Hessian Derivation

October 2, 2023

To solve equation

$$
\begin{cases}
\dot{\mathbf{y}} = \mathbf{z} & \text{(1a)} \\
0 = \mathbf{J}(\mathbf{y})\mathbf{z} + \mathbf{g}(\mathbf{y}) & \text{(1b)}
\end{cases}
$$

we need the Jacobian

$$
\tilde{\mathbf{J}}|_{(\mathbf{y},\mathbf{z})} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{H}(\mathbf{y}) \otimes \mathbf{z} + \mathbf{J}(\mathbf{y}) & \mathbf{J}(\mathbf{y}) \end{bmatrix} \tag{2}
$$

where

$$
\mathbf{y} = \begin{bmatrix} \theta_{PV_1} \\ \theta_{PV_2} \\ \vdots \\ \theta_{PV_{n_{PV}}} \\ \theta_{PQ_1} \\ \theta_{PQ_2} \\ \vdots \\ \theta_{PQ_{n_{PQ}}} \\ V_{PQ_1} \\ V_{PQ_2} \\ \vdots \\ V_{PQ_{n_{PQ}}} \end{bmatrix}.
$$

$\mathbf{H}(\mathbf{y}) \otimes \mathbf{z}$ can be written as

$$
\begin{aligned}
&\left(\frac{\partial \boldsymbol{\nabla} \mathbf{g}_i(\mathbf{y})^\top}{\partial \mathbf{y}_j} \right)_{ij} \otimes \mathbf{z} \\
=&\left(\frac{\partial \boldsymbol{\nabla} \mathbf{g}_i(\mathbf{y})^\top}{\partial \mathbf{y}_j} \mathbf{z} \right)_{ij} \\
=&\left(\sum_k \frac{\partial^2 \mathbf{g}_i(\mathbf{y})}{\partial \mathbf{y}_k \partial \mathbf{y}_j} \mathbf{z}_k \right)_{ij}
\end{aligned} \tag{3}
$$

where $\otimes$ denotes the Kronecker product.

Complex power injection is the equation

$$
G^s(X) = S^{\text{bus}} + S_{\text{d}} - C_{\text{g}} S_{\text{g}}.
$$

The Jacobian matrix can be divided into four parts

$$
\mathbf{J}(\mathbf{y}) = \begin{bmatrix} \operatorname{Re}\left\{ \left(\frac{\partial G_i^s}{\partial \theta_j} \right)_{i \in [PV,PQ],\, j \in [PV,PQ]} \right\} & \operatorname{Re}\left\{ \left(\frac{\partial G_i^s}{\partial V_j} \right)_{i \in [PV,PQ],\, j \in [PQ]} \right\} \\ \operatorname{Im}\left\{ \left(\frac{\partial G_i^s}{\partial \theta_j} \right)_{i \in [PQ],\, j \in [PV,PQ]} \right\} & \operatorname{Im}\left\{ \left(\frac{\partial G_i^s}{\partial V_j} \right)_{i \in [PQ],\, j \in [PQ]} \right\} \end{bmatrix}
$$

According to (3), we have, for the upper left part,

$$(\mathbf{H}(\mathbf{y})\otimes\mathbf{z})_{\text{u,l}} = \text{Re}\left\{\left(\sum_{k=1}^{n_{\text{PV}}+n_{\text{PQ}}} \frac{\partial^2 G_i^s}{\partial\theta_k\partial\theta_j}\mathbf{z}_k + \sum_{k=n_{\text{PV}}+n_{\text{PQ}}+1}^{n_{\text{PV}}+2n_{\text{PQ}}} \frac{\partial^2 G_i^s}{\partial V_k\partial\theta_j}\mathbf{z}_k\right)_{i\in[PV,PQ],\ j\in[PV,PQ]}\right\}.$$

Similarly, we have

$$(\mathbf{H}(\mathbf{y})\otimes\mathbf{z})_{\text{u,r}} = \text{Re}\left\{\left(\sum_{k=1}^{n_{\text{PV}}+n_{\text{PQ}}} \frac{\partial^2 G_i^s}{\partial\theta_k\partial V_j}\mathbf{z}_k + \sum_{k=n_{\text{PV}}+n_{\text{PQ}}+1}^{n_{\text{PV}}+2n_{\text{PQ}}} \frac{\partial^2 G_i^s}{\partial V_k\partial V_j}\mathbf{z}_k\right)_{i\in[PV,PQ],\ j\in[PQ]}\right\}.$$

$$(\mathbf{H}(\mathbf{y})\otimes\mathbf{z})_{\text{l,l}} = \text{Im}\left\{\left(\sum_{k=1}^{n_{\text{PV}}+n_{\text{PQ}}} \frac{\partial^2 G_i^s}{\partial\theta_k\partial\theta_j}\mathbf{z}_k + \sum_{k=n_{\text{PV}}+n_{\text{PQ}}+1}^{n_{\text{PV}}+2n_{\text{PQ}}} \frac{\partial^2 G_i^s}{\partial V_k\partial\theta_j}\mathbf{z}_k\right)_{i\in[PQ],\ j\in[PV,PQ]}\right\}.$$

$$(\mathbf{H}(\mathbf{y})\otimes\mathbf{z})_{\text{l,r}} = \text{Im}\left\{\left(\sum_{k=1}^{n_{\text{PV}}+n_{\text{PQ}}} \frac{\partial^2 G_i^s}{\partial\theta_k\partial V_j}\mathbf{z}_k + \sum_{k=n_{\text{PV}}+n_{\text{PQ}}+1}^{n_{\text{PV}}+2n_{\text{PQ}}} \frac{\partial^2 G_i^s}{\partial V_k\partial V_j}\mathbf{z}_k\right)_{i\in[PQ],\ j\in[PQ]}\right\}.$$

According to Matpower manual, we have derived the expressions for

$$G^s_{\theta\theta}(\dot\theta) = \frac{\partial}{\partial\theta}\left(G^s_\theta\dot\theta\right) = \left(\sum_k \frac{\partial^2 G_i^s}{\partial\theta_k\partial\theta_j}\dot\theta_k\right)_{ij}$$

$$G^s_{V\theta}(\dot V) = \frac{\partial}{\partial\theta}\left(G^s_V\dot V\right) = \left(\sum_k \frac{\partial^2 G_i^s}{\partial V_k\partial\theta_j}\dot V_k\right)_{ij}$$

$$G^s_{\theta V}(\dot\theta) = \frac{\partial}{\partial V}\left(G^s_\theta\dot\theta\right) = \left(\sum_k \frac{\partial^2 G_i^s}{\partial\theta_k\partial V_j}\dot\theta_k\right)_{ij}$$

$$G^s_{VV}(\dot V) = \frac{\partial}{\partial V}\left(G^s_V\dot V\right) = \left(\sum_k \frac{\partial^2 G_i^s}{\partial V_k\partial V_j}\dot V_k\right)_{ij}$$

where $\dot\theta, \dot V \in \mathbb{R}^{\text{nbus}}$. $\dot\theta_{[PV]} = \mathbf{z}_{[1:PV]}$. $\dot\theta_{[PQ]} = \mathbf{z}_{[PV+1:PV+PQ]}$. $\dot V_{[PQ]} = \mathbf{z}_{[PV+PQ:PV+2PQ]}$. The other elements of $\dot\theta$ and $\dot V$ equal zero.

Finally,

$$(\mathbf{H}(\mathbf{y})\otimes\mathbf{z})_{\text{u,l}} = \text{Re}\left\{\left(G^s_{\theta\theta}(\dot\theta) + G^s_{V\theta}(\dot V)\right)_{i\in[PV,PQ],\ j\in[PV,PQ]}\right\}.$$

$$(\mathbf{H}(\mathbf{y})\otimes\mathbf{z})_{\text{u,r}} = \text{Re}\left\{\left(G^s_{\theta V}(\dot\theta) + G^s_{VV}(\dot V)\right)_{i\in[PV,PQ],\ j\in[PQ]}\right\}.$$

$$(\mathbf{H}(\mathbf{y})\otimes\mathbf{z})_{\text{l,l}} = \text{Im}\left\{\left(G^s_{\theta\theta}(\dot\theta) + G^s_{V\theta}(\dot V)\right)_{i\in[PQ],\ j\in[PV,PQ]}\right\}.$$

$$(\mathbf{H}(\mathbf{y})\otimes\mathbf{z})_{\text{l,r}} = \text{Im}\left\{\left(G^s_{\theta V}(\dot\theta) + G^s_{VV}(\dot V)\right)_{i\in[PQ],\ j\in[PQ]}\right\}.$$

Given vector A and B,

$$C = A_{\text{D}}B = B_{\text{D}}A$$

We have rule

$$\frac{\partial C}{\partial X} = A_{\text{D}}\frac{\partial B}{\partial X} + B_{\text{D}}\frac{\partial A}{\partial X}$$

where the subscript D denotes the diagonal matrix formulated by the vector.

Note that

$$(A_{\mathrm{D}}B)_{\mathrm{D}} = A_{\mathrm{D}}B_{\mathrm{D}}, \quad (\alpha \cdot B)_{\mathrm{D}} \neq \alpha \cdot B_{\mathrm{D}}$$

where α is a non-diagonal matrix.

Since

$$G^s_\theta = jV_{\mathrm{D}}(I^*_{\mathrm{D}} - Y^* V^*_{\mathrm{D}})$$

$$G^s_{\mathcal{V}} = V_{\mathrm{D}}(I^*_{\mathrm{D}} + Y^* V^*_{\mathrm{D}})\mathcal{V}^{-1}_{\mathrm{D}}$$

$$\frac{\partial I}{\partial \mathcal{V}} = YE_{\mathrm{D}}, \quad \frac{\partial I^*}{\partial \theta} = (-jY^* V^*_{\mathrm{D}})$$

$$\frac{\partial V}{\partial \mathcal{V}} = E_{\mathrm{D}}, \quad \frac{\partial V}{\partial \theta} = jV_{\mathrm{D}},$$

$$\frac{\partial E}{\partial \mathcal{V}} = 0, \quad \frac{\partial E}{\partial \theta} = jE_{\mathrm{D}}, \quad \frac{\partial E^*}{\partial \theta} = -jE^*_{\mathrm{D}}, \quad E = \mathcal{V}^{-1}_{\mathrm{D}} V$$

$$\frac{\partial I}{\partial \mathcal{V}} = YE_{\mathrm{D}}, \quad \frac{\partial I}{\partial \theta} = jYV_{\mathrm{D}}$$

where $\mathcal{V} = |V|$, we have

$$\begin{aligned}
G^s_{\theta\theta}(\lambda) =& \frac{\partial}{\partial \theta}(G^s_\theta \lambda) \\
=& \frac{\partial}{\partial \theta}(jV_{\mathrm{D}}(I^*_{\mathrm{D}} - Y^* V^*_{\mathrm{D}})\lambda) \\
=& jV_{\mathrm{D}}\left(\lambda_{\mathrm{D}} \underbrace{(-jY^* V^*_{\mathrm{D}})}_{\frac{\partial I^*}{\partial \theta}} - Y^* \lambda_{\mathrm{D}} \underbrace{(-jV^*_{\mathrm{D}})}_{\frac{\partial V^*}{\partial \theta}}\right) + j[(I^*_{\mathrm{D}} - Y^* V^*_{\mathrm{D}})\lambda]_{\mathrm{D}} \underbrace{(jV_{\mathrm{D}})}_{\frac{\partial V}{\partial \theta}} \\
=& V_{\mathrm{D}}(\lambda_{\mathrm{D}} Y^* - Y^* \lambda_{\mathrm{D}}) V^*_{\mathrm{D}} - [(I^*_{\mathrm{D}} - Y^* V^*_{\mathrm{D}})\lambda]_{\mathrm{D}} V_{\mathrm{D}} \\
=& V_{\mathrm{D}} \mathcal{B} V^*_{\mathrm{D}} - [\mathcal{A}\lambda]_{\mathrm{D}} V_{\mathrm{D}}
\end{aligned}$$

$$\begin{aligned}
G^s_{\mathcal{V}\theta}(\lambda) =& \frac{\partial}{\partial \theta}(G^s_{\mathcal{V}} \lambda) \\
=& \frac{\partial}{\partial \theta}\left(V_{\mathrm{D}}(I^*_{\mathrm{D}} + Y^* V^*_{\mathrm{D}})\mathcal{V}^{-1}_{\mathrm{D}} \lambda\right) \\
=& \frac{\partial}{\partial \theta}(E_{\mathrm{D}} I^*_{\mathrm{D}} \lambda + V_{\mathrm{D}} Y^* E^*_{\mathrm{D}} \lambda) \\
=& E_{\mathrm{D}} \lambda_{\mathrm{D}} \underbrace{(-jY^* V^*_{\mathrm{D}})}_{\frac{\partial I^*}{\partial \theta}} + I^*_{\mathrm{D}} \lambda_{\mathrm{D}} \underbrace{jE_{\mathrm{D}}}_{\frac{\partial E}{\partial \theta}} + V_{\mathrm{D}} Y^* \lambda_{\mathrm{D}} \underbrace{(-jE^*_{\mathrm{D}})}_{\frac{\partial E^*}{\partial \theta}} + [Y^* E^*_{\mathrm{D}} \lambda]_{\mathrm{D}} \underbrace{(jV_{\mathrm{D}})}_{\frac{\partial V}{\partial \theta}} \\
=& jE_{\mathrm{D}}(I^*_{\mathrm{D}} \lambda_{\mathrm{D}} - \lambda_{\mathrm{D}} Y^* V^*_{\mathrm{D}}) - jV_{\mathrm{D}} Y^* \lambda_D E^*_{\mathrm{D}} + j[Y^* E^*_{\mathrm{D}} \lambda]_{\mathrm{D}} V_{\mathrm{D}} \\
=& j\mathcal{V}^{-1}_{\mathrm{D}} V_{\mathrm{D}} \lambda_{\mathrm{D}} (I^*_{\mathrm{D}} - Y^* V^*_{\mathrm{D}}) - jV_{\mathrm{D}} Y^* \lambda_{\mathrm{D}} \mathcal{V}^{-1}_{\mathrm{D}} V^*_{\mathrm{D}} + j\left[Y^* \mathcal{V}^{-1}_{\mathrm{D}} V^*_{\mathrm{D}} \lambda\right]_{\mathrm{D}} V_{\mathrm{D}} \\
=& j\mathcal{X} \lambda_{\mathrm{D}} \mathcal{A} - jV_{\mathrm{D}} \mathcal{W} \mathcal{Y} + j\mathcal{C} V_{\mathrm{D}}
\end{aligned}$$

$$
\begin{aligned}
G^s_{\theta\mathcal{V}}(\lambda) =& \frac{\partial}{\partial \mathcal{V}}(G^s_\theta \lambda) \\
=& \frac{\partial}{\partial \mathcal{V}}(jV_{\mathrm{D}}(I^*_{\mathrm{D}} - Y^* V^*_{\mathrm{D}})\lambda) \\
=& jV_{\mathrm{D}} \left(\lambda_{\mathrm{D}} \underbrace{Y^* E^*_{\mathrm{D}}}_{\frac{\partial I^*}{\partial \mathcal{V}}} - Y^* \lambda_{\mathrm{D}} \underbrace{E^*_{\mathrm{D}}}_{\frac{\partial V^*}{\partial \mathcal{V}}} \right) + j[(I^*_{\mathrm{D}} - Y^* V^*_{\mathrm{D}})\lambda]_{\mathrm{D}} \underbrace{E_{\mathrm{D}}}_{\frac{\partial V}{\partial \mathcal{V}}} \\
=& jV_{\mathrm{D}}(\lambda_{\mathrm{D}} Y^* - Y^* \lambda_{\mathrm{D}}) \mathcal{V}^{-1}_{\mathrm{D}} V^*_{\mathrm{D}} + j[(I^*_{\mathrm{D}} - Y^* V^*_{\mathrm{D}})\lambda]_{\mathrm{D}} \mathcal{V}^{-1}_{\mathrm{D}} V_{\mathrm{D}} \\
=& jV_{\mathrm{D}} \mathcal{B} \mathcal{Y} + j[\mathcal{A}\lambda]_{\mathrm{D}} \mathcal{X}
\end{aligned}
$$

$$
\begin{aligned}
G^s_{\mathcal{V}\mathcal{V}}(\lambda) =& \frac{\partial}{\partial \mathcal{V}}(G^s_{\mathcal{V}} \lambda) \\
=& \frac{\partial}{\partial \mathcal{V}}\big(V_{\mathrm{D}}(I^*_{\mathrm{D}} + Y^* V^*_{\mathrm{D}}) \mathcal{V}^{-1}_{\mathrm{D}} \lambda\big) \\
=& \frac{\partial}{\partial \mathcal{V}}(E_{\mathrm{D}} I^*_{\mathrm{D}} \lambda + V_{\mathrm{D}} Y^* E^*_{\mathrm{D}} \lambda) \\
=& E_{\mathrm{D}} \lambda_{\mathrm{D}} \underbrace{Y^* E^*_{\mathrm{D}}}_{\frac{\partial I^*}{\partial \mathcal{V}}} + I^*_{\mathrm{D}} \lambda_{\mathrm{D}} \cdot \underbrace{0}_{\frac{\partial E}{\partial \mathcal{V}}} + V_{\mathrm{D}} Y^* \lambda_{\mathrm{D}} \cdot \underbrace{0}_{\frac{\partial E^*}{\partial \mathcal{V}}} + [Y^* E^*_{\mathrm{D}} \lambda]_{\mathrm{D}} \underbrace{E_{\mathrm{D}}}_{\frac{\partial V}{\partial \mathcal{V}}} \\
=& \mathcal{V}^{-1}_{\mathrm{D}} V_{\mathrm{D}} \lambda_D Y^* \mathcal{V}^{-1}_{\mathrm{D}} V^*_{\mathrm{D}} + \big[Y^* \mathcal{V}^{-1}_{\mathrm{D}} V^*_{\mathrm{D}} \lambda\big]_{\mathrm{D}} \mathcal{V}^{-1}_{\mathrm{D}} V_{\mathrm{D}} \\
=& \mathcal{X} \mathcal{Z} \mathcal{Y} + \mathcal{C} \mathcal{X}
\end{aligned}
$$

where

$$
\begin{aligned}
\text{constant:} \quad \mathcal{X} &= e^{j\theta}_{\mathrm{D}} \\
\text{constant:} \quad \mathcal{Y} &= e^{-j\theta}_{\mathrm{D}} \\
\mathcal{Z} &= \lambda_{\mathrm{D}} Y^* \\
\mathcal{W} &= Y^* \lambda_{\mathrm{D}} \\
\text{constant:} \quad \mathcal{A} &= I^*_{\mathrm{D}} - Y^* V^*_{\mathrm{D}} \\
\mathcal{B} &= \mathcal{Z} - \mathcal{W} \\
\mathcal{C} &= [Y^* \mathcal{Y} \lambda]_{\mathrm{D}}
\end{aligned}
$$