Dynamic Personalized Offers while Learning Changing Tastes

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Abstract

Firms selling products to consumers realize that it takes time to learn consumers' preferences (say, by tracking their online behavior). What makes it even more challenging is that consumers' preferences may change over time, depreciating the value of acquired information. How should the firm personalize its offers and change them dynamically to learn as well as to adapt to changing tastes when it cannot commit to future behavior? How should consumers behave in light of these dynamic offers? I build a continuous-time bargaining model with one-sided incomplete information where a buyer's binary type is publicly revealed through Brownian motion and the binary type changes via a Poisson process. In equilibrium, firms will start with high prices which will only be accepted by high-type consumers with positive probability and as belief drifts below a certain threshold, the firm will offer the lowest price that will be accepted by both types immediately. Changing tastes have two effects: a level effect that leaves low value consumers less likely to accept a given offer and a slope effect so that the firm screens high value consumers faster. Hence type change benefits both types of consumers at a cost to the firm. If the firm is restricted to constant prices and can use the acquired information to select consumers, it is better off than under dynamic prices. The continuation bargaining process gets resolved slower under constant prices than under flexible prices, which makes consumers more willing to accept a given offer. At last, the firm benefits from knowing when the type changes even if the new type is an independent of the old type and the firm does not observe the new realization.

1 Introduction

Concerns about consumer privacy have become a focal point of debate in recent years. Firms, regardless of the industries they operate in, all seek to build consumer profiles through the data collected from browser cookies or third party platforms. The profiles are valuable to these firms as they can select their target consumers more effectively and extract more surplus with better personalized price offers. More importantly, such collection of consumer data and learning about consumer types is never one shot as consumers' tastes and valuations are constantly changing. For example, after Apple's implement of App Tracking Transparency Initiative in 2021, "when a user opts out of tracking, Facebook can still keep targeting that user with data it collected before the opt-out. But that data quickly loses value." (Haggin and Vranica, 2022) The change of consumer's value over time is thus an important factor in considering firms' learning and pricing decisions when approach potential customers.

How does the type change of consumer valuations affect firm's strategies and consumer's purchasing decisions? In many scenarios, a firm approaching new customers has little commitment power in their future selling and pricing decisions. The firm, with offers not taken by the consumer, may soon come back with a new offer. In general, the firm will start with a high price and gradually decrease it to screen high-value consumers through intertemporal price discrimination and exploit low-value consumers. With changing tastes, however, the possibility and the anticipation of it by both firms and consumers brings new dynamics to firms' pricing strategies and consumers' purchasing behavior. Low-value consumers have the option to wait for a high value shock and delay accepting offers, while the firm needs to trade off between waiting for more information about the consumer's current type and the depreciation of currently acquired information in case of a type change. Sometimes the firm may be restricted to charge the same price to all consumers, but they can still partially price discriminate consumers by selecting consumers, such as targeted advertising to differentiate awareness level based on collected information. Will the firm benefit from offering personalized prices to consumers based on the acquired information in comparison with the case that it can only offer the same price but use the information to select consumers?

To answer these questions, we build a bargaining model with one-sided incomplete information where the uninformed seller proposes all the offers to the informed buyer who only decides whether to accept it. Before the trade occurs, the seller can gradually learn about the consumer's current type through a exogenous news source, while the type itself may also change over time. To analyze the equilibrium, we first extend the model of Daley and Green (2020) by introducing exogenous termination to the bargaining model with news and find the relationship between exogenous (continuation) payoff conditional on termination and the current payoff function. Then we model the exogenous termination specifically as an endogenous type transition where the continuation game will be bargaining with the new type. Specifically, we consider an informed buyer of binary types regarding his valuation of a product with unit demand and an uninformed seller making frequent offers to the buyer. The seller can learn about the buyer's current value via a Brownian diffusion process. With a Poisson arrival process the buyer's type may change.

With exogenous termination, the equilibrium features the "skimming property" common in the literature: the seller would offer high prices when the belief of buyer being high type is high, which will only be accepted by the high type buyer with positive probability. High type mixes between accepting and rejecting, while low type always rejects. Over time, the belief will go down with updating from both the news process and buyer's rejection decision. Seller would gradually decrease price as belief drifts down and offer the lowest price that both type buyers would accept immediately when the belief is sufficiently low. This lowest price is determined by low type's continuation payoff, which gives low type buyer payoff equivalent to his "outside option". As a result, the type transition benefits both types of buyers at the cost of the uninformed seller. There are two effects of the type change: the level effect that the seller needs to give up some surplus to the low type buyer to make the trade; and the slope effect that the seller screens the high type at a faster rate, which also benefits both type buyers.

If the seller can only charge the same price over time but can select when (and whether) to sell, this fixed price is set at the highest price that the low type buyer is willing to accept and seller simply decides when to make such an offer to the buyer (and trade). Given the same continuation payoff conditional on exogenous termination, the fixed pricing and flexible pricing gives the seller the same equilibrium payoff. This is a generalization of the "Coasian force" as in Daley and Green (2020) that without commitment power the seller can be improve her payoff by changing prices based on collected information. The low type buyer has the same payoff whereas the high type buyer is better off under flexible price equilibrium than under fixed price one.

A key factor in driving the equivalence result above is that the continuation payoff is realized immediately upon type transition, so the continuation "game" is identical for both fixed pricing and flexible pricing. When the continuation game is endogenous type transition, however, the payoff may not be realized immediately and the continuation bargaining process may differ. Instead of equivalence, the seller payoff under flexible price lies between two fixed price frameworks when we expect delay in the continuation bargaining. The delay in continuation bargaining with the new type will affect the expected payoff upon transition and hence buyer's acceptance decision. The equilibrium will be a fixed point of the mapping from continuation payoff to current expected payoff, and the lowest offered price will be determined endogenously. The fixed price framework has two slightly different interpretations under type transition. The first is that the seller cannot revoke the price offer upon transition whereas the second one can. These two feature no difference with exogenous termination as the game simply ends upon event arrival. With type transition and delay in the continuation bargaining, however, they describe different scenarios. In comparison with flexible pricing equilibrium, the seller is better off than the first benchmark and is worse off than the second benchmark. Intuitively, given the same termination payoffs, the high type buyer benefits from the screening process with experimentation offers under flexible pricing. However, under endogenous type transition, this means that the "outside option" of the future bargaining process upon transition is more valuable for the buyer, so he would demand more from the seller in the current bargaining stage. As a result, the seller is better off using a fixed price if she can take back the offer upon transition. On the other hand, the flexible pricing is still better than the fixed price for the seller if she cannot take back the fixed price offer. This is linked to our last result on private transition that the timing information of type transition event itself and the ability to act upon such information is valuable for the seller.

Lastly, we show that if the type transition event itself is privately observed by the buyer, which cannot be modelled as the exogenous termination, the seller is worse off than under public transition even if the new type is independent of the old one. Without observing the transition event, the belief of the seller will always be adjusted continuously. The high type buyer can more easily pretend to be the low type, which gives the low type more power to ask for a lower price. As a result, the seller gives up more surplus when the transition is privately observed by the buyer.

The model studied in this paper is also relevant for other cases with learning of changing types. For example, consider a venture capital (VC) trying to acquire a start-up pharmaceutical company which produces Covid-19 vaccines. The start-up firm privately knows its own value, while VC can conduct investigations of the firm and learn firm's value over time. In the meantime VC may negotiate offers with the firm. Sometime in the future, new vaccines or new biochemical technologies may enter the market and change the firm's value. The venture capital may observe such environment change, but it does not know the new value of the firm and needs to investigate again. Hence, the change in firm's value affects the both the firm's continuation value of waiting and VC's value of the acquired information of the firm.

2 Literature Review

This paper contributes to the literature with one sided incomplete information (Ausubel, Cramton, and Deneckere, 2002; Fuchs and Skrzypacz, 2020). Fuchs and Skrzypacz (2010) considers the arrival of exogenous termination events while Daley and Green (2020) considers the arrival of public news. The reduced form of public transition case in this paper combines these two channels and extends the finding in Daley and Green (2020) that the uninformed firm cannot benefit from adjusting prices freely. The equilibrium structure that the uninformed party screens the informed party gradually using early tempting offers and thus induces delay and inefficiency is commonly observed in the literature (Deneckere and Liang, 2006; Fudenberg et al., 1991; Gerardi, Maestri, and Monzon, 2020; Gul and Sonnenschein, 1988). In bilateral bargaining where the uninformed party makes all the offers, the most disadvantaged type of the informed party is usually exploited all the surplus. Say, if buyer is the informed party, the uninformed seller will start with some high prices and gradually screen out high type buyers, known as the "skimming property" (Deneckere and Liang, 2006; Fuchs and Skrzypacz, 2010; Fudenberg et al., 1991). Along the path, seller updates her belief through buyer's rejection decisions and becomes more confident that buyer is a lower type. The "skimming property" also exists in our setting where the seller gradually learns about a consumer who through some exogenous news source.

The equilibrium definition builds on Daley and Green (2020) and adds to the literature that models bargaining with one-sided private information directly in continuous time. In comparison, Ortner (2020) adds uninformed party's time varying cost to the bargaining. Daley and Green (2020) studies public learning of the private information itself. Lomys (2020) studies public learning of their outside options. Dilmé (2021) analyzes the role of discounting in shaping the bargaining outcome. This paper contributes to the literature by analyzing the interaction between gradual learning of the current private information and expected arrival of new one. It highlights how the expectation of future learning and type transition affects the value of currently acquired information and shapes the bargaining dynamics. In comparison with Daley and Green (2020), this paper extends their interesting result that the zero benefit from ability to negotiate a better price as a manifestation of "Coasian force" still exists when there is exogenous Poisson termination event. This indifference result breaks, however, under the public endogenous type transition case where the continuation bargaining process conditional on Poisson arrival matters. We highlight the role of the discrete jump in belief from observing the type transition event in shaping this result that the firm benefits from ability to adjust prices.

The private transition case in this paper contributes to the literature on information acquisition of an evolving type (Kremer, Schreiber, and Skrzypacz, 2020). Ortner (2020) also considers a evolving private information but without learning. In contrast, this paper considers gradual learning of the evolving type and finds that the public observability of the transition time plays a role even if the new draw is independent.

3 Model

In the benchmark setup, we describe the model with public transition that both the buyer and seller observe whether the type transition occurs but only the buyer privately observes the new type. The private transition case is the same except that the transition event itself is also privately observed by the buyer.

3.1 Setup

A seller tries to sell a durable asset of type $\theta \in \{L, H\}$ to a buyer. The buyer privately knows the asset type. Let β_0 denote the prior probability that the seller assigns to the asset type $\theta = H$ at time $t = 0$. In the general setup, we allow for interdependent values. Specifically, the seller's cost of parting with the asset is C^θ, where we normalize $C^L = 0 < C^H$, while the buyer's value for the asset is V^θ, with

$V^H \geq V^L$.[1] There is common knowledge of gains from trade: $V^\theta > C^\theta$ for each θ. Both players are risk-neutral, expected utility maximizers.

The game is played in continuous time, starting at $t = 0$ with infinite horizon. At each instant t, the seller proposes a price offer p to the buyer. If the buyer accepts an offer of p at time t, the transaction is executed and the game ends. The payoffs to the seller and the buyer respectively are $e^{-rt}(p - C^\theta)$ and $e^{-rt}(V^\theta - p)$, where $r > 0$ is the common discount rate. An important feature is that the asset type is not fixed over time. With Poisson arrival rate $\lambda > 0$, the asset type changes and a new type is drawn according to the prior probability β_0. Both seller and buyer know if the asset type changes, but only the buyer observes the new type.

The uninformed seller can also learn about current asset type through a exogenous news source. Specifically, public signal about the asset's current type is revealed via a Brownian diffusion process X_t, which satisfies $X_0 = 0$ and evolves according to

$$dX_t = \mu^\theta dt + \sigma dB_t,$$

where $B = \{B_t, \mathcal{F}_t, 0 \leq t \leq \infty\}$ is the standard Brownian motion process . At each time t, the entire history of news, $\{X_s, 0 \leq s \leq t\}$, is observable to both players. For simplicity, assume that every time a new type is drawn at time t, the news process is refreshed such that $X_t = 0$. The parameters μ^H, μ^L, and σ are common knowledge and $\mu^H \geq \mu^L$. Define the signal-to-noise ratio $\gamma \equiv (\mu^H - \mu^L)/\sigma > 0$. When $\gamma = 0$, the news is completely uninformative, or the seller does not have access to the exogenous news source. Larger values of γ imply more informative news.

The timing of the game is as follows. At each instant, if the type transition occurs, then bargaining process restarts with prior belief and reset the news process. Otherwise game continues and there is arrival of news about the buyer. The seller makes an offer based on the history of information and actions, and then the buyer decides whether to accept offer or not. If the buyer accepts the offer, the game ends; if the buyer rejects, the game continues to the next instant.

3.2 Equilibrium Conditions

Following the literature, we focus on Markovian equilibria with respect to the uninformed party's belief, or public belief in the this case and where the high type buyer's payoff is non-decreasing with respect to this belief. As the game is stationary every time the type transition occurs and the transition is publicly observable, we can focus on the time period before the transition while taking into account that type transition may happen at some random point in the future. We can first find the equilibrium and construct the equilibrium payoffs as functions of the continuation payoffs conditional on type transition, and then we fully solve the equilibrium using a fixed point algorithm. As the continuation

[1] We can also interpret θ as consumer's type, and the cost of serving the consumer is higher for the high type.

payoffs are from the continuation bargaining process which is stationary every time type transition occurs, the equilibrium payoff functions are the same for every bargaining process following the type transition. That is, we consider the reduced form representation that with Poisson arrival rate $\lambda > 0$, the game is exogenously ended, leaving payoffs U_O^θ to type-θ buyer and $\Pi_O \geq 0$ to the seller, where $U_O^H \geq U_O^L \geq 0$.[2] The payoffs here can be interpreted as termination payoffs when some event ends the game or continuation payoffs when some public event changes the continuation bargaining process. We will discuss in details how these payoffs affect the both parties' equilibrium payoffs and how they are determined.

For the equilibrium concept, we adopt the one developed by Daley and Green (2020). The formal components of the equilibrium are defined in Appendix B.1. Here we give a brief overview of conditions. On the equilibrium path, buyer is sequentially rational to seller's price offer process and the seller updates her belief in consistence with the buyer's strategy and available information. Off path refinement is imposed with discrete time analog that the buyer optimally responds to any offer taking seller's future strategy as given. Seller would make price offers maximizing her payoff.

Following the literature, we focus on stationary equilibria, using the uninformed seller's belief as the state variable. At any instant t before trade occurs, the seller assigns a probability, $\beta_t \in [0,1]$, to $\theta = H$. We will use the log-likelihood ratio of the probability to represent the belief, denoted as $Z_t \equiv \ln(\beta_t/(1-\beta_t)) \in \bar{\mathbb{R}}$. The belief z is also known to the buyer, so it will be the state variable. We can also write seller's belief of buyer being high type at state z as $\beta(z) = e^z/(1+e^z)$. The seller will choose a offer process $\mathcal{P} = \{P(z) \in \mathbb{R}_+ | z \in \bar{\mathbb{R}}\}$ as a best response to buyer's strategies, the type-θ buyer will choose the acceptance probability $s^\theta(z,p) \in [0,1]$ as a best response to seller's offer p at state z, which aggregates to the distribution over stopping time, $S^\theta : \bar{\mathbb{R}}_+ \to [0,1]$. The seller and type-$\theta$ buyer's payoffs at belief z are denoted $\Pi(z)$ and $U^\theta(z)$ respectively.

Now we can state the optimization problems faced by the seller and buyer and how they choose their optimal strategies. The optimal strategies are pinned down by (i) maximizing the instantaneous expected payoff at each instant taken the continuation game as given, and (ii) maximizing the total expected discounted payoff with intertemporal tradeoffs by also taking into account the effect of exogenous news and type transition in the continuation game. First, as in the literature, it can be shown that the seller will never make a offer that gives low type buyer payoff better than his outside option. In this case, the outside option is the expected continuation payoff of the type transition. Denote $\kappa^\theta \equiv \frac{\lambda}{\lambda+r}\left(U_O^\theta + \Pi_O\right)$ as the expected discounted total termination surplus with a current low type asset. Denote $\tilde{V}^L \equiv V^L - \frac{\lambda}{\lambda+r} U_O^L$ as the price that gives the low type buyer his "outside option", which is his valuation minus the expected discounted termination payoff.

LEMMA 1. *The seller never makes an offer lower than $\tilde{V}^L$.*

[2] In the endogenous type transition case where the new type is independently drawn according to the prior β_0, $U_O^H = U_O^L$. Here we allow generalization of this specific case.

This lemma gives a lower bound of the equilibrium price offers. An immediate result is that both types of buyers would accept an offer below or equal to $\tilde{V}^L$ immediately. Therefore, we can just focus on price offers weakly above $\tilde{V}^L$.

Belief consistency: at each instant when transition occurs, the seller's belief of buyer type jumps to the prior z_0. Before transition occurs (or between any two adjacent transitions), at each instant right after an offer is made, the seller's belief has to be consistent with buyer's strategy. Let $\tilde{z}(z,p)$ be the seller's updated belief if her offer of p is rejected in state z, then the Bayesian updating states

$$\tilde{z}(z,p) = z + \ln \frac{1-s^H(z,p)}{1-s^L(z,p)}.$$

Over time, the seller's updated belief takes into account not only the past rejection decisions of the buyer, but also the news received. In particular, the seller's belief z at instant t can be described by the process

$$Z_t = z_0 + \underbrace{\frac{\gamma}{\sigma}\left(X_t - \frac{\mu^H+\mu^L}{2}t\right)}_{\text{update from news}} + \underbrace{\ln\left(\frac{1-S^H_{t^-}}{1-S^L_{t^-}}\right)}_{\text{update from buyer rejections}} .$$

Buyer's problem: at each instant right after an offer is made, the buyer's acceptance strategy maximizes his expected payoff given the continuation payoff function, i.e. $s^\theta(z,p)$ solves

$$\max_s s(V^\theta - p) + (1-s)U^\theta\left(\tilde{z}(z,p)\right).$$

Over time, the buyer maximizes the expected discounted payoffs with expectation of future news and type transition, i.e. every stopping time τ in the support of S^θ solves

$$U^\theta(z) = \sup_t E^\theta_z\left[e^{-(\lambda+r)t}(V^\theta - p_t)\right] + (1-e^{-(\lambda+r)t})\frac{\lambda}{\lambda+r}U^\theta_O,$$

where E^θ_z is the expectation with respect to the process Z starting from state z and conditional on the type being θ. The first component is the expectation over buyer type of the probability that the transition has yet to happen before the stopping time τ times the discounted payoff of buying the product at price p_τ. The second component is the expected discounted payoff in case of exogenous termination. Any stopping time in the mixed strategy of the seller needs to maximize her expected payoff.

Seller's problem: Similarly, at each instant, the seller optimally chooses price offer $P(z)$ given state z and continuation game. If the seller offers p in state z, it will either be accepted, earning her $p - C^\theta$ from type-θ buyer, or be rejected, earning her the continuation value from the post rejection belief $\tilde{z}(z,p)$. The offer $P(z)$ is optimal if

$$P(z) \in \arg\max_{p} \beta(z)\left[s^H(z,p)(p-C^H)+\left(1-s^H(z,p)\right)\Pi(\tilde{z}(z,p))\right] \\ +(1-\beta(z))\left[s^L(z,p)p+\left(1-s^L(z,p)\right)\Pi(\tilde{z}(z,p))\right],$$

This condition is the optimality condition at each instant. We can write seller's expected payoff in any given state z as:

$$\Pi(z) \equiv \beta(z)E_z^H\left[\int_0^\infty e^{-rt}\left(e^{-\lambda t}(P(Z_t)-C^H)+(1-e^{-\lambda t})\Pi_O\right)dS_{t^-}^H\right] \\ +(1-\beta(z))E_z^L\left[\int_0^\infty e^{-rt}\left(e^{-\lambda t}P(Z_t)+(1-e^{-\lambda t})\Pi_O\right)dS_{t^-}^L\right],$$

It depends on the acceptance decision of two type buyers and the expected arrival of type transition. Specifically, the updated belief $\tilde{z}(z,p)$ and continuation payoff take into account of the rejection decision of buyers while the value from news acquisition and type transition is included in the form of seller's equilibrium payoff function Π. The next condition specifies intertemporal optimality condition that the seller makes optimal use of the option to wait for more information or type transition. For all stopping time τ in the support of S^θ,

$$\Pi(z) \geq E_z\left[e^{-(\lambda+r)\tau}\Pi(\hat{Z}_\tau)+(1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}\Pi_O\right].$$

We make some assumptions for the benchmark setup. The equilibrium bargaining dynamics will depend on whether there is a "gap" between seller's cost and buyer's value and whether there are positive trade surpluses for both types compared to "outside option".

Assumption 1. *The* ***static lemons condition (SLC)*** *holds that* $C^H > V^L$.

Assumption 2. *The* ***low surplus condition (LSC)*** *holds that* $V^L > \kappa^L$, *and the* ***high surplus condition (HSC)*** *holds that* $V^H - C^H > \kappa^H$.

Assumption 1 guarantees the static adverse selection. Assumption 2 guarantees the trade surplus with both types is higher than the expected discounted trade surplus from waiting for exogenous termination, so it's always efficient to trade immediately with the current type than waiting for exogenous termination and getting the outside option. With complete information where seller extracts all the surplus by making all the offers, this assumption guarantees that the seller is willing to trade with current type than simply waiting for the termination.

Given the above the equilibrium conditions, we can characterize the equilibrium structure given the expected payoffs $\{\Pi_O, U_O^H, U_O^L\}$ from the continuation bargaining process upon type transition. Then we can construct the mapping $\Phi : \mathbb{R}_+^3 \to \mathbb{R}_+^3$ such that $\{\Pi_0, U_0^H, U_0^L\} = \Phi\left(\Pi_O, U_O^H, U_O^L\right)$, where Π_0 is seller's ex-ante expected payoff, and U_0^θ is type-θ buyer's ex-ante expected payoff. Φ describes the equilibrium behavior that maps continuation payoffs to ex-ante expected equilibrium payoffs, with Φ^S, Φ^H, Φ^L referring to the seller, high type buyer and low type buyer's payoffs in the image of Φ.

3.3 Fixed Price Framework

The personalized pricing is often viewed as unfair or unacceptable(Poort and Zuiderveen Borgesius, 2021). If firms are constrained to charge a single price rather than personalized pricing, they may engage in other form of price discrimination using the collected information. One natural way is to select consumers using consumer data through targeted advertising. In the context of our model, if the seller charges a single price and simply decides when to make this offer depending on her updated belief, will the seller be worse off than personalized pricing? We leave aside the uninteresting case where the seller always charges the high type buyer's willingness to pay and wait for the low type to change to the high type. Instead we consider the scenario where this single price offer is acceptable to both types of buyers so that they will not simply wait for the transition to occur. The trade will happen immediately once such offer is made, so the seller will optimal choose the highest fixed price that buyer types of buyers are willing to accept immediately.

To understand the role of news process and type transition in shaping the results, we once again first analyze the single price framework when the continuation payoff conditional on type transition is exogenously given, which extends the due diligence benchmark as in Daley and Green (2020) without type transition. The price is exogenously fixed at the highest price that the low type buyer is willing to accept immediately. It is easy to show that this fixed price is also the optimal fixed price the seller would choose. The seller observes the news, so she only decides when to make the offer and complete the transaction. When the type transition is endogenous, however, this fixed price charged by the seller is endogenously determined by the anticipation of the future bargaining process conditional on type transition. The fixed price framework can have two different interpretations depending on whether the seller can make use of the knowledge of transition happening, which we will discuss in details when talking about equilibrium.

When the type is fixed and there is no future event, so with the seller having all the bargaining power, the fixed price is simply set at V^L. In this paper, with Poisson arrival of the new event, the low type always has the option to wait for the event instead of accepting the current offer. Therefore, this highest price depends on the expected arrival of type transition. With "outside option" given by the continuation payoff for the low type buyer, the highest price is the same as equilibrium one $\tilde{V}^L$. Similarly, we can construct a mapping for the fixed price benchmark $\Psi : \mathbb{R}^3_+ \to \mathbb{R}^3_+$ such that $\{\Pi^{F,0}, U_H^{F,0}, U_L^{F,0}\} = \Psi\left(\Pi_O, U_O^H, U_O^L\right)$. Ψ^S, Ψ^H, Ψ^L will refer to the seller, high type buyer and low type buyer's payoffs in the image of Ψ.

4 Exogenous Termination

To solve the equilibrium and pin down the strategies of the buyer and seller, we first consider the reduced form representation with the continuation payoff conditional on type transition being exogenously given, and construct the mappings from exogenous continuation payoff to the expected payoff for each player at each state/belief. This is the reduced-form way of modelling some exogenous event arrival (Fuchs and Skrzypacz, 2010). We show that given the same set of exogenous continuation payoffs upon the arrival of the event, the seller gets the same payoff from a flexible pricing scheme and from a fixed pricing scheme. This speaks to the generalized idea of "Coasian force" that when there is no commitment power of the seller and the time interval between two adjacent offer periods vanishes, the competition with the future self drives all the screening surplus to the high type buyer. Therefore, the seller does not benefit from adjusting prices based on collected information.

THEOREM 1. *Given a set of exogenous continuation payoffs, there exists a cutoff belief $\hat{\beta} \in (0,1)$ such that*

1. *when prior belief $\beta_0 \leq \hat{\beta}$, all players earn the same payoff $\Phi = \Psi$ under two frameworks;*
2. *when prior belief $\beta_0 > \hat{\beta}$, the seller and low type buyer earn the same payoff $\Phi^i = \Psi^i$ for $i = S, L$, but high type buyer is better off $\Phi^H > \Psi^H$ under flexible pricing framework.*

Under exogenous termination, seller and low type buyer always have the same payoff while high type buyer is better off under flexible pricing equilibrium than under the fixed price one when there is delay in bargaining. When the prior belief is low enough such that there is no delay, the seller would find it optimal not to wait for news or type transition, but instead make a trade with the current type immediately. The seller with all the bargaining power will exploit low type buyer and offers the price that will give low type buyer exactly the expected discounted payoff from his "outside option" U_O^L in both flexible and fixed price equilibria. There is no experimentation stage or earlier higher offers, so all players earn the same payoff under these two frameworks. When the prior belief is high, however, the seller finds it optimal to screen the high type buyer with early higher price offers. Compared to the fixed price framework, these initial experimentation offers speed up the bargaining process, so the high type buyer receives the same price offer sooner, and is hence better off. However, all these experimentation benefits are taken by the high type buyer, The next subsections will show how we can construct the equilibrium mappings under two frameworks.

4.1 Flexible Pricing Equilibrium

Now we first solve the flexible pricing equilibrium and construct the equilibrium payoffs. At each instant, the acceptance payoff is strictly higher for the high type seller with higher valuation, whereas the option value of waiting is higher for the low type with better separating signals and value from transition. Hence, the high type has strictly stronger incentive to accept the same offer at each instant. If, at one instant, high type accepts with probability 1 and low type accepts with probability less than 1, then next instant the seller knows for sure the remaining buyer is low type and would offer a strictly lower price (with a discrete jump). Therefore, this cannot happen in the equilibrium. Instead, the equilibrium is unique under some refinement conditions and features gradual screening of the high type, known as the "skimming property".[3] The seller starts with high price offers that are only accepted by the high type with positive probability and gradually decreases prices with decreasing belief. The seller will offer the lowest price $\tilde{V}^L$, which will be accepted by both types of buyers immediately, when the belief is lower than some threshold ζ.

Along the equilibrium path, the low type buyer plays a pure strategy that he stops the first time belief reaches ζ whereas the high type mixes among a range of stopping times. Therefore, for high beliefs $z > \zeta$, only the high type buyer accepts offers, which implies that rejection is a positive signal that $\theta = L$. Therefore, along the path, the seller's belief Z conditional on rejections, has additional downward drift compared to the belief that updates solely based on news. The additional downward drift of Z, is the hazard rate of the high type's acceptance, denoted as $q(z) \equiv dS^H_{t^-} / \left(1 - S^H_{t^-}\right) \geq 0$. Therefore, conditional on offers being rejected, the belief process Z updates according to

$$dZ_t = \underbrace{\frac{\gamma}{\sigma}\left(dX_t - \frac{\mu^H + \mu^L}{2}dt\right)}_{\hat{Z}_t\text{: Updating from news}} - \underbrace{q(Z_t)dt}_{\text{Updating from rejections}} . \tag{1}$$

For beliefs $z > \zeta$, the characterization of seller's value function evolution can be divided into several parts. First with a type transition rate of λ, the Poisson event occurs that leaves her the additional benefit of $\Pi_O - \Pi(z)$. Second, she trades with high type buyer at rate $\beta(z)q(z)$, which leads to a net flow payoff of $P(z) - C^H - \Pi(z)$. Lastly, if the offer is rejected, she earns the option value of waiting for more informative news, where the updated belief has expected drift $\left(\gamma^2/2\right)(2\beta(z) - 1) - q(Z_t)$ and volatility γ. The seller's value function can be characterized by the sum of these flow payoffs

$$\begin{aligned} r\Pi(z) = & \underbrace{\lambda(\Pi_O - \Pi(z))}_{\text{Evolution due to type transition}} + \underbrace{\frac{\gamma^2}{2}(2\beta(z) - 1)\Pi'(z) + \frac{\gamma^2}{2}\Pi''(z)}_{\text{Evolution due to news}} \\ & + q(z)\underbrace{\left(\beta(z)(P(z) - C^H - \Pi(z)) - \Pi'(z)\right)}_{\text{Net benefit of screening at z}}. \end{aligned} \tag{2}$$

The first two terms on the right-hand side of (2) is the evolution of the seller's value arising from type transition and news. The third term is the additional value she derives from trade with the high type, which is the product of the trade rate, $q(z)$, and the net benefit of screening high types. This net benefit of screening in equilibrium must be 0 in equilibrium as a manifestation of "Coasian force" that

[3] The proof of the existence and uniqueness of the equilibrium is shown in the appendix.

the seller does not benefit from the intertemporal price discrimination when the commitment power vanishes under frequent offers as in Daley and Green (2020).

With only seller's value function in the simplified differential equation, we can solve it as

$$\Pi(z) = K_1 \frac{e^{az}}{1+e^z} + \frac{\lambda}{\lambda+r}\Pi_O,$$

where $a = \frac{1}{2}\left(1 - \sqrt{1+8(\lambda+r)/\gamma^2}\right)$ and $K_1 = \left(\frac{V^L-\kappa^L}{1-a}\right)^{1-a}\left(\frac{V^L-\kappa^L-C^H}{a}\right)^{a}$. The seller optimally chooses when to stop screening by choosing the stopping belief

$$\hat{\zeta} = \ln\left(\frac{V^L-\kappa^L}{V^L-\kappa^L-C^H}\right) + \ln\left(\frac{a}{1-a}\right),$$

and correspondingly $\hat{\beta} = \beta(\hat{\zeta})$. The seller's value function is decreasing in belief z as high type buyer has higher cost to serve. The type transition affects seller's payoffs not only directly through the expected arrival of continuation payoff ($\frac{\lambda}{\lambda+r}\Pi_O$), but also indirectly through how fast the seller screens the high type buyer (K_1). We can further solve the game and construct the equilibrium mapping from continuation payoffs to ex ante expected payoffs of both buyer and seller under the flexible pricing equilibrium. Specifically, the mapping Φ from continuation payoffs $\{\Pi_O, U_O^H, U_O^L\}$ to ex-ante expected payoff can be described as

$$\begin{aligned}
\Phi^S\left(\Pi_O, U_O^H, U_O^L\right) &= \begin{cases} \tilde{V}^L - \beta(z_0)C^H & \text{if } z_0 \le \hat{\zeta}, \\ K_1 \frac{e^{az_0}}{1+e^{z_0}} + \frac{\lambda}{\lambda+r}\Pi_O & \text{if } z_0 > \hat{\zeta} \end{cases} \\
\Phi^H\left(\Pi_O, U_O^H, U_O^L\right) &= \begin{cases} V^H - \tilde{V}^L & \text{if } z_0 \le \hat{\zeta}, \\ V^H - C^H - K_1 a e^{(a-1)z_0} - \frac{\lambda}{\lambda+r}\Pi_O & \text{if } z_0 > \hat{\zeta}, \end{cases} \\
\Phi^L\left(\Pi_O, U_O^H, U_O^L\right) &= V^L - \tilde{V}^L.
\end{aligned} \tag{3}$$

With the seller making all the offers and grabbing all the bargaining power, the low type buyer loses "all" the possible surplus and just earns the value of the "outside option", i.e. the expected discounted continuation payoff. He will either wait for the lowest price offer to be made, or for the Poisson event to arrive. Therefore, his payoff is not affected by the bargaining process or prior belief z_0 and only depends on his own continuation value U_O^L. When the prior belief is low enough, the seller and high type buyer's payoffs also just depend on low type buyer's continuation value. The seller finds it optimal to make the deal with the current buyer than to wait and learn more about buyer's type with the risk of type transition. The offer she makes needs to make the low type also willing to accept in order to make it credible. When the prior belief is relatively high, however, the seller would find it worthwhile to learn and screen high types gradually with higher expected cost. The continuation value of the seller would push up her equilibrium payoff at each belief and have a level effect on both seller and high type buyer's payoffs. Both low type buyer and seller's continuation value also affect the value of waiting for the news and hence the slope of seller and high type buyer's payoff functions. The results can be summarized by the following proposition.

PROPOSITION 1. *In the flexible pricing equilibrium mapping Φ, given a set of continuation payoffs (Π_O, U_O^H, U_O^L),*

1. *when prior belief $\beta_0 \leq \hat{\beta}$, both seller and buyer's payoffs only depend on U_O^L;*
2. *when prior belief $\beta_0 > \hat{\beta}$, the seller and high type buyer's payoffs depend on both Π_O and U_O^L while low type buyer's payoff only depends on U_O^L.*

When the belief is low and the seller's optimal strategy is to trade immediately, she would exploit the low type buyer and offer the price that gives him value of "outside option". Both type buyers would accept immediately, so the binding factor in determining their payoffs is low type's continuation payoff. When the belief is high, low type buyer's situation does not change, being left with his "outside option". Seller and high type buyer, however, are also affected by seller's continuation payoff. The anticipation of both seller and low type buyer's continuation payoffs affect the speed of initial screening process through their effect on the option value of waiting. Seller's offer, taking into account high type buyer's response, needs to make herself indifferent between trading sooner or faster. Therefore, the high type buyer's continuation does not really affect the bargaining payoffs as long as it is not too high.

4.2 Fixed Price Equilibrium

In the fixed pricing framework where this single price is acceptable to both types of consumers, the seller would optimally choose the highest such price possible to maximize profit. The seller can use the collected information to select when (and whether) she wants to make the offer. As there is only a single price and this price is better for both types of consumers than simply waiting for transition, trade happens immediately once the seller makes the offer. Therefore the fixed price game reduces to a stopping problem for the seller with the additional component of type transition. Her belief about the buyer type evolves over time based on news process and she has the option to make the trade at the fixed price $\tilde{V}^L$ at any time. Without a history of buyer's rejection decisions, the seller's belief process can be described by $\hat{Z}_t$ alone. For the seller's problem, without the type transition consideration, she trades off making deals sooner against waiting for more precise information about the buyer. The seller would benefit from low belief so that she is more confident that the buyer is of low type and the expected cost for the seller is low. With type transition, however, the value of acquired information is higher when the belief is low as the uncertain prospect of type transition would make the currently acquired information useless. On the hand, when the seller is more "pesimisitic" that the buyer is more likely to be high type with higher expected selling cost, the seller is less willing to give the offer as the type transition brings a chance of trading with a more promising partner. Hence, she chooses a stopping time T to solve

$$\sup_t E[e^{-(\lambda+r)T}(\tilde{V}^L - \beta(\hat{Z}_T)C^H)] + (1 - e^{-(\lambda+r)T})\frac{\lambda}{\lambda+r}\Pi_O.$$

The solution to this optimal stopping problem is a simply threshold strategy: stop the first time that the belief process $\hat{Z}$ drifts weakly below some threshold ζ^f. For any $z \leq \zeta^f$, the seller's value is $\Pi_F(z) = \tilde{V}^L - \beta(z)C^H$. For $z > \zeta^f$, $\hat{Z}$ evolves according to

$$d\hat{Z}_t = \frac{\gamma}{\sigma}\left(dX_t - \frac{\mu^H + \mu^L}{2}dt\right), \tag{4}$$

which differs from the belief updating under flexible pricing equilibrium, without the additional updating from buyer's rejection decisions. The seller's value function is described by the following differential equation

$$r\Pi_F(z) = \underbrace{\lambda(\Pi_O - \Pi_F(z))}_{\text{Evolution due to type transition}} + \underbrace{\frac{\gamma^2}{2}(2\beta(z)-1)\Pi_F'(z) + \frac{\gamma^2}{2}\Pi_F''(z)}_{\text{Evolution due to news}}, \tag{5}$$

which is identical to the evolution of seller's value function as under flexible pricing framework with the value of screening being 0 in equilibrium (see equation 2). Naturally we arrive at the value function of the seller, given by

$$\Pi_F(z) = K_2\frac{e^{az}}{1+e^z} + \frac{\lambda}{\lambda+r}\Pi_O,$$

with $K_2 = K_1$. The optimal stopping strategy for the seller is to offer $\tilde{V}^L$ once the belief drifts weakly below $\zeta_f = \hat{\zeta}$. Given the seller's stopping strategy and news process, we can further pin down the value functions for both type buyers and construct the mapping from continuation payoffs as

$$\begin{aligned}
\Psi^S\left(\Pi_O, U_O^H, U_O^L\right) &= \begin{cases} \tilde{V}^L - \beta(z)C^H & \text{if } z_0 \leq \zeta_f, \\ K_1\frac{e^{az_0}}{1+e^{z_0}} + \frac{\lambda}{\lambda+r}\Pi_O & \text{if } z_0 > \zeta_f, \end{cases} \\
\Psi^H\left(\Pi_O, U_O^H, U_O^L\right) &= \begin{cases} V^H - \tilde{V}^L & \text{if } z_0 \leq \zeta_f, \\ \frac{\lambda}{\lambda+r}U_O^H + e^{a(z_0-\zeta^f)}\left(V^H - V^L + \frac{\lambda}{\lambda+r}(U_O^L - U_O^H)\right) & \text{if } z_0 > \zeta_f, \end{cases} \\
\Psi^L\left(\Pi_O, U_O^H, U_O^L\right) &= \frac{\lambda}{\lambda+r}U_O^L.
\end{aligned} \tag{6}$$

PROPOSITION 2. *In the fixed pricing equilibrium mapping Ψ, given a set of continuation payoffs $\{\Pi_O, U_O^H, U_O^L\}$,*

1. *when prior belief $\beta_0 \leq \hat{\beta}$, both seller and buyer's payoffs only depend on U_O^L;*

2. *when prior belief $\beta_0 > \hat{\beta}$, the seller payoffs depends on both Π_O and U_O^L, high type buyer's payoff depends on all three continuation payoffs, and low type buyer's payoff only depends on U_O^L.*

When the belief is low enough, there is no delay in the bargaining and the seller gets the advantage of making the offer, so the continuation payoffs only affect seller and buyer through the price that depends only on the low type buyer's "outside option". When the belief is high that waiting to learn about the buyer is better for the seller, each player's own continuation payoff starts to kick in through the uncertain arrival of type transition. The low type buyer is still left with his "outside option", with all remaining surplus being extracted. For seller and high type buyer, however, their payoffs are affected by the learning process from the news. The value of accrued information versus further learning is again affected by the uncertain prospect of type transition, and therefore seller's continuation payoff Π_O would affect both seller and high type buyer's current payoffs through the impact on learning process. Finally, high type is no longer given the option to accept early offers and play mixed strategies, so his payoff is not governed by the indifference condition of rejecting and waiting for lowest price offer. As a result, his own continuation payoff U_O^H would affect his expected payoff through the type transition event.

Now we can compare the flexible price and fixed price equilibrium mappings to see how continuation payoffs affect ex-ante expected payoffs differently. Given the same continuation payoffs upon type transition, the seller and low type buyer enjoy the same payoff under flexible pricing and fixed pricing. A key factor in shaping the result is that the lowest price under flexible price equilibrium is the same as the optimal fixed price that both type consumers would accept under fixed price equilibrium. This price, $\tilde{V}^L$ makes the low type buyer indifferent between immediate acceptance and waiting for type transition, which leaves him the value of "outside option" under all beliefs. Therefore, the low type would earn the same payoff. For the seller, she can make a deal with the buyer sooner with screening of initially higher price offers under flexible pricing. However, the seller earns no benefit from such gradual screening due to the competition with future self and earns the same payoff as under fixed pricing, which is a manifestation of Coasian force (Daley and Green, 2020). On the other hand, this gradual screening leaves all the experimentation surplus to the high type buyer. Along the screening process, the high type buyer is indifferent between accepting a earlier higher price and waiting for lower prices with a chance of type transition. In comparison with the fixed price framework, however, the learning process is speed up by the screening and the lowest price offer $\tilde{V}^L$ appear earlier. Hence, with HSC assumption that the outsider option of high type is not too high, the high type buyer is strictly better off when there is delay in the bargaining process.

In general, for the screening process under flexible pricing, the seller faces the tradeoff between trading with the buyer sooner at a higher price and waiting for more information to reveal about the buyer type. The value of such option to wait and learn is deteriorated by the potential arrival of type transition before making the trade in the future. As the seller gets more confident that the buyer is of low type so that the expected selling is low, she also faces an additional risk of type transition that may end the current negotiation process and hence destroy the value of accrued information. Such threat is also more pressing as belief drifts down and information becomes more valuable, which would speed up the screening process of the seller. Under fixed pricing, the seller does not have the experimentation option, but still faces the problem of deterioration of information with further learning when type transition exists.

5 Endogenous Type Transition

The analysis so far shows that the Poisson event arrival with exogenous continuation payoffs leaves the uninformed seller indifferent between negotiating prices based on the news and rejection decisions and selecting the buyer with a fixed price. A key factor for this indifference result is that negotiation is resolved immediately upon the arrival of type transition or the exogenous event. Effectively, it is as if there is no delay in the continuation bargaining game upon transition. With the same termination payoffs exogenously given, both the seller and the low type buyer would have the same expected discounted payoffs under flexible and fixed pricing and the details of continuation bargaining process following Poisson event would not matter. When we specifically model the type transition as endogenous, however, the continuation payoffs come from the bargaining game with the new type. For simplicity, we assume the extreme case that the new buyer type is independent of the old one, drawn according to the prior probability β_0.[4] We will argue that the results also holds for the intermediate case where the new type is positively correlated with the old one.

Specifically, assume that the asset type is not fixed. With Poisson arrival rate $\lambda > 0$, a new type is drawn. Both seller and buyer know if a new type is drawn, but only the buyer observes the new type. The seller assigns probability β_0 to $\theta = H$ each time the new type is drawn.

5.1 Flexible Price Equilibrium

First we analyze the flexible price equilibrium with type transition. Whenever transition occurs, the public belief jumps back to prior β_0, so the continuation payoff will be the ex-ante expected payoff of the new bargaining process for all players. The players' continuation payoffs $\{\Pi_O, U_O^H, U_O^L\}$ are determined by their ex-ante expected payoffs of each new bargaining process, so the whole game can be solved using a fixed point algorithm. Specifically, the equilibrium solution will be a fixed point $\{\Pi_O, U_O^H, U_O^L\}$ to the system of functions

$$\begin{aligned} \Pi_O &= \Phi^S(\Pi_O, U_O^H, U_O^L), \\ U_O^H = U_O^L &= \beta_0 \Phi^H(\Pi_O, U_O^H, U_O^L)) + (1-\beta_0)\Phi^L(\Pi_O, U_O^H, U_O^L). \end{aligned} \tag{7}$$

[4]The no type transition or fixed type case can be thought of as the other extreme case where the new type is the same as the old type.

The optimal threshold chosen by the seller is determined by the continuation payoffs which in turn depend on prior belief z_0. Hereafter, we will write $\hat{\zeta}(z_0)$ as the optimal threshold when prior belief is z_0.

When the prior is low such that $z_0 \leq \hat{\zeta}(z_0)$, there is no delay in bargaining. The seller offers the highest acceptable price (for the low type) to the buyer, which will be accepted immediately. The lowest equilibrium price is given by

$$\tilde{V}^L = V^L - \frac{\lambda}{r}\beta_0\left(V^H - V^L\right).$$

The seller needs to compensate the low type buyer for the "outside option", i.e. waiting for the type to change, with expected discounted value of $\frac{\lambda}{r}\beta_0\left(V^H - V^L\right)$. The continuation payoffs are given by

$$\Pi_O = V^L - \beta_0\left(C^H + \frac{\lambda}{r}(V^H - V^L)\right),$$
$$U_O^H = U_O^L = \frac{\lambda + r}{r}\beta_0\left(V^H - V^L\right).$$

In this case, the endogenous continuation payoffs will act as if the exogenous ones. Both parties receive the payoffs right upon the Poisson arrival and are not affected by the news and bargaining process. Their payoffs, however, are affected by type transition as the acceptable price for the low type buyer depends on how fast he expects the type change may come. The seller will exploit the buyer with the bargaining power from making offers. Therefore, high and low type buyer's payoffs are affected by their own and each other's valuations V^H and V^L due to probabilistic transition into the other type, not by the seller's payoff. Specifically, such probabilistic transition into high type gives the low type buyer leverage to ask for and earn a positive surplus with a positive expected continuation payoff. Seller's payoff is also affected by the conceding in surplus due to type transition.

When the prior is high such that $z_0 > \hat{\zeta}(z_0)$, there will be delay in the continuation bargaining game. The lowest equilibrium price satisfies

$$\tilde{V}^L < V^L - \frac{\lambda}{r}\beta_0\left(V^H - V^L\right).$$

When the prior belief is high, the low type buyer finds to more likely to become a high type conditional on type transition, and hence has a more valuable "outside option". Naturally he would ask for a more favorable price. The higher prior also implies the continuation bargaining process would takes longer to resolve, which would speed up the screening of the seller. We find the continuation payoffs as:

$$\Pi_O = \frac{\lambda + r}{r} K_1 \frac{e^{az_0}}{1+e^{z_0}},$$
$$U_O^H = U_O^L = \frac{V^H - C^H - K_1 a e^{(a-1)z_0} - \frac{\lambda}{r} K_1 \frac{e^{az_0}}{1+e^{z_0}}}{1 - \frac{\lambda}{\lambda+r} e^{-z_0}}.$$

With anticipation of delay in the continuation bargaining game, the low type buyer now is also influenced by the speed of the screening process in the future. With probabilistic transition into high type, the current low type buyer may also benefit from the experimentation offers by the seller in the future. As the future screening process of the seller is affected not only by low type's acceptable price (which is again subject to anticipation of future type transition) but also by her own expected selling cost, so are the payoffs of both types of buyers.

Combining the results for both low and high prior belief, we arrive at the following proposition.

PROPOSITION 3. *In comparison with the fixed buyer type bargaining game (no type transition), the seller is worse off with type transition while both types of buyers are better off.*

Intuitively, the type transition gives the low type a chance to become a high type and gain more surplus. This creates an endogenous positive continuation payoff conditional on type transition, and grants low type more leverage to ask for a lower price. As a result, the seller needs to concede more surplus even with all the bargaining power. In addition, when the prior belief is high such that there is a expected delay whenever type transition causes the bargaining game to restart, the seller has more incentives to speed up the screening process. When the beliefs drifts down and the seller is more confident that the buyer is of low type, the threat of potential type transition and prolonged new bargaining process following that would make the acquired information useless and puts pressure on the seller to make a earlier deal. This speed-up of screening also benefits the buyer.

5.2 Fixed Price Equilibrium

Now we discuss the fixed price equilibrium under type transition. The fixed price framework has two different interpretations under type transition. The optimal fixed price is still set at the highest fixed price that the low type buyer will accept immediately. Under the first interpretation, once a seller makes an offer, it stays there regardless of whether type transition occurs. Under the second one, upon transition, seller can revoke the offer and deliberate before making the same offer. These two interpretations have no difference under exogenous termination as the continuation payoffs are realized immediately upon Poisson arrival with no delay or future interactions. They show no difference in the equilibrium seller behavior either. However, the different off-path considerations under these two interpretations would make the acceptable fixed price (denoted by $\bar{P}$ under endogenous type transition) different. When there is a difference, we will use $\bar{P}^1$ to denote the fixed price under first interpretation and $\bar{P}^2$ to denote the second one. We will discuss and compare both of them.

When the prior is low such that $z_0 \leq \hat{\zeta}(z_0)$, the seller would not delay in making the offer $\bar{P}_1$ to the buyer upon transition. The seller offers such price offer $\bar{P}_1$, which will be accepted by both types of buyers immediately. The two interpretations have no difference. This fixed price $\bar{P}$ is pinned down by the condition that low type buyer is indifferent between accepting immediately and waiting for transition to high type to accept the same offer. The high type arrives with Poisson rate $\lambda\beta_0$, so the price offer is given by

$$\begin{aligned} V^L - \bar{P} &= \int_0^\infty e^{-rt}\lambda\beta_0 e^{-\lambda\beta_0 t}\left(V^H - \bar{P}\right)dt \\ \Rightarrow \bar{P} &= P^0 \equiv V^L - \frac{\lambda\beta_0}{r}\left(V^H - V^L\right). \end{aligned}$$

Since $V^L - \bar{P} = U_O^L$ as under the flexible pricing, the compensation to the low type buyer for his outside option is the same. Therefore the equivalence result of seller and low type buyer's payoffs still holds between the flexible price equilibrium and fixed price one.

When the prior is high such that $z_0 > \hat{\zeta}(z_0)$, there is delay for the seller to learn and make the fixed price offer upon transition. Under the first interpretation, the fixed price offer is still equal to $\bar{P}^1 = P^0$ as the offer is still in place after transition and the low type buyer faces the same tradeoff. The ex-ante expected payoffs can simply be derived from the fixed price mapping Ψ by replacing $U_O^H = U_O^L = \beta_0 V^H + (1-\beta_0)V^L - \bar{P}^1$.

Under the second interpretation, this fixed price $\bar{P}^2$ is determined by the condition that the low type buyer is willing to accept immediately rather than waiting for transition to high type and waiting for the same offer. The key difference is the fixed price will not be offered immediately following type transition. Equivalently, the low type buyer is indifferent between accepting immediately and waiting to transition earning the ex-ante expected payoff. The price $\bar{P}_2$ depends on the fixed point argument as the off-path continuation stopping problem determines low type's highest acceptable price.

$$V^L - \bar{P}^2 = \Psi^L(\Pi_O, U_O^H, U_O^L)$$

Comparing these two fixed prices with the lowest equilibrium price $\tilde{V}^L$ under flexible pricing, we have $\bar{P}^1 < \tilde{V}^L < \bar{P}^2$. The first inequality means that the fixed price $\bar{P}_1$ has to be set lower than the equilibrium lowest price under flexible pricing. Intuitively, by giving up the revoking option of the fixed price offer to the buyer, the seller cannot leverage on the information of type transition itself. Instead, the low type buyer can now make use of the option to wait for transition to high type after receiving a price offer, and hence demands a more favorable price. The fixed price $\bar{P}^1 = P^0$ is related to our results under private transition where the seller cannot observe the transition event itself and therefore cannot leverage on it. The second inequality means that the fixed price $\bar{P}_2$ can be set higher than the equilibrium lowest price under flexible pricing. Intuitively, by keeping the revoking option of the fixed price offer, the seller retains the option to learn in case of type transition. As shown in the exogenous termination section, the fixed pricing game takes longer to resolve, so the expected discounted payoff from the continuation game is lower for the buyer. Therefore, the low type buyer can be compensated less for his outside option and the fixed price can be set higher. Next, we can further compare the welfare of the seller and both types of buyers.

THEOREM 2. *Under endogenous type transition, in comparison with the flexible price equilibrium,*

1. *If $z_0 \leq \hat{\zeta}(z_0)$ (for low prior beliefs), the seller and low type buyer earns the same payoff under two fixed price equilibria, while the high type buyer earns a lower payoff.*
2. *If $z_0 > \hat{\zeta}(z_0)$ (for high prior beliefs), the seller earns a lower payoff under the first fixed price equilibrium, and a higher payoff under the second one. The opposite holds for the low type buyer. The high type buyer earns a lower payoff under the second fixed price equilibrium.*

When the prior belief is low, the payoffs are realized immediately when transition occurs, so the two fixed prices are the same as the equilibrium lowest price under flexible pricing. As a result, the comparison of the ex-ante payoffs just follow the comparison of the flexible and fixed pricing mappings. When the belief is high, however, the difference in high type buyer's ex-ante expected payoff imply different continuation payoffs of both type buyers, which will affect the fixed point. We can use the equilibrium fixed point as the continuation payoff and apply them to the fixed price benchmarks, but such a fixed price will not be accepted by the low type under the first interpretation and is not the highest (optimal) one under the second interpretation.

The second fixed price equilibrium is to provide better contrast with the flexible pricing under endogenous type transition. In both cases, the seller can observe and act on the mere knowledge of whether the type transition occurs. The welfare improvement for the seller under fixed pricing comes from the worse outcome from the continuation game. For each bargaining phase with exogenous termination, the high type buyer is worse off under fixed prices without the experimentation offers from the seller. From the low type buyer's perspective, this lowers the expected continuation payoff he may get from type transition. Therefore in the current bargaining stage, the low type buyer would ask for less surplus and the seller is better off. In a sense, the seller benefits from "committing" to continue using the fixed price in the continuation game. Therefore, even if the new type is not an independently drawn according to the prior probability, but is instead positively correlated with the old type, we can still expect such effect to exist and that the seller benefits from using the collected information to select the buyer than adjusting prices.

The first fixed price equilibrium, as we mentioned, is linked to the private transition case. The seller is unable to either act on or observe the transition event itself. In both cases, the fixed price or the lowest equilibrium price are the same as P^0. The seller is worse off in these two cases compared with the flexible pricing equilibrium. This highlights the value of information about whether type changes or not.

5.3 Uninformative News

To understand how the news process and type transition shape the equilibrium, consider the case with type transition alone. We analyze the case by making the news completely uninformative ($\gamma \to 0$).

PROPOSITION 4. *Take $\gamma \to 0$, two fixed price interpretations will be the same. In addition, in comparison with flexible pricing equilibrium,*

1. *For high prior belief β_0, the seller and both buyer types earn the same payoff under fixed pricing.*
2. *For low prior belief β_0, the seller and low type buyer earn the same payoff under fixed pricing, while the high type buyer is worse off.*

It is straightforward that the two fixed price interpretations converge to the same one, as the seller does not learn anything without news nor experimentation offers. Therefore, whether she can revoke the offer upon type transition does not matter as each new bargaining process is the same with the same prior belief. As usual, with optimistic belief that the buyer is of low type, the seller is happy to offer something attractive to both types, so in equilibrium the future transition never takes place and flexible and fixed pricing give the seller the same payoff. The news process only changes the threshold belief. The more interesting case is when the belief is high such that delay occurs in equilibrium. Without exogenous news, seller can only learn through buyer's rejection decisions and screens out the high type using a delay instead of experimentation offers. The only equilibrium belief that ever appears is the prior belief β_0 and a threshold belief at which the seller is indifferent between trading early and delaying for some period. Although the transition itself pushes down the acceptance upper bound for the low type buyer and makes the seller worse off by giving lower price offers, there is no benefit of making a higher offer. The competition among the seller at each instant drives the only interim belief to be her indifferent belief, which gives the her 0 expected payoff. In comparison with fixed price equilibrium where the seller never makes the offer without further learning, the equilibrium trade surplus all goes to the high type buyer, leaving the seller the same 0 payoff.

6 Privately Observed Transition

Now we consider the case where the type transition is privately observed by the buyer. In the public transition case, once the transition occurs, the stale news becomes completely obsolete, so the seller has a strong incentive to take advantage of accrued news prior type transition. With private transition, the seller would use all previous signals to update belief but puts less weight on stale news as time goes on.

The incentive to make use of accrued news becomes weaker. Specifically, for the private transition case, assume that with Poisson arrival rate λ the type of the buyer is drawn according to the prior probability β_0, which is privately observed by the buyer.

First we will pin down the belief process solely based on news without considering the rejection decisions of the seller. Assume that with Poisson arrival rate λ, there is a new draw of the type with prior probability p_0. Both the newly drawn type and the draw event itself are privately observed by the buyer. The evolution of the log-likelihood ratio of the belief can be described as

$$d\hat{Z}_t = \frac{\gamma}{\sigma}\left(dX_t - \frac{\mu^H + \mu^L}{2}dt\right) + \lambda\left(\frac{\beta(z_0)}{\beta(Z_t)} - \frac{1-\beta(z_0)}{1-\beta(Z_t)}\right)dt \tag{8}$$

Different from the public transition case, the increment of the belief process $\hat{Z}_t$ in no longer independent, but instead depends on the state $\hat{Z}_t$ itself. What's more, the evolution features "reversion to the prior" which incorporates the additional updating from the type transition possibility. Therefore, the drift term of the belief process is non-monotone. As the belief becomes more extreme to identify either high or low type, the updating from type transition favors the direction of the prior belief β_0. To analyze the flexible pricing equilibrium, it would be easier if we can first pin down the lowest price the seller may offer and the strategies taken by one of the buyer types. We have a similar lemma as Lemma 1.

LEMMA 2. *The seller would never make an offer lower than P^0, while the low type buyer would never accept an offer higher than P^0.*

This lemma states that the lowest equilibrium price the seller may offer is also the highest price acceptable to the low type buyer. This implies that the equilibrium is similar as before that the seller offers P^0 when the belief is low enough, which will be accepted by both types immediately. When the belief is relatively high, the seller would screen the buyer with experimentation high prices which will only be accepted by the high type buyer with positive density. Along the way the seller would become more confident that the buyer is of low type before she makes the offer P^0. The other observation is that the lowest price P^0 is also the fixed price offer under the first interpretation where the seller cannot make use of the type transition information to revoke her offer. Given the hazard rate of the high type's acceptance as $\tilde{q}(z) \geq 0$, the belief process of the seller has an additional drift term updating based on high type buyer's acceptance decisions:

$$d\hat{Z}_t = \frac{\gamma}{\sigma}\left(dX_t - \frac{\mu^H + \mu^L}{2}dt\right) + \lambda\left(\frac{\beta(z_0)}{\beta(Z_t)} - \frac{1-\beta(z_0)}{1-\beta(Z_t)}\right)dt - q(z_t)dt \tag{9}$$

For beliefs higher than the threshold, the characterization of seller's value function evolution can be divided into several parts. First with a type transition rate of λ, the Poisson event occurs that leaves her the additional benefit of $\frac{\beta_0 - \beta(z)}{\beta(z)(1-\beta(z))}\Pi'(z)$. The difference from the public transition case is that the effect of type transition comes from its effect on belief updating. Therefore, the effect is reflected in the differential equation as its effect through how fast it changes the belief and $\Pi'(z)$. Second, she trades with high type buyer at rate $\beta(z)q(z)$, which leads to a net flow payoff of $P(z) - C^H - \Pi(z)$. Lastly, if the offer is rejected, she earns the option value of waiting for more informative news, where the updated belief has expected drift $\left(\gamma^2/2\right)(2\beta(z)-1) - q(Z_t)$ and volatility γ. The seller's value function can be characterized by the sum of these flow payoffs

$$\begin{aligned} r\Pi(z) =& \lambda \frac{\beta_0 - \beta(z)}{\beta(z)(1-\beta(z))}\Pi'(z) + \frac{\gamma^2}{2}(2\beta(z)-1)\Pi'(z) + \frac{\gamma^2}{2}\Pi''(z) \\ & + q(z)\left(\beta(z)(P(z) - C^H - \Pi(z)) - \Pi'(z)\right) \end{aligned} \tag{10}$$

The zero net benefit of screening result still holds, so the equivalence in seller's payoff between fixed and flexible price also holds. Without explicit observation of the type transition, the belief of the seller changes continuously instead of having discrete jumps and hence restores the "Coasian force". In addition, it can be shown that the seller has a lower expected payoff when we compare the equilibrium payoff with that under public transition. This again emphasizes the value of making use of the type transition information for the seller in negotiation with the informed buyer.

7 Conclusion

In this paper, we analyze the optimal dynamic pricing strategies of a firm when it gradually collects information of a consumer with changing tastes. Specifically, we study a bargaining model with one-sided incomplete information and interdependent values. We consider how gradual learning of the informed party's private type and the transition in the underlying type would affect the bargaining dynamics. In general, we find that the existence of transition benefits the informed seller at the cost of the buyer. The type transition makes the lower type buyer harder to please and pushes the seller to give experimentation offers faster. When the seller can observe and make use of the mere information of whether type transition occurs, she is better off charging a fixed price and using the information to select the buyer than tailoring the price offers to the buyer. Finally, the uninformed seller benefits from knowing when the transition occurs even if each transition is a completely new draw. In a sense, the information of whether the buyer's type has changed is more valuable than the gradual learning of buyer's current type when the seller cannot commit to future frequent offers.

The environment we study in this project is that the firm cannot commit to its future pricing strategies. It would be interesting to analyze another environment where the seller has some commitment power. A closer contrast with this paper would be the case where the seller can only write contracts contingent only on the belief dimension not on the time dimension. More interestingly, would the firm voluntarily commit to no further learning under some beliefs?

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A Proofs and Detailed Steps

Proof of Lemma 1. Suppose the lowest price offer provided by the seller $\underline{P}$ is below $\tilde{V}^L$. Then we claim that $\tilde{V}^L$ or any offer close to it is accepted with probability 1 by both types. Denote $\varepsilon = \tilde{V}^L - \underline{P} > 0$. Consider a time interval with length Δt. As the most favorable offer made by the seller is at most $\underline{P}$, the highest expected payoff possible for type θ if rejecting an offer of $\tilde{V}^L - \varepsilon r \Delta t$ is

$$
\begin{aligned}
& e^{-r\Delta t} e^{-\lambda \Delta t}(V^\theta - \underline{P}) + \int_0^{\Delta t} e^{-rt} U_O^\theta \lambda e^{-\lambda t} dt \\
=& e^{-(\lambda+r)\Delta t}(V^\theta - \underline{P}) + (1 - e^{-(\lambda+r)\Delta t}) \frac{\lambda}{\lambda + r} U_O^\theta \\
\leq & V^\theta - \tilde{V}^L - \varepsilon(1 - e^{-(\lambda+r)\Delta t}) \\
\approx & V^\theta - \tilde{V}^L - \varepsilon(\lambda + r)\Delta t \\
< & V^\theta - \tilde{V}^L - \varepsilon r \Delta t
\end{aligned}
$$

The argument applies when $\Delta t \to 0$. Hence, the seller can raise the price offer and still have her offer accepted with probability 1. □

Proof of Proposition 1. In this proof, we construct the equilibrium mapping under flexible pricing. In a $\Sigma(\zeta, q)$ profile, along the path, the low type buyer plays a pure strategy $\tau^H = T(\zeta)$ whereas the high type mixes. If $t > T(\zeta)$, trade occurs by time t with probability 1. Hence, in (34), the evolution of Z in this event is off-path. Likewise for $\tilde{z}(z, p)$ if $p \leq \tilde{V}^L$ in (40). For $z > \zeta$, only the high type accepts offers, meaning rejection is a (weakly) negative signal that $\theta = H$. Therefore, along the equilibrium path, the seller's belief conditional on rejection Z, has additional downward drift compared to the belief that updates solely based on news $\hat{Z}$. For $t < T(\zeta)$, the low type always rejects ($S_{t^-}^L = 0$), therefore $Q_t = -\ln(1 - S_{t^-}^H)$ and $dQ_t = dS_{t^-}^H / (1 - S_{t^-}^H) = q(Z_t)dt$. Hence, the additional downward drift of Z, relative to $\hat{Z}$, is the hazard rate of the high type's acceptance, $q(z) \geq 0$, with $dZ_t = d\hat{Z}_t - q(Z_t)dt$.

First we characterize the seller's value function. For $z > \zeta$, the seller trades at rate $\beta(z)q(z)$, which leads to a net payoff of $P(z) - C^H - \Pi(z)$. If the high type rejects, the seller receives the discounted expected continuation payoff, and Z evolves according to

$$
dZ_t = \frac{\gamma}{\sigma}\left(dX_t - \frac{\mu^H + \mu^L}{2} dt\right) - q(Z_t)dt,
$$

which has drift $\left(\gamma^2/2\right)(2\beta(z) - 1) - q(Z_t)$ given the seller's information and volatility γ. Therefore the seller's value function satisfies

$$
\begin{aligned}
r\Pi(z) =& \lambda(\Pi_O - \Pi(z)) + q(z)\beta(z)(P(z) - C^H - \Pi(z)) \\
& + \left(\frac{\gamma^2}{2}(2\beta(z) - 1) - q(z)\right)\Pi'(z) + \frac{\gamma^2}{2}\Pi''(z).
\end{aligned}
$$

Collecting the q terms gives

$$
\begin{aligned}
r\Pi(z) = \quad & \underbrace{\lambda(\Pi_O - \Pi(z))}_{\text{Evolution due to type transition}} \quad + \underbrace{\frac{\gamma^2}{2}(2\beta(z)-1)\Pi'(z) + \frac{\gamma^2}{2}\Pi''(z)}_{\text{Evolution due to news}} \\
& + q(z)\underbrace{\left(P(z) - C^H - \Pi(z) - \Pi'(z)\right)}_{\Gamma(z)\equiv\text{Net benefit of screening at z}}.
\end{aligned}
$$

The first two terms on the right-hand side of (2) is the evolution of the seller's value arising from type transition and news as in (5) for the fixed price game. The third term is the additional value she derives from trade with the high type, which is the product of the trade rate, $q(z)$, and the net benefit of this screening, denoted $\Gamma(z)$. The next lemma demonstrates that the result of zero screening benefit in equilibrium still holds as in Daley and Green (2020).

LEMMA 3. *If the proposed strategies and beliefs constitute an equilibrium, the net benefit of screening must be zero at all beliefs below the threshold, i.e.* $\Gamma(z) = 0$ *for all* $z > \zeta$.

Proof of Lemma 3. Suppose that $\Sigma(\beta, q)$ is an equilibrium. By construction, U^H is non-increasing (see Lemma 7) and above β, Π is $\mathcal{C}^2$. (B3) applied to arbitrarily short time τ implies that for all $z > \beta$,

$$
(\lambda + r)\Pi(z) \geq \lambda\Pi_O + \frac{\gamma^2}{2}(2\beta(z)-1)\Pi'(z) + \frac{\gamma^2}{2}\Pi''(z). \tag{11}
$$

That is, if (11) were violated at $z < \beta$, then there exists $\varepsilon > 0$ such that (11) is violated over the interval $(z-\varepsilon, z+\varepsilon)$. Let $\tau_\varepsilon = \inf\left\{t : \hat{Z}_t \notin (z-\varepsilon, z+\varepsilon)\right\}$, then by Dynkin's formula, (B3) is violated as

$$
\begin{aligned}
&\left(1 - e^{-(\lambda+r)\tau_\varepsilon}\right)\frac{\lambda}{\lambda+r}\Pi_O + E_z\left[e^{-(\lambda+r)\tau_\varepsilon}\Pi(\hat{Z}_{\tau_\varepsilon})\right] \\
=&\Pi(z) + E_z\left[\int_0^{\tau_\varepsilon} e^{-(\lambda+r)s}\left((\mathcal{A} - \lambda - r)\Pi(\hat{Z}_s) + \lambda\Pi_O\right)ds\right] > \quad \Pi(z),
\end{aligned}
$$

where $\mathcal{A}$ is the characteristic operator of $\hat{Z}$ under $\mathcal{Q}$ that $\mathcal{A}\Pi(z) = \frac{\gamma^2}{2}(2\beta(z)-1)\Pi'(z) + \frac{\gamma^2}{2}\Pi''(z)$. Notice that combining (11) with (2) implies that $q(z)\Gamma(z) \geq 0$ for all $z < \beta$.

Next, suppose that $\Gamma(z) > 0$ for some $z > \beta$. In $\Sigma(\beta, q)$, the belief does not jump, so $\Pi(z) = J(z,z)$. Observing that $\Gamma(z) = J_2(z,z)$, it follows that for small enough $\varepsilon > 0$, $J(z, z+\varepsilon) > J(z,z) = \Pi(z)$, violating (B2). Therefore, if $\Sigma(\beta, q)$ is an equilibrium, then $q(z)\Gamma(z) = 0$ for all $z < \beta$. Hence, at all $z < \beta$, (2) reduces to (12), and for any β, Π must have the form given by (15).

At the threshold β, $\Pi(\beta) = \tilde{V}^L - \beta(z)C^H$, $\Pi'(\beta) = -\beta'(z)C^H$, and $U^H(\beta) = V^H - \tilde{V}^L$. As $\beta'(z)C^H = (1-\beta(z))C^H$, we can get $\Gamma(\beta) = 0$. For an arbitrary q on $z < \beta$, let $G_L^q(z)$ be the expected payoff of a low type who rejects all offers until $Z_t \geq \beta$ (i.e., $G_L^q(z) = \frac{\lambda}{\lambda+r}U_O^L + E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[C^H + \frac{\lambda}{\lambda+r}(U_O^H - U_O^L)\right]$). Let q^* denote the expression for q given in (19) and Z^* be the belief process that is consistent with q^*.

By construction, for all $z < \beta$,

$$(1-\beta(z))(V^L - G_L^{q^*}(z) - \Pi(z)) + \Pi'(z) = 0.$$

Above we established that $\Gamma(z) \le 0$ and $q(z)\Gamma(z) = 0$ for all $z < \beta$. Suppose there exists a $\Sigma(\beta,q)$-equilibrium, with $z_0 < \beta$ and $\Gamma(z_0) < 0$. By continuity of $G_L^q(= U^L)$, W and Π', there exists an open interval around z_0 on which $\Gamma < 0$. Let I be the union of all such intervals $T^c \equiv (-\infty, \beta] \setminus I$. To satisfy $q(z)\Gamma(z) = 0$, then $q(z) = 0$ on I. Given W from (15), $\Gamma(z) = 0$ on I^c implies U^L on I^c must be equal to P as given by (16), and further that $q = q^*$ on the interior of I^c. Hence, $q(z) \le q(z^*)$ for almost all $z < \beta$. Therefore, starting from any $Z_0 = Z_0^* = z \le \beta$, $Z_t \le Z_t^*$ for all $t \le \inf\{s : Z_s^* \ge \beta\}$. It follows that $U^L(z) = G_L^q(z) \le G_L^{q^*}(z)$, which then implies

$$\Gamma(z) = (1-\beta(z))(V^L - U^L(z) - \Pi(z)) + \Pi'(z) \ge 0,$$

which gives a contradiction.Hence, if $\Sigma(\beta,q)$ is an equilibrium, $\Gamma(z) = 0$ for all $z < \beta$. □

At all $z < \zeta$, (2) reduces to

$$r\Pi(z) = \lambda(\Pi_O - \Pi(z)) + \frac{\gamma^2}{2}(2\beta(z)-1)\Pi'(z) + \frac{\gamma^2}{2}\Pi''(z), \tag{12}$$

Any solution to the ODE has the form

$$\Pi(z) = K_1 \frac{e^{az}}{1+e^z} + K_2^* \frac{e^{u_2 z}}{1+e^z} + \frac{\lambda}{\lambda+r}\Pi_O, \tag{13}$$

where $(a, u_2) = \frac{1}{2}\left(1 \pm \sqrt{1+8(\lambda+r)/\gamma^2}\right)$ and K_1, K_2^* are constants. For any candidate threshold ζ, these constants are pinned down by the observation that (i) the buyer's payoff is uniformly bounded between 0 and $V^H + \Pi_O$ (which implies $K_2^* = 0$), (ii) $\Pi(\zeta) = V(\zeta) - \tilde{V}^L$ (which pins down $K_1 = (1+e^\zeta)\left(V(\zeta) - C^H - \kappa^L\right)e^{-a\zeta}$, as a function of ζ). It follows that the buyer's payoff under an arbitrary threshold policy ζ is

$$\Pi(z|\zeta) = (1+e^\zeta)\left(V(\zeta) - C^H - \kappa^L\right)e^{-a\zeta}\frac{e^{az}}{1+e^z} + \frac{\lambda}{\lambda+r}\Pi_O.$$

Buyer takes the termination payoff Π_O as given and chooses the optimal threshold that maximizes $\Pi(z|\zeta)$ over all possible values of ζ. The first order condition $\frac{d}{d\zeta}\Pi(z|\zeta) = 0$ gives

$$e^\zeta(a-1) - a\frac{(C^H + \kappa^L) - V^L}{V^H - (C^H + \kappa^L)} = 0, \tag{14}$$

and has a unique real solution

$$\hat{\zeta} = \ln\left(\frac{(C^H + \kappa^L) - V^L}{V^H - (C^H + \kappa^L)}\right) + \ln\left(\frac{a}{a-1}\right),$$

which gives $K_1 = \left(\frac{(C^H+\kappa^L)-V^L}{a-1}\right)^{1-a}\left(\frac{V^H-(C^H+\kappa^L)}{a}\right)^{a}$.

Given $\hat{\zeta}$ and K_1, the buyer's value function is given by

$$\Pi(z) = \begin{cases} V(z) - \tilde{V}^L & \text{if } z \geq \hat{\zeta} \\ K_1 \frac{e^{az}}{1+e^z} + \frac{\lambda}{\lambda+r}\Pi_O & \text{if } z < \hat{\zeta}, \end{cases} \tag{15}$$

The offers are pinned down by $\Gamma(z) = 0$, which gives the offer process and type-L seller's value function

$$\begin{aligned} U^L(z) = P(z) &= V^L - \Pi(z) + \frac{\Pi'(z)}{1-\beta(z)} \\ &= \begin{cases} \tilde{V}^L & \text{if } z \geq \hat{\zeta} \\ V^L + K_1(a-1)e^{az} - \frac{\lambda}{\lambda+r}\Pi_O & \text{if } z < \hat{\zeta}, \end{cases} \end{aligned} \tag{16}$$

whereas type-H seller's value function is given by

$$\begin{aligned} U^H(z) &= E_z^H\left[e^{-(\lambda+r)T(\zeta)}(\tilde{V}^L - C^H) + (1 - e^{-(\lambda+r)T(\zeta)})\frac{\lambda}{\lambda+r}U_O^H\right] \\ &= \frac{\lambda}{\lambda+r}U_O^H + E_z^H\left[e^{-(\lambda+r)T(\zeta)}(\tilde{V}^L - C^H - \frac{\lambda}{\lambda+r}U_O^H)\right] \\ &= \frac{\lambda}{\lambda+r}U_O^H, \end{aligned} \tag{17}$$

which is independent of z. This is captured by the observation that the upper bound of the price offer is the lowest price high type seller would accept and that the upper bound is set to make the high type indifferent between accepting it and waiting for the news. Hence, the expected payoff of the high type seller is equal to the discounted ex-ante seller expected payoff.

Now we determine the rate of trade $q(z)$ for all $z < \zeta$. For type-L seller, Z evolves according to

$$dZ_t = \left(q(Z_t) - \frac{\gamma^2}{2}\right)dt + \gamma dB_t,$$

so type-L seller's value function satisfies

$$(\lambda + r)U^L(z) = \lambda U_O^L + \left(q(z) - \frac{\gamma^2}{2}\right)U^{H'}(z) + \frac{\gamma^2}{2}U^{H''}(z). \tag{18}$$

Hence we have

$$
\begin{aligned}
q(z) &= \frac{(\lambda+r)U^L(z) - \lambda U_O^L + \frac{\gamma^2}{2}U^{H'}(z) - \frac{\gamma^2}{2}U^{H''}(z)}{U^{H'}(z)} \\
&= \frac{\gamma^2}{2C_1^*}e^{-az}\left(V^L - \frac{\lambda}{\lambda+r}(\Pi_O + U_O^L)\right)
\end{aligned} \tag{19}
$$

The final part of the characterization is a description of the seller's reaction to off-path offers in arbitrary state z. The type-θ seller accepts an offer $\hat{p} \neq P(z)$ in state z with probability $s^\theta(z,\hat{p})$, where

$$
\begin{aligned}
s^H(z,\hat{p}) &= s^L(z,\hat{p}) = 0, \quad \text{if } \hat{p} < P(z); \\
s^H(z,\hat{p}) &= s^L(z,\hat{p}) = 1, \quad \text{if } \hat{p} \geq \tilde{V}^L.
\end{aligned} \tag{20}
$$

For $z < \zeta$ and $\hat{p} \in (P(z), \tilde{V}^L)$, then $s^H(z,\hat{p}) = 0$ and

$$
s^L(z,\hat{p}) = 1 - e^z \left[\frac{\hat{p} + \frac{\lambda}{\lambda+r}\Pi_O - V^L}{K_1(a-1)}\right]^{-\frac{1}{a}}. \tag{21}
$$

A Bayesian consistent belief starting from z given a rejection of such an offer $\hat{p}$ is therefore (from (R2))

$$
\hat{z} = \frac{1}{a}\ln\left[\frac{\hat{p} + \frac{\lambda}{\lambda+r}\Pi_O - V^L}{K_1(a-1)}\right] = U^{L^{-1}}(\hat{p}). \tag{22}
$$

□

That is, $\hat{z}$ is the unique solution to $U^L(z) = \hat{p}$. Therefore, the low type is indifferent and willing to mix between accepting $\hat{p}$ and rejecting to earn $U^L(\hat{z})$ in the continuation play. The continuation value includes both bargaining with the current type and expected payoff from exogenous termination.

Proof of Proposition 2. Hence, she chooses a stopping time T to solve

$$
\begin{aligned}
&\sup_t \; E[e^{-rT}(V(\hat{Z}_T) - \tilde{V}^L)e^{-\lambda T}] + \int_0^T e^{-rt}\Pi_O \lambda e^{-\lambda t}dt \\
=&\sup_t \; E[e^{-(\lambda+r)T}(V(\hat{Z}_T) - \tilde{V}^L)] + (1 - e^{-(\lambda+r)T})\frac{\lambda}{\lambda+r}\Pi_O.
\end{aligned} \tag{23}
$$

The solution to the fixed price game is a threshold policy: stop the first time $\hat{Z}$ is weakly above some threshold ζ^f. For any $z \geq \zeta^f$, the buyer's value is $\Pi_F(z) = V(z) - \tilde{V}^L$. For $z < \zeta^f$, $\hat{Z}$ evolves according to

$$
d\hat{Z}_t = \frac{\gamma}{\sigma}\left(dX_t - \frac{\mu^H + \mu^L}{2}dt\right),
$$

and the buyer's value function is given by

$$
\Pi_F(z) \approx e^{-(\lambda+r)dt}E_z\left(\Pi_F(z + d\hat{Z})\right) + (1 - e^{-(\lambda+r)dt})\frac{\lambda}{\lambda+r}\Pi_O,
$$

Applying Ito's lemma to $\Pi_F(\hat{Z}_t)$, we arrive at the following ordinary differential equation (ODE) for the buyer's value function

$$r\Pi_F(z) = \underbrace{\lambda(\Pi_O - \Pi_F(z))}_{\text{Evolution due to type transition}} + \underbrace{\frac{\gamma^2}{2}(2\beta(z)-1)W^{F'}(z) + \frac{\gamma^2}{2}W^{F''}(z)}_{\text{Evolution due to news}},$$

which is identical to the evolution of buyer's value function in equilibrium as in (2).

The solution is given by

$$\Pi_F(z) = K_1^F \frac{e^{az}}{1+e^z} + \frac{\lambda}{\lambda+r}\Pi_O, \tag{24}$$

where $K_1^F = (1+e^{\zeta})\left(V(\zeta) - C^H - \kappa^L\right)e^{-a\zeta}$. It follows that the buyer's payoff under an arbitrary threshold policy ζ is

$$\Pi_F(z|\zeta) = (1+e^{\zeta})\left(V(\zeta) - C^H - \kappa^L\right)e^{-a\zeta}\frac{e^{az}}{1+e^z} + \frac{\lambda}{\lambda+r}\Pi_O.$$

The optimal threshold maximizes $\Pi_F(z|\zeta)$ over all possible values of ζ: $\zeta^f = \ln\frac{(C^H+\kappa^L)-V^L}{V^H-(C^H+\kappa^L)} + \ln\left(\frac{a}{a-1}\right) = \hat{\zeta}$.

LEMMA 4. *The solution to the fixed price game is that the buyer completes the trade if and only if her belief is weakly above* ζ^f.

Proof of Lemma 4. We will verify the solution that $\tau = \inf\{t : \hat{Z} \geq \zeta^f\}$ is optimal using the variational inequalities for optimal stopping (Oksendal 2007, Theorem 10.4.1). By construction, the buyer's value function is twice continuously differentiable (i.e., $\mathcal{C}^2$) almost everywhere. It is clear that conditions (i), (iii), (iv), (v), (vii), (viii) and (ix) in the verification theorem hold. It remains to verify that (ii) $\Pi(z) \geq g(z) \equiv V(z) - C^H$ for all $z \leq \zeta^f$, and that (vi) $[\mathcal{A} - (\lambda+r)]\left[\Pi(z) - \frac{\lambda}{\lambda+r}\Pi_O\right] \leq 0$ for all $z \geq \zeta^f$. For (ii), it's the same as (B1) as we get the same $\Pi(z)$ in equilibrium, the proof will be delegated to proof of Theorem 3. For (vi), for $z \geq \zeta^f > \ln\frac{\tilde{V}^L + \frac{\lambda}{\lambda+r}\Pi_O - V^L}{V^H - \tilde{V}^L - \frac{\lambda}{\lambda+r}\Pi_O}$,

$$\begin{aligned}
[\mathcal{A} - (\lambda+r)]\left[\Pi(z) - \frac{\lambda}{\lambda+r}\Pi_O\right] =& [\mathcal{A} - (\lambda+r)]\left[V(\beta) - \tilde{V}^L - \frac{\lambda}{\lambda+r}\Pi_O\right] \\
=& \frac{\gamma^2}{2}(2\beta(z)-1)V'(z) + \frac{\gamma^2}{2}V''(z) \\
& - (\lambda+r)\left[V(\beta) - \tilde{V}^L - \frac{\lambda}{\lambda+r}\Pi_O\right] \\
=& -(\lambda+r)\left[V(\beta) - \tilde{V}^L - \frac{\lambda}{\lambda+r}\Pi_O\right] \\
<& 0.
\end{aligned}$$

□

The seller's payoff is pinned down by the expected time taken for the updated belief to pass the threshold and expected arrival of termination.

$$U_F^L(z) = \hat{E}_z^H \left[e^{-(\lambda+r)T(\zeta^f)} \left(\tilde{V}^L - C^H\right) + \left(1 - e^{-(\lambda+r)T(\zeta^f)}\right) \frac{\lambda}{\lambda+r} U_O^H \right] = \frac{\lambda}{\lambda+r} U_O^H.$$

$$\begin{aligned} U_F^H(z) &= \hat{E}_z^L \left[e^{-(\lambda+r)T(\zeta^f)} \tilde{V}^L + \left(1 - e^{-(\lambda+r)T(\zeta^f)}\right) \frac{\lambda}{\lambda+r} U_O^L \right] \\ &= \frac{\lambda}{\lambda+r} U_O^L + \hat{E}_z^L \left[e^{-(\lambda+r)T(\zeta^f)} \right] \left(\tilde{V}^L - \frac{\lambda}{\lambda+r} U_O^L \right) \end{aligned}$$

Here the $\hat{E}_z^\theta$ is expectation with respect to the probability law of the process $\hat{Z}$ starting from z and conditional on θ, so we can calculate the low type seller's payoff at arbitrary belief z is

$$U_F^H(z) = \begin{cases} \tilde{V}^L & \text{if } z \geq \zeta^f, \\ \frac{\lambda}{\lambda+r} U_O^L + e^{a(z-\zeta^f)} \left(\tilde{V}^L - \frac{\lambda}{\lambda+r} U_O^L \right) & \text{if } z < \zeta^f \end{cases}$$ □

Proof of Proposition 3. Denote buyer's ex-ante expected payoff as $\Pi_O \equiv \Pi(z_0)$ and type-θ's ex-ante expected payoff as $U_0^\theta \equiv U^\theta(z_0)$. The termination/continuation payoff of the buyer would simply be $\Pi_O = \Pi_O$, whereas the continuation payoff of both type sellers would be $U_O^H = U_O^L = \Pi_0 \equiv \beta_0 U_0^L + (1-\beta_0) U_0^H$. The equilibrium solution will be a fixed point $\{\Pi_O, U_0^L, U_0^H\}$ to the system of functions

$$\begin{aligned} \Pi_O &= \begin{cases} V(z_0) - C^H - \frac{\lambda}{\lambda+r}\Pi_0 & \text{if } z_0 \geq \hat{\zeta}, \\ K_1 \frac{e^{az_0}}{1+e^{z_0}} + \frac{\lambda}{\lambda+r}\Pi_O & \text{if } z_0 < \hat{\zeta} \end{cases} \\ U_0^L &= \frac{\lambda}{\lambda+r}\Pi_0, \\ U_0^H &= \begin{cases} C^H + \frac{\lambda}{\lambda+r}\Pi_0 & \text{if } z_0 \geq \hat{\zeta}, \\ V^L + K_1(a-1)e^{az_0} - \frac{\lambda}{\lambda+r}\Pi_O & \text{if } z_0 < \hat{\zeta} \end{cases} \end{aligned}$$

where $K_1 = \left(\frac{(C^H + \frac{\lambda}{\lambda+r}(\Pi_0+\Pi_O)) - V^L}{a-1} \right)^{1-a} \left(\frac{V^H - (C^H + \frac{\lambda}{\lambda+r}(\Pi_0+\Pi_O))}{a} \right)^a$. As the threshold is determined by Π_O and Π_0 which both depend on prior belief z_0, I will abuse notation a bit and write $\hat{\zeta}(z_0)$ as the optimal threshold when prior belief is z_0.

If $z_0 \geq \hat{\zeta}(z_0)$, there is no delay in bargaining. Buyer offers the lowest acceptable price (for the high type) to seller, which will be accepted immediately.

$$\begin{aligned} \Pi_0 &= \beta_0 \frac{\lambda}{\lambda+r}\Pi_0 + (1-\beta_0)\left[C^H + \frac{\lambda}{\lambda+r}\Pi_0 \right] \\ \Rightarrow \Pi_0 &= \frac{\lambda+r}{r}(1-\beta_0)C^H \end{aligned}$$

$$
\begin{aligned}
\Pi_O &= V(z_0) - \left[1 + \frac{\lambda}{r}(1-\beta_0)\right] C^H \\
\hat{\zeta} &= \ln \frac{C^H - V^L + \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 C^H)}{V^H - C^H - \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 C^H)} + \ln\left(\frac{a}{a-1}\right) \\
K_1 &= \left(\frac{(C^H + \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 C^H)) - V^L}{a-1}\right)^{1-a} \left(\frac{V^H - (C^H + \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 C^H))}{a}\right)^{a}
\end{aligned}
$$

When the prior belief is high enough such that there is no delay in the restart of the bargaining process, the endogenous continuation payoff will act as if the exogenous one. Both parties receive the payoff right upon the Poisson arrival and are not affected by the news and bargaining process. High and low type seller's payoffs are only affect by their own reservation values C^H and C^L, not by the payoff of the buyer. Mathematically, low and high type seller's continuation payoff or ex-ante expected payoff can be found as a fixed point of seller's payoff functions alone. Buyer's payoff is also determined by the fixed point of two type sellers.

If $z_0 < \hat{\zeta}(z_0)$, we can express the ex-ante expected seller payoff as:

$$
\begin{aligned}
\Pi_0 &= \beta_0 \frac{\lambda}{\lambda+r}\Pi_0 + (1-\beta_0)\left[V^L + K_1(a-1)e^{az_0} - \frac{\lambda}{\lambda+r}\Pi_O\right] \\
\Rightarrow \Pi_0 &= \frac{V^L + K_1(a-1)e^{az_0} - \frac{\lambda}{\lambda+r}\Pi_O}{1 + \frac{r}{\lambda+r}e^{z_o}}
\end{aligned}
$$

As Π_0 affects K_1 through $\tilde{V}^L$, we can solve the K_1 from

$$
\begin{aligned}
K_1 = &\left(\frac{(C^H + \frac{\lambda}{\lambda+r(1+e^{z_0})}(V^L + K_1 a e^{az_0}) - V^L}{a-1}\right)^{1-a} \\
&\cdot \left(\frac{V^H - (C^H + \frac{\lambda}{\lambda+r(1+e^{z_0})}(V^L + K_1 a e^{az_0}))}{a}\right)^{a}
\end{aligned} \tag{25}
$$

and with K_1 given by the above expression we can solve

$$
\begin{aligned}
\Pi_O &= \frac{\lambda+r}{r}K_1 \frac{e^{az_0}}{1+e^{z_0}}, \\
\Pi_0 &= \frac{V^L + K_1(a-1)e^{az_0} - \frac{\lambda}{r}K_1\frac{e^{az_0}}{1+e^{z_0}}}{1 + \frac{r}{\lambda+r}e^{z_0}}, \\
\hat{\zeta} &= \ln \frac{C^H - V^L + \frac{\lambda}{\lambda+r(1+e^{z_0})}(V^L + K_1 a e^{az_0})}{V^H - C^H - \frac{\lambda}{\lambda+r(1+e^{z_0})}(V^L + K_1 a e^{az_0})} + \ln\left(\frac{a}{a-1}\right)
\end{aligned} \tag{26}
$$

Mathematically, the fixed point $\{\Pi_O, U_0^H, U_0^L\}$ need to be solved using the three simultaneous equations. Buyer's continuation payoff has a level effect on both buyer and high type's payoff. All three continuation payoffs also affect the slope of the bargaining process.

The rate of trade $q(z)$ for all $z < \zeta$ and seller's reaction to off-path offers $s^{\theta}(z, \hat{p})$ can be derived by replacing the continuation payoff by the endogenous one. $\square$

Proof of Theorem 2. If $z_0 \geq \hat{\zeta}(z_0)$, there is no delay for the buyer to make the offer upon transition. Buyer offers the lowest acceptable price (for the high type) to seller, which will be accepted immediately. The two interpretations have no difference. This lowest price $\bar{P}$ is determined by the condition that high type seller is willing to accept immediately rather than waiting to transition to low type and accepting the same offer.

$$\begin{aligned}\bar{P} - C^H &= \int_0^{\infty} e^{-rt} \lambda (1-\beta_0) e^{-\lambda(1-\beta_0)t} \bar{P} dt \\ \Rightarrow \bar{P} &= \left(1 + \frac{\lambda}{r}(1-\beta_0)\right) C^H.\end{aligned}$$

As $\bar{P} = \Pi_0$, the equivalence result of buyer and high type seller's payoffs still holds between the equilibrium and fixed price benchmark.

If $z_0 < \hat{\zeta}(z_0)$, however, there is delay for the buyer to learn and make the fixed price offer. Under the first interpretation, the fixed price offer is still equal to $\bar{P}^1 = \left(1 + \frac{\lambda}{r}(1-\beta_0)\right) C^H$ as the offer is still in place after transition. The ex-ante expected payoffs can simply be derived from the fixed price mapping by replacing $U_O^H = U_O^L = \bar{P}^1 - \beta_0 C^H$.

Under the second interpretation, this lowest price $\bar{P}^2$ is determined by the condition that high type seller is willing to accept immediately rather than waiting to transition to low type and waiting for the same offer. Equivalently, high type seller is indifferent between accepting immediately and waiting to transition earning low type's ex-ante payoff. The price is also determined a fixed point argument.

$$\bar{P}^2 - C^H = \frac{\lambda}{\lambda + r} \Pi^{F^0}$$

The conditions for Π^{F^0} can be pinned down by

$$\begin{aligned}W^{F^0} &= K_1 \frac{e^{az_0}}{1+e^{z_0}} + \frac{\lambda}{\lambda+r} W^{F^0} \\ U_{F,0}^L &= \frac{\lambda}{\lambda+r} \Pi^{F^0}, \\ U_{F,0}^H &= \frac{\lambda}{\lambda+r} \Pi^{F^0} + e^{a(z_0-\zeta^f)} C^H, \\ \Pi^{F^0} &= \beta_0 U_{F,0}^L + (1-\beta_0) U_{F,0}^H, \\ K_1 &= (1+e^{\zeta}) \left(V(\zeta) - C^H - \frac{\lambda}{\lambda+r} \left(W^{F^0} + \Pi^{F^0} \right) \right) e^{-a\zeta}, \\ \zeta^f &= \ln \frac{\left(C^H + \frac{\lambda}{\lambda+r} \left(W^{F^0} + \Pi^{F^0} \right) \right) - V^L}{V^H - \left(C^H + \frac{\lambda}{\lambda+r} \left(W^{F^0} + \Pi^{F^0} \right) \right)} + \ln \left(\frac{a}{a-1} \right)\end{aligned}$$

$\square$

Proof of Proposition 4. I will start by taking $\gamma \to 0$ and analyzing the equilibrium for any given belief z, and then I will specifically focus on the ex-ante expected payoffs for prior belief z_0. As the (continuation) equilibrium behavior depends on whether the prior belief is above the threshold, the equilibrium discussion will divide into two cases. Take $\gamma \to 0$, $z_0 > \hat{\zeta}(z_0)$ will be reduced to $z_0 > \ln \frac{P^0 - V^L}{V^H - P^0}$.

If the prior belief is such that $z_0 > \ln \frac{P^0 - V^L}{V^H - P^0}$,

1. the threshold $\hat{\zeta}$ goes to $\ln \frac{C^H - V^L + \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 C^H)}{V^H - C^H - \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 C^H)}$;
2. for $z < \hat{\zeta}$, the rate of trade goes to ∞; for $z = \hat{\zeta}$, the rate of trade goes to 0;
3. the buyer's payoff converges uniformly to $\max\{\frac{\lambda}{\lambda+r}(V(z_0) - P^0), V(z) - P^0\}$;
4. the low type's payoff converges pointwise to $V^L - \frac{\lambda}{\lambda+r}(V(z_0) - P^0)$ below $\hat{\zeta}$ and to P^0 above $\hat{\zeta}$;
5. efficiency loss converges to 0 above $\hat{\zeta}$ but remains positive below $\hat{\zeta}$.

If the prior belief is such that $z_0 < \ln \frac{P^0 - V^L}{V^H - P^0}$,

1. the threshold $\hat{\zeta}$ goes to $\ln \frac{C^H + \frac{\lambda(1-P_0)}{\lambda(1-P_0)+r} V^L - V^L}{V^H - C^H - \frac{\lambda(1-P_0)}{\lambda(1-P_0)+r} V^L}$;
2. for $z \leq \hat{\zeta}$, the rate of trade goes to ∞; for $z = \hat{\zeta}$, the rate of trade goes to 0;
3. the buyer's payoff converges uniformly to $\max\{0, V(z) - C^H - \frac{\lambda(1-P_0)}{\lambda(1-P_0)+r} V^L\}$;
4. the low type's payoff converges pointwise to V^L below $\hat{\zeta}$ and to $C^H + \frac{\lambda(1-P_0)}{\lambda(1-P_0)+r} V^L$ above $\hat{\zeta}$;
5. efficiency loss converges to 0 above $\hat{\zeta}$ but remains positive below $\hat{\zeta}$.

What about the fixed price benchmarks? Similarly, the cutoff condition $z_0 > \zeta^f$ will be reduced to $z_0 > \ln \frac{P^0 - V^L}{V^H - P^0}$. If the prior belief is such that $z_0 > \ln \frac{P^0 - V^L}{V^H - P^0}$, two interpretations are the same.

1. the threshold ζ^f goes to $\ln \frac{C^H - V^L + \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 C^H)}{V^H - C^H - \frac{\lambda}{\lambda+r}(V(z_0) - \beta_0 C^H)}$;
2. the buyer's payoff converges uniformly to $\max\{\frac{\lambda}{\lambda+r}(V(z_0) - P^0), V(z) - P^0\}$;
3. the low type's payoff converges pointwise to $\frac{\lambda}{\lambda+r} P^0$ below ζ^f and to P^0 above ζ^f;
4. efficiency loss converges to 0 above ζ^f but remains positive below ζ^f.

If the prior belief is such that $z_0 < \ln \frac{P^0 - V^L}{V^H - P^0}$, under the first interpretation,

1. the threshold ζ^f goes to $\ln \frac{P^0 - V^L}{V^H - P^0}$;
2. the buyer's payoff converges uniformly to $\max\{0, V(z) - P^0\}$;

3. the low type's payoff converges pointwise to 0 below ζ^f and to P^0 above ζ^f;

4. efficiency loss converges to 0 above ζ^f but remains positive below ζ^f.

Under the second interpretation,

1. the threshold ζ^f goes to $\ln \frac{C^H - V^L}{V^H - C^H}$;

2. the buyer's payoff converges uniformly to $\max\{0, V(z) - C^H\}$;

3. the low type's payoff converges pointwise to 0 below ζ^f and to C^H above ζ^f;

4. efficiency loss converges to 0 above ζ^f but remains positive below ζ^f.

Without arrival of exogenous information, the low prior makes the buyer unwilling to offer the price where both types would accept upon transition, so the continuation values for both parties go to 0. This will push down high type's lowest acceptable price and the threshold.

The above analyses consider the equilibrium behavior and payoffs for each belief z and different prior belief z_0. However, some beliefs are not reachable in equilibrium when $\gamma \to 0$. In summary, the limiting equilibrium behavior is as follows:

1. For all $z_0 > \hat{\zeta}(z_0)$, (or $z_0 > \ln \frac{P^0 - V^L}{V^H - P^0}$,) the buyer would offer P^0 and the seller would accept with probability 1. The buyer earns payoff $V(z_0) - P^0$ and the type-θ seller earns payoff $P^0 - C^\theta$.

2. For all $z_0 < \hat{\zeta}(z_0)$, (or $z_0 < \ln \frac{P^0 - V^L}{V^H - P^0}$,), the buyer would offer V^L. The high type rejects and the low type mixes such that the belief is $\hat{\zeta}(z_0)$ following a rejection. The buyer earns 0 expected payoff.

3. Following a reject at $z_0 < \hat{\zeta}(z_0)$, there is delay of length τ satisfying $V^L = \tilde{V}^L - C^H + E[e^{-r\tau}]C^H$, after which the buyer offers $\tilde{V}^L$.

The limiting payoff for buyer and seller under fixed price benchmark are the same when $z_0 > \hat{\zeta}(z_0)$. When $z_0 < \hat{\zeta}(z_0)$, however, the buyer never makes the fixed price offer, so she earns the same 0 payoff as in equilibrium. She does not benefit from the ability of adjusting offers. However, low type also earns 0 payoff as opposed to V^L in equilibrium. □

Proofs of private type transition. First we will pin down the belief process solely based on news without considering the rejection decisions of the seller. Assume that with Poisson arrival rate λ, there is a new draw of the type with prior probability p_0. Both the newly drawn type and the draw event itself are privately observed by the seller. Consider the time period from t to $t + dt$. When the type transition is privately observed, the likelihood the buyer assigns to $\theta_t = H$ versus $\theta_t = L$ follows from

Bayes' rule as

$$
\begin{aligned}
&\frac{\Pr(\theta_{t+dt}=H|dX_t)}{\Pr(\theta_{t+dt}=H|dX_t)}\\
=&\frac{\beta_t[e^{-\lambda dt}+(1-e^{-\lambda dt})\beta_0]e^{-\frac{(dX_t-\mu^H dt)^2}{2\sigma^2 dt}}+\cdots}{(1-\beta_t)[e^{-\lambda dt}+(1-e^{-\lambda dt})(1-\beta_0)]e^{-\frac{(dX_t-\mu^L dt)^2}{2\sigma^2 dt}}+\cdots}\\
&\frac{(1-\beta_t)(1-e^{-\lambda dt})\beta_0\int_0^{dt}\frac{1}{dt}e^{-\frac{(dX_t-\mu^L s-\mu^H(dt-s))^2}{2\sigma^2 dt}}ds}{\beta_t(1-e^{-\lambda dt})(1-\beta_0)\int_0^{dt}\frac{1}{dt}e^{-\frac{(dX_t-\mu^H s-\mu^L(dt-s))^2}{2\sigma^2 dt}}ds}\\
\Longrightarrow d\hat{Z}_t\equiv& d\ln\frac{\Pr(\theta_{t+dt}=H|dX_t)}{\Pr(\theta_{t+dt}=H|dX_t)}\\
\approx&\frac{\gamma}{\sigma}\left(dX_t-\frac{\mu^H+\mu^L}{2}dt\right)+\lambda\left(\frac{\beta_0}{\beta_t}-\frac{1-\beta_0}{1-\beta_t}\right)dt
\end{aligned}
\tag{27}
$$

Solving the buyer's optimal stopping problem under fixed price benchmark, we arrive the the following ODE

$$
r\Pi(z)=\lambda\frac{\beta_0-\beta(z)}{\beta(z)(1-\beta(z))}\Pi'(z)+\frac{\gamma^2}{2}(2\beta(z)-1)\Pi'(z)+\frac{\gamma^2}{2}\Pi''(z). \tag{28}
$$

There is no explicit solution to this, but can check and compare with the equilibrium case.

Now we analyze the equilibrium case. We have a similar lemma as Lemma 1.

LEMMA 5. *The buyer would never make an offer greater than $\hat{p}=P^0$.*

Proof of Lemma 5. First, the type-H seller will never accept offers $p\le C^H$. The never accepting $p<C^H$ part is straightforward as the seller would earn negative payoff. For the never accepting $p=C^H$ part, this is based on the simple observation that a type-H seller could turn into a type-L seller with positive probability. If the type-L seller earns 0 expected payoff, it has to be that the buyer always offer $C^L=0$. This cannot be an equilibrium, however, as seller has incentive to deviate by offering C^H at the next instant with all type-L sellers accepting $C^L=0$ at time $t=0$. As this is not an equilibrium, the type-L seller would earn positive expected payoff, and so does the type-H seller.

Second, the buyer would never offer $p\ge V^H$. The never offering $p>V^H$ part is straightforward as the seller would for sure earn negative payoff. An offer of V^H will only be made when only type H seller is left, but in this case without uncertainty about seller type an offer of $V^H-\varepsilon$ with ε sufficiently small will also be accepted for sure if offered. There exists a cutoff $\hat{p}\in(C^H,V^H)$ such that

If $p\ge\hat{p}$, then $s^H(z,p)=s^L(z,p)=1$.

Suppose that $\hat{p}>P^0$. Then we claim that $\hat{p}$ or any offer close to it is accepted with probability 1 by both types. Denote $\varepsilon=\hat{p}-P^0$. Consider a time interval with length Δt. As the most favorable offer made by the seller is at most $\hat{p}$, the highest expected payoff possible for type θ if rejecting an offer of

$\hat{p} - \frac{1}{2}\varepsilon r\Delta t$ is

$$\begin{aligned}
& e^{-r\Delta t} e^{-\lambda\Delta t}(\hat{p} - C^\theta) + \int_0^{\Delta t} e^{-rt}(\hat{p} - \beta_0 C^H)\lambda e^{-\lambda t} dt \\
=& e^{-(\lambda+r)\Delta t}(\hat{p} - C^\theta) + (1 - e^{-(\lambda+r)\Delta t})\frac{\lambda}{\lambda+r}(\hat{p} - \beta_0 C^H) \\
=& \hat{p} - C^\theta - (1 - e^{-(\lambda+r)\Delta t})\left[\frac{r}{\lambda+r}\hat{p} - C^\theta + \frac{\lambda}{\lambda+r}\beta_0 C^H\right] \\
\leq& \hat{p} - C^\theta - (1 - e^{-(\lambda+r)\Delta t})\left[\frac{r}{\lambda+r}(P^0 + \varepsilon) - C^H + \frac{\lambda}{\lambda+r}\beta_0 C^H\right] \\
<& \hat{p} - C^\theta - \frac{1}{2}\varepsilon r\Delta t
\end{aligned}$$

The argument applies when $\Delta t \to 0$. Hence, the buyer can lower the offer and still have her offer accepted with probability 1. □

LEMMA 6. *The high type seller would never accept an offer lower than $\hat{p} = P^0$.*

Proof of Lemma 6. From Lemma 5, there will be a highest price buyer may offer, say $\bar{p} \leq P^0$. If buyer offers $\bar{p}$, low type will accept with probability 1 as he would not receive a better offer. Then the buyer's belief will jump to $\beta = 1$ if high type seller does not accept with probability 1. As we maintain the refinement that low type or equivalently offer is non-decreasing in belief, the offer will still be $\bar{p}$ the next instant, which will be accepted by the low type with probability 1 in case of any type transition. Therefore, the offer will stay at $\bar{p}$ after it is offered.

The offer $\bar{p}$ has to make the high type willing to accept immediately than waiting for transitioning into low type.

$$\begin{aligned}
& \bar{p} - C^H \geq \int_0^\infty e^{-rt}\bar{p}\lambda(1-\beta_0)e^{-\lambda(1-\beta_0)t} dt \\
\Longrightarrow & \bar{p} - C^H \geq \frac{\lambda(1-\beta_0)}{\lambda(1-\beta_0)+r}\bar{p} \\
\Longrightarrow & \bar{p} \geq P^0
\end{aligned}$$

□

Based on Lemma 5 and Lemma 6, we can guess that the equilibrium is similar as before that the buyer offers P^0 when the belief is high enough, which will be accepted by both types immediately. When the belief is relatively low, the buyer would offer something less which will only be accepted with positive density. Denote the hazard rate of the low type's acceptance as $\tilde{q}(z) \geq 0$. Given the belief β_t (and z_t) at time t, the belief at time $t + dt$ is given by

$$\begin{aligned}
& \frac{\Pr(\theta_{t+dt} = H | dX_t)}{\Pr(\theta_{t+dt} = H | dX_t)} \\
=& \frac{\beta_t[e^{-\lambda dt} + (1 - e^{-\lambda dt})\beta_0]e^{-\frac{(dX_t - \mu^H dt)^2}{2\sigma^2 dt}} + \cdots}{(1-\beta_t)[e^{-\lambda dt} + (1 - e^{-\lambda dt})(1-\beta_0)]e^{-\frac{(dX_t - \mu^L dt)^2}{2\sigma^2 dt}} + \cdots}
\end{aligned}$$

$$
\begin{aligned}
&\frac{(1-\beta_t)(1-e^{-\lambda dt})\beta_0\int_0^{dt}\frac{1}{dt}e^{-\frac{(dX_t-\mu^L s-\mu^H(dt-s))^2}{2\sigma^2 dt}}e^{-q(z_t)s}ds}{\beta_t(1-e^{-\lambda dt})(1-\beta_0)\int_0^{dt}\frac{1}{dt}e^{-\frac{(dX_t-\mu^H s-\mu^L(dt-s))^2}{2\sigma^2 dt}}e^{-q(z_t)s}ds}\\
\Longrightarrow\ & d\hat{Z}_t\equiv d\ln\frac{\Pr(\theta_{t+dt}=H|dX_t)}{\Pr(\theta_{t+dt}=H|dX_t)}\\
\approx&\frac{\gamma}{\sigma}\left(dX_t-\frac{\mu^H+\mu^L}{2}dt\right)+q(z_t)dt+\lambda\left(\frac{\beta_0}{\beta_t}-\frac{1-\beta_0}{1-\beta_t}\right)dt
\end{aligned}
$$

We arrive at the the following ODE

$$
\begin{aligned}
r\Pi(z)=&q(z)(1-\beta(z))(V^L-P(z)-\Pi(z))+\lambda\frac{\beta_0-\beta(z)}{\beta(z)(1-\beta(z))}\Pi'(z)\\
&+q(z)\Pi'(z)+\frac{\gamma^2}{2}(2\beta(z)-1)\Pi'(z)+\frac{\gamma^2}{2}\Pi''(z)\\
=&\lambda\frac{\beta_0-\beta(z)}{\beta(z)(1-\beta(z))}\Pi'(z)+\frac{\gamma^2}{2}(2\beta(z)-1)\Pi'(z)+\frac{\gamma^2}{2}\Pi''(z)\\
&+q(z)\big((1-\beta(z))(V^L-P(z)-\Pi(z))+\Pi'(z)\big)
\end{aligned}
$$

The zero net benefit of screening result in Lemma 3 still holds, so the equivalence result between fixed and flexible price also holds. In addition, it can be shown that the seller has a lower expected payoff when we compare the equilibrium payoff with that under public transition. □

B Formal Notations of Equilibrium

B.1 Equilibrium definition

In this section, we define formally the equilibrium of the model following a slight generalization of the definition as in Daley and Green, 2020. The main conditions are sequential optimality of both the seller and the buyer, and the belief consistency of the uninformed seller. The added refinement is stationarity and sequential optimality of the buyer for any deviating offers off path. The seller chooses an offer process $P=\{p_t:t\in\mathbb{R}_+\}$ as a best response to her belief of the buyer's type and the buyer's strategy of when to accept the offer. The type-θ buyer chooses a potentially mixed strategy, described by the cumulative distribution function (CDF) over stopping times $S^\theta:\bar{\mathbb{R}}_+\to[0,1]$ ($\bar{\mathbb{R}}$ stands for the extended real numbers). The seller's belief of buyer type β is consistent with the news process, the buyer's strategy of acceptance, and the process of type transition.

Buyer Optimality – Consider a complete probability space $\{\Omega,\mathcal{F},\mathcal{P}\}$, for a given offer process $P=\{p_t:t\in\mathbb{R}_+\}$ by the seller, adapted to history $\{\mathcal{H}_t\}$, the buyer faces a stopping problem: when to accept the seller's offer. A pure strategy for the type-θ buyer is a stopping time $\tau^\theta(\omega):\Omega\to\bar{\mathbb{R}}_+$ of the filtration $\{\mathcal{H}_t\}$, where ω denotes an arbitrary element of Ω, and $\bar{\mathbb{R}}_+=\mathbb{R}_+\cup\{\infty\}$ is the set of extended non-negative real numbers. Let $\mathcal{T}$ be the set of all such stopping times. The type-θ buyer's stopping problem is

$$\sup_{\tau\in\mathcal{T}} E^{\theta}\left[e^{-(\lambda+r)\tau}(p_{\tau}-C^{\theta})\right]+(1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}U_{O}^{\theta}. \tag{BP$^\theta$}$$

A mixed strategy for the buyer is a distribution over $\mathcal{T}$, which can be represented by the CDF it endows over the type-θ buyer's acceptance time for each sample path of news process, denoted S^{θ}. Formally, $S^{\theta}=\{S_{t}^{\theta},0\le t\le\infty\}$ is a stochastic process that is (i) adapted to $\{\mathcal{H}_{t}\}$, (ii) right-continuous, and (iii) satisfies $0\le S_{t}^{\theta}\le S_{t'}^{\theta}\le 1$ for all $t\le t'$. The function $S^{\theta}(\omega)$ is the CDF over the type-θ seller's acceptance time on $\bar{\mathbb{R}}_{+}$ along the sample path $X(\omega,\theta)$.

Let $\mathcal{S}^{\theta}=supp(S^{\theta})$ as the support of type-θ buyer's stopping times. We say that S^{θ} solves (BP$^{\theta}$) if all $\tau\in\mathcal{S}^{\theta}$ solve (BP$^{\theta}$).

Condition 1 (Buyer Optimality). *The type-θ buyer's strategy S^{θ}, solves (BP$^{\theta}$).*

Belief Consistency – At any time t before trade has yet occurred, the seller assigns a probability, $\beta_{t}\in[0,1]$, to the buyer being of high type $\theta=H$. We will represent belief using the injective log-likelihood ratio to have the linear form, denoted $Z_{t}\equiv\ln(\beta_{t}/(1-\beta_{t}))\in\bar{\mathbb{R}}$. We will mainly keep track of the belief process between any two adjacent type transitions. Upon transition, the belief consistency requires that the belief jumps to the prior β_{0}.

The seller's belief at time t is conditioned on the history of news and the fact that the buyer has rejected all past offers. These two sources of information can be separately in additive form. Let f_{t}^{θ} be the density of X_{t} conditional on θ, which is normally distributed with mean μ and variance $\sigma^{2}t$ (and with $f_{0}^{H}=f_{0}^{L}$ being the Dirac delta function). Let $S_{t^{-}}^{\theta}\equiv\lim_{s\uparrow t}S_{s}^{\theta}$ (which is well defined for $t>0$ given that S^{θ} is bounded and non-decreasing), and specify that $S_{0^{-}}^{\theta}=0$. The belief "at time t" means *before* observing the buyer's decision at time t, represented by left limits. If $S_{t^{-}}^{L}\cdot S_{t^{-}}^{H}<1$ (i.e., given the history at time t, there is positive probability that the buyer has not yet accepted an offer), then the probability the seller assigns to $\theta=H$ follows from Bayes' rule as

$$\frac{\beta_{0}f_{t}^{H}(X_{t})(1-S_{t^{-}}^{H})}{\beta_{0}f_{t}^{H}(X_{t})(1-S_{t^{-}}^{H})+(1-\beta_{0})f_{t}^{L}(X_{t})(1-S_{t^{-}}^{L})}. \tag{29}$$

Taking the log-likelihood ratio of (29) results in

$$Z_{t}=\underbrace{\ln\left(\frac{\beta_{0}}{1-\beta_{0}}\right)+\ln\left(\frac{f_{t}^{H}(X_{t})}{f_{t}^{L}(X_{t})}\right)}_{\hat{Z}_{t}}+\underbrace{\ln\left(\frac{1-S_{t^{-}}^{H}}{1-S_{t^{-}}^{L}}\right)}_{Q_{t}}. \tag{30}$$

As seen in (30), working in log-likelihood space enables us to represent Bayesian updating as a linear process, and the seller's belief as the sum of two information sources, $Z_{t}=\hat{Z}_{t}+Q$. The term $\hat{Z}_{t}$ is the belief updating only based on news, $\{X_{s}:0\le s\le t\}$, starting from $\hat{Z}_{0}=z_{0}=\ln(\beta_{0}/(1-\beta_{0}))$. The term Q is the stochastic process that keeps track of the information conveyed in equilibrium through buyer's rejection of all past offers.

Condition 2 (Belief Consistency). *For all t such that $S^L_{t^-} \cdot S^H_{t^-} < 1$, Z_t is given by (30).*

Stationarity. – Keeping up with the literature, we focus on stationary equilibria, using the uninformed party's belief as the state variable. Specifically, we use z when referring to the state variable under stochastic process Z (i.e., if $Z_t = z$, then the game is "in state z, at time t"). We can also write seller's belief of buyer being high type at state z as $\beta(z) = e^z/(1+e^z)$.

Condition 3 (Stationarity). *The seller's offer in state z is given by $P(z)$, where $P : \bar{\mathbb{R}} \to \mathbb{R}$ is a Borel measurable function, and Z is a time homogeneous $\mathcal{H}_t$-Markov process.*

DEFINITION 1. *An **S-candidate** is a quadruple (P, S^L, S^H, Z) satisfying Conditions 1-3.*

In any S-candidate, the value functions for each player depend only on the current state. Let $U^\theta(z)$ denote the expected payoff for the type-θ buyer given state z. That is, for any $\tau \in \mathcal{S}^\theta$,

$$U^\theta(z) \equiv E^\theta_z \left[e^{-(\lambda+r)\tau}(V^\theta - p_\tau) + (1 - e^{-(\lambda+r)\tau}) \frac{\lambda}{\lambda + r} U^\theta_0 \right],$$

where E^θ_z is the expectation with respect to the probability law of the process Z starting from $Z_0 = z$ and conditional on θ, which we denote by $\mathcal{Q}^\theta_z$. Similarly, let $\Pi(z)$ denote the expected payoff to the seller in any given state z:

$$\begin{aligned} \Pi(z) \equiv + \beta(z) E^H_z \left[\int_0^\infty e^{-rt} \left(e^{-\lambda t}(P(Z_t) - C^H) + (1 - e^{-\lambda t})\Pi_O \right) dS^H_{t^-} \right] \\ (1 - \beta(z)) E^L_z \left[\int_0^\infty e^{-rt} \left(e^{-\lambda t} P(Z_t) + (1 - e^{-\lambda t})\Pi_O \right) dS^L_{t^-} \right], \end{aligned} \tag{31}$$

where $\beta(z) = e^z/(1+e^z)$ and Π_O is seller's ex-ante expected payoff whenever a new type is drawn.

Response to Any Offer. – Take an S-candidate, and suppose that in state z, the seller offers p, which may or not be the prescribed offer $P(z)$. Let $s^\theta(z, p)$ be the probability that the type-θ buyer accepts the offer. Motivated by discrete time notions of sequential rationality and belief consistency in equilibrium concepts such as Perfect Bayesian Equilibrium, we require the following.

First, the type-L buyer will never accept offers $p > \tilde{V}^L$, as the buyer can always wait and wait for the transition. Second, the seller would never offer $p < 0$ as she would earn negative payoff. Hence, there exists a cutoff $\tilde{V}^L \in [\tilde{V}^L, V^H]$ such that

(R1) If $p \leq \tilde{V}^L$, then $s^H(z, p) = s^L(z, p) = 1$.

Lemma 1 is derived from limit of the discrete analog of the game. For offers higher than $\tilde{V}^L$, the buyer responds optimally and the seller's belief updates consistently with these responses as follows. Let $\tilde{z}(z, p)$ be the seller's updated belief if her offer of p is rejected in state z.

(R2) If $p > \tilde{V}^L$, then $\beta\left(\tilde{z}(z,p)\right) = \frac{\beta(z)\left(1-s^H(z,p)\right)}{\beta(z)(1-s^H(z,p))+(1-\beta(z))(1-s^L(z,p))}$.

Hence, the buyer's choice is whether to accept p or reject and get $U^\theta\left(\tilde{z}(z,p)\right)$.

(R3) If $p > \tilde{V}^L$, then $s^\theta(z,p) \in \arg\max_\sigma \sigma(V^\theta - p) + (1-\sigma)U^\theta\left(\tilde{z}(z,p)\right)$.

If $p = P(z) > \tilde{V}^L$ is on the equilibrium path, then (R2) and (R3) are implied by *belief consistency* and *seller optimality*.

Condition 4 (Response to any offer). *For any z, $\{s^H(z,\cdot), s^L(z,\cdot), \tilde{z}(z,\cdot)\}$ satisfy (R1)–(R3).*

Seller Optimality. – If the seller offers p in state z, either it will be accepted, earning her $p - C^\theta$ from type-θ buyer, or rejected, earning her the continuation value from the post rejection belief $\tilde{z}(z,p)$. The prescribed offer $P(z)$ is optimal if

$$
\begin{aligned}
P(z) \in \arg\max_p \; & \beta(z)s^H(z,p)(V^H - p) + (1-\beta(z))\,s^L(z,p)(p - C^H) \\
& + \left(1 - \left(\beta(z)s^H(z,p) + (1-\beta(z))s^L(z,p)\right)\right)\Pi(\tilde{z}(z,p)),
\end{aligned}
\tag{32}
$$

Condition (32) requires that the seller makes offers to maximize her expected payoff given belief z at each instant. The updated belief $\tilde{z}(z,p)$ and continuation payoff take into account of the rejection decision of different buyers but not the news acquisition or event arrival. The next condition requires that the seller makes optimal use of this option to wait and learn. In particular, we require that for all stopping times $\tau \in \mathcal{T}$,

$$
\Pi(z) \geq E_z\left[e^{-(\lambda+r)\tau}\Pi(\hat{Z}_\tau) + (1 - e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}\Pi_O\right], \tag{33}
$$

where $\hat{Z}$ is the belief process updating solely based on news. If there was no news, $\hat{Z}_t$ would be constant over time and therefore $E_z\left[\Pi(\hat{Z}_\tau)\right] = \Pi(z)$ for any τ, meaning that (33) implies $\Pi(z) \geq \frac{\lambda}{\lambda+r}\Pi_O$ for all z. However, there are important additional implications with news. For instance, any upward kink in the seller's value function violates (33), since the seller could improve her payoff by waiting in a neighborhood around the kink.

Condition 5 (Seller Optimality). *P and Π as defined by (31), satisfy (32)-(33) for all z.*

DEFINITION 2. *An equilibrium is a profile $(P, S^L, S^H, Z, s^H, s^L, \tilde{z})$ that satisfies Conditions 1-5.*

DEFINITION 3. *An S-candidate (P, S^L, S^H, Z) is supported by $(s^H, s^L, \tilde{z})$ if $(P, S^L, S^H, Z, s^H, s^L, \tilde{z})$ is an equilibrium. An S-candidate (P, S^L, S^H, Z) can be supported if there exists a $(s^H, s^L, \tilde{z})$ such that $(P, S^L, S^H, Z, s^H, s^L, \tilde{z})$ is an equilibrium.*

PROPOSITION 5. *An S-candidate endowing value functions (Π, U^H, U^L), with U^H non-increasing can be supported as an equilibrium if and only if, for all z,*

$$\Pi(z) \geq \tilde{V}^L - \beta(z)C^H, \tag{B1}$$

and

$$\Pi(z) \geq \max_{z' \geq z} \left\{ \frac{\beta(z') - \beta(z)}{\beta(z')}(V^H - C^H - U^H(z')) + \frac{\beta(z)}{\beta(z')}\Pi(z') \right\}, \tag{B2}$$

and for all $\tau \in \mathcal{T}$,

$$\Pi(z) \geq E_z \left[e^{-(\lambda+r)\tau}\Pi(\hat{Z}_\tau) \right] + \left(1 - e^{-(\lambda+r)\tau}\right) \frac{\lambda}{\lambda + r}\Pi_O. \tag{B3}$$

Proof of Proposition 5. We first establish basic properties that hold in any monotone equilibrium and in any monotone *S*-candidate satisfying (B1)-(B3). We then show that (B1)-(B3) are necessary and sufficient for an *S*-candidate to be supportable as an equilibrium if U^H and U^L are non-increasing.

Basic Facts: In either (a) a monotone equilibrium or (b) a monotone *S*-candidate satisfying (B1)-(B3), the following hold.

FACT 1. *Let $\bar{P} \equiv \inf_z P(z)$. For any θ and z, if $\bar{P} < V^\theta$, then $U^\theta(z) \in [V^\theta - P(z), V^\theta - \bar{P}]$.*

The lower bound is by buyer optimality; the upper bound is by feasibility and definition of $\bar{P}$.

FACT 2. $\bar{P} \geq \tilde{V}^L$.

For (a) by requirement of (R1) and (32). For (b) if there exists z with $P(z) < \tilde{V}^L$, then Fact 1 implies that $\beta(z)U^H(z) + (1-\beta(z))U^L(z) \geq \beta(z)V^H + (1-\beta(z))V^L - P(z) > \beta(z)V^H + (1-\beta(z))V^L - \tilde{V}^L$. The highest possible expected surplus is $\beta(z)V^H + (1-\beta(z))V^L - \beta(z)C^H \geq \Pi(z) + \beta(z)U^H(z) + (1-\beta(z))U^L(z)$, implying $\Pi(z) < \tilde{V}^L - \beta(z)C^H$ and contradicting (B1).

FACT 3. $\bar{P} = \min_z P(z) = \tilde{V}^L$.

If the low type never trades and waits for the type change, he guarantees expected payoff $\frac{\lambda}{\lambda+r}U_0^H$; if he trades, he gets at most $V^L - \bar{P} \leq V^L - \tilde{V}^L = \frac{\lambda}{\lambda+r}U_0^L$ by Fact 2. If Fact 3 does not hold, then by Fact 2 and buyer optimality, the low type never trades and the probability of trade with the current type goes to 0 as $z \to -\infty$ and thus $\Pi(z) \to \frac{\lambda}{\lambda+r}\Pi_O < \tilde{V}^L$ from Assumption 2. For (a) by (R1) the seller can profitably deviate by offering $\tilde{V}^L$ for z large enough, violating (32). For (b) (B1) is violated for z large enough.

FACT 4. *There exists $\beta > -\infty$ such that $P(z) = V^H - U^H(z) = \tilde{V}^L$ for all $z < \beta$ and $P(z) \geq V^H - U^H(z) < \tilde{V}^L$ for all $z > \beta$.*

Immediate from Facts 1 and 3 and U^H non-increasing.

FACT 5. $S_t^L \leq \mathbf{1}_{\{t \geq T(\beta)\}}$, *where $T(\beta) \equiv \inf\{t : Z_t \leq \beta\}$.*

The low type rejects $P(z)$ for all $z > \beta$, which is immediate from Fact 4 and ***buyer optimality***.

FACT 6. *Prior to $T(\beta)$, Q is a weakly decreasing process.*

Follows from Fact 5 and ***belief consistency****.*

FACT 7. *For all z, $U^H(z) = \frac{\lambda}{\lambda+r}U_O^H + E_z^H\left[e^{-(\lambda+r)T(\beta)}\right]\left[V^H - \frac{\lambda}{\lambda+r}(U_O^H - U_O^L)\right]$.*

For all $z \geq \beta$,the fact is immediate from Fact 4. Hence, if the fact were false, there exists a state $z < \beta$ such that $U^L(z) > \frac{\lambda}{\lambda+r}U_O^L + E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[C^H + \frac{\lambda}{\lambda+r}(U_O^H - U_O^L)\right]$ by ***seller optimality****. Then there exists a state $z' \in [z, \beta)$ in which the low type trades with probability 1, but at a price $P(z') > \tilde{V}^L$. But then, by Fact 5 and* ***belief consistency****, rejection at $Z_t = z'$ leads to a belief of $Z_{t^+} = \infty$ and an offer of $\tilde{V}^L$. Hence, the low type would do better to reject at z', generating a contradiction of* ***seller optimality****.*

FACT 8. *U^L is continuous.*

Immediate for $z > \beta$. Suppose that U^L is discontinuous at some $z_1 \leq \beta$. Then by Fact 7, Z must also be discontinuous at z_1. The monotonicity of Q (Fact 6) implies that Z can only have upward jumps, so $U^L(z_1^-) = U^L(z_2)$ for some "jump-to" point $z_2 > z_1$. Note, U^L is non-decreasing, so $U^L(z_2) \geq U^L(z_1^+) \geq U^L(z_1^-) = U^L(z_2)$, contradicting a discontinuity of U^L at z_1.

Sufficiency of (B1)-(B3): Fix a monotone S-candidate satisfying (B1)-(B3). For any z and $p \geq \tilde{V}^L$, set $s^H(z,p) = s^L(z,p) = 1$. For any z and $p > \tilde{V}^L$, set $s^H(z,p) = 0$ and

$$s^L(z,p) = \frac{\beta(\tilde{z}(z,p)) - \beta(z)}{\beta(\tilde{z}(z,p))(1-\beta(z))},$$

with $\tilde{z}(z,p) = \max\{z, \max\{z' : U^L(p) = z'\}\}$, which is well defined by U^L non-decreasing and continuous (Fact 8).

We now need to verify that Conditions 4 and 5 are satisfied. Each part of Condition 4, (R1)-(R3), is satisfied by construction. (B3) trivially implies (33) as being identical. The final step is to verify that (32) is satisfied. Fix any z. An offer of $p \geq \tilde{V}^L$ generates a payoff of $V(z) - p \leq V(z) - \tilde{V}^L \leq \Pi(z)$ by (B1), so $P(z)$ is weakly better for the buyer than any such p. An offer of $p > \tilde{V}^L$ generates a payoff of

$$J(z, \tilde{z}(z,p)) \leq \max_{z' \geq z} J(z,z') \leq \Pi(z),$$

where the first inequality is by definition of maximum and $\tilde{z}(z,p) \geq z$, and the second inequality is (B3). Hence, Condition 5 is satisfied.

Necessity of (B1)-(B3): Fix a monotone equilibrium. (33) trivially implies (B3) as being identical. Next, if (B1) were violated at some z, then the buyer could improve her payoff (therefore violating (32)) by offering $\tilde{V}^L$ in state z: by (R1), both types accept, producing a payoff of $V(z) - \tilde{V}^L > \Pi(z)$.

For (B2), note that for all z and $p \in [U^L(z), \tilde{V}^L)$: (i) $s^H(z,p) = 0$ by Fact 5. Hence, by (R2) and (R3), $\tilde{z}(z,p) \geq z$ and $\tilde{z}(z,p) \in U^{L^{-1}}(p)$ (can be verified by checking each of the three cases where

$s^L(z,p)=0$, $s^L(z,p)=1$ and $s^L(z,p)\in(0,1)$ are high type's best responses respectively), which is nonempty by U^L continuous (Fact 8). Therefore, solving (R2) for $s^L(z,p)$ gives

$$s^L(z,p)=\frac{\beta(\tilde{z}(z,p))-\beta(z)}{\beta(\tilde{z}(z,p))(1-\beta(z))},$$

and the buyer's payoff from offering p is $J(z,\tilde{z}(z,p))$. Denote $\tilde{\mathscr{Z}}(z)\equiv\{z':\exists p\in[U^L(z),\tilde{V}^L)$ such that $\tilde{z}(z,p)=z'\}$, which is the set of beliefs that can be reached by rejection of an offer $p\in[U^L(z),\tilde{V}^L)$ at state z. Condition (32) implies that

$$\begin{aligned}\Pi(z)&\geq\max_{p\in[U^L(p),\tilde{V}^L)}\left\{\frac{\beta(\tilde{z}(z,p))-\beta(z)}{\beta(\tilde{z}(z,p))}(V^L-p)+\frac{\beta(z)}{\beta(\tilde{z}(z,p))}\Pi(\tilde{z}(z,p))\right\}\\&=\max_{z'\in\tilde{\mathscr{Z}}(z)}\left\{\frac{\beta(z')-\beta(z)}{\beta(z')}(V^L-U^L(z'))+\frac{\beta(z)}{\beta(z')}\Pi(z')\right\}.\end{aligned}\tag{A4}$$

Hence, the final step is to show that the maximum is not improved if we replace $\tilde{\mathscr{Z}}(z)$ with $\{z':z'\geq z\}$ in (A4). Notice, U^L non-decreasing and $\tilde{z}(z,p)\in U^{L^{-1}}(p)$ imply that $\tilde{\mathscr{Z}}(z)\subseteq\{z':z'\geq z\}$. As U^L is continuous, non-decreasing and bounded above by $\tilde{V}^L$, it is sufficient to show that $J(z,z')=J(z,z'')$ for all z',z'' in $U^{L^{-1}}(p)$. To do so, suppose that $z'<z''$ and $U^L(z')=U^L(z'')$. Function U^L non-decreasing and continuous (Fact 8) imply that $z',z''\in[z_1,z_2]$ such that $U^L(z)=U^L(z')$ if and only if $z\in[z_1,z_2]$. Hence, by Fact 7, $E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]$ is constant on $[z_1,z_2]$, and strictly higher for all $z>z_2$. Fact 6 then implies that $\tilde{z}(z,P(z))=z_2$ for all $z\in[z_1,z_2]$ (that is, rejecting the equilibrium offer in any state $z\in[z_1,z_2]$ leads to a jump in the belief to z_2: otherwise, $E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]$ would not be constant on the interval. They must jump to the same belief: $\geq z_2$ from $\tilde{z}(z,p)\geq z$, $\leq z_2$ from (R2) and (R3) that the type-L seller would reject otherwise). It follows that

$$\begin{aligned}J(z,z')&=\frac{\beta(z')-\beta(z)}{\beta(z')}(V^L-U^L(z'))+\frac{\beta(z)}{\beta(z')}\Pi(z')\\&=\frac{\beta(z')-\beta(z)}{\beta(z')}(V^L-U^L(z'))+\frac{\beta(z)}{\beta(z')}J(z',z_2)\\&=\frac{\beta(z')-\beta(z)}{\beta(z')}(V^L-U^L(z))+\frac{\beta(z)}{\beta(z')}\left[\frac{\beta(z_2)-\beta(z')}{\beta(z_2)}(V^L-U^L(z))+\frac{\beta(z')}{\beta(z_2)}\Pi(z_2)\right]\\&=\frac{\beta(z_2)-\beta(z)}{\beta(z_2)}(V^L-U^L(z))+\frac{\beta(z)}{\beta(z_2)}\Pi(z_2)\\&=J(z,z_2)\end{aligned}$$

and

$$
\begin{aligned}
J(z,z'') &= \frac{\beta(z'')-\beta(z)}{\beta(z'')}(V^L - U^L(z'')) + \frac{\beta(z)}{\beta(z'')}\Pi(z'') \\
&= \frac{\beta(z'')-\beta(z)}{\beta(z'')}(V^L - U^L(z'')) + \frac{\beta(z)}{\beta(z'')}J(z'',z_2) \\
&= J(z,z_2).
\end{aligned}
\qquad \square
$$

DEFINITION 4. *Henceforth, we refer to any equilibrium (S-candidate) in which U^H is non-increasing as a **monotone equilibrium** (S-candidate).*

Proposition 5 is useful both for proving existence and uniqueness. It reduces verifying that a monotone S-candidate is an equilibrium to checking three variational inequalities on the resulting Π (i.e., (B1)-(B3)). It also facilitates ruling out other types of candidate equilibria.

That equilibrium conditions imply (B1) and (B3) is straightforward; they must be satisfied in any equilibrium regardless of whether $U^L$is non-decreasing. Monotonicity of U^L is used to establish the necessity of (B2). The expression being maximized on the RHS of (B2) is the buyer's payoff from offering $p = U^L(z')$ in state z if the post-rejection belief, $\tilde{z}(z,p)$, will be z'. Define this expression as $J(z,z')$:

$$
J(z,z') \equiv \underbrace{\frac{\beta(z')-\beta(z)}{\beta(z')}}_{\text{Prob. offer accepted}} \underbrace{(V^L - U^L(z'))}_{\text{Payoff if accepted}} + \underbrace{\frac{\beta(z)}{\beta(z')}}_{\text{Prob. rejected}} \underbrace{\Pi(z')}_{\text{Continuation Payoff}} .
$$

That (B2) is required is easiest to see when U^L is strictly increasing below C^H. In this case, for all $p \in [U^L(z), \tilde{V}^L)$, there exists a unique $z' \geq z$ such that $U^L(z') = p$. Then, (R2) and (R3) immediately imply that $\tilde{z}\left(z, U^L(z')\right)$ must be z'. Thus, if (B2) was violated and U^L is strictly increasing, there exists a profitable deviation for the buyer, i.e., (R2) and (32) cannot be simultaneously satisfied.

B.2 Equilibrium Construction

I characterize the equilibrium using a pair (ζ, q), where ζ is the threshold above which the buyer offers $P(z) = \tilde{V}^L$ and trade is immediate. When $z < \zeta$, the buyer offers some $P(z) > \tilde{V}^L$, which the high type rejects and the low type accepts at a state specific hazard rate, $q(z)$ (i.e., proportional to time). In any state $z < \zeta$, the buyer's offer is such that the low type is indifferent between accepting $P(z)$ or waiting until $\tilde{V}^L$ is offered. The next definition gives a formal description of such a profile (ζ, q). Let $V(z) \equiv \beta(z)V^H + (1-\beta(z))V^L$ be the expected value of the asset when the buyer's log-likelihood belief is z.

DEFINITION 5. *For $\zeta \in \bar{\mathbb{R}}$ and measurable function $q : (-\infty, \zeta) \to \mathbb{R}_+$, let $T(\zeta) \equiv \inf\{t : Z_t \geq \zeta\}$ and $\Sigma(\zeta, q)$ be the strategy profile and belief process:*

$$Z_t = \begin{cases} \hat{Z}_t + \int_0^t q(Z_s)ds & \text{if } t < T(\zeta), \\ \text{arbitrary} & \text{otherwise} \end{cases} \tag{34}$$

$$S_t^H = \begin{cases} 0 & \text{if } t < T(\zeta), \\ 1 & \text{otherwise} \end{cases} \tag{35}$$

$$S_t^L = \begin{cases} 1 - e^{-\int_0^t q(Z_s)ds} & \text{if } t < T(\zeta), \\ 1 & \text{otherwise} \end{cases} \tag{36}$$

$$P(z) = \begin{cases} \tilde{V}^L & \text{if } z \geq \zeta, \\ \frac{\lambda}{\lambda+r} U_O^L + E_z^L \left[e^{-(\lambda+r)T(\zeta)} \right] \left(\tilde{V}^L - \frac{\lambda}{\lambda+r} U_O^L \right) & \text{otherwise} \end{cases} \tag{37}$$

$$s^H(z,p) = \begin{cases} 1 & \text{if } p \geq \tilde{V}^L, \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

$$s^L(z,p) = \begin{cases} 1 & \text{if } p \geq \tilde{V}^L, \\ \frac{\beta(P^{-1}(p)) - \beta(z)}{\beta(P^{-1}(p))(1-\beta(z))} & \text{if } p \in (P(z), \tilde{V}^L), \\ 0 & \text{otherwise} \end{cases} \tag{39}$$

$$\tilde{z}(z,p) = \begin{cases} \text{arbitrary} & \text{if } p \geq \tilde{V}^L, \\ P^{-1}(p) & \text{if } p \in (P(z), \tilde{V}^L), \\ z & \text{otherwise} \end{cases} \tag{40}$$

Based on the definition of such $\Sigma(\zeta, q)$ pair, the equilibrium payoff of both type sellers can be constructed, with high type payoff being constant and low type payoff being non-decreasing.

LEMMA 7. *In any $\Sigma(\zeta, q)$ profile, the seller's payoffs are given by*

$$U^H(z) = \frac{\lambda}{\lambda+r} U_O^H$$
$$U^L(z) = P(z) = \frac{\lambda}{\lambda+r} U_O^L + E_z^L \left[e^{-(\lambda+r)T(\zeta)} \right] \left(\tilde{V}^L - \frac{\lambda}{\lambda+r} U_O^L \right).$$

where high type's payoff is constant and low type's payoff is non-decreasing and strictly increasing below ζ.

Proof of Lemma 7. For $z \geq \beta$, immediately we have $U^L(z) = P(z) = \tilde{V}^L$, which is non-decreasing. For

$z < \beta$ and any strategy $\tau \leq T(\beta)$ the low type's payoff is

$$
\begin{aligned}
&\left(1-e^{-(\lambda+r)\tau}\right)\frac{\lambda}{\lambda+r}U_O^L + E_z^L\left[e^{-(\lambda+r)\tau}P(Z_\tau)\right] \\
=&\left(1-e^{-(\lambda+r)\tau}\right)\frac{\lambda}{\lambda+r}U_O^L \\
&+E_z^L\left[e^{-(\lambda+r)\tau}\left(\frac{\lambda}{\lambda+r}U_O^L + E_{Z_\tau}^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[C^H+\frac{\lambda}{\lambda+r}(U_O^H-U_O^L)\right]\right)\right] \\
=&\frac{\lambda}{\lambda+r}U_O^L + E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[C^H+\frac{\lambda}{\lambda+r}(U_O^H-U_O^L)\right] \\
=&P(z),
\end{aligned}
$$

by (37). Finally, $P(z) = \frac{\lambda}{\lambda+r}U_O^L + E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[C^H+\frac{\lambda}{\lambda+r}(U_O^H-U_O^L)\right]$ is strictly increasing at all $z < \beta$ because Z from (34) has continuous sample path with probability 1. □

The next theorem proves that there exists such an equilibrium and that the equilibrium is the unique "reasonable" stationary equilibrium. The imposed refinement is that the seller's value function is non-decreasing in the public belief, which means that good news is never harmful to the seller.

THEOREM 3. *There exists a unique pair (β, q) such that the strategies and beliefs above constitute an equilibrium of the bargaining game with event arrival. In addition, this is the unique stationary equilibrium in which the seller's value function is non-decreasing.*

Proof of existence in Theorem 3. The characterization above shows that there exists a unique candidate $\Sigma(\beta, q)$. We just need to verify that the found candidate is well defined and satisfies the equilibrium conditions as in Proposition 5.

To prove verify that the candidate is well defined, first we show the candidate admits a unique strong solution to (34) for any $t \leq T(\beta)$. $\hat{Z}_t$ is linear in X_t given by $\hat{Z}_t = \hat{Z}_t + \frac{\gamma}{\sigma}\left(X_t - \frac{\mu^H+\mu^L}{2}t\right)$. Since we are looking for solutions to $Z_t = \hat{Z}_t + Q_t$ and $Q_t = \int_0^t q(Z_s)ds$, it is sufficient to show that there exists a unique solution to

$$
Q_t = \int_0^t q(\hat{Z}_s + Q_s)ds. \tag{41}
$$

With candidate $q(z)$ given in (19), we can write (41) in differential form:

$$
dQ_t = e^{-a(\hat{Z}_s+Q_s)}, Q_0 = 0, \tag{42}
$$

where $\kappa^L = \frac{\gamma^2}{2C_1^*}\left(V^L - \frac{\lambda}{\lambda+r}\left(\Pi_O + U_O^L\right)\right)$. For each (t,ω), (42) is a separable ODE, which has a unique solution $Q_t = \frac{1}{a}\ln\left(1+\kappa^L a\int_0^t e^{a\hat{Z}_s}ds\right)$. Hence, for any $t \leq T(\beta)$, there exists unique solutions for Q_t and Z_t, and thus a unique solution to (34). Given Z, the remaining objects (S^H, S^L, P) are all well defined.

Now we verify the equilibrium conditions: Conditions 1-3 for S-candidate and Conditions (B1)-(B3). Condition 2 follows from plugging (35) and (36) into (30) to get (34). Condition 3 follows directly from (34). Condition 1 requires seller optimality for both high and low types. For high type, $\mathcal{S}^H = \{T(\beta)\}$ from (35) and $P(z) \leq C^H + \frac{\lambda}{\lambda+r}U_O^H$ from (37). Therefore,

$$
\begin{aligned}
&\sup_{\tau\in\mathcal{T}} E^H\left[e^{-(\lambda+r)\tau}(p_\tau - C^H)\right] + (1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}U_O^H \\
\leq &\sup_{\tau\in\mathcal{T}} e^{-(\lambda+r)\tau}\frac{\lambda}{\lambda+r}U_O^H + (1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}U_O^H \\
= &\frac{\lambda}{\lambda+r}U_O^H = U^H(z),
\end{aligned}
$$

which verifies that S^H solves (SP_H).

For low type, by construction $U^L(z) = \frac{\lambda}{\lambda+r}U_O^L + E_z^L\left[e^{-(\lambda+r)T(\beta)}\right]\left[C^H + \frac{\lambda}{\lambda+r}(U_O^H - U_O^L)\right]$. Let $\mathcal{T}(\beta) \equiv \mathcal{T} \cap \{\tau : \tau \leq T(\beta)\}$, i.e., the set of all stopping times such that $\tau \leq T(\beta)$. As P is bounded above by $\tilde{V}^L$ and delay is costly,

$$
\begin{aligned}
&E_z^L\left[e^{-(\lambda+r)\tau}P(Z_\tau)\right] + (1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}U_O^L \\
\leq &\frac{\lambda}{\lambda+r}U_O^L + E_z^L\left[e^{-(\lambda+r)\tau}\right]\left[C^H + \frac{\lambda}{\lambda+r}(U_O^H - U_O^L)\right] \\
\leq &U^L(z),
\end{aligned}
$$

for any $\tau \in \mathcal{T} \setminus \mathcal{T}(\beta)$. As $\mathcal{S}^L \subseteq \mathcal{T}(\beta)$, to verify that S_L solves (SP_L), it suffices to show that for any $\tau \in \mathcal{T}(\beta)$, $E_z^L\left[e^{-(\lambda+r)\tau}P(Z_\tau)\right] + (1-e^{-(\lambda+r)\tau})\frac{\lambda}{\lambda+r}U_O^L = U^L(z)$.

Let $f_L(t,z) \equiv e^{-(\lambda+r)t}\left(P(z) - \frac{\lambda}{\lambda+r}U_O^L\right)$ and note that f_L is C^2 for all $z \neq \beta$. Conditional on $\theta = L$ and $t < T(\beta)$, Z evolves according to

$$
dZ_t = \left(q(Z_t) - \frac{\gamma^2}{2}\right)dt + \gamma dB_t.
$$

By Dynkin's formula, for any $\tau \in \mathcal{T}(\beta)$,

$$
E_z^L\left[f_L(\tau, Z_\tau)\right] = f_L(0,z) + E_z^L\left[\int_0^\tau \mathcal{A}^L f_L(s, Z_s)ds\right],
$$

where $\mathcal{A}^L$ is the characteristic operator for the process $Y_t = (t, Z_t)$ under $\mathcal{Q}^L$, i.e.,

$$\begin{aligned}
\mathcal{A}^L f_L(t,z) &= \frac{\partial f_L}{\partial t} + \left(q(z) - \frac{\gamma^2}{2}\right)\frac{\partial f_L}{\partial z} + \frac{1}{2}\gamma^2\frac{\partial^2 f_L}{\partial z^2} \\
&= e^{-(\lambda+r)t}\left[-(\lambda+r)\left(P(z)\frac{\lambda}{\lambda+r}U_O^L\right) + \left(q(z)-\frac{\gamma^2}{2}\right)P'(z) + \frac{\gamma^2}{2}P''(z)\right] \\
&= e^{-(\lambda+r)t}\left[-(\lambda+r)\left(U^L\frac{\lambda}{\lambda+r}U_O^L\right) + \left(q(z)-\frac{\gamma^2}{2}\right)U^{H'}(z) + \frac{\gamma^2}{2}U^{H''}(z)\right] \\
&= 0,
\end{aligned}$$

where the second inequality comes from (19). Hence, $E_z^L\left[f_L(\tau, Z_\tau)\right] = U^L(z)$ for any $\tau \in \mathcal{T}(\beta)$.

For (B1), when $z > \beta$, $\Pi(z) = V(z) - \tilde{V}^L$ by construction. When $z \le \beta$,

$$\begin{aligned}
&\Pi(\beta - x) - (V(\beta - x) - \tilde{V}^L) \\
=&\frac{\left(e^{(1-a)x} + (a-1)e^x - a\right)(V^H - \hat{C}^H)(\hat{C}^H - V^L)}{e^x(a-1)(V^H - \hat{C}^H) + a(\hat{C}^H - V^L)}
\end{aligned}$$

The denominator is positive by Assumptions 1 and 2. The numerator is positive as $e^{(1-a)x} + (a-1)e^x - a > 0$ for all $x > 0, a > 1$.

For (B2), when $z \le \beta$,

$$\frac{\partial}{\partial z'}J(z,z') = \frac{K_1^* * a(a-1)e^{az'}(e^{z-z'}-1)}{1+e^z} > 0, \forall z' \in (z,\beta).$$

Since $J(z,z')$ is decreasing in z' and $\Pi(z) = J(z,z)$, $\Pi(z) = J(z,z) = \sup_{z' \in (z,\beta)} J(z,z')$. As $J(z,z') = V(z) - \tilde{V}^L = \Pi(z)$ for all $z' \ge \beta$, we have $\Pi(z) \ge J(z,z')$ for all $z' \ge z$. When $z > \beta$, $U^L(z) = \tilde{V}^L$ and $J(z,z') = V(z) - \tilde{V}^L = \Pi(z)$ for all $z' \ge z > \beta$.

For (B3), let $f_B(t,z) = e^{-(\lambda+r)t}\left(\Pi(z) - \frac{\lambda}{\lambda+r}\Pi_O\right)$, f_B is C^2 for all $z \ne \beta$. For any stopping time τ such that $E_z[\tau] < \infty$, using Dynkin's formula, we have

$$E_z\left[f_B(\tau, \hat{Z}_\tau)\right] = f_B(0,z) + E_z\left[\int_0^\tau \mathcal{A}^B f_B(t,\hat{Z}_t)dt\right],$$

where $\mathcal{A}^B$ is the characteristic operator of the process $(t, \hat{Z}_t)$ under $\mathcal{Q}^B$ starting from $\hat{Z}_0 = z$, i.e., $\mathcal{A}^B f(t,z) = \frac{\partial f}{\partial t} + \frac{\gamma^2}{2}(2\beta(z)-1)\frac{\partial f}{\partial z} + \frac{\gamma^2}{2}\frac{\partial^2 f}{\partial z^2}$. From (12), $\mathcal{A}^B f_B = 0$ for all $z < \beta$. For $z > \beta$, $\Pi(z) = V(z) - \tilde{V}^L$. As $\frac{\gamma^2}{2}(2\beta(z)-1)V'(z) + \frac{\gamma^2}{2}V''(z) = 0$, $\mathcal{A}^B f_B = -e^{-(\lambda+r)t}(\lambda+r)\left(V(z) - \tilde{V}^L\right) > 0$. Hence, $\mathcal{A}^B f_B \le 0$, and for any stopping time, $\Pi(z) - \frac{\lambda}{\lambda+r}\Pi_O = f(0,z) \ge E_z[f_B(\tau, Z_\tau)] = E_z\left[e^{-(\lambda+r)\tau}\left(\Pi(z) - \frac{\lambda}{\lambda+r}\Pi_O\right)\right]$.□

Proof of uniqueness in Theorem 3. We will refer equilibrium as the monotone equilibrium (i.e., U^L is non-decreasing).

LEMMA 8. *In any equilibrium, there exists $\beta < \infty$ such that $P(z) = U^L(z) = \tilde{V}^L$ if and only $z \ge \beta$.*

Proof. Same as Fact 4 in the proof of Proposition 5. □

By Lemma 8, in any equilibrium there exists a β such that the buyer offers C^H (and the seller accepts with probability 1) if and only if $z \geq \beta$. Consider equilibrium play for $t < T(\beta)$, by Lebesgue's decomposition for monotone functions (cf. Proposition 5.4.5, Bogachev 2007), we can decompose Q into two process:

$$Q = Q^{abs} + Q^{sing}$$

where Q^{abs} is an absolutely continuous process and Q^{sing} is non-decreasing process with $dQ_t^{sing} = 0$ almost everywhere. We have shown in Theorem 3 that the equilibrium is unique among those in which Q is absolutely continuous. Now it suffices to rule out equilibria in a singular trading intensity.

First note that Q^{sing} can be further decomposed into a continuous non-decreasing process and a non-decreasing jump process. Thus, a singularity can take one of two forms. Either, (i) a jump from some z_0 to some $z_1 > z_0$ or (ii) trading intensity of greater than dt at some isolated z_0.

LEMMA 9. *In any equilibrium, if W is C^2 on any interval (z_1, z_2), then for all $z \in (z_1, z_2)$,*

$$(\mathcal{A} - (\lambda + r))\Pi(z) \leq -\lambda \Pi_O, \tag{43}$$

where $\mathcal{A}$ is the characteristic operator of $\hat{Z}$ under $\mathcal{Q}$.

Proof. If (43) is violated at some $z \in (z_1, z_2)$, then since W is C^2 on the interval, there exists $\varepsilon > 0$ such that (43) is violated over the interval $(z - \varepsilon, z + \varepsilon)$. Let $\tau_\varepsilon = \inf\{t : \hat{Z}_t \notin (z - \varepsilon, z + \varepsilon)\}$, then by Dynkin's formula,

$$\begin{aligned} E_z\left[e^{-(\lambda+r)\tau_\varepsilon}\Pi(\hat{Z}_{\tau_\varepsilon})\right] &= \Pi(z) + E_z\left[\int_0^{\tau_\varepsilon} e^{-(\lambda+r)s}(\mathcal{A} - (\lambda + r))\Pi(\hat{Z}_s)ds\right] \\ &> \Pi(z) - \left(1 - e^{(\lambda+r)\tau_\varepsilon}\right)\frac{\lambda}{\lambda + r}\Pi_O, \end{aligned}$$

which violates (B3). □

LEMMA 10. *In any equilibrium, if W is differentiable at z, then $\Gamma(z) \leq 0$.*

Proof. Fix z, and assume that $\Pi'(z)$ exists. If $z \geq \beta$, then by Lemma 8 $U^L(z') = \tilde{V}^L$ and $\Pi(z) = V(z') - \tilde{V}^L$ for all $z' \geq z$. Substituting in these expressions, one finds that $J(z, z') = V(z) - \tilde{V}^L$ for all $z' \geq z$, and $\Gamma(z) = 0$.

For $z < \beta$, if Z does not jump following a rejection at z, then $q(z)\Gamma(z) \leq 0$ by the argument in the proof of Lemma 3. Finally if Z does jump from z to $z' > z$, then $\Pi(z) = J(z, z')$. Using envelope theorem,

$$\Pi'(z) = J_1(z, z') = \frac{\beta'(z)}{\beta(z')}\left(\Pi(z') + U^L(z') - V^L\right).$$

Since the low type must be indifferent, $U^L(z) = U^L(z')$. As

$$\Pi(z') = \frac{\beta(z')}{\beta(z)}(U^L(z') + \Pi(z) - V^L) - U^L(z') + V^L,$$

we have

$$\Pi'(z) = \frac{\beta'(z)}{\beta(z)}\left(\Pi(z) + U^L(z) - V^L)\right),$$

which implies $\Gamma(z) = 0$. $\square$

LEMMA 11. *In any equilibrium, $\beta > \underline{z}$ and $\Pi(z) \geq E_z\left[e^{-(\lambda+r)\hat{T}(\beta)}(V(\beta) - \tilde{V}^L)\right] + (1 - e^{-(\lambda+r)\hat{T}(\beta)})\frac{\lambda}{\lambda+r}\Pi_O > \frac{\lambda}{\lambda+r}\Pi_O$, where $\hat{T}(\beta) = \inf\{t \geq 0 : \hat{Z}_t \geq \beta\}$ and $\hat{Z}_0 = z$.*

Proof. For any $z_1 > \underline{z}$, the strategy of waiting for news for all $z < z_1$ and offering $\tilde{V}^L$ for all $z \geq z_1$ is feasible for the buyer and starting from any z, generates a payoff of

$$E_z\left[e^{-(\lambda+r)\hat{T}(z_1)}\left(V(z_1) - \tilde{V}^L - \frac{\lambda}{\lambda+r}\Pi_O\right)\right] + \frac{\lambda}{\lambda+r}\Pi_O > \frac{\lambda}{\lambda+r}\Pi_O.$$

Hence, the buyer's equilibrium payoff must be at least as large as when $z_1 = \beta$. If $\beta < \underline{z}$, then $\Pi(z) = V(\beta) - C^H < \frac{\lambda}{\lambda+r}\Pi_O$, which cannot be true. $\square$

LEMMA 12. *In any equilibrium, $U^L(z) = \frac{\lambda}{\lambda+r}U^L_O + E^L_z\left[e^{-(\lambda+r)T(\beta)}\right]\left[C^H + \frac{\lambda}{\lambda+r}(U^H_O - U^L_O)\right]$.*

Proof. Same as Fact 7 in the proof of Proposition 5. $\square$

LEMMA 13. *In any equilibrium: (i) U^L is non-decreasing, (ii) U^L is continuous, and (iii) W is continuous.*

Proof. (i) is by assumption. (ii) is the same as Fact 8 in the proof of Proposition 5. For (iii), we will first show that $\Pi(z_0^-) < \Pi(z_0^+)$ violates (B2). Starting from $z_0 - \varepsilon$, the buyer can offer $p = U^L(z_0 + \varepsilon)$ and trade with arbitrarily small probability at price which is bounded above by $\tilde{V}^L$ with payoff arbitrarily close to $\Pi(z_0^+)$, which violates (B2). Since U^L is continuous, if $\Pi(z_0^-) > \Pi(z_0^+)$, then $\Pi(z_0^-) = J(z_0, z_1)$ for some $z_1 > z_0$ (i.e., Z must jump upward as it approaches z_0 from the left). But J is continuous in its first argument and therefore $\Pi(z_0^+) < J(z_0, z_1)$ violating (B2). $\square$

LEMMA 14. *In any equilibrium, Q has continuous sample paths (i.e., there cannot exist an atom of trade with only with the low type).*

Proof. Suppose that starting from $Z_t = z_0$, the equilibrium belief process jumps to $Z_{t^+} = \alpha > z_0$. By Lemma 12, it must be that $U^L(z_0) = U^L(\alpha)$ and then U^L non-decreasing implies that $U^L(z) = U^L(z_0)$ for all $z \in (z_0, \alpha)$. Moreover, there must exist a $z_1 > \alpha$ such that Z evolves continuously in the interval

(α, z_1) (otherwise $Z_{t^+} \neq \alpha$). Stationarity then implies that α be a reflecting barrier for the belief process conditional on rejection starting from any $Z_t \geq \alpha$. We claim that these equilibrium dynamics require the following properties:

(i) $(\mathcal{A} - (\lambda + r))\Pi(z) = -\lambda \Pi_O$ and $\Gamma(z) \leq 0$ for all $z \in (\alpha, z_1)$;

(ii) $\Gamma(z) = 0$ for all $z \in (z_0, \alpha)$;

(iii) $U^{H'}(\alpha) = 0$;

(iv) W is C^2 at α.

(i) follow from the proof of Lemma 3. For (ii), the buyer's payoff starting from any $z \in (z_0, \alpha)$ is given by

$$\Pi(z) = J(z, \alpha) = \frac{\beta(\alpha) - \beta(z)}{\beta(\alpha)}(V^L - U^L(\alpha)) + \frac{\beta(z)}{\beta(\alpha)}\Pi(\alpha). \tag{44}$$

Since $\alpha \in \arg\max_{z' \geq z} J(z, z')$, envelope theorem yields

$$\begin{aligned}
\Pi'(z) = J_1(z, \alpha) &= \frac{\beta'(z)}{\beta(\alpha)}\left(\Pi(\alpha) + U^L(\alpha) - V^L\right) \\
&= \frac{\beta'(z)}{\beta(z)}\left(\Pi(z) + U^L(\alpha) - V^L\right) \\
&= \frac{\beta'(z)}{\beta(z)}\left(\Pi(z) + U^L(z) - V^L\right),
\end{aligned}$$

where the third inequality comes from replacing $\Pi(\alpha)$. This implies (ii). For (iii), $U^L(\alpha^-) = 0$ is implied by $U^L(z) = U^L(\alpha)$ for all $z \in (z_0, \alpha)$ and $U^{H'}(\alpha^+) = 0$ is implied by the reflecting barrier and smooth pasting. For (iv), $\mathcal{C}^1$ follows form *Robin* condition

$$\Pi'(\alpha^+) = \frac{\beta'(\alpha)}{\beta(\alpha)}\left(\Pi(\alpha) + U^L(\alpha) - V^L\right), \tag{45}$$

where $\beta'(\alpha)/\beta(\alpha)$ is the (unconditional) intensity at which the seller accepts at α and the second term is the difference of buyer's payoff following acceptance versus rejection. Differentiating (44) and taking limit $z \uparrow \alpha$ yields that $\Pi(\alpha^-)$ is equal to $\Pi(\alpha^+)$ in (45). For $\mathcal{C}^2$, if $\Pi''(\alpha^+) < \Pi''(\alpha^-)$ then $(\mathcal{A} - (\lambda + r))\Pi(z) > -\lambda \Pi_O$ in a neighborhood just below α, which violates (43). If $\Pi''(\alpha^+) > \Pi''(\alpha^-)$, then

$$\begin{aligned}
\Gamma'(\alpha^+) &= -\beta'(\alpha)\left(\Pi(\alpha) + U^L(\alpha) - V^L\right) - (1 - \beta(\alpha))\left(\Pi'(\alpha) + U^{H'}(\alpha)\right) + \Pi''(\alpha^+) \\
&= \Gamma'(\alpha^-) + \Pi''(\alpha^+) - \Pi''(\alpha^-) \\
&> 0,
\end{aligned}$$

where the second equality uses (iii) and the final inequality contradicts $\Gamma(z) \leq 0$ in (i). Hence, (i)-(iv) are established.

We claim that (i)-(iv) requires $\Pi(\alpha) \leq \frac{\lambda}{\lambda+r}\Pi_O$, which contradicts Lemma 11. First, (ii)-(iv) imply $\Gamma(\alpha) = 0$. Therefore to satisfy $Gamma(z) \leq 0$ for the neighborhood above α as in (i) requires $\Gamma'(\alpha) \leq 0$.

$$\begin{aligned}
\Gamma'(\alpha) \leq 0 &\iff -\beta'(\alpha)\left(\Pi(\alpha) + U^L(\alpha) - V^L\right) - (1-\beta(\alpha))\Pi'(\alpha) + \Pi''(\alpha) \leq 0 \\
&\iff (2p(\alpha)-1)\Pi'(\alpha) + \Pi(\alpha) \leq \beta(\alpha)\Gamma(\alpha) \\
&\iff \mathcal{A}\Pi(\alpha) \leq 0 \\
&\iff \Pi(\alpha) \leq \frac{\lambda}{\lambda+r}\Pi_O,
\end{aligned}$$

where the last one comes from (i) and (iv). $\square$

LEMMA 15. *There cannot exist an isolated point of singular trading intensity.*

Proof. We first prove that FB must be $\mathcal{C}^2$ at any such α. Since there are no jumps and α is an isolated point, Q is absolutely continuous in a neighborhood of α. Hence, there exists $\varepsilon > 0$ such that

$$(\mathcal{A} - (\lambda+r))\Pi(z) = -\lambda\Pi_O, \forall z \in N_\varepsilon(\alpha) \setminus \alpha. \tag{46}$$

By Lemma 13, U^L and W are continuous. Therefore, if $dQ_t > 0$ at $Z_t = \alpha$, it must be that

$$(1-\beta(\alpha))(V^L - U^L(\alpha) - \Pi(\alpha)) + \Pi'(\alpha^+). \tag{47}$$

To prove that W must be $\mathcal{C}^1$ at α, suppose that $\Pi'(\alpha^-) < \Pi'(\alpha^+)$. Starting from $Z_t = \alpha$, let $\tau_\varepsilon = \inf\{s \geq t : |\hat{Z}_s - \hat{Z}_t| \geq \varepsilon\}$. Let $\Delta \equiv \Pi'(\alpha^+) - \Pi'(\alpha^-) > 0$.An extension of Ito's formula (Harrison 2013, Proposition 4.12) gives

$$\begin{aligned}
e^{-(\lambda+r)\tau}\left(\Pi(Z_{\tau_\varepsilon}) - \frac{\lambda}{\lambda+r}\Pi_O\right) =& \Pi(\alpha) - \frac{\lambda}{\lambda+r}\Pi_O + \frac{1}{2}\gamma^2\Delta l(\tau_\varepsilon, \alpha) \\
&+ \int_0^{\tau_\varepsilon} e^{-(\lambda+r)s}(\mathcal{A} - (\lambda+r))\left(\Pi(Z_s) - \frac{\lambda}{\lambda+r}\Pi_O\right) \\
&+ \int_0^{\tau_\varepsilon} e^{-(\lambda+r)s}(\mathcal{A} - (\lambda+r))\gamma\Pi'(Z_s)dB_s.
\end{aligned}$$

Taking the expectation over the sample paths, we obtain a violation of (B3):

$$\begin{aligned}
E_z[e^{-(\lambda+r)\tau}\left(\Pi(Z_{\tau_\varepsilon}) - \frac{\lambda}{\lambda+r}\Pi_O\right)] =& \Pi(\alpha) + \frac{1}{2}\sigma^2\Delta E_\alpha[l(\tau_\varepsilon, \alpha)] \\
&+ \Pi(\alpha) + \frac{1}{2}\sigma^2\Delta\int_0^{\tau_\varepsilon} p_0(s, \alpha)ds > \Pi(\alpha),
\end{aligned}$$

where $p_0(s,\cdot)$ is the density of Z_s starting from $Z_0 = \alpha$. Next, suppose that $\Pi'(\alpha^-) > \Pi'(\alpha^+)$. Then

$$\begin{aligned}\Gamma(\alpha^-) &= (1-\beta(\alpha))(V^L - U^L(\alpha) - \Pi(\alpha)) + \Pi'(\alpha^-)\\ &> (1-\beta(\alpha))(V^L - U^L(\alpha) - \Pi(\alpha)) + \Pi'(\alpha^+)\\ &= \Gamma(\alpha^+) = 0,\end{aligned}$$

which violates Lemma 10 in a neighborhood below α. Thus, we have established that W is $\mathcal{C}^1$ at α.

For $\mathcal{C}^2$, since (43) holds with equality at α^+ and W is $\mathcal{C}^1$ at α, if $\Pi''(\alpha^-) > \Pi''(\alpha^+)$ then (43) is violated in a neighborhood below α. If $\Pi''(\alpha^-) < \Pi''(\alpha^+)$, then it must be that (43) holds with strict inequality in a neighborhood below α, which violates (46). Hence, the smoothness of W is established at α.

An isolated singularity at α means that for $t \le \tau_\varepsilon$, Q_t^{sing} increases only at times t where $Z_t = \alpha$. Thus, Q_t^{sing} is proportional to the local time of Z_t at α (Harrison 2013, Section 1.2), which we denote by $l_\alpha^Z(t)$. For $t \le \tau_\varepsilon$, Z evolves according to

$$Z_t = \hat{Z}_t + Q_t^{abs} + \delta l_\alpha^Z(t). \tag{48}$$

Harrison and Shepp (1981) show that (48) has a unique solution if and only if $|\delta| \le 1$, in which case Z is distributed as skew Brownian motion (SBM) with δ capturing the degree of skewness. If $\delta = 1$, then Z has a reflecting boundary at α, where for $\delta = 0$ there is no singularity at α and Z is a standard Ito diffusion. By Lemma 12, SBM involves a kink in the low type's value function at α, namely

$$\gamma U^{H'}(\alpha^+) = (1-\gamma)U^{H'}(\alpha^-), \tag{49}$$

where $\gamma = (1+\delta)/2$ (see Kolb 2016). Since Q must be monotonically increasing, it is sufficient to rule out any $\delta > 0$. There are three exhaustive cases to consider.

First, suppose $U^{H'}(\alpha^+) = U^{H'}(\alpha^-) = 0$. Then we have $\Gamma(\alpha) = 0$, and (43) holds in a neighborhood around α. Using an argument as in proof of Lemma 14 shows that $\Pi(\alpha) \le 0$, which yields a contradiction. Second, suppose $U^{H'}(\alpha^+) = U^{H'}(\alpha^-) \ne 0$, then (49) requires $\gamma = \frac{1}{2}$, which implies $\delta = 0$, contradicting the isolated singularity at α. Third, suppose $U^{H'}(\alpha^+) \ne U^{H'}(\alpha^-)$. By U^L non-decreasing, $U^{H'}(\alpha^-), U^{H'}(\alpha^+) \le 0$. Further, (49) and $\delta > 0$ imply that $U^{H'}(\alpha^-) > U^{H'}(\alpha^+) > 0$. As $\Gamma(\alpha) = 0$ and $\Gamma(z) \le 0$ as required by Lemma 10, we have $\Gamma'(\alpha^-) \ge 0$. As Γ' is strictly decreasing in $U^{H'}$, $U^{H'}(\alpha^+) < U^{H'}(\alpha^-)$ implies that $\Gamma'(\alpha^+) > \Gamma'(\alpha^-) \ge 0$. Since $\Gamma(\alpha) = 0$, $\Gamma(z) > 0$ for z in the neighborhood just above α, violating Lemma 10. Hence, a contradiction arises in all cases, and therefore there cannot exist an isolated point of singular trading intensity. □

The two key features of the constructed equilibrium are (i) a completion threshold β above which trade takes place immediately at a price of $\tilde{V}^L$, which have been shown in the equilibrium definition section, and (ii) for all $z < \beta$, trade takes place at a rate proportional to time. The above lemmas have shown that any singular component that corresponds to trade at a rate "faster" than dt, which can take the form of an atom (i.e., a jump in Z) or local time (e.g, a reflecting boundary), cannot be sustained in equilibrium. Therefore, Q process must be absolutely continuous as in (ii). $\square$