EE323 Digital Signal Processing

Lab 4: Discrete Fourier Transform

> **SID:** 11812214
>
> **Name:** 任振裕

4.1 Introduction

This laboratory will introduce the Discrete Fourier Transform (DFT) and the associated sampling and windowing effects.

The corresponding equations needed for this lab are listed below:

> *DTFT: Discrete Time Fourier Transform.*
>
> $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{1}$$
>
> *IDTFT: Inverse Discrete Time Fourier Transform.*
>
> $$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\mathrm{d}\omega \tag{2}$$
>
> *DFT: Discrete Fourier Transform.*
>
> $$X_N[k] = \sum_{n=0}^{N-1} x_N[K]e^{j2\pi kn/N} \tag{3}$$
>
> *IDFT: Inverse Discrete Fourier Transform.*
>
> $$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_N(k)e^{j2\pi kn/N} \tag{4}$$

The main content for this lab is：

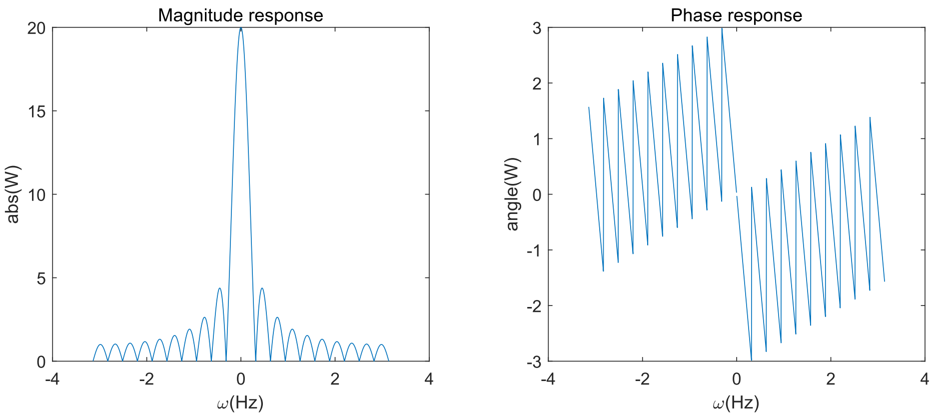
4.2 Deriving the DFT from the DTFT

4.2.3 Windowing Effects

- Phase and magnitude response of $W(e^{j\omega})$:

  `q1.m`

```
clear;
N=20;
w=-pi:pi/1000:pi;
W = (w~=0).*exp(-j*w*(N-1)/2).*sin(w*N/2)./(sin(w/2))+(w == 0)*N;
subplot(121),plot(w,abs(W)),xlabel('\omega(Hz)'),ylabel('abs(W)'),title('Magnitude response');
subplot(122),plot(w,angle(W)),xlabel('\omega(Hz)'),ylabel('angle(W)'),title('Phase response');
```



- An expression for $X(e^{j\omega})$:

  $\because x[n] = cos(\frac{\pi n}{4}) = \frac{1}{2}(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n})$ and $Ft\{e^{j\omega_0 n}\} = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$

  $\therefore$ We have:

  $X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \pi\delta(\omega - \frac{\pi}{4} - 2\pi l) + \sum_{l=-\infty}^{\infty} \pi\delta(\omega + \frac{\pi}{4} - 2\pi l)$

  More specifically,

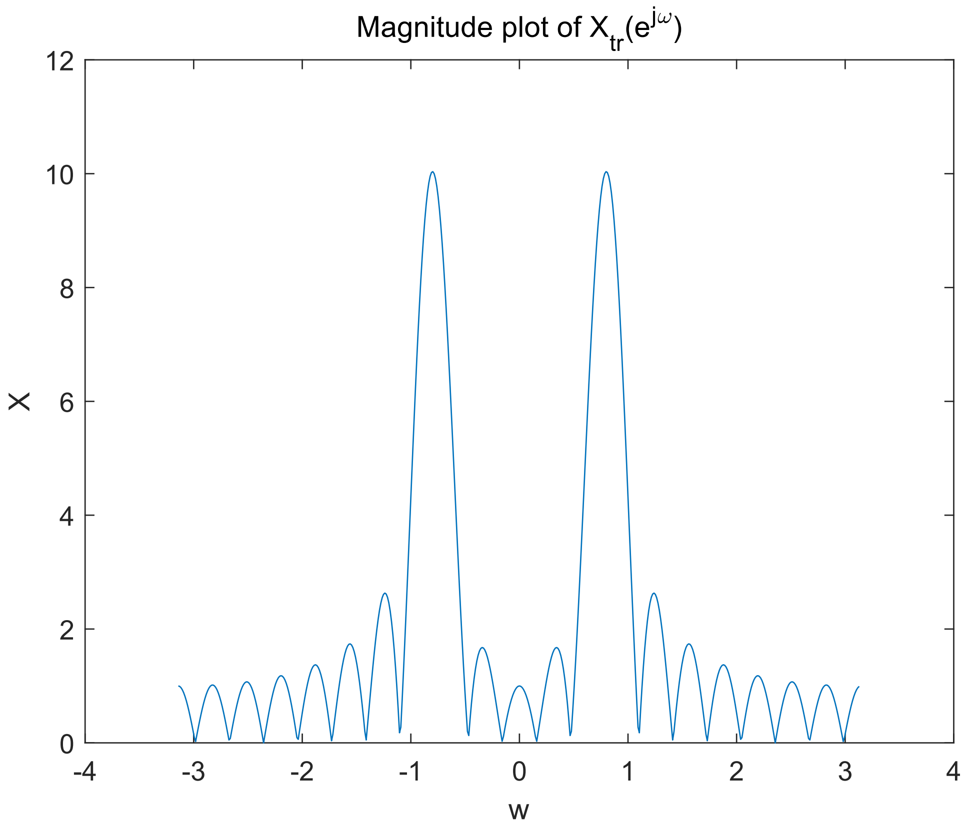
$$X(e^{j\omega}) = \pi\delta(\omega - \tfrac{\pi}{4}) + \pi\delta(\omega + \tfrac{\pi}{4}), \qquad -\pi \le \omega \le \pi \tag{5}$$

  And $X(e^{j\omega})$ has a period of 2π.

- Magnitude plot of $X_{tr}(e^{j\omega})$:

  `q2.m`

```
clear;
% Truncate the signal
n=0:19;
x=cos(pi/4*n);
[X,w]=DTFT(x,512);
plot(w,abs(X)),xlabel('w'),ylabel('X'),title('Magnitude plot of X_{tr}(e^{j\omega})');
```



- Difference between $|X_{tr}(e^{j\omega})|$ and $|X(e^{j\omega})|$:
  - According to previously derived equation (5), it could be observed that $|X(e^{j\omega})|$ is composed of two impulses in one period which are separately centered at $\frac{\pi}{4}$ and $-\frac{\pi}{4}$.
  - However, according to the above figure (magnitude plot of $X_{tr}(e^{j\omega})$), it can be observed that $|X_{tr}(e^{j\omega})|$ is composed of two time-shiftings of a sinc waves. And actually, these sinc waves are centered at $\frac{\pi}{4}$ and $-\frac{\pi}{4}$ respectively.

    Proof for this argument is quite straightforward:

    According to the multiplication property of DTFT, we have:

$$\begin{aligned} X_{tr}(e^{j\omega}) &= Ft\{x[n] \times w[n]\} \\ &= X(e^{j\omega}) \circledast W(e^{j\omega}) \\ &= (\pi\delta(\omega - \frac{\pi}{4}) + \pi\delta(\omega + \frac{\pi}{4})) \circledast W(e^{j\omega}) \\ &= \pi W(e^{j(\omega-\pi/4)}) + \pi W(e^{j(\omega+\pi/4)}) \end{aligned} \tag{6}$$

    Moreover, we can derive that $W(e^{j\omega})$ is a sinc wave:

$$\begin{aligned} W\left(e^{j\omega}\right) &= \sum_{n=-\infty}^{\infty} w[n]e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n} \\ &= \begin{cases} \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}, & \text{for } \omega \neq 0, \pm 2\pi, \ldots \\ N, & \text{for } \omega = 0, \pm 2\pi, \ldots \end{cases} \end{aligned} \tag{7}$$

$$W\left(e^{j\omega}\right) = \frac{e^{-j\omega N/2}}{e^{-j\omega/2}} \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}} = e^{-j\omega(N-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

  Proof completed.

- Effects of using a different length of the window:

  According to equation (6) ,(7), using different length of the window will both change the amplitude and zero points of $X_{tr}(e^{j\omega})$.

4.3 The Discrete Fourier Transform

4.3.1 Computing the DFT

Section A: DFTsum

- MATLAB codes:

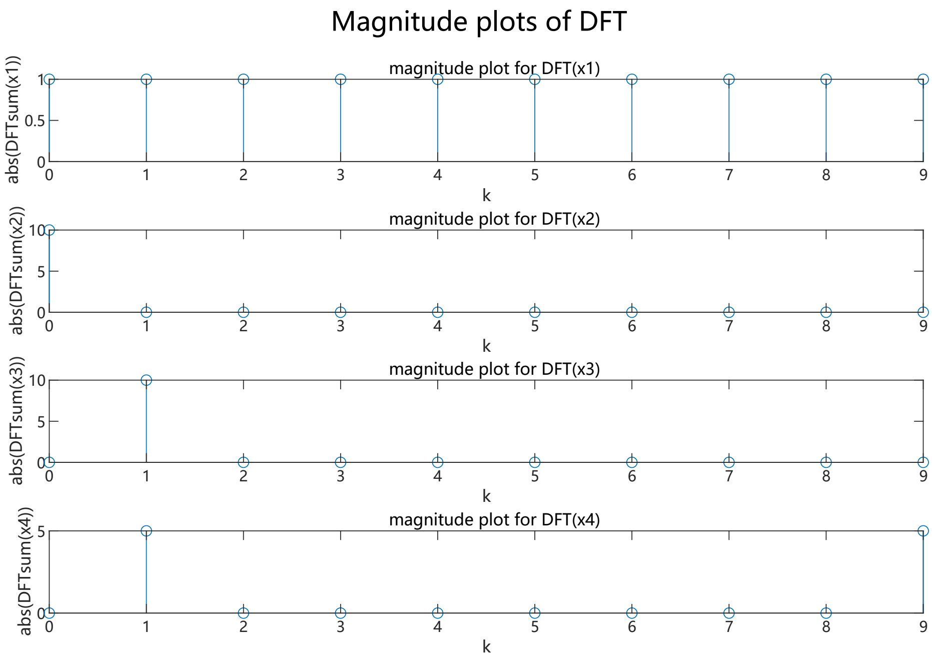
  `DFTsum.m`

  ```
  function X = DFTsum(x)
  % X_N[k]=\sum_{n=0}^{N-1}x[n]e^{-j*2*pik*n/N}
  % X(k)=\sum_{n=1}^Nx(n)e^(-j*2*pi*(k-1)*(n-1)/N), 1≤k≤N
  N = length(x);
  X = zeros(1,N);
  for k=1:N
  for n=1:N
      X(k)= X(k)+x(n)*exp(-j*2*pi*(k-1)*(n-1)/N);
  end
  end
  end
  ```

  `magnitudePlots.m`

  ```
  clear;
  n=[0:9];%N=10;
  x1=(n==0);
  x2=ones(1,10);
  x3=exp(j*2*pi*n/10);
  x4=cos(2*pi*n/10);
  figure(1)% DFT plots
  sgtitle('Magnitude plots')
  subplot(411),stem([1:10],abs(DFTsum(x1))),xlabel('k'),ylabel('abs(DFTsum(x1)
  )'),title('magnitude plot for DFT(x1)')
  subplot(412),stem([1:10],abs(DFTsum(x2))),xlabel('k'),ylabel('abs(DFTsum(x2)
  )'),title('magnitude plot for DFT(x2)')
  subplot(413),stem([1:10],abs(DFTsum(x3))),xlabel('k'),ylabel('abs(DFTsum(x3)
  )'),title('magnitude plot for DFT(x3)')
  subplot(414),stem([1:10],abs(DFTsum(x4))),xlabel('k'),ylabel('abs(DFTsum(x4)
  )'),title('magnitude plot for DFT(x4)')
  ```

-



- Closed-form analytical expressions for DFT：

  According to equation (3), we have ($N = 10$):

  - Case 1: $x[n] = \delta[n]$

    $X[k] = \sum_{n=0}^{9} \delta[n] e^{\frac{-j2\pi kn}{N}} = 1, \qquad k = 0, 1, \ldots, 9$

  - Case 2: $x[n] = 1$

    $X[k] = \sum_{n=0}^{9} e^{\frac{-j2\pi kn}{N}} = 10\delta[k], \qquad k = 0, 1, \ldots, 9$

  - Case 3: $x[n] = e^{j2\pi n/10}$

    $X[k] = \sum_{n=0}^{9} e^{\frac{j2\pi(1-k)n}{N}} = 10\delta[k-1], \qquad k = 0, 1, \ldots, 9$

  - Case 4: $x[n] = cos(2\pi n/10)$

    $X[k] = \sum_{n=0}^{9} cos(2\pi n/10) e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{9} \frac{1}{2}(e^{j2\pi n/10} + e^{-j2\pi n/10}) e^{\frac{-j2\pi kn}{N}}$

    $= \frac{1}{2} \sum_{n=0}^{9} e^{\frac{j2\pi(1-k)n}{N}} + \frac{1}{2} \sum_{n=0}^{9} e^{\frac{j2\pi(-1-k)n}{N}} = 5\delta[k-1] + 5\delta[k+1]$

    $\because \; k = 0, 1, \ldots, 9$

    We have: $X[k] = 5\delta[k-1], \qquad k = 0, 1, \ldots, 9$

    > Case 2, 3, 4 all use the property below to simplify the computation:
    >
    > $$W_N = e^{-j*2\pi/N}$$
    >
    > $$\sum_N W_N^k = \begin{cases} N, & k = mN \\ 0, & otherwise \end{cases}$$

Section B: IDFTsum

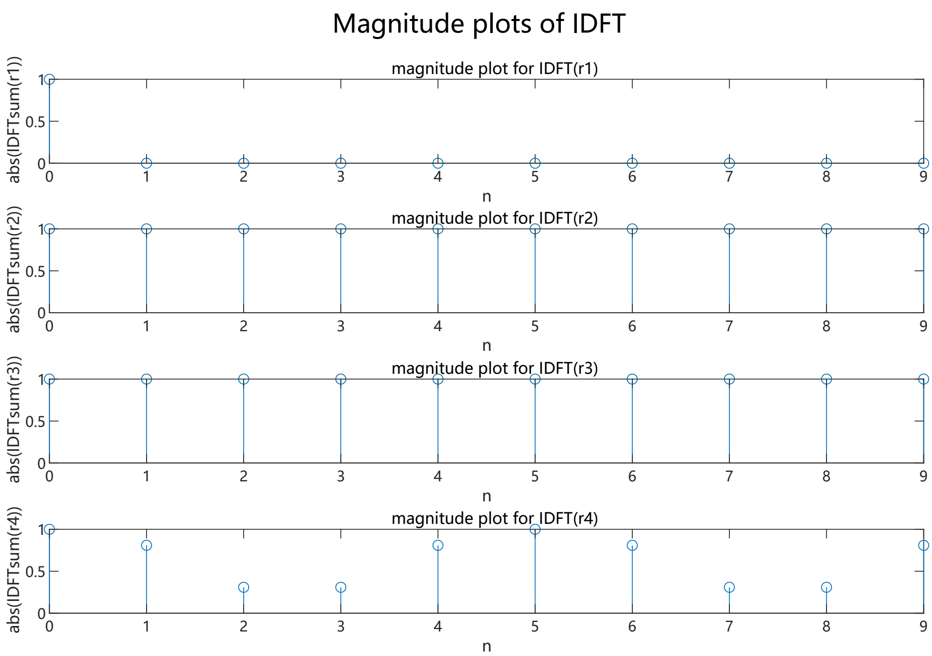
- MATLAB codes:

  `IDFTsum.m`

```
function x = IDFTsum(X)
% x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X_N(k)e^{j2\pi kn/N}
% x(n)=(1/N)\sum_{k=1}^{N}X(k)*exp(j*2*pi*(k-1)*(n-1)/N),1≤n≤N
N = length(X);
x = zeros(1,N);
for n=1:N
for k=1:N
    x(n)=x(n)+(1/N)*sum(X(k)*exp(j*2*pi*(k-1)*(n-1)/N));
end
end
end
```

Add below codes to previous `magnitudePlots.m`:

```
figure(2)% IDFT plots
r1=IDFTsum(X1);r2=IDFTsum(X2);r3=IDFTsum(X3);r4=IDFTsum(X4);
sgtitle('Magnitude plots of IDFT')
subplot(411),stem([0:9],abs(r1)),xlabel('n'),ylabel('abs(IDFTsum(r1))'),title('magnitude plot for IDFT(r1)')
subplot(412),stem([0:9],abs(r2)),xlabel('n'),ylabel('abs(IDFTsum(r2))'),title('magnitude plot for IDFT(r2)')
subplot(413),stem([0:9],abs(r3)),xlabel('n'),ylabel('abs(IDFTsum(r3))'),title('magnitude plot for IDFT(r3)')
subplot(414),stem([0:9],abs(r4)),xlabel('n'),ylabel('abs(IDFTsum(r4))'),title('magnitude plot for IDFT(r4)')
```

- 

- Analysis: `IDFTsum.m` successfully recovers the original signal, which means that the design of `IDFTsum.m` is right.

4.3.2 Matrix Representation of the DFT

Section A: DFTmatrix

- MATLAB codes:

`DFTmatrix.m`

```
function A = DFTmatrix(N)
A=zeros(N,N);
for k=1:N
for n=1:N
    A(k,n)=exp(-j*2*pi*(k-1)*(n-1)/N);
end
end
end
```

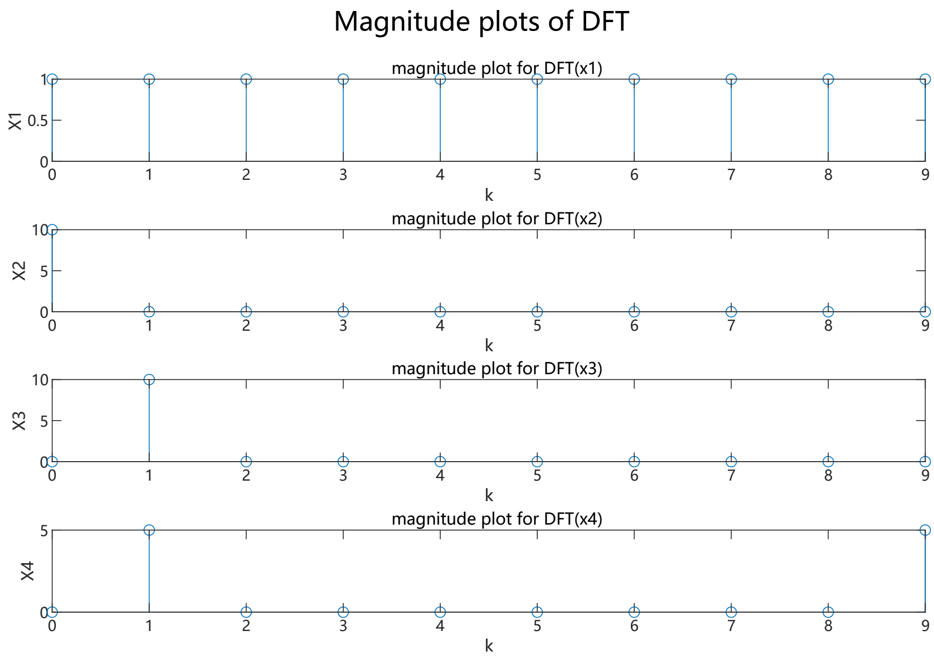
- Print-outs for matrix **A** for N=5:

```
>> A=DFTmatrix(5)

A =

   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   1.0000 + 0.0000i   0.3090 - 0.9511i  -0.8090 - 0.5878i  -0.8090 + 0.5878i   0.3090 + 0.9511i
   1.0000 + 0.0000i  -0.8090 - 0.5878i   0.3090 + 0.9511i   0.3090 - 0.9511i  -0.8090 + 0.5878i
   1.0000 + 0.0000i  -0.8090 + 0.5878i   0.3090 - 0.9511i   0.3090 + 0.9511i  -0.8090 - 0.5878i
   1.0000 + 0.0000i   0.3090 + 0.9511i  -0.8090 + 0.5878i  -0.8090 - 0.5878i   0.3090 - 0.9511i
```

- Magnitude plots for DFT's:

`q4_3_2.m`

```
clear;
n=[0:9]';%N=10;
x1=(n==0);x2=ones(10,1);x3=exp(j*2*pi*n/10);x4=cos(2*pi*n/10);
A=DFTmatrix(10);
X1=A*x1;X2=A*x2;X3=A*x3;X4=A*x4;
figure(1)% DFT plots
sgtitle('Magnitude plots of DFT')
subplot(411),stem([1:10]',abs(X1)),xlabel('k'),ylabel('X1'),title('magnitude plot for DFT(x1)')
subplot(412),stem([1:10]',abs(X2)),xlabel('k'),ylabel('X2'),title('magnitude plot for DFT(x2)')
subplot(413),stem([1:10]',abs(X3)),xlabel('k'),ylabel('X3'),title('magnitude plot for DFT(x3)')
subplot(414),stem([1:10]',abs(X4)),xlabel('k'),ylabel('X4'),title('magnitude plot for DFT(x4)')
```



- To calculate DFT using matrix product, we need N^2 multiples, which means a computing complexity of $O(n^2)$.

Section B: IDFTmatrix

- MATLAB codes:

  `IDFTmatrix.m`

```
function B = IDFTmatrix(N)
A=DFTmatrix(N);
B=(1/N)*A';
end
```

- Print-outs for matrix **B** for N=5:

```
>> B=IDFTmatrix(5)

B =

   0.2000 + 0.0000i   0.2000 + 0.0000i   0.2000 + 0.0000i   0.2000 + 0.0000i   0.2000 + 0.0000i
   0.2000 + 0.0000i   0.0618 + 0.1902i  -0.1618 + 0.1176i  -0.1618 - 0.1176i   0.0618 - 0.1902i
   0.2000 + 0.0000i  -0.1618 + 0.1176i   0.0618 - 0.1902i   0.0618 + 0.1902i  -0.1618 - 0.1176i
   0.2000 + 0.0000i  -0.1618 - 0.1176i   0.0618 + 0.1902i   0.0618 - 0.1902i  -0.1618 + 0.1176i
   0.2000 + 0.0000i   0.0618 - 0.1902i  -0.1618 - 0.1176i  -0.1618 + 0.1176i   0.0618 + 0.1902i
```

- Print out the elements of $\mathbf{C} = \mathbf{BA}$:

```
>> C=DFTmatrix(5)*IDFTmatrix(5)

C =

   1.0000 + 0.0000i  -0.0000 + 0.0000i  -0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
  -0.0000 - 0.0000i   1.0000 + 0.0000i  -0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
  -0.0000 - 0.0000i  -0.0000 - 0.0000i   1.0000 + 0.0000i  -0.0000 + 0.0000i  -0.0000 + 0.0000i
   0.0000 - 0.0000i   0.0000 - 0.0000i  -0.0000 - 0.0000i   1.0000 - 0.0000i  -0.0000 - 0.0000i
   0.0000 - 0.0000i   0.0000 - 0.0000i  -0.0000 - 0.0000i  -0.0000 - 0.0000i   1.0000 + 0.0000i
```

- Analytical expression for the elements of $\mathbf{B}$:

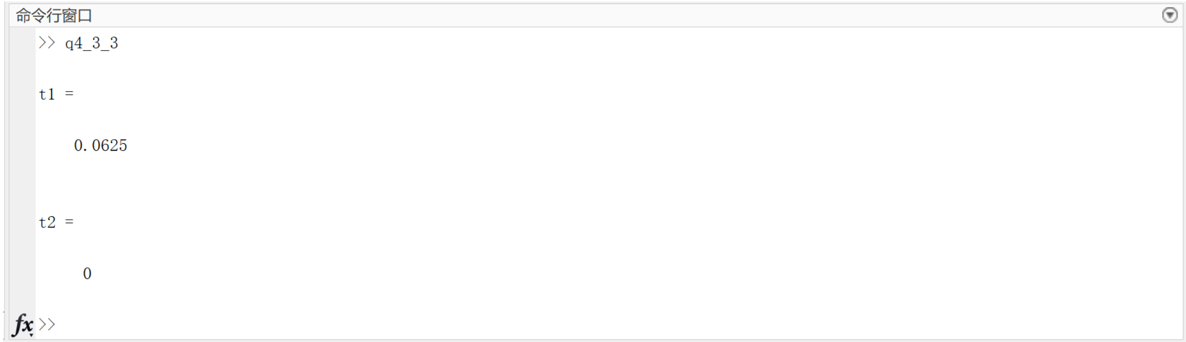
$$B_{kn} = \frac{1}{N} e^{j2\pi(k-1)(n-1)/N}$$

4.3.3 Computation Time Comparison

- MATLAB codes to test the cputime;

```
clear;
n=0:511;x=cos(2*pi*n);A=DFTmatrix(512);
% DFTsum     t1
start1=cputime;X=DFTsum(x);end1=cputime;t1=end1-start1;display(t1);
x=x';
% DFTmatrix     t2
start2=cputime;X=A*x;end2=cputime;t2=end2-start2;display(t2);
```

- Running Result:



- Analysis:
  - Method using DFTsum costs 0.0625s to calculate the DFT.
  - However, method using DFTmatrix almost costs no time.
  - After several repetitions, the same conclusion hold that the cputime of DFTmatrix is almost zero while method using DFTsum approximately cost 0.4—0.6s time.
  - Although the complexity of these two methods are same, method using DFTsum is explicitly lower than method using DFTmatrix. This is because that `DFTsum.m` uses two "for" loops in the algorithm which runs quite slow in MATLAB while the other method only involves the matrix operations which is exact the strength of MATLAB.

4.4 Conclusion & Experience

4.4.1 Conclusion.

- DFT actually uses $\omega = 2\pi k/N$ to sample the DTFT.
- There are two ways to calculate DFT and IDFT:
  - Use DFTsum to calculate:

$$\text{(DFT)} \qquad X_N[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$\text{(inverse DFT)} \qquad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_N[k] e^{j2\pi kn/N}$$

  - Use DFTmatrix to calculate:

$$(\text{DFT}):\begin{cases} X & = Ax \\ X_k & = \sum_{n=1}^{N} A_{kn} x_n \\ A_{kn} & = e^{-j2\pi(k-1)(n-1)/N} \end{cases}$$

$$(\text{inverse DFT}):\begin{cases} x & = BX \\ x_n & = \sum_{k=1}^{N} B_{kn} X_k \\ B_{kn} & = \frac{1}{N} e^{j2\pi(k-1)(n-1)/N} \end{cases}$$

- In MATLAB programming, we should always try to avoid using any "for" loops to make the program running faster.

4.4.2 Experience.

- Writing down the relative equations could help me understand the principles of this lab better.
- We should always try our best to avoid using "for" loops in MATLAB programming.