数学２D演習　第8回

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[1]　復習

略

[2]

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[3]　和の公式

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[4]

略

[5]　Γ 関数の漸近展開

(1)

$t = x\tau$ と変数変換すると、

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty e^{-t} t^x \mathrm{d}t \\
&= \int_0^\infty e^{-t + x\ln t} \mathrm{d}t \\
&= \int_0^\infty e^{-x\tau + x\ln x\tau} x \mathrm{d}\tau \quad (t = x\tau) \\
&= \int_0^\infty e^{(\ln\tau - \tau)x} x^{x+1} \mathrm{d}\tau \\
&= x^{x+1} \int_0^\infty e^{(\ln\tau - \tau)x} \mathrm{d}\tau
\end{aligned}$$

と変形できるので、

$$f(\tau) = \ln\tau - \tau$$

(2)

$\partial_\tau f(\tau) = \dfrac{1}{\tau} - 1 \geq 0$ とすると、$\tau \leq 1$ であるので、$f(\tau)$ が最大となるときの τ は、

$$\tau_0 = 1$$

また、

$$\partial_{\tau\tau} f(\tau) = \partial_\tau\Big(\frac{1}{\tau} - 1\Big) = -\frac{1}{\tau^2}$$

よって、$\tau = \tau_0$ における 2 次までの Taylor 展開は、

$$\begin{aligned}
f(\tau) &\approx f(\tau_0) + \partial_\tau|_{\tau=\tau_0}(\tau - \tau_0) + \frac{1}{2}\partial_{\tau\tau}|_{\tau=\tau_0}(\tau - \tau_0)^2 \\
&= -1 + \ln 1 + 0 + \frac{1}{2} \times (-\frac{1}{1^2}) \times (\tau - 1)^2 \\
&= -1 - \frac{1}{2}(\tau - 1)^2
\end{aligned}$$

これを (1) の式に代入すると、

$$\begin{aligned}
\Gamma(x+1) &= x^{x+1}\int_0^\infty e^{(\ln\tau - \tau)x}\mathrm{d}\tau \\
&\approx x^{x+1}\int_0^\infty e^{\left(-1-\frac{1}{2}(\tau-1)^2\right)x}\mathrm{d}\tau \\
&= x^{x+1}e^{-x}\int_0^\infty e^{-\frac{x}{2}(\tau-1)^2}\mathrm{d}\tau \\
&\approx x^{x+1}e^{-x}\int_0^\infty e^{-\frac{x}{2}\tau^2}\mathrm{d}\tau \\
&= x^{x+1}e^{-x}\sqrt{\frac{2\pi}{x}} \\
&= x^x e^{-x}\sqrt{2\pi x}
\end{aligned}$$

(3)

$\Gamma(n+1) = n!$ より、(2) の結果と比較して、

$$n! = n^x e^{-n}\sqrt{2\pi n}$$

両辺自然対数とって、

$$\begin{aligned}
\ln n! &\approx \ln\left(n^x e^{-n}\sqrt{2\pi n}\right) \\
&= n\ln n - n + \frac{1}{2}\ln 2\pi n
\end{aligned}$$

以上より、Staring の近似式

$$\ln n! = n\ln n - n + \frac{1}{2}\ln 2\pi n$$

が導かれた。

(4)

$f(\tau) = -\tau + \ln\tau = -1 + \xi^2$ の両辺微分すると、

$$-1 + \frac{1}{\tau} = -2\xi\frac{\mathrm{d}\xi}{\mathrm{d}\tau} \tag{4.1}$$

$\tau=\tau(\xi)=a_0+a_1\xi+a_2\xi^2+a_3\xi^3$ として、式 (4.1) の両辺を $\tau=1$ すなわち $\xi=0$ の周りで Taylor 展開することを考えると、

$$
\begin{aligned}
(\text{左辺}) &= -1+\frac{1}{\tau} \\
&\approx -1+\frac{1}{\tau(0)}-\frac{1}{\tau(0)^2}\partial_\xi\tau(0)\xi+\frac{1}{2}\big(\frac{2}{\tau(0)^3}\partial_\xi\tau(0)\partial_\xi\tau(0)-\frac{1}{\tau(0)^2}\partial_\xi^2\tau(0)\big)\xi^2 \\
&\quad +\frac{1}{6}\big(-\frac{6}{\tau(0)^4}\partial_\xi\tau(0)\big(\partial_\xi\tau(0)\big)^2+\frac{4}{\tau(0)^3}\partial_\xi^2\tau(0)\partial_\xi\tau(0)+\frac{2}{\tau(0)^3}\partial_\xi^2\tau(0)\partial_\xi\tau(0)-\frac{1}{\tau(0)^2}\partial_\xi^3\tau(0)\big)\xi^3 \\
&= -1+\frac{1}{a_0}-\frac{a_1}{a_0^2}\xi+\frac{a_1^2-a_0a_2}{a_0^3}\xi^2-\frac{a_1^3-2a_0a_1a_2+a_0^2a_3}{{a_0}^4}\xi^3 \\
(\text{右辺}) &= -2\xi\frac{\mathrm{d}\xi}{\mathrm{d}\tau} \\
&= -2\xi\frac{1}{\partial_\xi\tau} \\
&\approx -2\xi\left(\frac{1}{\partial_\xi\tau(0)}-\frac{1}{\partial_\xi\tau(0)^2}\partial_\xi^2\tau(0)\xi+\frac{1}{2}\frac{2}{\partial_\xi\tau(0)^3}\partial_\xi^2\tau(0)^2\xi^2-\frac{1}{2}\frac{1}{\partial_\xi\tau(0)^2}\partial_\xi^3\tau(0)\xi^2\right) \\
&= -\frac{2}{a_1}\xi+\frac{4a_2}{{a_1}^2}\xi^2-\frac{8{a_2}^2-6a_1a_3}{{a_1}^3}\xi^3
\end{aligned}
$$

両辺の係数を比べると、

$$
\begin{cases}
-1+\frac{1}{a_0}=0 \\
-\frac{a_1}{{a_0}^2}=-\frac{2}{a_1} \\
\frac{a_1^2-a_0a_2}{a_0^3}=\frac{4a_2}{{a_1}^2} \\
-\frac{a_1^3-2a_0a_1a_2+a_0^2a_3}{a_0^4}=-\frac{8{a_2}^2-6a_1a_3}{{a_1}^3}
\end{cases}
\qquad \therefore \quad
\begin{cases}
a_0=1 \\
a_1=\sqrt{2} \\
a_2=\frac{2}{3} \\
a_3=\frac{1}{9\sqrt{2}}
\end{cases}
$$

よって

$$
\begin{aligned}
\Gamma(x+1) &= x^{x+1}\int_0^\infty e^{f(\tau)x}\mathrm{d}\tau \\
&= x^{x+1}\int_{-\infty}^\infty e^{x(1-\xi^2)}\frac{\mathrm{d}\tau}{\mathrm{d}\xi}\mathrm{d}\xi \\
&= x^{x+1}e^{-x}\int_{-\infty}^\infty e^{x(1-\xi^2)}\Big(\sqrt{2}+\frac{4}{3}\xi+\frac{1}{3\sqrt{2}}xi^2+\cdots\Big)\mathrm{d}\xi \\
&= x^{x+1}e^{-x}\Big(\sqrt{2}\sqrt{\frac{\pi}{x}}+0+\frac{1}{3\sqrt{2}}\frac{1}{2x}\frac{\pi}{x}+\cdots\Big) \\
&= x^xe^{-x}\sqrt{2\pi x}\big(1+\frac{1}{12x}+\cdots\big)
\end{aligned}
$$

[6]　d 次元球の表面積・体積

(1)

ガウス積分を用いると、

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_d e^{-a({x_1}^2+{x_2}^2+\cdots+{x_d}^2)} \\
&= \int_{-\infty}^{\infty} \mathrm{d}x_1 e^{-a{x_1}^2} \int_{-\infty}^{\infty} \mathrm{d}x_2 e^{-a{x_2}^2} \cdots \int_{-\infty}^{\infty} \mathrm{d}x_d e^{-a{x_d}^2} \\
&= \left(\frac{\pi}{a}\right)^{\frac{d}{2}}
\end{aligned}
$$

(2)

$S_d(r) = s_d r^{d-1}$ と書けるので、

$$
\begin{aligned}
I &= \int_0^{\infty} \mathrm{d}r S_d(r) e^{-ar^2} \\
&= \int_0^{\infty} \mathrm{d}r s_d r^{d-1} e^{-ar^2} \\
&= s_d \int_0^{\infty} \mathrm{d}r r^{d-1} e^{-ar^2}
\end{aligned}
$$

d が偶数の時、

$$
\begin{aligned}
I &= s_d \int_0^{\infty} \mathrm{d}r r^{d-1} e^{-ar^2} \\
&= s_d \left(\frac{\mathrm{d}}{\mathrm{d}(-a)}\right)^{\frac{d}{2}-1} \int_0^{\infty} \mathrm{d}r r e^{-ar^2} \\
&= s_d \left(\frac{\mathrm{d}}{\mathrm{d}(-a)}\right)^{\frac{d}{2}-1} \left(\frac{1}{2a}\right) \\
&= s_d \left(\frac{d}{2}-1\right)! \frac{1}{2a^{\frac{d}{2}}} \\
&= s_d \frac{1}{2a^{\frac{d}{2}}} \Gamma\left(\frac{d}{2}\right)
\end{aligned}
$$

d が奇数の時、

$$
\begin{aligned}
I &= s_d \int_0^{\infty} \mathrm{d}r r^{d-1} e^{-ar^2} \\
&= s_d \Big(\frac{\mathrm{d}}{\mathrm{d}(-a)}\Big)^{\frac{d-1}{2}} \int_0^{\infty} \mathrm{d}r e^{-ar^2} \\
&= s_d \Big(\frac{\mathrm{d}}{\mathrm{d}(-a)}\Big)^{\frac{d-1}{2}} \frac{1}{2}\sqrt{\frac{\pi}{a}} \\
&= s_d \frac{\sqrt{\pi}}{2} \times \frac{1}{2} \times \frac{3}{2} \times \frac{5}{2} \times \cdots \times \frac{d-2}{2} \times \frac{1}{a^{\frac{d}{2}}} \\
&= s_d \frac{1}{2a^{\frac{d}{2}}} \Gamma\Big(\frac{1}{2}\Big) \times \frac{1}{2} \times \frac{3}{2} \times \frac{5}{2} \times \cdots \times \frac{d-2}{2} \quad (\because \Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}) \\
&= s_d \frac{1}{2a^{\frac{d}{2}}} \Gamma\Big(\frac{3}{2}\Big) \times \frac{3}{2} \times \frac{5}{2} \times \cdots \times \frac{d-2}{2} \quad (\because z\Gamma(z) = \Gamma(z+1)) \\
&= s_d \frac{1}{2a^{\frac{d}{2}}} \Gamma\Big(\frac{5}{2}\Big) \times \frac{5}{2} \times \cdots \times \frac{d-2}{2} \\
&= \cdots \\
&= s_d \frac{1}{2a^{\frac{d}{2}}} \Gamma\Big(\frac{d}{2}\Big)
\end{aligned}
$$

これは d が偶数の時に等しい。よって、

$$
I = s_d \frac{1}{2a^{\frac{d}{2}}} \Gamma\Big(\frac{d}{2}\Big)
$$

(3)

(1) と (2) を比較して、

$$
s_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\big(\frac{d}{2}\big)}
$$

従って、

$$
S_d(r) = s_d r^{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\big(\frac{d}{2}\big)} r^{d-1}
$$

(4)

体積は、

$$
\begin{aligned}
V_d(r) &= \int_0^{d} \mathrm{d}r S_d(r) \\
&= \int_0^{r} \mathrm{d}r' \frac{2\pi^{\frac{d}{2}}}{\Gamma\big(\frac{d}{2}\big)} r'^{d-1} \\
&= \frac{2\pi^{\frac{d}{2}}}{\Gamma\big(\frac{d}{2}\big)} \Big[\frac{1}{d} r'^{d}\Big]_0^r \\
&= \frac{2\pi^{\frac{d}{2}}}{\Gamma\big(d\frac{d}{2}\big)} r^{d}
\end{aligned}
$$

(5)

(3)、(4) より、

$$V_d(r) = \frac{r}{d} S_d(r)$$

$d = 2$ のとき、

$$S_2(r) = \frac{2\pi}{\Gamma(1)} r = 2\pi r$$

$$V_2(r) = \frac{2\pi}{\Gamma(2)} r^2 = \pi r^2$$

$d = 3$ のとき、

$$S_3(r) = \frac{2\pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} r^2 = \frac{2\pi^{\frac{3}{2}}}{\frac{\sqrt{\pi}}{2}} r^2 = 4\pi r^2$$

$$V_3(r) = \frac{r}{d} S_d(r) = \frac{4}{3} \pi r^3$$