Appendix B

Plate Buckling Under Uniaxial Compression

B.1 Wide and Slender Struts

Consider the axial compression of a wide, thin plate, of thickness t, with similar length and width dimensions, a and b respectively (see Figure. B.1a). Having simply supported (pinned) ends and unsupported sides, the strut deflects laterally with uniform curvature and surface strains in opposite sense. The cross-section $b \times t$ remains rectangular during bending when biaxial stresses σ_1 and σ_2 exist in the plane of the plate as shown.

The lateral stress σ_2 arises from there being no strain in the 2-direction:

$$
\begin{aligned}
\varepsilon_2 &= 0 = (1/E)(\sigma_2 - \nu\sigma_1), \Rightarrow \sigma_2 = \nu\sigma_1 \\
\varepsilon_1 &= (1/E)(\sigma_1 - \nu\sigma_2) = (1 - \nu^2)(\pm\,\sigma_1)/E
\end{aligned}
\tag{B.1}
$$

In a slender strut, where b is small compared to a, the presence of lateral strains ε_2 results in *anticlastic curvature*, i.e. the cross-section does not remain rectangular. The distortion arises from opposing ε_2 within the tensile and compressive surfaces (the Poisson effect). Here, as $\sigma_2 = 0$, the axial and lateral surface strains are $\varepsilon_1 = (\pm\sigma_1)/E$ and $\varepsilon_2 = -\nu(\pm\sigma_1)/E$. Comparing ε_1 with equation (B.1) reveals a difference in ε_1 between slender and wide struts. Consequently, it becomes necessary to modify the flexure equation for a wide strut by the factor $(1 - \nu^2)$. From Figure B.1b, the curvature is expressed in reciprocal form:

$$\mathrm{d}^2 w/\mathrm{d}x^2 = 1/R = \varepsilon_1/z$$

Substituting from equation (B.1), with $M/I = \sigma_1/z$ (the sign of σ_1 depends upon sign of z),

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{1-\nu^2}{E}\frac{\sigma_1}{z} = \frac{(1-\nu^2)M}{EI}$$

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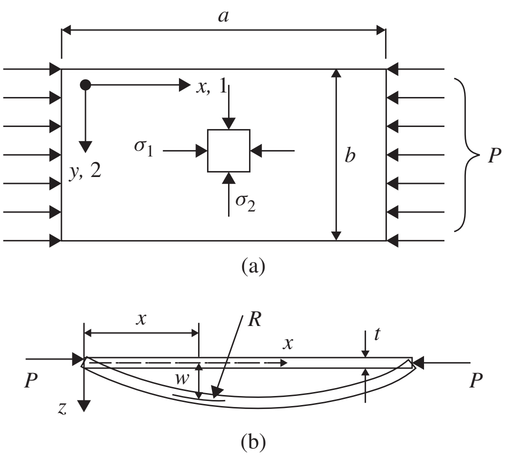

Figure B.1 Buckling of a wide plate strut

With a bending moment $M = -Pw$, the solution to the critical buckling load for a pinned-end plate is

$$P_{\mathrm{cr}} = \frac{\pi^2 EI}{(1 - \nu^2)a^2}$$

Substituting $P_{cr} = \sigma_{cr} bt$ and $I = Ak^2$ leads to the critical buckling stress in the section

$$\sigma_{\mathrm{cr}} = \frac{\pi^2 E}{(1 - \nu^2)(a/k)^2} \tag{B.2a}$$

Other edge fixings may be accounted for by rewriting equation (B.2a) in terms of an equivalent length L_e as

$$\sigma_{\mathrm{cr}} = \frac{\pi^2 E}{(L_e/k)^2} \tag{B.2b}$$

in which the equivalent lengths are:

$$L_e = a(1 - \nu^2)^{1/2} \quad \text{for pinned ends} \tag{B.3a}$$

$$L_e = (a/2)(1 - \nu^2)^{1/2} \quad \text{for fixed ends} \tag{B.3b}$$

$$L_e = (a/\sqrt{2})(1 - \nu^2)^{1/2} \quad \text{for pinned-fixed ends} \tag{B.3c}$$

$$L_e = 2a(1 - \nu^2)^{1/2} \quad \text{for fixed-free ends} \tag{B.3d}$$

Moreover, an inelastic buckling stress for wide-plate struts may be estimated from equation (B.2b) from these equivalent lengths when E in equation (B.2b) is replaced by the tangent modulus E_T. In this book we write the plate buckling stress formulae in an alternative form. Setting $k^2 = t^2/12$ in equation (B.2a) reveals the form adopted:

$$\sigma_{cr} = \frac{\pi^2}{12}\left(\frac{b}{a}\right)^2\left(\frac{E}{1-\nu^2}\right)\left(\frac{t}{b}\right)^2 = KE\left(\frac{t}{b}\right)^2 \tag{B.4}$$

where the buckling coefficient K depends upon the plate geometry, the elastic constant ν and the manner in which its ends are supported (see equations (B.3a–d)).

B.2 Plates with Supported Sides

More generally, with the unidirectional compression of thin, wide plates the sides may be free, simply supported or clamped. In the case of free, or unsupported, sides we return to the wide-strut problem considered above. Compressive loading of thin, wide plates, as opposed to slender struts, refers to plates whose side lengths a and b (see Figure B.1) do not differ by more than an order of magnitude. The theory outlined in Appendix C accounts for buckling of thin plates with similar geometry under biaxial compression when their four sides have various types of edge support. We may derive from this theory the following reduction to axial compression as required here.

B.2.1 Simple Supports

Under a unidirectional compression, the critical buckling stress σ_1, for a plate with all four sides simply supported, follows from setting $\sigma_2 = 0\,(\beta = 0)$ in equation (C.1). This gives [1]:

$$\sigma_{cr} = \frac{D\pi^2}{t}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2\left(\frac{a}{m}\right)^2 = \frac{D\pi^2}{tb^2}\left(\frac{mb}{a} + \frac{n^2 a}{mb}\right)^2 \tag{B.5a}$$

where $D = Et^3/[12(1-\nu^2)]$ is the flexural stiffness, and t, a and b are the plate thickness, length and breadth, respectively. The number of half-waves of buckling in the 1- and 2-directions are denoted by m and n, respectively. For example, the simply supported plate in Figure B.2 has buckled with a single half-wave ($n = 1$) in the 2-direction and three half-waves ($m = 3$) in the 1-direction.

When $n = 1$ in equation (B.5a), m will correspond to a minimum in the buckling stress expression:

$$\sigma_{cr} = \frac{D\pi^2}{tb^2}\left(\frac{m}{r} + \frac{r}{m}\right)^2 \tag{B.5b}$$

where $r = a/b$. Differentiating for the m-values that minimise equation (B.5b) with integral r:

$$\mathrm{d}\sigma_{cr}/\mathrm{d}m = 2(m/r + r/m)(1/r - r/m^2) = 0$$

$$m/r^2 + 1/m - 1/m - r^2/m^3 = 0$$

$$m^4 - r^4 = 0$$

$$(m - r)(m + r)(m^2 + r^2) = 0$$

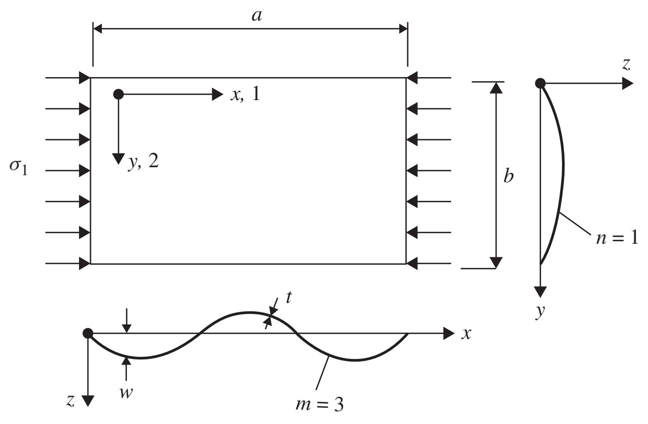

Figure B.2 Buckling of a thin, simply supported plate under uniaxial compression

The condition $m = r$ implies that the plate will buckle into an integral number of square cells $b \times b$ each under the same stress. That is, from equation (B.5b),

$$\sigma_{cr} = \frac{4D\pi^2}{tb^2} = \frac{\pi^2 E}{3(1-\nu^2)}\left(\frac{t}{b}\right)^2 \tag{B.6a,b}$$

which has a similar form to the wide strut buckling stress in equation (B.4). However, K will now take a different value depending upon r and m. Figure B.3 provides the buckling stress when both integer and non-integer values of r are taken with values of $m = 1$, 2, 3 and 4 in equation (B.5b). The trough in each curve corresponds to equation (B.6a). At the intersections of these curves, where m has been increased by one successively:

$$[m/r + r/m] = [(m+1)/r + r/(m+1)]$$
$$\therefore r = \sqrt{m(m+1)}$$

from which $r = \sqrt{2}$ for $m = 1, r = \sqrt{6}$ for $m = 2, r = \sqrt{12}$ for $m = 3$, etc., as shown.

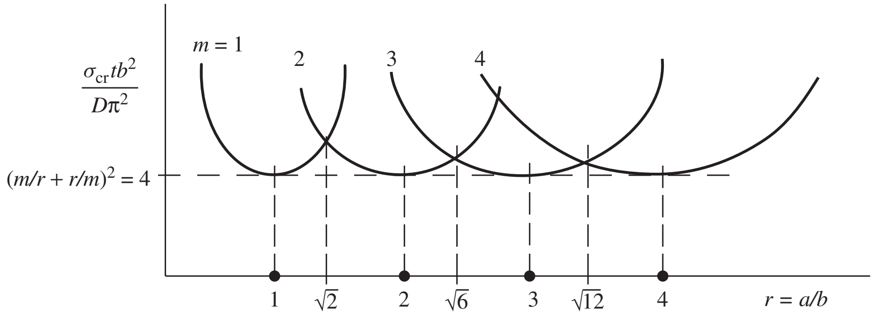

Figure B.3 Effect of r and m on uniaxial buckling stress

B.2.2 Other Edge Fixings

Various methods account for the effect of constraining the plate's edges. The simplest of these [2] generalise equation (B.6a) by introducing elastic constraint coefficients K as follows:

$$\sigma_{\mathrm{cr}} = \frac{K_r D\pi^2}{tb^2} = K_e E\left(\frac{t}{b}\right)^2 \tag{B.7a,b}$$

where equation (B.7a) applies to any edge fixings when

$$K_r = (m/r)^2 + p + q(r/m)^2 \tag{B.7c}$$

Clearly, with edge rotational restraint factors $p = 2$ and $q = 1$, equation (B.7c) contains equation (B.5b) in the special case of simple supports. The dependence of K_r upon the edge restraint factors (p and q), the plate aspect ratio ($r = a/b$) and the buckling mode m, has been established experimentally in certain cases. For example, Table B.1 applies to a plate with fixed sides. As r increases, the effect of edge restraint lessens and K_r approaches the minimum value of 4 as found from equation (B.6a) for a plate with simply supported edges. A more convenient graphical approach employs equation (B.7b) within design curves that provide the restraint coefficient $K_e = K_r \times \pi^2/[12(1 - \nu^2)]$ for both uniaxial and biaxial loadings [3]. These curves present the ratio between the critical elastic buckling stress $(\sigma_{cr})_e$ for a plate with a particular edge fixings to the buckling stress for a simply supported plate (i.e. equation (B.6b)) with a similar aspect ratio $r = a/b$. Figure B.4 shows how this stress ratio varies with r for clamped and various mixed-edge fixings for plates under uniaxial compression.

To apply Figure B.4, for example, let a 5 mm thick aluminium alloy plate with $a = 400$ mm and $b = 200$ have compression applied through a rigid clamp to its shorter sides. The buckling stress is required when the longer side is simply supported. Firstly, the reference stress, used to normalise the ordinate in Figure B.4, is given by equation (B.6b):

$$\sigma_{\mathrm{cr}} = \frac{\pi^2 E}{3(1 - \nu^2)}\left(\frac{t}{b}\right)^2 = \frac{\pi^2 \times 70000}{3(1 - 0.33^2)}\left(\frac{5}{200}\right)^2 = 25.9\ \mathrm{MPa}$$

in which the elastic constants are $E = 70$ GPa and $\nu = 0.33$. Reading from the appropriate graph in Figure B.4 gives the elastic buckling coefficient for $a/b = 2$:

$$\eta = (\sigma_{\mathrm{cr}})_e/\sigma_{\mathrm{cr}} = 1.22$$

Table B.1 Restraint coefficients for a plate with fixed edges

$r = a/b$	0.75	1.0	1.5	2.0	2.5	3.0
K_r	11.69	10.07	8.33	7.88	7.57	7.37

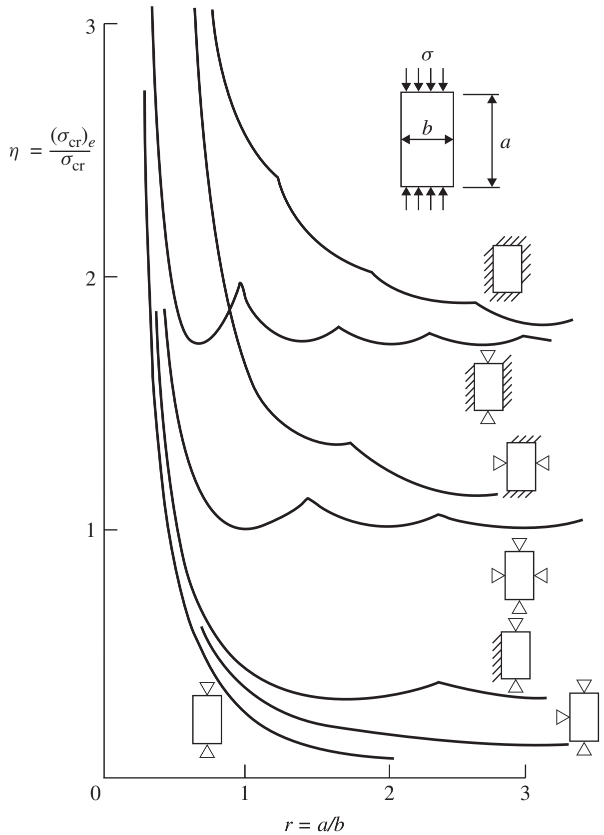

Figure B.4 Plate buckling under compression

Hence, the critical, elastic buckling stress follows as

$$(\sigma_{cr})_e = \eta\,\sigma_{cr} = 1.22 \times 25.9 = 31.6 \text{ MPa}$$

B.3 Inelastic Buckling

In thicker plates the buckling stress $(\sigma_{cr})_e$, calculated from Figure B.4 and equation (B.6b), in the manner of the previous example, may exceed the yield stress σ_y of the plate material. The solution will be invalid because linear elasticity cannot then be assumed at the critical stress level. Elastic buckling analyses employ the elastic compressive modulus which is constant up to the proportional limit. A correction for plastic buckling accounts for the decreasing gradient to the flow curve as the stress increases within the plastic range (see Figure B.5a).

In a deflected, slender strut a stress gradient arises where the sum of direct and bending stresses amounts to a greater compression on its concave side. Where the elastic limit is exceeded a tangent modulus E_T (see Figure B.5b) is often employed with Euler's strut

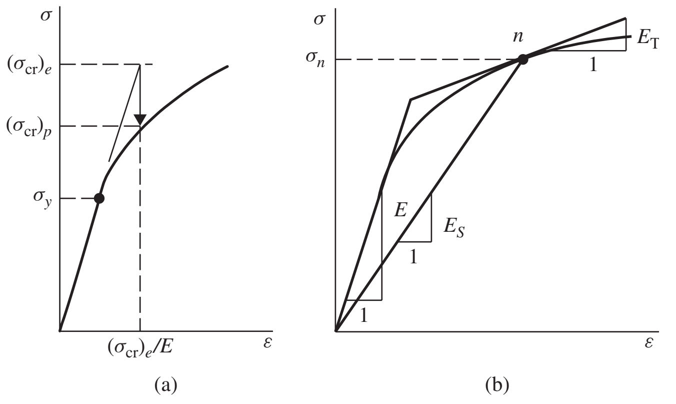

Figure B.5 Tangent and secant moduli

theory to provide an average, compressive stress at buckling. Equally, E_T may replace E for an account of plastic buckling of thin plates, particularly where through-thickness stress gradients are less severe. Figure B.5a shows that where plasticity has occurred it reduces the buckling stress to a lower level $(\sigma_{\mathrm{cr}})_p$. Figure B.5b shows how both the tangent and secant moduli, E_T and E_S respectively, are used to account for a plastic stress level. The secant modulus gives the total strain at a reference point n as

$$\varepsilon_n = \sigma_n / E_S \tag{B.8a}$$

The Ramberg–Osgood description [4] to a stress–strain curve gives the total strain under a plastic stress level as

$$\varepsilon = \sigma/E + \alpha(\sigma/E)^m \tag{B.8b}$$

Combining equations (B.8a,b) at the reference stress σ_n gives

$$E/E_S - 1 = \alpha(\sigma_n/E)^{m-1} \tag{B.9a}$$

Differentiating equation (B.8b) gives the gradient $E_T = \mathrm{d}\sigma/\mathrm{d}\varepsilon$ at point n:

$$(1/m)(E/E_T - 1) = \alpha(\sigma/E)^{m-1} \tag{B.9b}$$

Now let σ_n be the stress level at which $E_T = E/2$. Equation (B.9b) gives

$$\alpha = (1/m)(\sigma_n/E)^{1-m} \tag{B.9c}$$

Substituting equation (B.9c) into equation (B.8b) and multiplying through by E/σ_n leads to a normalised stress–total strain relationship

$$\varepsilon = \frac{\sigma_n}{E}\left[\frac{\sigma}{\sigma_n} + \frac{1}{m}\left(\frac{\sigma}{\sigma_n}\right)^m\right] \tag{B.10}$$

Table B.2 m-values for aluminium (L and DTD - grades) and steel (S - grades)

Material	L71	L73	DTD687	L65	S520	S514	S515	S134	S96
m	16	17	15	29	4.9	17	8.4	7	105
Form	S	S	P	B	S	S	S	B	B

where σ_n and m are material properties found from fitting equation (B.10) to a stress–strain curve. Table B.2 gives typical m-values that lie in the range 16–29 for aluminium alloys and 5–105 for steels [5]. The m-value depends upon whether the material is in sheet (S), plate (P) or bar (B) form.

Figure B.5a shows that the elastic strain under $(\sigma_{cr})_e$ is $\varepsilon = (\sigma_{cr})_e/E$. Substituting this value for ε in equation (B.10) allows $\sigma = (\sigma_{cr})_p$ to be found. Design data [6] adopt a graphical solution to $(\sigma_{cr})_p$, employing a *plasticity reduction factor* $\mu < 1$ in

$$\mu = (\sigma_{\text{cr}})_p/(\sigma_{\text{cr}})_e \tag{B.11}$$

Both m and σ_n influence μ in the manner shown in Figure B.6. When applying Figure B.6, firstly, we find from Figure B.4, the critical elastic buckling stress $(\sigma_{\text{cr}})_e$. This determines the ratio $(\sigma_{\text{cr}})_e/\sigma_n$ from which a μ-value is found from Figure B.6. Equation (B.11) is then employed to find $(\sigma_{\text{cr}})_p$.

Note that equation (B.6) may be used to find $(\sigma_{\text{cr}})_e$ for a simply supported plate subjected to uniaxial compression across its shorter sides. For example, consider the buckling of a 5 mm thick S520 steel plate with sides $a = 600$ mm and $b = 300$ mm under this condition. Given this material's properties ($E = 210$ GPa, $v = 0.27$, $m = 4.9$, $\sigma_y = 300$ MPa and $\sigma_n = 450$ MPa), we find from equation (B.6b) the critical elastic buckling stress $\sigma_{\text{cr}} = (\sigma_{\text{cr}})_e$:

$$(\sigma_{\text{cr}})_e = \frac{\pi^2 E}{3(1-v^2)}\left(\frac{t}{b}\right)^2 = \frac{\pi^2(210 \times 10^3)}{3(1-0.27^2)}\left(\frac{5}{200}\right)^2 = 465.75 \text{ MPa}$$

As this stress exceeds $\sigma_y = 300$ MPa, it requires a correction for plasticity. The abscissa in Figure B.6 is $465.75/450 = 1.035$, from which its ordinate $\mu = 0.83$ applies to $m = 4.9$.

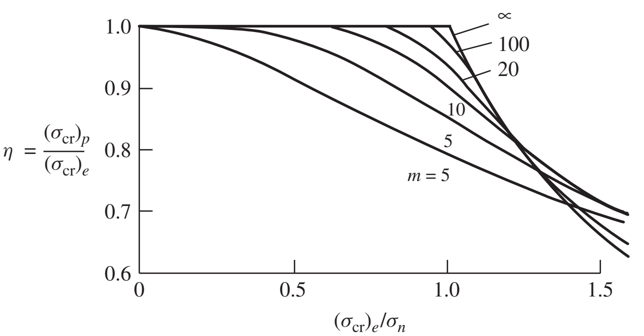

Figure B.6 Plasticity reduction factor for a simply supported plate

Hence, from equation (B.11), the critical plastic buckling stress becomes

$$(\sigma_{\mathrm{cr}})_p = \mu(\sigma_{cr})_e = 0.83 \times 465.75 = 386.57 \text{ MPa}$$

We should note here that Figure B.6 is constructed for a plate having all four sides simply supported in a material with Poisson's ratios 0.3 and 0.5 within the elastic and plastic ranges, respectively. Fortunately, the greatest possible deviations in v (from 0.25 to 0.35) can have only a negligible influence upon the reduction factor and therefore this effect may be ignored. The elastic Poisson's ratio effect in Figure B.4 should, however, be accounted for. The effect of the plate supports upon the reduction factor μ has been discussed further in [7, 8]. This amounts to a vertical shift in the curves in Figure B.6 when the sides are supported by other means. For example, if this plate has clamped edges at both ends and along both sides, we read from the upper curve in Figure B.4 the following stress ratio, corresponding to the integral number of buckled panels (i.e. $r = a/b = 2$) in the sheet:

$$\eta = (\sigma_{\mathrm{cr}})_e/\sigma_{\mathrm{cr}} = 2$$

As the denominator has been identified with 465.75 MPa for a simply supported plate, the required elastic buckling stress for the clamped plate follows as $(\sigma_{cr})_e = 2 \times 465.75 = 931.5$ MPa. Strictly speaking, three further corrections should be applied to this stress magnitude:

(i) $(\sigma_{\mathrm{cr}})_e$, as calculated above, should be multiplied by the factor $(1 - 0.3^2)/(1 - v^2)$ for a Poisson's ratio different from 0.3 (the basis of Figure B.4). This gives

$$(\sigma_{\mathrm{cr}})_e = \frac{1 - 0.30^2}{1 - 0.27^2} \times 931.5 = 914.3 \text{ MPa}$$

(ii) $(\sigma_{\mathrm{cr}})_{\mathrm{e}}$ in (i) above should then be corrected for plasticity using Figure B.6 with an abscissa value of $914.3/450 = 2.03$. This gives $\mu \approx 0.62$, at $m = 4.9$, for a plate with simply supported edges.

(iii) With clamped plate edges, a downward shift in the respective curve in Figure B.6 has shown [6] that $\mu \approx 0.58$, i.e. a multiplying factor of 0.935 applies to the plasticity reduction factor found in (ii). Hence, the critical plastic buckling stress is estimated as

$$(\sigma_{\mathrm{cr}})_p = \mu(\sigma_{\mathrm{cr}})_e = 0.58 \times 914.3 = 530.3 \text{ MPa}$$

In the event of a near-ultimate strength value being found from this procedure the implication is that, in the manner of its fixing, this plate cannot buckle without a significant amount of accompanying plasticity. Elastic-plastic buckling at a lesser critical stress would be more likely to arise in a plate of reduced thickness.

B.4 Post-Buckling

When a plate has buckled the load may be increased further as the axial compressive stress increases in the material adjacent to the side supports (see Figure B.7). Only a

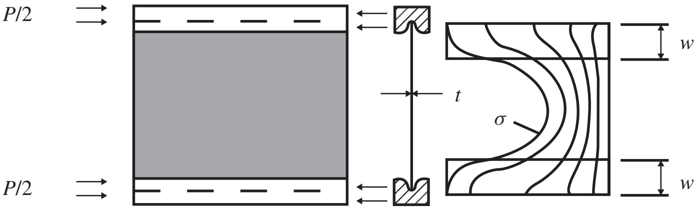

Figure B.7 Stress distribution in a buckled plate

slight increase in axial stress occurs in the central buckled material. Consequently, if we assume that the whole load P is carried by two edge strips of effective width $2w$, over which σ is assumed constant, the load supported becomes

$$P = 2wt\sigma \tag{B.12a}$$

With the edges of our equivalent elastic plate all simply supported, the critical buckling stress becomes [2]

$$\sigma_{cr} = \frac{\pi^2 E}{3(1-\nu^2)}\left(\frac{t}{2w}\right)^2 \tag{B.12b}$$

from which w may be found once σ_{cr} attains the yield stress σ_y:

$$w = \frac{\pi t}{2}\sqrt{\frac{E}{3\sigma_y(1-\nu^2)}} \tag{B.12c}$$

Taking $\nu = 0.3$ in equation (B.12c) gives

$$w = 0.95t\sqrt{E/\sigma_y} \tag{B.13a}$$

Hence when $b = 2w$ the full plate width is used most effectively in post-buckling. With alternative support along the longer side, the asymptotic values of the critical elastic buckling stress ratio apply. These appear in Figure B.4, in which equation (B.12b) defines the denominator for the ordinate $\eta = (\sigma_{cr})_e/\sigma_{cr}$. For example, when one long side is simply supported and the other parallel side is free, the second lowest plot in Figure B.6 shows that the critical stress ratio is 0.106. This modifies the semi-effective width for the plate:

$$w = 0.95t\sqrt{0.106E/\sigma_y} = 0.31t\sqrt{E/\sigma_y} \tag{B.13b}$$

Experiment [9] has shown that the coefficient in equation (B.13a) is nearer 0.85. However, when the ‘plate’ forms the compressive surface of a box section in bending the coefficient is 1.14.

References

[1] Vinson, J. R. *Structural Mechanics*, Wiley, 1974.
[2] Timoshenko, S. P. and Gere, J. *Theory of Elastic Stability*, 2nd edition, McGraw-Hill, 1961.

[3] ESDU 72019, Buckling of flat isotropic plates under uniaxial and biaxial loading, August 1972.
[4] Ramberg, W. and Osgood, W. R. Description of stress-strain curves by three parameters, NACA Tech Note 902, April 1943.
[5] ESDU 76016, Generalisation of smooth, continuous stress-strain curves for metallic materials, May 1985.
[6] ESDU 83044, Plasticity correction factors for plate buckling, December 1983.
[7] Gerard, G. and Becker, H. *Handbook of Structural Stability*, Part I - Buckling of flat plates, NACA Tech Note 3781, October 1954.
[8] Gerard, G. *Introduction to Structural Stability Theory*, McGraw-Hill, 1962.
[9] Yu, W., *Cold Formed Steel Structures*, McGraw-Hill, 1973.