{-\lambda t} dt \\ =&\sup_T E[e^{-(\lambda+r)T}(V(\hat{Z}_T) - \tilde{K}_H)] + (1 - e^{-(\lambda+r)T})\frac{\lambda}{\lambda+r} V^O. \end{aligned} \tag{23}$$

The solution to the fixed price game is a threshold policy: stop the first time $\hat{Z}$ is weakly above some threshold ζ_d. For any $z \geq \zeta_d$, the buyer's value is $W^F(z) = V(z) - \tilde{K}_H$. For $z < \zeta_d$, $\hat{Z}$ evolves according to

$$d\hat{Z}_t = \frac{\gamma}{\sigma}\left(dX_t - \frac{\mu_H + \mu_L}{2} dt\right),$$

and the buyer's value function is given by

$$W^F(z) \approx e^{-(\lambda+r)dt} E_z\left(W^F(z + d\hat{Z})\right) + (1 - e^{-(\lambda+r)dt})\frac{\lambda}{\lambda+r} V^O,$$

Applying Ito's lemma to $W^F(\hat{Z}_t)$, we arrive at the following ordinary differential equation (ODE) for the buyer's value function

$$rW^F(z) = \underbrace{\lambda(V^O - W^F(z))}_{\text{Evolution due to type transition}} + \underbrace{\frac{\gamma^2}{2}(2\beta(z) - 1)W^{F'}(z) + \frac{\gamma^2}{2} W^{F''}(z)}_{\text{Evolution due to news}}, \tag{24}$$

which is identical to the evolution of buyer's value function in equilibrium as in (11).

The solution is given by

$$W^F(z) = C_1^F \frac{e^{u_1 z}}{1 + e^z} + \frac{\lambda}{\lambda + r} V^O, \tag{25}$$

where $C_1^F = (1+e^{\zeta})(V(\zeta) - K_H - \kappa)e^{-u_1\zeta}$. It follows that the buyer's payoff under an arbitrary threshold policy ζ is

$$W^F(z|\zeta) = (1+e^{\zeta})(V(\zeta) - K_H - \kappa)e^{-u_1\zeta}\frac{e^{u_1 z}}{1+e^z} + \frac{\lambda}{\lambda+r}V^O.$$

The optimal threshold maximizes $W^F(z|\zeta)$ over all possible values of ζ: $\zeta_d = \ln\frac{(K_H+\kappa)-V_L}{V_H-(K_H+\kappa)} + \ln\left(\frac{u_1}{u_1-1}\right) = \hat{\zeta}$.

LEMMA 4. *The solution to the fixed price game is that the buyer completes the trade if and only if her belief is weakly above ζ_d.*

The seller's payoff is pinned down by the expected time taken for the updated belief to pass the threshold and expected arrival of termination.

$$\Pi_H^F(z) = \hat{E}_z^H\left[e^{-(\lambda+r)T(\zeta_d)}\left(\tilde{K}_H - K_H\right) + \left(1-e^{-(\lambda+r)T(\zeta_d)}\right)\frac{\lambda}{\lambda+r}K_H^O\right] = \frac{\lambda}{\lambda+r}K_H^O.$$

$$\begin{aligned}\Pi_L^F(z) &= \hat{E}_z^L\left[e^{-(\lambda+r)T(\zeta_d)}\tilde{K}_H + \left(1-e^{-(\lambda+r)T(\zeta_d)}\right)\frac{\lambda}{\lambda+r}K_L^O\right]\\ &= \frac{\lambda}{\lambda+r}K_L^O + \hat{E}_z^L\left[e^{-(\lambda+r)T(\zeta_d)}\right]\left(\tilde{K}_H - \frac{\lambda}{\lambda+r}K_L^O\right)\end{aligned}$$

Here the $\hat{E}_z^{\theta}$ is expectation with respect to the probability law of the process $\hat{Z}$ starting from z and conditional on θ, so we can calculate the low type seller's payoff at arbitrary belief z is

$$\Pi_L^F(z) = \begin{cases}\tilde{K}_H & \text{if } z \geq \zeta_d,\\ \frac{\lambda}{\lambda+r}K_L^O + e^{u_1(z-\zeta_d)}\left(\tilde{K}_H - \frac{\lambda}{\lambda+r}K_L^O\right) & \text{if } z < \zeta_d\end{cases}$$

PROPOSITION 2. *In the fixed price benchmark, the equilibrium mapping is*

$$\Phi_1^F\left(V^O, K_H^O, K_L^O\right) = \begin{cases}V(z_0) - K_H - \frac{\lambda}{\lambda+r}K_H^O & \text{if } z_0 \geq \zeta_d,\\ C_1^{F^*}\frac{e^{u_1 z_0}}{1+e^{z_0}} + \frac{\lambda}{\lambda+r}V^O & \text{if } z_0 < \zeta_d\end{cases}$$

$$\Phi_2^F\left(V^O, K_H^O, K_L^O\right) = \frac{\lambda}{\lambda+r}K_H^O,$$

$$\Phi_3^F\left(V^O, K_H^O, K_L^O\right) = \begin{cases}K_H + \frac{\lambda}{\lambda+r}K_H^O & \text{if } z_0 \geq \zeta_d,\\ \frac{\lambda}{\lambda+r}K_L^O + e^{u_1(z_0-\zeta_d)}\left(K_H + \frac{\lambda}{\lambda+r}(K_H^O - K_L^O)\right) & \text{if } z_0 < \zeta_d\end{cases}$$

where $\zeta_d = \hat{\zeta}$ *and* $C_1^{F^*} = C_1^*$.

Now we can compare the equilibrium mapping and fixed price mapping to see the difference how continuation payoffs affect ex-ante expected payoffs. With slight abuse of notation, $\Phi\left(V^O, K_H^O, K_L^O\right) = \Phi^F\left(V^O, K_H^O, K_L^O\right) = \Phi\left(K_H^O\right)$ if $z_0 \geq \hat{\zeta}$. When the belief is high, there is no delay in bargaining, so the only effect from continuation game is high type's continuation payoff which pushes up the lowest price he is willing to take. If $z_0 < \hat{\zeta}$,

$$
\begin{aligned}
\Phi_1\left(V^O, K_H^O, K_L^O\right) &= \Phi_1^F\left(V^O, K_H^O, K_L^O\right) = \Phi_1\left(V^O, K_H^O\right), \\
\Phi_2\left(V^O, K_H^O, K_L^O\right) &= \Phi_2^F\left(V^O, K_H^O, K_L^O\right) = \Phi_2\left(K_H^O\right), \\
\Phi_3\left(V^O, K_H^O, K_L^O\right) &= \Phi_3\left(V^O, K_H^O\right), \\
\Phi_3^F\left(V^O, K_H^O, K_L^O\right) &= \Phi_3^F\left(V^O, K_H^O, K_L^O\right).
\end{aligned}
$$

Low type seller's continuation payoff only affects low type's ex-ante expected payoff under the fixed price benchmark when belief is low. In the equilibrium mapping, however, buyer's competition with her future self makes her give tempting offers which the low type seller accepts with positive intensity. Hence, with LSC, it can be shown that low type seller is strictly better off with earlier offers pushed up and his payoff does not depend on his own continuation payoff.

Buyer and high type seller enjoy the same payoff as in equilibrium. The fixed price is the lowest price the high type is indifferent between acceptance and waiting for exogenous termination, which leaves him the expected discounted continuation payoff for all beliefs. The buyer has no benefit of gradual screening due to the competition with future self and earns the same payoff as in equilibrium, which is a function of both buyer and high type seller's continuation payoffs. The buyer continuation value, however, does not have a level effect but only a slope effect on low type's payoff as low type's payoff is just the expected discounted time to get the fixed price offer. Instead, the level effect on low type's payoff comes from its own continuation payoff.

4.4 Full Commitment Benchmark

In this subsection, I will consider a full commitment benchmark where the buyer can design and commit to a contract. First ignore the exogenous news process and the possibility of exogenous termination. It is never optimal to offer a price where neither type accepts due to discounting as the buyer just front load her strategies. At any given belief β, as low type seller always have more incentives to accept a offer, it is one of the cases where both types accept for sure, only low type accept for sure or low type accepts with positive density/probability. It is not optimal for the buyer to have low type accept with probability less than 1 either, as the buyer can increase the earlier price by ε to break the indifference. In order to separate these two types, the buyer at time t' can offer an contract that gives $e^{-r\Delta t}K_H$ at time t' and K_H at time $t'+\Delta t$. Low type will accept at time t' and high type will accept at time $t'+\Delta t$. The buyer will maximize

$$\max_{\Delta t}\ \beta e^{-r\Delta t}(V_H - K_H) + (1-\beta)(V_L - e^{-r\Delta t}K_H)$$

The optimal solution is given by $\Delta t = \infty$ if $\beta < \frac{K_H}{V_H}$ and $\Delta t = 0$ if $\beta > \frac{K_H}{V_H}$. Note that this includes the case of offering a price accepted immediately by both types.

Now I will introduce just news into this contract. The buyer can commit ex ante to offer the above contract at any belief β. Both the above contract and equilibrium behavior features two segments. Define $\hat{\beta} \equiv \frac{e^{\hat{\xi}}}{1+e^{\hat{\xi}}}$. The buyer would offer K_H to the seller which will be accept immediately for sure if $\beta \geq \max\{\frac{K_H}{V_H}, \hat{\beta}\}$ under both the optimal contract without news and the equilibrium under news. Therefore, there is no benefit from learning with the news. By comparing the two cutoffs, $\frac{K_H}{V_H} > \hat{\beta} \iff u_1 > \frac{K_H}{V_L}$. If $u_1 \geq \frac{K_H}{V_L}$, the contract always dominates the equilibrium payoff for the buyer, so she would just offer this contract. If $u_1 < \frac{K_H}{V_L}$, the equilibrium payoff is higher for the buyer when $\frac{V_L^{1/u}}{C_1^{1/u}+V_L^{1/u}} < \beta < \hat{\beta}$, so she would offer the contract when the belief lies outside of the region.

5 Type Transition

The analysis shows that the new event arrival with exogenous termination payoff does not affect the result that the uninformed buyer does not benefit from negotiating based on the news and rejection decision. Next I will model the exogenous termination specifically as type transition and treat the termination payoff as the endogenous continuation payoff of bargaining with the new type. I will show that the result changes when the termination payoff is endogenously determined.

Specifically, assume that the asset type is not fixed. With Poisson arrival rate $\lambda > 0$, a new type is drawn. Both seller and buyer know if a new type is drawn, but only the seller observes the new type. Let $\beta_0 \in (0,1)$ denote the prior probability that the buyer assigns to $\theta = H$ each time the new type is drawn. Now we need to adjust our definition of belief consistency a bit.

I will mainly keep track of the belief process between any adjacent type transitions. Upon transition, the belief consistency requires that the belief jumps to the prior β_0.

First we analyze the equilibrium results with type transition. Whenever transition occurs, the belief jumps back to prior β_0, so the continuation payoff will be the ex-ante expected payoff of the bargaining process for all players. Denote buyer's ex-ante expected payoff as $W^0 \equiv W(z_0)$ and type-θ's ex-ante expected payoff as $\Pi_\theta^0 \equiv \Pi_\theta(z_0)$. The termination/continuation payoff of the buyer would simply be $V^O = W^0$, whereas the continuation payoff of both type sellers would be $K_H^O = K_L^O = \Pi^0 \equiv \beta_0 \Pi_H^0 + (1-\beta_0)\Pi_L^0$. The equilibrium solution will be a fixed point $\{W^0, \Pi_H^0, \Pi_L^0\}$ to the system of functions

$$
\begin{aligned}
W^0 &= \begin{cases} V(z_0) - K_H - \frac{\lambda}{\lambda+r}\Pi^0 & \text{if } z_0 \geq \hat{\zeta}, \\ C_1^* \frac{e^{u_1 z_0}}{1+e^{z_0}} + \frac{\lambda}{\lambda+r} W^0 & \text{if } z_0 < \hat{\zeta} \end{cases} \\
\Pi_H^0 &= \frac{\lambda}{\lambda+r}\Pi^0, \\
\Pi_L^0 &= \begin{cases} K_H + \frac{\lambda}{\lambda+r}\Pi^0 & \text{if } z_0 \geq \hat{\zeta}, \\ V_L + C_1^*(u_1-1)e^{u_1 z_0} - \frac{\lambda}{\lambda+r} W^0 & \text{if } z_0 < \hat{\zeta} \end{cases}
\end{aligned}
$$

where $C_1^* = \left(\frac{(K_H + \frac{\lambda}{\lambda+r}(\Pi^0+W^0)) - V_L}{u_1-1}\right)^{1-u_1} \left(\frac{V_H - (K_H + \frac{\lambda}{\lambda+r}(\Pi^0+W^0))}{u_1}\right)^{u_1}$. As the threshold is determined by W^0 and Π^0 which both depend on prior belief z_0, I will abuse notation a bit and write $\hat{\zeta}(z_0)$ as the optimal threshold when prior belief is z_0.

If $z_0 \geq \hat{\zeta}(z_0)$, there is no delay in bargaining. Buyer offers the lowest acceptable price (for the high type) to seller, which will be accepted immediately.

$$
\begin{aligned}
\Pi^0 &= \beta_0 \frac{\lambda}{\lambda+r}\Pi^0 + (1-\beta_0)\left[K_H + \frac{\lambda}{\lambda+r}\Pi^0\right] \\
\Rightarrow \Pi^0 &= \frac{\lambda+r}{r}(1-\beta_0)K_H \\
W^0 &= V(z_0) - \left[1 + \frac{\lambda}{r}(1-\beta_0)\right] K_H \\
\hat{\zeta} &= \ln \frac{K_H - V_L + \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 K_H)}{V_H - K_H - \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 K_H)} + \ln\left(\frac{u_1}{u_1-1}\right) \\
C_1^* &= \left(\frac{(K_H + \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 K_H)) - V_L}{u_1-1}\right)^{1-u_1} \left(\frac{V_H - (K_H + \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 K_H))}{u_1}\right)^{u_1}
\end{aligned}
$$

When the prior belief is high enough such that there is no delay in the restart of the bargaining process, the endogenous continuation payoff will act as if the exogenous one. Both parties receive the payoff right upon the Poisson arrival and are not affected by the news and bargaining process. High and low type seller's payoffs are only affect by their own reservation values K_H and K_L, not by the payoff of the buyer. Mathematically, low and high type seller's continuation payoff or ex-ante expected payoff can be found as a fixed point of seller's payoff functions alone. Buyer's payoff is also determined by the fixed point of two type sellers.

If $z_0 < \hat{\zeta}(z_0)$, we can express the ex-ante expected seller payoff as:

$$\Pi^0 = \beta_0 \frac{\lambda}{\lambda + r}\Pi^0 + (1-\beta_0)\left[V_L + C_1^*(u_1-1)e^{u_1 z_0} - \frac{\lambda}{\lambda+r}W^0\right]$$

$$\Rightarrow \Pi^0 = \frac{V_L + C_1^*(u_1-1)e^{u_1 z_0} - \frac{\lambda}{\lambda+r}W^0}{1+\frac{r}{\lambda+r}e^{z_o}}$$

As Π^0 affects C_1^* through $\tilde{K}_H$, we can solve the C_1^* from

$$C_1^* = \left(\frac{(K_H + \frac{\lambda}{\lambda+r(1+e^{z_0})}(V_L + C_1^* u_1 e^{u_1 z_0}) - V_L}{u_1 - 1}\right)^{1-u_1} \cdot \left(\frac{V_H - (K_H + \frac{\lambda}{\lambda+r(1+e^{z_0})}(V_L + C_1^* u_1 e^{u_1 z_0}))}{u_1}\right)^{u_1} \tag{26}$$

and with C_1^* given by the above expression we can solve

$$\begin{aligned} W^0 &= \frac{\lambda+r}{r}C_1^*\frac{e^{u_1 z_0}}{1+e^{z_0}}, \\ \Pi^0 &= \frac{V_L + C_1^*(u_1-1)e^{u_1 z_0} - \frac{\lambda}{r}C_1^*\frac{e^{u_1 z_0}}{1+e^{z_0}}}{1+\frac{r}{\lambda+r}e^{z_0}}, \\ \hat{\zeta} &= \ln\frac{K_H - V_L + \frac{\lambda}{\lambda+r(1+e^{z_0})}(V_L + C_1^* u_1 e^{u_1 z_0})}{V_H - K_H - \frac{\lambda}{\lambda+r(1+e^{z_0})}(V_L + C_1^* u_1 e^{u_1 z_0})} + \ln\left(\frac{u_1}{u_1-1}\right) \end{aligned} \tag{27}$$

Mathematically, the fixed point $\{W^0, \Pi_L^0, \Pi_H^0\}$ need to be solved using the three simultaneous equations. Buyer's continuation payoff has a level effect on both buyer and high type's payoff. All three continuation payoffs also affect the slope of the bargaining process.

The rate of trade $q(z)$ for all $z < \zeta$ and seller's reaction to off-path offers $\sigma_\theta(z, \hat{p})$ can be derived by replacing the continuation payoff by the endogenous one.

5.1 Fixed Price Benchmark

Now I will discuss the fixed price benchmark under type transition. The fixed price benchmark has two different interpretations under type transition. The fixed price is still set at the lowest fixed price that the high type will accept immediately. Under the first interpretation, once a buyer makes an offer, it stays there regardless of type transition. Under the second one, however, upon transition, buyer can revoke the offer and deliberate before making the same offer. These two interpretations have no difference with exogenous termination as the continuation payoffs are realized upon Poisson arrival with no delay or future interactions. They show little difference in the equilibrium buyer behavior either as buyer makes an offer which will be accepted by the seller immediately. However, the different off-path considerations under these two interpretations would make the fixed price (denoted by $\bar{P}$) different. When there is a difference, I will use $\bar{P}^1$ to denote the fixed price under first interpretation and $\bar{P}^2$ to denote the second one. I will discuss and compare both of them. $\{W^{F_1^0}, \Pi_L^{F_1^0}, \Pi_H^{F_1^0}\}$ are used to denote the ex-ante expected payoffs under the first interpretation and $\{W^{F_2^0}, \Pi_L^{F_2^0}, \Pi_H^{F_2^0}\}$ are used to denote the second.

If $z_0 \geq \hat{\zeta}(z_0)$, there is no delay for the buyer to make the offer upon transition. Buyer offers the lowest acceptable price (for the high type) to seller, which will be accepted immediately. The two interpretations have no difference. This lowest price $\bar{P}$ is determined by the condition that high type seller is willing to accept immediately rather than waiting to transition to low type and accepting the same offer.

$$\begin{aligned}
\bar{P} - K_H &= \int_0^\infty e^{-rt}\lambda(1-\beta_0)e^{-\lambda(1-\beta_0)t}\bar{P}dt \\
\Rightarrow \bar{P} &= \left(1 + \frac{\lambda}{r}(1-\beta_0)\right)K_H.
\end{aligned}$$

As $\bar{P} = \Pi^0$, the equivalence result of buyer and high type seller's payoffs still holds between the equilibrium and fixed price benchmark.

If $z_0 < \hat{\zeta}(z_0)$, however, there is delay for the buyer to learn and make the fixed price offer. Under the first interpretation, the fixed price offer is still equal to $\bar{P}^1 = \left(1 + \frac{\lambda}{r}(1-\beta_0)\right)K_H$ as the offer is still in place after transition. The ex-ante expected payoffs can simply be derived from the fixed price mapping by replacing $K_H^O = K_L^O = \bar{P}^1 - \beta_0 K_H$.

Under the second interpretation, this lowest price $\bar{P}^2$ is determined by the condition that high type seller is willing to accept immediately rather than waiting to transition to low type and waiting for the same offer. Equivalently, high type seller is indifferent between accepting immediately and waiting to transition earning low type's ex-ante payoff. The price is also determined a fixed point argument.

$$\bar{P}^2 - K_H = \frac{\lambda}{\lambda + r}\Pi^{F^0}$$

The conditions for Π^{F^0} can be pinned down by

$$\begin{aligned}
W^{F^0} &= C_1^* \frac{e^{u_1 z_0}}{1+e^{z_0}} + \frac{\lambda}{\lambda+r} W^{F^0} \\
\Pi_H^{F^0} &= \frac{\lambda}{\lambda+r}\Pi^{F^0}, \\
\Pi_L^{F^0} &= \frac{\lambda}{\lambda+r}\Pi^{F^0} + e^{u_1(z_0-\zeta_d)} K_H, \\
\Pi^{F^0} &= \beta_0 \Pi_H^{F^0} + (1-\beta_0)\Pi_L^{F^0}, \\
C_1 &= (1+e^{\zeta})\left(V(\zeta) - K_H - \frac{\lambda}{\lambda+r}\left(W^{F^0} + \Pi^{F^0}\right)\right) e^{-u_1 \zeta}, \\
\zeta_d &= \ln \frac{\left(K_H + \frac{\lambda}{\lambda+r}\left(W^{F^0}+\Pi^{F^0}\right)\right) - V_L}{V_H - \left(K_H + \frac{\lambda}{\lambda+r}\left(W^{F^0}+\Pi^{F^0}\right)\right)} + \ln\left(\frac{u_1}{u_1 - 1}\right)
\end{aligned}$$

PROPOSITION 3. *If $z_0 \geq \hat{\zeta}(z_0)$ (for high prior beliefs), the comparison of ex-ante expected payoffs are $W^0 = W^{F_1^0} = W^{F_2^0}$, $\Pi_H^0 = \Pi_H^{F_1^0} = \Pi_H^{F_2^0}$, $\Pi_L^0 > \Pi_L^{F_1^0} = \Pi_L^{F_2^0}$. If $z_0 < \hat{\zeta}(z_0)$ (for low prior beliefs), the comparison of ex-ante expected beliefs are $W^{F_1^0} < W^0 < W^{F_2^0}$, $\Pi_H^{F_1^0} > \Pi_H^0 > \Pi_H^{F_2^0}$, $\Pi_L^0 > \Pi_L^{F_2^0}$.*

When the belief is high, the payoffs are realized immediately if transition occurs, the high type's continuation payoff and his lowest acceptable price will be the same in equilibrium as in fixed price benchmarks. The comparison of the ex-ante payoffs just follow the comparison of the mapping functions. When the belief is low, however, the lowest fixed price acceptable to the high will be different from the equilibrium upper bound of price offers. Instead, we have $\bar{P}^1 > K_H + \frac{\lambda}{\lambda+r}\Pi^0 > \bar{P}^2$. The first fixed price is determined with no revoking price offer, so it does not depend on fixed point argument. The second fixed price and the equilibrium one depends on fixed point argument as the continuation bargaining process or off-path continuation stopping problem determines high type's lowest acceptable price.

With endogenous type transition, the difference in low type's ex-ante expected payoffs imply different continuation payoffs of both type sellers, which will affect the fixed point. We can use the equilibrium fixed point as the continuation payoff and apply them to the fixed price benchmarks, but such a fixed price will not be accepted by the high type under the first interpretation and is not the lowest one under the second interpretation.

5.2 Uninformative News

To understand how the news process and type transition shape the equilibrium, consider the case with type transition alone. I analyze the case by making the news completely uninformative ($\gamma \to 0$). Denote $\approx{K}_H \equiv \left(1+\frac{\lambda}{r}(1-\beta_0)\right) K_H$.

PROPOSITION 4. *Take $\gamma \to 0$, two fixed price interpretations will be the same. If $z_0 > \ln \frac{\approx{K}_H - V_L}{V_H - \approx{K}_H}$, the comparison of ex-ante expected payoffs are $W^0 = W^{F^0}$, $\Pi_H^0 = \Pi_H^{F^0}$, $\Pi_L^0 = \Pi_L^{F^0}$. If $z_0 < \ln \frac{\approx{K}_H - V_L}{V_H - \approx{K}_H}$, the comparison of ex-ante expected beliefs are $W^0 = W^{F^0}$, $\Pi_H^0 = \Pi_H^{F^0}$, $\Pi_L^0 > \Pi_L^{F^0}$.*

With optimistic belief, the buyer is happy to offer something attractive to both types, so in equilibrium the future transition never takes place and equilibrium and fixed price gives buyer the same payoff. The focus is relatively low beliefs where delay occurs in equilibrium. Without exogenous news, buyer can only learn through seller's rejection decisions and screens out the low type using a delay. The only equilibrium belief that ever appears is the prior belief and threshold belief at which the buyer is indifferent. Although the transition itself pushes up the acceptance lower bound for the high type seller and makes the buyer worse off by giving higher offers, there is no benefit of making a lower offer. The competition among buyers at each instant drives the only interim belief to be buyer's indifferent belief, which gives the buyer 0 expected payoff. In comparison with fixed price benchmark where the buyer never makes the offer, the equilibrium trade surplus all goes to the low type, leaving the buyer the same 0 payoff.

When $\gamma > 0$ and $z_0 < \hat{\zeta}(z_0)$, buyer would benefit from the ability of adjusting prices while high type seller is worse off with lower equilibrium offers. In the bargaining process, when belief is relatively high (close to the threshold), high type seller values more of the current optimistic belief and associated high offer than buyer or low type when there is concern of future type transition. Once transition occurs, they need to engage in the bargaining process again and would take longer to get the same offer. The buyer would leverage on this and offer less to the high type. Such decrease of offer upper bound pushes down the price at each belief and makes the buyer better off at each belief z.

6 Privately Observed Transition

Now we consider the case where the type transition is privately observed by the seller. In the public transition case, once the transition occurs, the stale news becomes completely obsolete, so the buyer has a strong incentive to take advantage of accrued news prior type transition. With private transition, the buyer would use all news to update belief but puts less weight on stale news as time goes on. The incentive to make use of accrued news becomes weaker. For the private transition case, assume that with Poisson arrival rate λ type θ transitions to type $-\theta$, which is privately observed by the seller.

First we will pin down the belief process solely based on news without considering the rejection decisions of the seller. Assume that with Poisson arrival rate λ, there is a new draw of the type with prior probability p_0. Both the newly drawn type and the draw event itself are privately observed by the seller. Consider the time period from t to $t+dt$. When the type transition is privately observed, the likelihood the buyer assigns to $\theta_t = H$ versus $\theta_t = L$ follows from Bayes' rule as

$$
\begin{aligned}
&\frac{\Pr(\theta_{t+dt} = H|dX_t)}{\Pr(\theta_{t+dt} = H|dX_t)} \\
=&\frac{\beta_t[e^{-\lambda dt} + (1-e^{-\lambda dt})\beta_0]e^{-\frac{(dX_t-\mu_H dt)^2}{2\sigma^2 dt}} + \cdots}{(1-\beta_t)[e^{-\lambda dt} + (1-e^{-\lambda dt})(1-\beta_0)]e^{-\frac{(dX_t-\mu_L dt)^2}{2\sigma^2 dt}} + \cdots} \\
&\frac{(1-\beta_t)(1-e^{-\lambda dt})\beta_0 \int_0^{dt} \frac{1}{dt} e^{-\frac{(dX_t-\mu_L s-\mu_H(dt-s))^2}{2\sigma^2 dt}} ds}{\beta_t(1-e^{-\lambda dt})(1-\beta_0) \int_0^{dt} \frac{1}{dt} e^{-\frac{(dX_t-\mu_H s-\mu_L(dt-s))^2}{2\sigma^2 dt}} ds} \\
\Longrightarrow\ & d\hat{Z}_t \equiv d\ln \frac{\Pr(\theta_{t+dt} = H|dX_t)}{\Pr(\theta_{t+dt} = H|dX_t)} \\
\approx&\frac{\gamma}{\sigma}\left(dX_t - \frac{\mu_H+\mu_L}{2}dt\right) + \lambda\left(\frac{\beta_0}{\beta_t} - \frac{1-\beta_0}{1-\beta_t}\right)dt
\end{aligned} \tag{28}
$$

Solving the buyer's optimal stopping problem under fixed price benchmark, we arrive the the following ODE

$$
rW(z) = \lambda\frac{\beta_0 - \beta(z)}{\beta(z)(1-\beta(z))}W'(z) + \frac{\gamma^2}{2}(2\beta(z)-1)W'(z) + \frac{\gamma^2}{2}W''(z). \tag{29}
$$

There is no explicit solution to this, but can check and compare with the equilibrium case.

Now we analyze the equilibrium case. We have a similar lemma as Lemma 1.

LEMMA 5. *The buyer would never make an offer greater than $\hat{p} = \tilde{\tilde{K}}_H$.*

Proof of Lemma 5. First, the type-H seller will never accept offers $p \leq K_H$. The never accepting $p < K_H$ part is straightforward as the seller would earn negative payoff. For the never accepting $p = K_H$ part, this is based on the simple observation that a type-H seller could turn into a type-L seller with positive probability. If the type-L seller earns 0 expected payoff, it has to be that the buyer always offer $K_L = 0$. This cannot be an equilibrium, however, as seller has incentive to deviate by offering K_H at the next instant with all type-L sellers accepting $K_L = 0$ at time $t = 0$. As this is not an equilibrium, the type-L seller would earn positive expected payoff, and so does the type-H seller.

Second, the buyer would never offer $p \geq V_H$. The never offering $p > V_H$ part is straightforward as the seller would for sure earn negative payoff. An offer of V_H will only be made when only type H seller is left, but in this case without uncertainty about seller type an offer of $V_H - \varepsilon$ with ε sufficiently small will also be accepted for sure if offered. There exists a cutoff $\hat{p} \in (K_H, V_H)$ such that

If $p \geq \hat{p}$, then $\sigma_H(z,p) = \sigma_L(z,p) = 1$.

Suppose that $\hat{p} > \tilde{\tilde{K}}_H$. Then we claim that $\hat{p}$ or any offer close to it is accepted with probability 1 by both types. Denote $\varepsilon = \hat{p} - \tilde{\tilde{K}}_H$. Consider a time interval with length Δt. As the most favorable offer made by the seller is at most $\hat{p}$, the highest expected payoff possible for type θ if rejecting an offer of $\hat{p} - \frac{1}{2}\varepsilon r \Delta t$ is

$$\begin{aligned}
&e^{-r\Delta t} e^{-\lambda \Delta t}(\hat{p} - K_\theta) + \int_0^{\Delta t} e^{-rt}(\hat{p} - \beta_0 K_H)\lambda e^{-\lambda t} dt \\
=&e^{-(\lambda+r)\Delta t}(\hat{p} - K_\theta) + (1 - e^{-(\lambda+r)\Delta t})\frac{\lambda}{\lambda + r}(\hat{p} - \beta_0 K_H) \\
=&\hat{p} - K_\theta - (1 - e^{-(\lambda+r)\Delta t})\left[\frac{r}{\lambda + r}\hat{p} - K_\theta + \frac{\lambda}{\lambda + r}\beta_0 K_H\right] \\
\leq&\hat{p} - K_\theta - (1 - e^{-(\lambda+r)\Delta t})\left[\frac{r}{\lambda + r}(\tilde{\tilde{K}}_H + \varepsilon) - K_H + \frac{\lambda}{\lambda + r}\beta_0 K_H\right] \\
<&\hat{p} - K_\theta - \frac{1}{2}\varepsilon r \Delta t
\end{aligned}$$

The argument applies when $\Delta t \to 0$. Hence, the buyer can lower the offer and still have her offer accepted with probability 1. □

LEMMA 6. *The high type seller would never accept an offer lower than $\hat{p} = \tilde{\tilde{K}}_H$.*

Proof of Lemma 6. From Lemma 5, there will be a highest price buyer may offer, say $\bar{p} \leq \tilde{\tilde{K}}_H$. If buyer offers $\bar{p}$, low type will accept with probability 1 as he would not receive a better offer. Then the buyer's belief will jump to $\beta = 1$ if high type seller does not accept with probability 1. As we maintain the refinement that low type or equivalently offer is non-decreasing in belief, the offer will still be $\bar{p}$ the next instant, which will be accepted by the low type with probability 1 in case of any type transition. Therefore, the offer will stay at $\bar{p}$ after it is offered.

The offer $\bar{p}$ has to make the high type willing to accept immediately than waiting for transitioning into low type.

$$
\begin{aligned}
&\bar{p} - K_H \geq \int_0^\infty e^{-rt} \bar{p} \lambda (1-\beta_0) e^{-\lambda(1-\beta_0)t} dt \\
\implies &\bar{p} - K_H \geq \frac{\lambda(1-\beta_0)}{\lambda(1-\beta_0)+r} \bar{p} \\
\implies &\bar{p} \geq \tilde{\tilde{K}}_H
\end{aligned}
$$

□

Based on Lemma 5 and Lemma 6, we can guess that the equilibrium is similar as before that the buyer offers $\tilde{\tilde{K}}_H$ when the belief is high enough, which will be accepted by both types immediately. When the belief is relatively low, the buyer would offer something less which will only be accepted with positive density. Denote the hazard rate of the low type's acceptance as $\tilde{q}(z) \geq 0$. Given the belief β_t (and z_t) at time t, the belief at time $t + dt$ is given by

$$
\begin{aligned}
&\frac{\Pr(\theta_{t+dt} = H | dX_t)}{\Pr(\theta_{t+dt} = H | dX_t)} \\
=&\frac{\beta_t [e^{-\lambda dt} + (1-e^{-\lambda dt})\beta_0] e^{-\frac{(dX_t - \mu_H dt)^2}{2\sigma^2 dt}} + \cdots}{(1-\beta_t)[e^{-\lambda dt} + (1-e^{-\lambda dt})(1-\beta_0)] e^{-\frac{(dX_t - \mu_L dt)^2}{2\sigma^2 dt}} + \cdots} \\
&\frac{(1-\beta_t)(1-e^{-\lambda dt})\beta_0 \int_0^{dt} \frac{1}{dt} e^{-\frac{(dX_t - \mu_L s - \mu_H(dt-s))^2}{2\sigma^2 dt}} e^{-q(z_t)s} ds}{\beta_t (1-e^{-\lambda dt})(1-\beta_0) \int_0^{dt} \frac{1}{dt} e^{-\frac{(dX_t - \mu_H s - \mu_L(dt-s))^2}{2\sigma^2 dt}} e^{-q(z_t)s} ds} \\
\implies & d\hat{Z}_t \equiv d \ln \frac{\Pr(\theta_{t+dt} = H | dX_t)}{\Pr(\theta_{t+dt} = H | dX_t)} \\
\approx & \frac{\gamma}{\sigma} \left(dX_t - \frac{\mu_H + \mu_L}{2} dt \right) + q(z_t) dt + \lambda \left(\frac{\beta_0}{\beta_t} - \frac{1-\beta_0}{1-\beta_t} \right) dt
\end{aligned}
$$

We arrive at the the following ODE

$$
\begin{aligned}
rW(z) =& q(z)(1-\beta(z))(V_L - P(z) - W(z)) + \lambda \frac{\beta_0 - \beta(z)}{\beta(z)(1-\beta(z))} W'(z) \\
& + q(z)W'(z) + \frac{\gamma^2}{2}(2\beta(z)-1)W'(z) + \frac{\gamma^2}{2}W''(z) \\
=& \lambda \frac{\beta_0 - \beta(z)}{\beta(z)(1-\beta(z))} W'(z) + \frac{\gamma^2}{2}(2\beta(z)-1)W'(z) + \frac{\gamma^2}{2}W''(z) \\
& + q(z)\left((1-\beta(z))(V_L - P(z) - W(z)) + W'(z)\right)
\end{aligned}
$$

The zero net benefit of screening result in Lemma 3 still holds, so the equivalence result between fixed and flexible price also holds. Without explicit observation of the type transition, the belief of the seller changes continuously and hence restores the "Coasian force". In addition, it can be shown that the seller has a lower expected payoff when we compare the equilibrium payoff with that under public transition.

7 Conclusion

In this paper, I study a bargaining model with one-sided incomplete information and interdependent values. Specifically, I consider how gradual learning of the informed party's private type affects the bargaining dynamics when the private type is not fixed. I find that the existence of transition benefits the informed buyer at the cost of the seller and the fixed price scheme is better for the seller than the flexible price scheme. Finally, the uninformed seller benefits from knowing when the transition occurs even if each transition is a completely new draw.

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Appendices

A Equilibrium Definition

Seller Optimality – For a given offer process P, adapted to $\{\mathcal{H}_t\}$, the seller faces a stopping problem: when (if ever) to accept the buyer's offer. A pure strategy for the type-θ seller is then a stopping time $\tau_\theta(\omega) : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ of the filtration $\{\mathcal{H}_t\}$, where ω denotes an arbitrary element of Ω. Let $\mathcal{T}$ be the set of all such stopping times. The type-θ seller's stopping problem is

$$(\mathrm{SP}_\theta) \qquad \sup_{\tau \in \mathcal{T}} E^\theta \left[e^{-(\lambda+r)\tau}(P_\tau - K_\theta) \right] + (1 - e^{-(\lambda+r)\tau}) \frac{\lambda}{\lambda + r} K_\theta^O,$$

where $K_\theta^0 = \Pi^0$ is seller's ex ante expected payoff whenever the type is newly drawn. A mixed strategy for the seller is a distribution over $\mathcal{T}$, which can be represented by the CDF it endows over the type-θ seller's acceptance time for each sample path of news, denoted S^θ. Formally, $S^\theta = \{S_t^\theta, 0 \le t \le \infty\}$ is a stochastic process that is (i) adapted to $\{\mathcal{H}_t\}$, (ii) right-continuous, and (iii) satisfies $0 \le S_t^\theta \le S_{t'}^\theta \le 1$ for all $t \le t'$. The function $S^\theta(\omega)$ is the CDF over the type-θ seller's acceptance time on $\mathbb{R}_+ \cup \{\infty\}$ along the sample path $X(\omega, \theta)$.

Let $\mathcal{S}^\theta = supp(S^\theta)$. We say that S^θ solves (SP_θ) if all $\tau \in \mathcal{S}^\theta$ solve (SP_θ).

Condition 1 (Seller Optimality)**.** *The seller's strategy, S^θ, solves (SP_θ).*

Belief Consistency – At any time t, if trade has not yet occurred, the buyer assigns a probability, $\beta_t \in [0,1]$, to $\theta = H$. Analytically, it is convenient to track the belief in terms of its log-likelihood ratio, denoted $Z_t \equiv \ln(\beta_t/(1-\beta_t)) \in \bar{\mathbb{R}}$ (i.e. the extended real numbers). This transformation from belief as a probability to a log-likelihood ratio is injective. We will mainly keep track of the belief process between any type transitions. Upon transition, the belief consistency requires that the belief jumps to the prior.

The buyer's belief at time t is conditioned on the history of news and the fact that the seller has rejected all past offers. It will be convenient to separate these two sources of information. Let f_t^θ be the density of X_t conditional on θ, which is normally distributed with mean $\mu_\theta t$ and variance $\sigma^2 t$ (and with $f_0^H = f_0^L$ being the Dirac delta function). Let $S_{t^-}^\theta \equiv \lim_{s \uparrow t} S_s^\theta$ (which is well defined for $t > 0$ given that S^θ is bounded and non-decreasing), and specify that $S_{0^-}^\theta = 0$. The belief "at time t" should be interpreted to mean *before* observing the seller's decision at time t, which is why left limits are appropriate. If $S_{t^-}^L \cdot S_{t^-}^H < 1$ (i.e., given the history at time t, there is positive probability that the seller has not yet accepted an offer), then the probability the buyer assigns to $\theta = H$ follows from Bayes' rule as

$$\frac{\beta_0 f_t^H(X_t)(1-S_{t^-}^H)}{\beta_0 f_t^H(X_t)(1-S_{t^-}^H)+(1-\beta_0)f_t^L(X_t)(1-S_{t^-}^L)}. \tag{30}$$

Taking the log-likelihood ratio of (30) results in

$$Z_t = \underbrace{\ln\left(\frac{\beta_0}{1-\beta_0}\right)+\ln\left(\frac{f_t^H(X_t)}{f_t^L(X_t)}\right)}_{\hat{Z}_t}+\underbrace{\ln\left(\frac{1-S_{t^-}^H}{1-S_{t^-}^L}\right)}_{Q_t}. \tag{31}$$

As seen in (1), working in log-likelihood space enables us to represent Bayesian updating as a linear process, and the buyer's belief as the sum of two components, $Z_t = \hat{Z}_t + Q$. Notice that the two component processes separate the two sources of information to the buyer. The term $\hat{Z}_t$ is the belief for a Bayesian who update *only based on news*, $\{X_s : 0 \leq s \leq t\}$, starting from $\hat{Z}_0 = Z_0 = \ln(\beta_0/(1-\beta_0))$. The term Q is the stochastic process that keeps track of the information conveyed in equilibrium by the fact that the seller has rejected all past offers.

Condition 2 (Belief Consistency)**.** *For all t such that $S_{t^-}^L \cdot S_{t^-}^H < 1$, Z_t is given by (1).*

Stationarity. – In keeping with the literature, we focus on stationary equilibria, using the uninformed party's belief as the state variable. We will use z when referring to the state variable as opposed to the stochastic process Z (i.e., if $Z_t = z$, then the game is "in state z, at time t").

Condition 3 (Stationarity)**.** *The buyer's offer in state z is given by $P(z)$, where $P : \bar{\mathbb{R}} \to \mathbb{R}$ is a Borel measurable function, and Z is a time homogeneous $\mathcal{H}_t$-Markov process.*

DEFINITION 2. *An **S-candidate** is a quadruple (P, S^L, S^H, Z) satisfying Conditions 1-3.*

In any S-candidate, the value functions for each player depend only on the current state. Let $\Pi_\theta(z)$ denote the expected payoff for the type-θ seller given state z. That is, for any $\tau \in \mathcal{S}^\theta$,

$$\Pi_\theta(z) \equiv E_z^\theta\left[e^{-(\lambda+r)\tau}(P_\tau - K_\theta) + (1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}\Pi^0\right],$$

where E_z^θ is the expectation with respect to the probability law of the process Z starting from $Z_0 = z$ and conditional on θ, which we denote by $\mathcal{Q}_z^\theta$. Formally, $\Pi^0 = \beta_0\Pi_H(z_0) + (1-$

$\beta_0)\Pi_L(z_0)$. Similarly, let $W(z)$ denote the expected payoff to the seller in any given state z:

$$
\begin{aligned}
W(z) \equiv\, & (1-\beta(z))E_z^L\left[\int_0^\infty e^{-rt}\left(e^{-\lambda t}(V_L - P(Z_t)) + (1-e^{-\lambda t})V^O\right)dS_{t^-}^L\right] \\
& + \beta(z)E_z^H\left[\int_0^\infty e^{-rt}\left(e^{-\lambda t}(V_H - P(Z_t)) + (1-e^{-\lambda t})V^O\right)dS_{t^-}^H\right],
\end{aligned} \tag{32}
$$

where $\beta(z) = e^z/(1+e^z)$ and $V^O = W^0 = W(z_0)$ is buyer's ex-ante expected payoff whenever a new type is drawn.

Response to Any Offer. – Take an S-candidate, and suppose that in state z, the buyer offers p, which may or not be the prescribed offer $P(z)$. Let $\sigma_\theta(z,p)$ be the probability that the type-θ seller accepts the offer. Motivated by discrete time notions of sequential rationality and belief consistency in equilibrium concepts such as PBE, we require the following.

First, the type-H seller will never accept offers $p < K_H + \frac{\lambda}{\lambda+r}\Pi^0$, as the seller can always wait and wait for the transition. Second, the buyer would never offer $p > V_H$ as she would earn negative payoff. Hence, there exists a cutoff $\tilde{K}_H \in [K_H + \frac{\lambda}{\lambda+r}\Pi^0, V_H]$ such that

(R1) If $p \geq \tilde{K}_H$, then $\sigma_H(z,p) = \sigma_L(z,p) = 1$.

Lemma 1 is derived from limit of the discrete analog of the game. For offers less than $\tilde{K}_H$, the seller responds optimally and the buyer's belief updates consistently with these responses as follows. Let $\tilde{z}(z,p)$ be the buyer's updated belief if her offer of p is rejected in state z.

(R2) If $p < \tilde{K}_H$, then $\beta\left(\tilde{z}(z,p)\right) = \frac{\beta(z)(1-\sigma_H(z,p))}{\beta(z)(1-\sigma_H(z,p))+(1-\beta(z))(1-\sigma_L(z,p))}$.

Hence, the seller's choice is whether to accept p or reject and get $\Pi_\theta\left(\tilde{z}(z,p)\right)$.

(R3) If $p < \tilde{K}_H$, then $\sigma_\theta(z,p) \in \arg\max_\sigma \sigma(p-K_\theta) + (1-\sigma)\Pi_\theta\left(\tilde{z}(z,p)\right)$.

If $p = P(z) < \tilde{K}_H$, then (R2) and (R3) are implied by *belief consistency* and *seller optimality*.

Condition 4 (Response to any offer)**.** *For any z, $\{\sigma_H(z,\cdot), \sigma_L(z,\cdot), \tilde{z}(z,\cdot)\}$ satisfy (R1)–(R3).*

Buyer Optimality. – If the buyer offers p in state z, either it will be accepted, earning her $V_\theta - p$ from type θ seller, or rejected, earning her the continuation value from the post rejection belief $\tilde{z}(z,p)$. The prescribed offer $P(z)$ is optimal if

$$
\begin{aligned}
P(z) \in \arg\max_p\, & \beta(z)\sigma_H(z,p)(V_H - p) + (1-\beta(z))\,\sigma_L(z,p)(V_L - p) \\
& + \left(1 - \left(\beta(z)\sigma_H(z,p) + (1-\beta(z))\sigma_L(z,p)\right)\right)W(\tilde{z}(z,p)),
\end{aligned} \tag{W1}
$$

Condition (W1) requires that the buyer makes offers to maximize her expected payoff given belief z at each instant. The updated belief $\tilde{z}(z,p)$ and continuation payoff take into account of the rejection decision of different sellers but not the news acquisition or event arrival. The next condition requires that the buyer makes optimal use of this option to wait and learn. In particular, we require that for all $\tau \in \mathcal{T}$,

$$W(z) \geq E_z\left[e^{-(\lambda+r)\tau} W(\hat{Z}_\tau) + (1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}W^0\right]. \tag{W2}$$

Recall that $\hat{Z}$ is the belief process updating solely based on revelation of the news. If there was no news, $\hat{Z}_t$ would be constant over time and therefore $E_z\left[W(\hat{Z}_\tau)\right] = W(z)$ for any τ, meaning that (W2) implies $W(z) \geq \frac{\lambda}{\lambda+r}W^0$ for all z. However, with news, the condition has important additional implications. For instance, any upward kink in the buyer's value function violates (W2), since the buyer could improve her payoff by waiting in a neighborhood around the kink.

Condition 5 (Buyer Optimality)**.** *P and W as defined by (2), satisfy (W1)-(W2) for all z.*

DEFINITION 3. *An equilibrium is a profile $(P, S^L, S^H, Z, \sigma_H, \sigma_L, \tilde{z})$ that satisfies Conditions 1-5.*

DEFINITION 4. *An S-candidate (P, S^L, S^H, Z) is supported by $(\sigma_H, \sigma_L, \tilde{z})$ if $(P, S^L, S^H, Z, \sigma_H, \sigma_L, \tilde{z})$ is an equilibrium. An S-candidate (P, S^L, S^H, Z) can be supported if there exists a $(\sigma_H, \sigma_L, \tilde{z})$ such that $(P, S^L, S^H, Z, \sigma_H, \sigma_L, \tilde{z})$ is an equilibrium.*

PROPOSITION 5. *An S-candidate endowing value functions (W, Π_H, Π_L), with Π_L non-decreasing can be supported as an equilibrium if and only if, for all z,*

$$W(z) \geq V(z) - \tilde{K}_H, \tag{B1}$$

and

$$W(z) \geq \max_{z' \geq z}\left\{\frac{\beta(z') - \beta(z)}{\beta(z')}(V_L - \Pi_L(z')) + \frac{\beta(z)}{\beta(z')}W(z')\right\}, \tag{B2}$$

and for all $\tau \in \mathcal{T}$,

$$W(z) \geq E_z\left[e^{-(\lambda+r)\tau} W(\hat{Z}_\tau)\right] + \left(1 - e^{-(\lambda+r)\tau}\right)\frac{\lambda}{\lambda+r}W^0. \tag{B3}$$

DEFINITION 5. *Henceforth, we refer to any equilibrium (S-candidate) in which Π_L is non-decreasing as a **monotone equilibrium** (S-candidate).*

Proposition 5 is useful both for proving existence and uniqueness. It reduces verifying that a monotone S-candidate is an equilibrium to checking three variational inequalities on the resulting W (i.e., (B1)-(B3)). It also facilitates ruling out other types of candidate equilibria.

That equilibrium conditions imply (B1) and (B3) is straightforward; they must be satisfied in any equilibrium regardless of whether $\Pi_L$is non-decreasing. Monotonicity of Π_L is used to establish the necessity of (B2). The expression being maximized on the RHS of (B2) is the buyer's payoff from offering $p = \Pi_L(z')$ in state z if the post-rejection belief, $\tilde{z}(z,p)$, will be z'. Define this expression as $J(z,z')$:

$$J(z,z') \equiv \underbrace{\frac{\beta(z')-\beta(z)}{\beta(z')}}_{\text{Prob. offer accepted}} \underbrace{(V_L - \Pi_L(z'))}_{\text{Payoff if accepted}} + \underbrace{\frac{\beta(z)}{\beta(z')}}_{\text{Prob. rejected}} \underbrace{W(z')}_{\text{Continuation Payoff}} .$$

That (B2) is required is easiest to see when Π_L is strictly increasing below K_H. In this case, for all $p \in [\Pi_L(z), \tilde{K}_H)$, there exists a unique $z' \geq z$ such that $\Pi_L(z') = p$. Then, (R2) and (R3) immediately imply that $\tilde{z}(z, \Pi_L(z'))$ must be z'. Thus, if (B2) was violated and Π_L is strictly increasing, there exists a profitable deviation for the buyer, i.e., (R2) and (W1) cannot be simultaneously satisfied.

B Proofs

Proof of Lemma 1. Suppose that $\tilde{K}_H > \tilde{K}_H$. Then we claim that $\tilde{K}_H$ or any offer close to it is accepted with probability 1 by both types. Denote $\varepsilon = \tilde{K}_H - \tilde{K}_H > 0$. Consider a time interval with length Δt. As the most favorable offer made by the seller is at most $\tilde{K}_H$, the highest expected payoff possible for type θ if rejecting an offer of $\tilde{K}_H - \varepsilon r \Delta t$ is

$$\begin{aligned}
& e^{-r\Delta t} e^{-\lambda \Delta t} (\tilde{K}_H - K_\theta) + \int_0^{\Delta t} e^{-rt} K_\theta^O \lambda e^{-\lambda t} dt \\
=& e^{-(\lambda+r)\Delta t} (\tilde{K}_H - K_\theta) + (1 - e^{-(\lambda+r)\Delta t}) \frac{\lambda}{\lambda + r} K_\theta^O \\
\leq& \tilde{K}_H - K_\theta - \varepsilon (1 - e^{-(\lambda+r)\Delta t}) \\
\approx& \tilde{K}_H - K_\theta - \varepsilon (\lambda + r) \Delta t \\
<& \tilde{K}_H - K_\theta - \varepsilon r \Delta t
\end{aligned}$$

The argument applies when $\Delta t \to 0$. Hence, the buyer can lower the offer and still have her offer accepted with probability 1. $\square$

PROOF OF PROPOSITION 5. We first establish basic properties that hold in any monotone equilibrium and in any monotone S-candidate satisfying (B1)-(B3). We then show that (B1)-(B3) are necessary and sufficient for an S-candidate to be supportable as an equilibrium if Π_H and Π_L are non-decreasing.

Basic Facts: In either (a) a monotone equilibrium or (b) a monotone S-candidate satisfying (B1)-(B3), the following hold.

FACT 1. *Let $\bar{P} \equiv \sup_z P(z)$. For any θ and z, if $\bar{P} \geq K_\theta$, then $\Pi_\theta(z) \in [P(z) - K_\theta, \bar{P} - K_\theta]$.*

*The lower bound is by **seller optimality**; the upper bound is by feasibility and definition of $\bar{P}$.*

FACT 2. $\bar{P} \leq \tilde{K}_H$.

For (a) by requirement of (R1) and (W1). For (b) if there exists z with $P(z) > \tilde{K}_H$, then Fact 1 implies that $\beta(z)\Pi_H(z) + (1-\beta(z))\Pi_L(z) \geq P(z) - \beta(z)K_H > \tilde{K}_H - \beta(z)K_H$. The highest possible expected surplus is $V(z) - \beta(z)K_H \geq W(z) + \beta(z)\Pi_H(z) + (1-\beta(z))\Pi_L(z)$, implying $W(z) < V(z) - \tilde{K}_H$ and contradicting (B1).

FACT 3. $\bar{P} = \max_z P(z) = \tilde{K}_H$.

If the high type never trades and waits for the type change, he guarantees expected payoff $\frac{\lambda}{\lambda+r}K_H^0$; if he trades, he gets at most $\bar{P} - K_H \leq \tilde{K}_H - K_H = \frac{\lambda}{\lambda+r}K_H^0$ by Fact 2. If Fact 3 does not hold, then by Fact 2 and seller optimality, the high type never trades and the probability of trade with the current type goes to 0 as $z \to \infty$ and thus $W(z) \to \frac{\lambda}{\lambda+r}V^0 < V_H - \tilde{K}_H$ from Assumption 2. For (a) by (R1) the buyer can profitably deviate by offering $\tilde{K}_H$ for z large enough, violating (W1). For (b) (B1) is violated for z large enough.

FACT 4. *There exists $\beta < \infty$ such that $P(z) = \Pi_L(z) = \tilde{K}_H$ for all $z > \beta$ and $P(z) \leq \Pi_L(z) < \tilde{K}_H$ for all $z < \beta$.*

Immediate from Facts 1 and 3 and Π_L non-decreasing.

FACT 5. $S_t^H \leq \mathbf{1}_{\{t \geq T(\beta)\}}$, *where $T(\beta) \equiv \inf\{t : Z_t \geq \beta\}$.*

*The high rejects $P(z)$ for all $z < \beta$, which is immediate from Fact 4 and **seller optimality**.*

FACT 6. *Prior to $T(\beta)$, Q is a weakly increasing process.*

*Follows from Fact 5 and **belief consistency**.*

FACT 7. *For all z, $\Pi_L(z) = \frac{\lambda}{\lambda+r}K_L^O + E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[K_H + \frac{\lambda}{\lambda+r}(K_H^O - K_L^O)\right]$.*

*For all $z \geq \beta$,the fact is immediate from Fact 4. Hence, if the fact were false, there exists a state $z < \beta$ such that $\Pi_L(z) > \frac{\lambda}{\lambda+r}K_L^O + E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[K_H + \frac{\lambda}{\lambda+r}(K_H^O - K_L^O)\right]$ by **seller optimality**. Then there exists a state $z' \in [z,\beta)$ in which the low type trades with probability 1, but at a price $P(z') < \tilde{K}_H$. But then, by Fact 5 and **belief consistency**, rejection at $Z_t = z'$ leads to a belief of $Z_{t^+} = \infty$ and an offer of $\tilde{K}_H$. Hence, the low type would do better to reject at z', generating a contradiction of **seller optimality**.*

FACT 8. *Π_L is continuous.*

Immediate for $z > \beta$. Suppose that Π_L is discontinuous at some $z_1 \leq \beta$. Then by Fact 7, Z must also be discontinuous at z_1. The monotonicity of Q (Fact 6) implies that Z can only have upward jumps, so $\Pi_L(z_1^-) = \Pi_L(z_2)$ for some "jump-to" point $z_2 > z_1$. Note, Π_L is non-decreasing, so $\Pi_L(z_2) \geq \Pi_L(z_1^+) \geq \Pi_L(z_1^-) = \Pi_L(z_2)$, contradicting a discontinuity of Π_L at z_1.

Sufficiency of (B1)-(B3): Fix a monotone S-candidate satisfying (B1)-(B3). For any z and $p \geq \tilde{K}_H$, set $\sigma_H(z,p) = \sigma_L(z,p) = 1$. For any z and $p < \tilde{K}_H$, set $\sigma_H(z,p) = 0$ and

$$\sigma_L(z,p) = \frac{\beta(\tilde{z}(z,p)) - \beta(z)}{\beta(\tilde{z}(z,p))(1-\beta(z))},$$

with $\tilde{z}(z,p) = \max\{z, \max\{z' : \Pi_L(p) = z'\}\}$, which is well defined by Π_L non-decreasing and continuous (Fact 8).

We now need to verify that Conditions 4 and 5 are satisfied. Each part of Condition 4, (R1)-(R3), is satisfied by construction. (B3) trivially implies (W2) as being identical. The final step is to verify that (W1) is satisfied. Fix any z. An offer of $p \geq \tilde{K}_H$ generates a payoff of $V(z) - p \leq V(z) - \tilde{K}_H \leq W(z)$ by (B1), so $P(z)$ is weakly better for the buyer than any such p. An offer of $p < \tilde{K}_H$ generates a payoff of

$$J(z,\tilde{z}(z,p)) \leq \max_{z' \geq z} J(z,z') \leq W(z),$$

where the first inequality is by definition of maximum and $\tilde{z}(z,p) \geq z$, and the second inequality is (B3). Hence, Condition 5 is satisfied.

Necessity of (B1)-(B3): Fix a monotone equilibrium. (W2) trivially implies (B3) as being identical. Next, if (B1) were violated at some z, then the buyer could improve her payoff (therefore violating (W1)) by offering $\tilde{K}_H$ in state z: by (R1), both types accept, producing a payoff of $V(z) - \tilde{K}_H > W(z)$.

For (B2), note that for all z and $p \in [\Pi_L(z), \tilde{K}_H)$: (i) $\sigma_H(z,p) = 0$ by Fact 5. Hence, by (R2) and (R3), $\tilde{z}(z,p) \geq z$ and $\tilde{z}(z,p) \in \Pi_L^{-1}(p)$ (can be verified by checking each of the three cases where $\sigma_L(z,p) = 0$, $\sigma_L(z,p) = 1$ and $\sigma_L(z,p) \in (0,1)$ are high type's best responses respectively), which is nonempty by Π_L continuous (Fact 8). Therefore, solving (R2) for $\sigma_L(z,p)$ gives

$$\sigma_L(z,p) = \frac{\beta(\tilde{z}(z,p)) - \beta(z)}{\beta(\tilde{z}(z,p))(1-\beta(z))},$$

and the buyer's payoff from offering p is $J(z, \tilde{z}(z,p))$. Denote $\tilde{\mathcal{Z}}(z) \equiv \{z' : \exists p \in [\Pi_L(z), \tilde{K}_H)$ such that $\tilde{z}(z,p) = z'\}$, which is the set of beliefs that can be reached by rejection of an offer $p \in [\Pi_L(z), \tilde{K}_H)$ at state z. Condition (W1) implies that

$$\text{(A4)} \quad \begin{aligned} W(z) &\geq \max_{p \in [\Pi_L(p), \tilde{K}_H)} \left\{ \frac{\beta(\tilde{z}(z,p)) - \beta(z)}{\beta(\tilde{z}(z,p))}(V_L - p) + \frac{\beta(z)}{\beta(\tilde{z}(z,p))} W(\tilde{z}(z,p)) \right\} \\ &= \max_{z' \in \tilde{\mathcal{Z}}(z)} \left\{ \frac{\beta(z') - \beta(z)}{\beta(z')}(V_L - \Pi_L(z')) + \frac{\beta(z)}{\beta(z')} W(z') \right\}. \end{aligned}$$

Hence, the final step is to show that the maximum is not improved if we replace $\tilde{\mathcal{Z}}(z)$ with $\{z' : z' \geq z\}$ in (A4). Notice, Π_L non-decreasing and $\tilde{z}(z,p) \in \Pi_L^{-1}(p)$ imply that $\tilde{\mathcal{Z}}(z) \subseteq \{z' : z' \geq z\}$. As Π_L is continuous, non-decreasing and bounded above by $\tilde{K}_H$, it is sufficient to show that $J(z,z') = J(z,z'')$ for all z', z'' in $\Pi_L^{-1}(p)$. To do so, suppose that $z' < z''$ and $\Pi_L(z') = \Pi_L(z'')$. Function Π_L non-decreasing and continuous (Fact 8) imply that $z', z'' \in [z_1, z_2]$ such that $\Pi_L(z) = \Pi_L(z')$ if and only if $z \in [z_1, z_2]$. Hence, by Fact 7, $E_z^L \left[e^{-(\lambda+r)T(\beta)} \right]$ is constant on $[z_1, z_2]$, and strictly higher for all $z > z_2$. Fact 6 then implies that $\tilde{z}(z, P(z)) = z_2$ for all $z \in [z_1, z_2]$ (that is, rejecting the equilibrium offer in any state $z \in [z_1, z_2]$ leads to a jump in the belief to z_2: otherwise, $E_z^L \left[e^{-(\lambda+r)T(\beta)} \right]$ would not be constant on the interval. They must jump to the same belief: $\geq z_2$ from $\tilde{z}(z,p) \geq z$, $\leq z_2$ from (R2) and (R3) that the type-L seller would reject otherwise). It follows that

$$
\begin{aligned}
J(z,z') &= \frac{\beta(z')-\beta(z)}{\beta(z')}(V_L-\Pi_L(z'))+\frac{\beta(z)}{\beta(z')}W(z') \\
&= \frac{\beta(z')-\beta(z)}{\beta(z')}(V_L-\Pi_L(z'))+\frac{\beta(z)}{\beta(z')}J(z',z_2) \\
&= \frac{\beta(z')-\beta(z)}{\beta(z')}(V_L-\Pi_L(z))+\frac{\beta(z)}{\beta(z')}\left[\frac{\beta(z_2)-\beta(z')}{\beta(z_2)}(V_L-\Pi_L(z))+\frac{\beta(z')}{\beta(z_2)}W(z_2)\right] \\
&= \frac{\beta(z_2)-\beta(z)}{\beta(z_2)}(V_L-\Pi_L(z))+\frac{\beta(z)}{\beta(z_2)}W(z_2) \\
&= J(z,z_2)
\end{aligned}
$$

and

$$
\begin{aligned}
J(z,z'') &= \frac{\beta(z'')-\beta(z)}{\beta(z'')}(V_L-\Pi_L(z''))+\frac{\beta(z)}{\beta(z'')}W(z'') \\
&= \frac{\beta(z'')-\beta(z)}{\beta(z'')}(V_L-\Pi_L(z''))+\frac{\beta(z)}{\beta(z'')}J(z'',z_2) \\
&= J(z,z_2).
\end{aligned}
$$

□

Proof of Lemma 2. For $z \geq \beta$, immediately we have $\Pi_L(z) = P(z) = \tilde{K}_H$, which is non-decreasing. For $z < \beta$ and any strategy $\tau \leq T(\beta)$ the low type's payoff is

$$
\begin{aligned}
&\left(1-e^{-(\lambda+r)\tau}\right)\frac{\lambda}{\lambda+r}K_L^O+E_z^L\left[e^{-(\lambda+r)\tau}P(Z_\tau)\right] \\
=&\left(1-e^{-(\lambda+r)\tau}\right)\frac{\lambda}{\lambda+r}K_L^O \\
&+E_z^L\left[e^{-(\lambda+r)\tau}\left(\frac{\lambda}{\lambda+r}K_L^O+E_{Z_\tau}^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[K_H+\frac{\lambda}{\lambda+r}(K_H^O-K_L^O)\right]\right)\right] \\
=&\frac{\lambda}{\lambda+r}K_L^O+E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[K_H+\frac{\lambda}{\lambda+r}(K_H^O-K_L^O)\right] \\
=&P(z),
\end{aligned}
$$

by (6). Finally, $P(z) = \frac{\lambda}{\lambda+r}K_L^O+E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[K_H+\frac{\lambda}{\lambda+r}(K_H^O-K_L^O)\right]$ is strictly increasing at all $z < \beta$ because Z from (3) has continuous sample path with probability 1. □

Proof of Lemma 4. We will verify the solution that $\tau = \inf\{t : \hat{Z} \geq \zeta_d\}$ is optimal using the variational inequalities for optimal stopping (Oksendal 2007, Theorem 10.4.1). By construction, the buyer's value function is twice continuously differentiable (i.e., $\mathcal{C}^2$) almost everywhere. It is clear that conditions (i), (iii), (iv), (v), (vii), (viii) and (ix) in the verification theorem hold. It remains to verify that (ii) $W(z) \geq g(z) \equiv V(z) - K_H$ for all $z \leq \zeta_d$, and that (vi) $[\mathcal{A} - (\lambda + r)]\left[W(z) - \frac{\lambda}{\lambda+r}V^0\right] \leq 0$ for all $z \geq \zeta_d$. For (ii), it's the same as (B1) as we get the same $W(z)$ in equilibrium, the proof will be delegated to proof of Theorem 2. For (vi), for $z \geq \zeta_d > \ln \frac{\tilde{K}_H + \frac{\lambda}{\lambda+r}V^O - V_L}{V_H - \tilde{K}_H - \frac{\lambda}{\lambda+r}V^O}$,

$$
\begin{aligned}
[\mathcal{A} - (\lambda + r)]\left[W(z) - \frac{\lambda}{\lambda + r}V^0\right] =& [\mathcal{A} - (\lambda + r)]\left[V(\beta) - \tilde{K}_H - \frac{\lambda}{\lambda + r}V^0\right] \\
=& \frac{\gamma^2}{2}(2\beta(z) - 1)V'(z) + \frac{\gamma^2}{2}V''(z) \\
& - (\lambda + r)\left[V(\beta) - \tilde{K}_H - \frac{\lambda}{\lambda + r}V^0\right] \\
=& - (\lambda + r)\left[V(\beta) - \tilde{K}_H - \frac{\lambda}{\lambda + r}V^0\right] \\
<& 0.
\end{aligned}
$$

□

Proof of Lemma 3. Suppose that $\Sigma(\beta, q)$ is an equilibrium. By construction, Π_L is non-decreasing (see Lemma 2) and below β, W is $\mathcal{C}^2$. (B3) applied to arbitrarily short time τ implies that for all $z < \beta$,

$$
(\lambda + r)W(z) \geq \lambda V^O + \frac{\gamma^2}{2}(2\beta(z) - 1)W'(z) + \frac{\gamma^2}{2}W''(z). \tag{33}
$$

That is, if (33) were violated at $z < \beta$, then there exists $\varepsilon > 0$ such that (33) is violated over the interval $(z - \varepsilon, z + \varepsilon)$. Let $\tau_\varepsilon = \inf\left\{t : \hat{Z}_t \notin (z - \varepsilon, z + \varepsilon)\right\}$, then by Dynkin's formula, (B3) is violated as

$$
\begin{aligned}
&\left(1 - e^{-(\lambda+r)\tau_\varepsilon}\right)\frac{\lambda}{\lambda + r}V^O + E_z\left[e^{-(\lambda+r)\tau_\varepsilon}W(\hat{Z}_{\tau_\varepsilon})\right] \\
=& W(z) + E_z\left[\int_0^{\tau_\varepsilon} e^{-(\lambda+r)s}\left((\mathcal{A} - \lambda - r)W(\hat{Z}_s) + \lambda V^O\right)ds\right] > \quad W(z),
\end{aligned}
$$

where $\mathcal{A}$ is the characteristic operator of $\hat{Z}$ under $\mathcal{Q}$ that $\mathcal{A}W(z) = \frac{\gamma^2}{2}(2\beta(z) - 1)W'(z) + \frac{\gamma^2}{2}W''(z)$. Notice that combining (33) with (11) implies that $q(z)\Gamma(z) \geq 0$ for all $z < \beta$.

Next, suppose that $\Gamma(z) > 0$ for some $z < \beta$. In $\Sigma(\beta, q)$, the belief does not jump, so $W(z) = J(z, z)$. Observing that $\Gamma(z) = J_2(z, z)$, it follows that for small enough $\varepsilon > 0$, $J(z, z+$

$\varepsilon) > J(z,z) = W(z)$, violating (B2). Therefore, if $\Sigma(\beta, q)$ is an equilibrium, then $q(z)\Gamma(z) = 0$ for all $z < \beta$. Hence, at all $z < \beta$, (11) reduces to (12), and for any β, W must have the form given by (15).

At the threshold β, $W(\beta) = V(\beta) - \tilde{K}_H$, $W'(\beta) = V'(\beta)$, and $\Pi_L(\beta) = \tilde{K}_H$. As $V'(z) = (1-\beta(z))(V(z) - V_L)$, we can get $\Gamma(\beta) = 0$. For an arbitrary q on $z < \beta$, let $G_L^q(z)$ be the expected payoff of a low type who rejects all offers until $Z_t \geq \beta$ (i.e., $G_L^q(z) = \frac{\lambda}{\lambda+r}K_L^O + E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[K_H + \frac{\lambda}{\lambda+r}(K_H^O - K_L^O)\right]$). Let q^* denote the expression for q given in (19) and Z^* be the belief process that is consistent with q^*. By construction, for all $z < \beta$,

$$(1-\beta(z))(V_L - G_L^{q^*}(z) - W(z)) + W'(z) = 0.$$

Above we established that $\Gamma(z) \leq 0$ and $q(z)\Gamma(z) = 0$ for all $z < \beta$. Suppose there exists a $\Sigma(\beta, q)$-equilibrium, with $z_0 < \beta$ and $\Gamma(z_0) < 0$. By continuity of $G_L^q(= \Pi_L)$, W and W', there exists an open interval around z_0 on which $\Gamma < 0$. Let I be the union of all such intervals $T^c \equiv (-\infty, \beta] \setminus I$. To satisfy $q(z)\Gamma(z) = 0$, then $q(z) = 0$ on I. Given W from (15), $\Gamma(z) = 0$ on I^c implies Π_L on I^c must be equal to P as given by (16), and further that $q = q^*$ on the interior of I^c. Hence, $q(z) \leq q(z^*)$ for almost all $z < \beta$. Therefore, starting from any $Z_0 = Z_0^* = z \leq \beta$, $Z_t \leq Z_t^*$ for all $t \leq \inf\{s : Z_s^* \geq \beta\}$. It follows that $\Pi_L(z) = G_L^q(z) \leq G_L^{q^*}(z)$, which then implies

$$\Gamma(z) = (1-\beta(z))(V_L - \Pi_L(z) - W(z)) + W'(z) \geq 0,$$

which gives a contradiction.Hence, if $\Sigma(\beta, q)$ is an equilibrium, $\Gamma(z) = 0$ for all $z < \beta$. $\square$

Proof of Theorem 2. The characterization above shows that there exists a unique candidate $\Sigma(\beta, q)$. We just need to verify that the found candidate is well defined and satisfies the equilibrium conditions as in Proposition 5.

To prove verify that the candidate is well defined, first we show the candidate admits a unique strong solution to (3) for any $t \leq T(\beta)$. $\hat{Z}_t$ is linear in X_t given by $\hat{Z}_t = \hat{Z}_t + \frac{\gamma}{\sigma}\left(X_t - \frac{\mu_H+\mu_L}{2}t\right)$. Since we are looking for solutions to $Z_t = \hat{Z}_t + Q_t$ and $Q_t = \int_0^t q(Z_s)ds$, it is sufficient to show that there exists a unique solution to

$$Q_t = \int_0^t q(\hat{Z}_s + Q_s)ds. \tag{34}$$

With candidate $q(z)$ given in (19), we can write (34) in differential form:

$$dQ_t = e^{-u_1(\hat{Z}_s + Q_s)}, Q_0 = 0, \tag{35}$$

where $\kappa = \frac{\gamma^2}{2C_1^*}\left(V_L - \frac{\lambda}{\lambda+r}\left(V^O + K_L^O\right)\right)$. For each (t,ω), (35) is a separable ODE, which has a unique solution $Q_t = \frac{1}{u_1}\ln\left(1+\kappa u_1\int_0^t e^{u_1\hat{Z}_s}ds\right)$. Hence, for any $t \le T(\beta)$, there exists unique solutions for Q_t and Z_t, and thus a unique solution to (3). Given Z, the remaining objects (S^H, S^L, P) are all well defined.

Now we verify the equilibrium conditions: Conditions 1-3 for S-candidate and Conditions (B1)-(B3). Condition 2 follows from plugging (4) and (5) into (1) to get (3). Condition 3 follows directly from (3). Condition 1 requires seller optimality for both high and low types. For high type, $\mathcal{S}^H = \{T(\beta)\}$ from (4) and $P(z) \le K_H + \frac{\lambda}{\lambda+r}K_H^O$ from (6). Therefore,

$$
\begin{aligned}
&\sup_{\tau\in\mathcal{T}} E^H\left[e^{-(\lambda+r)\tau}(P_\tau - K_H)\right] + (1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}K_H^O \\
\le &\sup_{\tau\in\mathcal{T}} e^{-(\lambda+r)\tau}\frac{\lambda}{\lambda+r}K_H^O + (1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}K_H^O \\
= &\frac{\lambda}{\lambda+r}K_H^O = \Pi_H(z),
\end{aligned}
$$

which verifies that S^H solves (SP_H).

For low type, by construction $\Pi_L(z) = \frac{\lambda}{\lambda+r}K_L^O + E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[K_H + \frac{\lambda}{\lambda+r}(K_H^O - K_L^O)\right]$. Let $\mathcal{T}(\beta) \equiv \mathcal{T}\cap\{\tau : \tau \le T(\beta)\}$, i.e., the set of all stopping times such that $\tau \le T(\beta)$. As P is bounded above by $\tilde{K}_H$ and delay is costly,

$$
\begin{aligned}
&E_z^L\left[e^{-(\lambda+r)\tau}P(Z_\tau)\right] + (1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}K_L^O \\
\le &\frac{\lambda}{\lambda+r}K_L^O + E_z^L\left[e^{-(\lambda+r)\tau}\right]\left[K_H + \frac{\lambda}{\lambda+r}(K_H^O - K_L^O)\right] \\
\le &\Pi_L(z),
\end{aligned}
$$

for any $\tau \in \mathcal{T}\setminus\mathcal{T}(\beta)$. As $\mathcal{S}^L \subseteq \mathcal{T}(\beta)$, to verify that S_L solves (SP_L), it suffices to show that for any $\tau \in \mathcal{T}(\beta)$, $E_z^L\left[e^{-(\lambda+r)\tau}P(Z_\tau)\right] + (1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}K_L^O = \Pi_L(z)$.

Let $f_L(t,z) \equiv e^{-(\lambda+r)t}\left(P(z) - \frac{\lambda}{\lambda+r}K_L^O\right)$ and note that f_L is C^2 for all $z \ne \beta$. Conditional on $\theta = L$ and $t < T(\beta)$, Z evolves according to

$$
dZ_t = \left(q(Z_t) - \frac{\gamma^2}{2}\right)dt + \gamma dB_t.
$$

By Dynkin's formula, for any $\tau \in \mathcal{T}(\beta)$,

$$E_z^L \left[f_L(\tau, Z_\tau)\right] = f_L(0,z) + E_z^L \left[\int_0^\tau \mathcal{A}^L f_L(s, Z_s) ds\right],$$

where $\mathcal{A}^L$ is the characteristic operator for the process $Y_t = (t, Z_t)$ under $\mathcal{Q}^L$, i.e.,

$$\begin{aligned}
\mathcal{A}^L f_L(t,z) &= \frac{\partial f_L}{\partial t} + \left(q(z) - \frac{\gamma^2}{2}\right)\frac{\partial f_L}{\partial z} + \frac{1}{2}\gamma^2 \frac{\partial^2 f_L}{\partial z^2} \\
&= e^{-(\lambda+r)t}\left[-(\lambda+r)\left(P(z)\frac{\lambda}{\lambda+r}K_L^O\right) + \left(q(z) - \frac{\gamma^2}{2}\right)P'(z) + \frac{\gamma^2}{2}P''(z)\right] \\
&= e^{-(\lambda+r)t}\left[-(\lambda+r)\left(\Pi_L \frac{\lambda}{\lambda+r}K_L^O\right) + \left(q(z) - \frac{\gamma^2}{2}\right)\Pi_L'(z) + \frac{\gamma^2}{2}\Pi_L''(z)\right] \\
&= 0,
\end{aligned}$$

where the second inequality comes from (19). Hence, $E_z^L \left[f_L(\tau, Z_\tau)\right] = \Pi_L(z)$ for any $\tau \in \mathcal{T}(\beta)$.

For (B1), when $z > \beta$, $W(z) = V(z) - \tilde{K}_H$ by construction. When $z \leq \beta$,

$$\begin{aligned}
&W(\beta - x) - (V(\beta - x) - \tilde{K}_H) \\
=&\frac{\left(e^{(1-u_1)x} + (u_1 - 1)e^x - u_1\right)(V_H - \hat{K}_H)(\hat{K}_H - V_L)}{e^x(u_1 - 1)(V_H - \hat{K}_H) + u_1(\hat{K}_H - V_L)}
\end{aligned}$$

The denominator is positive by Assumptions 1 and 2. The numerator is positive as $e^{(1-u_1)x} + (u_1 - 1)e^x - u_1 > 0$ for all $x > 0, u_1 > 1$.

For (B2), when $z \leq \beta$,

$$\frac{\partial}{\partial z'} J(z,z') = \frac{C_1^* u_1(u_1-1)e^{u_1 z'}(e^{z-z'} - 1)}{1+e^z} > 0, \forall z' \in (z, \beta).$$

Since $J(z,z')$ is decreasing in z' and $W(z) = J(z,z)$, $W(z) = J(z,z) = \sup_{z' \in (z,\beta)} J(z,z')$. As $J(z,z') = V(z) - \tilde{K}_H = W(z)$ for all $z' \geq \beta$, we have $W(z) \geq J(z,z')$ for all $z' \geq z$. When $z > \beta$, $\Pi_L(z) = \tilde{K}_H$ and $J(z,z') = V(z) - \tilde{K}_H = W(z)$ for all $z' \geq z > \beta$.

For (B3), let $f_B(t,z) = e^{-(\lambda+r)t}\left(W(z) - \frac{\lambda}{\lambda+r}V^O\right)$, f_B is C^2 for all $z \neq \beta$. For any stopping time τ such that $E_z[\tau] < \infty$, using Dynkin's formula, we have

$$E_z\left[f_B(\tau, \hat{Z}_\tau)\right] = f_B(0,z) + E_z\left[\int_0^\tau \mathcal{A}^B f_B(t, \hat{Z}_t) dt\right],$$

where $\mathcal{A}^B$ is the characteristic operator of the process $(t, \hat{Z}_t)$ under Q^B starting from $\hat{Z}_0 = z$, i.e., $\mathcal{A}^B f(t,z) = \frac{\partial f}{\partial t} + \frac{\gamma^2}{2}(2\beta(z)-1)\frac{\partial f}{\partial z} + \frac{\gamma^2}{2}\frac{\partial^2 f}{\partial z^2}$. From (12), $\mathcal{A}^B f_B = 0$ for all $z < \beta$. For $z > \beta$, $W(z) = V(z) - \tilde{K}_H$. As $\frac{\gamma^2}{2}(2\beta(z)-1)V'(z) + \frac{\gamma^2}{2}V''(z) = 0$, $\mathcal{A}^B f_B = -e^{-(\lambda+r)t}(\lambda + r)\left(V(z) - \tilde{K}_H\right) > 0$. Hence, $\mathcal{A}^B f_B \leq 0$, and for any stopping time, $W(z) - \frac{\lambda}{\lambda+r}V^O = f(0,z) \geq E_z[f_B(\tau, Z_\tau)] = E_z\left[e^{-(\lambda+r)\tau}\left(W(z) - \frac{\lambda}{\lambda+r}V^O\right)\right]$. □

Proof of Theorem 2. We will refer equilibrium as the monotone equilibrium (i.e., Π_L is non-decreasing).

LEMMA 7. *In any equilibrium, there exists $\beta < \infty$ such that $P(z) = \Pi_L(z) = \tilde{K}_H$ if and only $z \geq \beta$.*

Proof. Same as Fact 4 in the proof of Proposition 5. □

By Lemma 7, in any equilibrium there exists a β such that the buyer offers K_H (and the seller accepts with probability 1) if and only if $z \geq \beta$. Consider equilibrium play for $t < T(\beta)$, by Lebesgue's decomposition for monotone functions (cf. Proposition 5.4.5, Bogachev 2007), we can decompose Q into two process:

$$Q = Q^{abs} + Q^{sing}$$

where Q^{abs} is an absolutely continuous process and Q^{sing} is non-decreasing process with $dQ_t^{sing} = 0$ almost everywhere. We have shown in Theorem 2 that the equilibrium is unique among those in which Q is absolutely continuous. Now it suffices to rule out equilibria in a singular trading intensity.

First note that Q^{sing} can be further decomposed into a continuous non-decreasing process and a non-decreasing jump process. Thus, a singularity can take one of two forms. Either, (i) a jump from some z_0 to some $z_1 > z_0$ or (ii) trading intensity of greater than dt at some isolated z_0.

LEMMA 8. *In any equilibrium, if W is C^2 on any interval (z_1, z_2), then for all $z \in (z_1, z_2)$,*

$$(\mathcal{A} - (\lambda + r))W(z) \leq -\lambda V^O, \tag{36}$$

where $\mathcal{A}$ is the characteristic operator of $\hat{Z}$ under Q.

Proof. If (36) is violated at some $z \in (z_1, z_2)$, then since W is C^2 on the interval, there exists $\varepsilon > 0$ such that (36) is violated over the interval $(z - \varepsilon, z + \varepsilon)$. Let $\tau_\varepsilon = \inf\{t : \hat{Z}_t \notin (z - \varepsilon, z +$

$\varepsilon)\}$, then by Dynkin's formula,

$$E_z\left[e^{-(\lambda+r)\tau_\varepsilon}W(\hat{Z}_{\tau_\varepsilon})\right] = W(z) + E_z\left[\int_0^{\tau_\varepsilon} e^{-(\lambda+r)s}(\mathcal{A}-(\lambda+r))W(\hat{Z}_s)ds\right]$$
$$> W(z) - \left(1-e^{(\lambda+r)\tau_\varepsilon}\right)\frac{\lambda}{\lambda+r}V^O,$$

which violates (B3). □

LEMMA 9. *In any equilibrium, if W is differentiable at z, then $\Gamma(z) \leq 0$.*

Proof. Fix z, and assume that $W'(z)$ exists. If $z \geq \beta$, then by Lemma 7 $\Pi_L(z') = \tilde{K}_H$ and $W(z) = V(z') - \tilde{K}_H$ for all $z' \geq z$. Substituting in these expressions, one finds that $J(z,z') = V(z) - \tilde{K}_H$ for all $z' \geq z$, and $\Gamma(z) = 0$.

For $z < \beta$, if Z does not jump following a rejection at z, then $q(z)\Gamma(z) \leq 0$ by the argument in the proof of Lemma 3. Finally if Z does jump from z to $z' > z$, then $W(z) = J(z,z')$. Using envelope theorem,

$$W'(z) = J_1(z,z') = \frac{\beta'(z)}{\beta(z')}\left(W(z') + \Pi_L(z') - V_L\right).$$

Since the low type must be indifferent, $\Pi_L(z) = \Pi_L(z')$. As

$$W(z') = \frac{\beta(z')}{\beta(z)}(\Pi_L(z') + W(z) - V_L) - \Pi_L(z') + V_L,$$

we have

$$W'(z) = \frac{\beta'(z)}{\beta(z)}\left(W(z) + \Pi_L(z) - V_L\right),$$

which implies $\Gamma(z) = 0$. □

LEMMA 10. *In any equilibrium, $\beta > \underline{z}$ and $W(z) \geq E_z\left[e^{-(\lambda+r)\hat{T}(\beta)}(V(\beta) - \tilde{K}_H)\right] + (1 - e^{-(\lambda+r)\hat{T}(\beta)})\frac{\lambda}{\lambda+r}V^O > \frac{\lambda}{\lambda+r}V^O$, where $\hat{T}(\beta) = \inf\{t \geq 0 : \hat{Z}_t \geq \beta\}$ and $\hat{Z}_0 = z$.*

Proof. For any $z_1 > \underline{z}$, the strategy of waiting for news for all $z < z_1$ and offering $\tilde{K}_H$ for all $z \geq z_1$ is feasible for the buyer and starting from any z, generates a payoff of

$$E_z\left[e^{-(\lambda+r)\hat{T}(z_1)}\left(V(z_1) - \tilde{K}_H - \frac{\lambda}{\lambda+r}V^O\right)\right] + \frac{\lambda}{\lambda+r}V^O > \frac{\lambda}{\lambda+r}V^O.$$

Hence, the buyer's equilibrium payoff must be at least as large as when $z_1 = \beta$. If $\beta < \underline{z}$, then $W(z) = V(\beta) - K_H < \frac{\lambda}{\lambda+r}V^O$, which cannot be true. $\square$

LEMMA 11. *In any equilibrium,* $\Pi_L(z) = \frac{\lambda}{\lambda+r}K_L^O + E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[K_H + \frac{\lambda}{\lambda+r}(K_H^O - K_L^O)\right]$.

Proof. Same as Fact 7 in the proof of Proposition 5. $\square$

LEMMA 12. *In any equilibrium: (i)* Π_L *is non-decreasing, (ii)* Π_L *is continuous, and (iii)* W *is continuous.*

Proof. (i) is by assumption. (ii) is the same as Fact 8 in the proof of Proposition 5. For (iii), we will first show that $W(z_0^-) < W(z_0^+)$ violates (B2). Starting from $z_0 - \varepsilon$, the buyer can offer $p = \Pi_L(z_0 + \varepsilon)$ and trade with arbitrarily small probability at price which is bounded above by $\tilde{K}_H$ with payoff arbitrarily close to $W(z_0^+)$, which violates (B2). Since Π_L is continuous, if $W(z_0^-) > W(z_0^+)$, then $W(z_0^-) = J(z_0, z_1)$ for some $z_1 > z_0$ (i.e., Z must jump upward as it approaches z_0 from the left). But J is continuous in its first argument and therefore $W(z_0^+) < J(z_0, z_1)$ violating (B2). $\square$

LEMMA 13. *In any equilibrium, Q has continuous sample paths (i.e., there cannot exist an atom of trade with only with the low type).*

Proof. Suppose that starting from $Z_t = z_0$, the equilibrium belief process jumps to $Z_{t^+} = \alpha > z_0$. By Lemma 11, it must be that $\Pi_L(z_0) = \Pi_L(\alpha)$ and then Π_L non-decreasing implies that $\Pi_L(z) = \Pi_L(z_0)$ for all $z \in (z_0, \alpha)$. Moreover, there must exist a $z_1 > \alpha$ such that Z evolves continuously in the interval (α, z_1) (otherwise $Z_{t^+} \neq \alpha$). Stationarity then implies that α be a reflecting barrier for the belief process conditional on rejection starting from any $Z_t \geq \alpha$. We claim that these equilibrium dynamics require the following properties:

(i) $(\mathcal{A} - (\lambda + r))W(z) = -\lambda V^O$ and $\Gamma(z) \leq 0$ for all $z \in (\alpha, z_1)$;

(ii) $\Gamma(z) = 0$ for all $z \in (z_0, \alpha)$;

(iii) $\Pi_L'(\alpha) = 0$;

(iv) W is C^2 at α.

(i) follow from the proof of Lemma 3. For (ii), the buyer's payoff starting from any $z \in (z_0, \alpha)$ is given by

$$W(z) = J(z, \alpha) = \frac{\beta(\alpha) - \beta(z)}{\beta(\alpha)}(V_L - \Pi_L(\alpha)) + \frac{\beta(z)}{\beta(\alpha)}W(\alpha). \tag{37}$$

Since $\alpha \in \arg\max_{z' \geq z} J(z,z')$, envelope theorem yields

$$\begin{aligned} W'(z) = J_1(z,\alpha) &= \frac{\beta'(z)}{\beta(\alpha)}\left(W(\alpha) + \Pi_L(\alpha) - V_L\right) \\ &= \frac{\beta'(z)}{\beta(z)}\left(W(z) + \Pi_L(\alpha) - V_L\right) \\ &= \frac{\beta'(z)}{\beta(z)}\left(W(z) + \Pi_L(z) - V_L\right), \end{aligned}$$

where the third inequality comes from replacing $W(\alpha)$. This implies (ii). For (iii), $\Pi_L(\alpha^-) = 0$ is implied by $\Pi_L(z) = \Pi_L(\alpha)$ for all $z \in (z_0, \alpha)$ and $\Pi_L'(\alpha^+) = 0$ is implied by the reflecting barrier and smooth pasting. For (iv), $\mathcal{C}^1$ follows form *Robin* condition

$$W'(\alpha^+) = \frac{\beta'(\alpha)}{\beta(\alpha)}\left(W(\alpha) + \Pi_L(\alpha) - V_L\right), \tag{38}$$

where $\beta'(\alpha)/\beta(\alpha)$ is the (unconditional) intensity at which the seller accepts at α and the second term is the difference of buyer's payoff following acceptance versus rejection. Differentiating (37) and taking limit $z \uparrow \alpha$ yields that $W(\alpha^-)$ is equal to $W(\alpha^+)$ in (38). For $\mathcal{C}^2$, if $W''(\alpha^+) < W''(\alpha^-)$ then $(\mathcal{A} - (\lambda + r))W(z) > -\lambda V^O$ in a neighborhood just below α, which violates (36). If $W''(\alpha^+) > W''(\alpha^-)$, then

$$\begin{aligned} \Gamma'(\alpha^+) &= -\beta'(\alpha)\left(W(\alpha) + \Pi_L(\alpha) - V_L\right) - (1 - \beta(\alpha))\left(W'(\alpha) + \Pi_L'(\alpha)\right) + W''(\alpha^+) \\ &= \Gamma'(\alpha^-) + W''(\alpha^+) - W''(\alpha^-) \\ &> 0, \end{aligned}$$

where the second equality uses (iii) and the final inequality contradicts $\Gamma(z) \leq 0$ in (i). Hence, (i)-(iv) are established.

We claim that (i)-(iv) requires $W(\alpha) \leq \frac{\lambda}{\lambda + r} V^O$, which contradicts Lemma 10. First, (ii)-(iv) imply $\Gamma(\alpha) = 0$. Therefore to satisfy *Gamma*$(z) \leq 0$ for the neighborhood above α as in (i) requires $\Gamma'(\alpha) \leq 0$.

$$\begin{aligned} \Gamma'(\alpha) \leq 0 &\iff -\beta'(\alpha)\left(W(\alpha) + \Pi_L(\alpha) - V_L\right) - (1 - \beta(\alpha))W'(\alpha) + W''(\alpha) \leq 0 \\ &\iff (2p(\alpha) - 1)W'(\alpha) + W(\alpha) \leq \beta(\alpha)\Gamma(\alpha) \\ &\iff \mathcal{A}W(\alpha) \leq 0 \\ &\iff W(\alpha) \leq \frac{\lambda}{\lambda + r} V^O, \end{aligned}$$

where the last one comes from (i) and (iv). □

LEMMA 14. *There cannot exist an isolated point of singular trading intensity.*

Proof. We first prove that FB must be $\mathcal{C}^2$ at any such α. Since there are no jumps and α is an isolated point, Q is absolutely continuous in a neighborhood of α. Hence, there exists $\varepsilon > 0$ such that

$$(\mathcal{A}-(\lambda+r))W(z) = -\lambda V^O, \forall z \in N_\varepsilon(\alpha)\setminus\alpha. \tag{39}$$

By Lemma 12, Π_L and W are continuous. Therefore, if $dQ_t > 0$ at $Z_t = \alpha$, it must be that

$$(1-\beta(\alpha))(V_L - \Pi_L(\alpha) - W(\alpha)) + W'(\alpha^+). \tag{40}$$

To prove that W must be $\mathcal{C}^1$ at α, suppose that $W'(\alpha^-) < W'(\alpha^+)$. Starting from $Z_t = \alpha$, let $\tau_\varepsilon = \inf\{s \geq t : |\hat{Z}_s - \hat{Z}_t| \geq \varepsilon\}$. Let $\Delta \equiv W'(\alpha^+) - W'(\alpha^-) > 0$.An extension of Ito's formula (Harrison 2013, Proposition 4.12) gives

$$\begin{aligned}
e^{-(\lambda+r)\tau}\left(W(Z_{\tau_\varepsilon}) - \frac{\lambda}{\lambda+r}V^O\right) =& W(\alpha) - \frac{\lambda}{\lambda+r}V^O + \frac{1}{2}\gamma^2\Delta l(\tau_\varepsilon,\alpha) \\
&+ \int_0^{\tau_\varepsilon} e^{-(\lambda+r)s}(\mathcal{A}-(\lambda+r))\left(W(Z_s) - \frac{\lambda}{\lambda+r}V^O\right) \\
&+ \int_0^{\tau_\varepsilon} e^{-(\lambda+r)s}(\mathcal{A}-(\lambda+r))\gamma W'(Z_s)dB_s.
\end{aligned}$$

Taking the expectation over the sample paths, we obtain a violation of (B3):

$$\begin{aligned}
E_z[e^{-(\lambda+r)\tau}\left(W(Z_{\tau_\varepsilon}) - \frac{\lambda}{\lambda+r}V^O\right)] =& W(\alpha) + \frac{1}{2}\sigma^2\Delta E_\alpha[l(\tau_\varepsilon,\alpha)] \\
&+ W(\alpha) + \frac{1}{2}\sigma^2\Delta\int_0^{\tau_\varepsilon} p_0(s,\alpha)ds > W(\alpha),
\end{aligned}$$

where $p_0(s,\cdot)$ is the density of Z_s starting from $Z_0 = \alpha$. Next, suppose that $W'(\alpha^-) > W'(\alpha^+)$. Then

$$\begin{aligned}
\Gamma(\alpha^-) &= (1-\beta(\alpha))(V_L - \Pi_L(\alpha) - W(\alpha)) + W'(\alpha^-) \\
&> (1-\beta(\alpha))(V_L - \Pi_L(\alpha) - W(\alpha)) + W'(\alpha^+) \\
&= \Gamma(\alpha^+) = 0,
\end{aligned}$$

which violates Lemma 9 in a neighborhood below α. Thus, we have established that W is $\mathcal{C}^1$ at α.

For $\mathcal{C}^2$, since (36) holds with equality at α^+ and W is $\mathcal{C}^1$ at α, if $W''(\alpha^-) > W''(\alpha^+)$ then (36) is violated in a neighborhood below α. If $W''(\alpha^-) < W''(\alpha^+)$, then it must be that (36) holds with strict inequality in a neighborhood below α, which violates (39). Hence, the smoothness of W is established at α.

An isolated singularity at α means that for $t \leq \tau_\varepsilon$, Q_t^{sing} increases only at times t where $Z_t = \alpha$. Thus, Q_t^{sing} is proportional to the local time of Z_t at α (Harrison 2013, Section 1.2), which we denote by $l_\alpha^Z(t)$. For $t \leq \tau_\varepsilon$, Z evolves according to

$$Z_t = \hat{Z}_t + Q_t^{abs} + \delta l_\alpha^Z(t). \tag{41}$$

Harrison and Shepp (1981) show that (41) has a unique solution if and only if $|\delta| \leq 1$, in which case Z is distributed as skew Brownian motion (SBM) with δ capturing the degree of skewness. If $\delta = 1$, then Z has a reflecting boundary at α, where for $\delta = 0$ there is no singularity at α and Z is a standard Ito diffusion. By Lemma 11, SBM involves a kink in the low type's value function at α, namely

$$\gamma \Pi_L'(\alpha^+) = (1-\gamma)\Pi_L'(\alpha^-), \tag{42}$$

where $\gamma = (1+\delta)/2$ (see Kolb 2016). Since Q must be monotonically increasing, it is sufficient to rule out any $\delta > 0$. There are three exhaustive cases to consider.

First, suppose $\Pi_L'(\alpha^+) = \Pi_L'(\alpha^-) = 0$. Then we have $\Gamma(\alpha) = 0$, and (36) holds in a neighborhood around α. Using an argument as in proof of Lemma 13 shows that $W(\alpha) \leq 0$, which yields a contradiction. Second, suppose $\Pi_L'(\alpha^+) = \Pi_L'(\alpha^-) \neq 0$, then (42) requires $\gamma = \frac{1}{2}$, which implies $\delta = 0$, contradicting the isolated singularity at α. Third, suppose $\Pi_L'(\alpha^+) \neq \Pi_L'(\alpha^-)$. By Π_L non-decreasing, $\Pi_L'(\alpha^-), \Pi_H'(\alpha^+) \leq 0$. Further, (42) and $\delta > 0$ imply that $\Pi_L'(\alpha^-) > \Pi_L'(\alpha^+) > 0$. As $\Gamma(\alpha) = 0$ and $\Gamma(z) \leq 0$ as required by Lemma 9, we have $\Gamma'(\alpha^-) \geq 0$. As Γ' is strictly decreasing in Π_L', $\Pi_L'(\alpha^+) < \Pi_L'(\alpha^-)$ implies that $\Gamma'(\alpha^+) > \Gamma'(\alpha^-) \geq 0$. Since $\Gamma(\alpha) = 0$, $\Gamma(z) > 0$ for z in the neighborhood just above α, violating Lemma 9. Hence, a contradiction arises in all cases, and therefore there cannot exist an isolated point of singular trading intensity. □

The two key features of the constructed equilibrium are (i) a completion threshold β above which trade takes place immediately at a price of $\tilde{K}_H$, which have been shown in the equilibrium definition section, and (ii) for all $z < \beta$, trade takes place at a rate proportional to time. The above lemmas have shown that any singular component that corresponds to trade at a rate "faster" than dt, which can take the form of an atom (i.e., a jump in Z) or local time (e.g, a reflecting boundary), cannot be sustained in equilibrium. Therefore, Q process must be absolutely continuous as in (ii). $\square$

Proof of Proposition 4. I will start by taking $\gamma \to 0$ and analyzing the equilibrium for any given belief z, and then I will specifically focus on the ex-ante expected payoffs for prior belief z_0. As the (continuation) equilibrium behavior depends on whether the prior belief is above the threshold, the equilibrium discussion will divide into two cases. Take $\gamma \to 0$, $z_0 > \hat{\zeta}(z_0)$ will be reduced to $z_0 > \ln \frac{\tilde{\tilde{K}}_H - V_L}{V_H - \tilde{\tilde{K}}_H}$.

If the prior belief is such that $z_0 > \ln \frac{\tilde{\tilde{K}}_H - V_L}{V_H - \tilde{\tilde{K}}_H}$,

1. the threshold $\hat{\zeta}$ goes to $\ln \frac{K_H - V_L + \frac{\lambda}{\lambda + r}(V(z_0) - \beta_0 K_H)}{V_H - K_H - \frac{\lambda}{\lambda + r}(V(z_0) - \beta_0 K_H)}$;

2. for $z < \hat{\zeta}$, the rate of trade goes to ∞; for $z = \hat{\zeta}$, the rate of trade goes to 0;

3. the buyer's payoff converges uniformly to $\max\{\frac{\lambda}{\lambda + r}(V(z_0) - \tilde{\tilde{K}}_H), V(z) - \tilde{\tilde{K}}_H\}$;

4. the low type's payoff converges pointwise to $V_L - \frac{\lambda}{\lambda + r}(V(z_0) - \tilde{\tilde{K}}_H)$ below $\hat{\zeta}$ and to $\tilde{\tilde{K}}_H$ above $\hat{\zeta}$;

5. efficiency loss converges to 0 above $\hat{\zeta}$ but remains positive below $\hat{\zeta}$.

If the prior belief is such that $z_0 < \ln \frac{\tilde{\tilde{K}}_H - V_L}{V_H - \tilde{\tilde{K}}_H}$,

1. the threshold $\hat{\zeta}$ goes to $\ln \frac{K_H + \frac{\lambda(1-P_0)}{\lambda(1-P_0)+r} V_L - V_L}{V_H - K_H - \frac{\lambda(1-P_0)}{\lambda(1-P_0)+r} V_L}$;

2. for $z \le \hat{\zeta}$, the rate of trade goes to ∞; for $z = \hat{\zeta}$, the rate of trade goes to 0;

3. the buyer's payoff converges uniformly to $\max\{0, V(z) - K_H - \frac{\lambda(1-P_0)}{\lambda(1-P_0)+r} V_L\}$;

4. the low type's payoff converges pointwise to V_L below $\hat{\zeta}$ and to $K_H + \frac{\lambda(1-P_0)}{\lambda(1-P_0)+r} V_L$ above $\hat{\zeta}$;

5. efficiency loss converges to 0 above $\hat{\zeta}$ but remains positive below $\hat{\zeta}$.

What about the fixed price benchmarks? Similarly, the cutoff condition $z_0 > \zeta_d$ will be reduced to $z_0 > \ln \frac{\overset{\approx}{K}_H - V_L}{V_H - \overset{\approx}{K}_H}$. If the prior belief is such that $z_0 > \ln \frac{\overset{\approx}{K}_H - V_L}{V_H - \overset{\approx}{K}_H}$, two interpretations are the same.

1. the threshold ζ_d goes to $\ln \frac{K_H - V_L + \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 K_H)}{V_H - K_H - \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 K_H)}$;
2. the buyer's payoff converges uniformly to $\max\{\frac{\lambda}{\lambda+r}(V(z_0) - \overset{\approx}{K}_H), V(z) - \overset{\approx}{K}_H\}$;
3. the low type's payoff converges pointwise to $\frac{\lambda}{\lambda+r}\overset{\approx}{K}_H$ below ζ_d and to $\overset{\approx}{K}_H$ above ζ_d;
4. efficiency loss converges to 0 above ζ_d but remains positive below ζ_d.

If the prior belief is such that $z_0 < \ln \frac{\overset{\approx}{K}_H - V_L}{V_H - \overset{\approx}{K}_H}$, under the first interpretation,

1. the threshold ζ_d goes to $\ln \frac{\overset{\approx}{K}_H - V_L}{V_H - \overset{\approx}{K}_H}$;
2. the buyer's payoff converges uniformly to $\max\{0, V(z) - \overset{\approx}{K}_H\}$;
3. the low type's payoff converges pointwise to 0 below ζ_d and to $\overset{\approx}{K}_H$ above ζ_d;
4. efficiency loss converges to 0 above ζ_d but remains positive below ζ_d.

Under the second interpretation,

1. the threshold ζ_d goes to $\ln \frac{K_H - V_L}{V_H - K_H}$;
2. the buyer's payoff converges uniformly to $\max\{0, V(z) - K_H\}$;
3. the low type's payoff converges pointwise to 0 below ζ_d and to K_H above ζ_d;
4. efficiency loss converges to 0 above ζ_d but remains positive below ζ_d.

Without arrival of exogenous information, the low prior makes the buyer unwilling to offer the price where both types would accept upon transition, so the continuation values for both parties go to 0. This will push down high type's lowest acceptable price and the threshold.

The above analyses consider the equilibrium behavior and payoffs for each belief z and different prior belief z_0. However, some beliefs are not reachable in equilibrium when $\gamma \to 0$. In summary, the limiting equilibrium behavior is as follows:

1. For all $z_0 > \hat{\zeta}(z_0)$, (or $z_0 > \ln \frac{\tilde{\tilde{K}}_H - V_L}{V_H - \tilde{\tilde{K}}_H}$,) the buyer would offer $\tilde{\tilde{K}}_H$ and the seller would accept with probability 1. The buyer earns payoff $V(z_0) - \tilde{\tilde{K}}_H$ and the type-θ seller earns payoff $\tilde{\tilde{K}}_H - K_\theta$.

2. For all $z_0 < \hat{\zeta}(z_0)$, (or $z_0 < \ln \frac{\tilde{\tilde{K}}_H - V_L}{V_H - \tilde{\tilde{K}}_H}$,), the buyer would offer V_L. The high type rejects and the low type mixes such that the belief is $\hat{\zeta}(z_0)$ following a rejection. The buyer earns 0 expected payoff.

3. Following a reject at $z_0 < \hat{\zeta}(z_0)$, there is delay of length τ satisfying $V_L = \tilde{K}_H - K_H + E[e^{-r\tau}]K_H$, after which the buyer offers $\tilde{K}_H$.

The limiting payoff for buyer and seller under fixed price benchmark are the same when $z_0 > \hat{\zeta}(z_0)$. When $z_0 < \hat{\zeta}(z_0)$, however, the buyer never makes the fixed price offer, so she earns the same 0 payoff as in equilibrium. She does not benefit from the ability of adjusting offers. However, low type also earns 0 payoff as opposed to V_L in equilibrium. $\square$