数学２D演習　第5回

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[1]　復習

(1)　$\log i$

nを整数として、

$$\begin{aligned}\log i &= \log\Big(\exp\big(i(\frac{\pi}{2}+2n\pi)\big)\Big)\\ &= i(\frac{\pi}{2}+2n\pi) \quad (n\in\mathbb{Z})\end{aligned}$$

(2)　$i^{\frac{1}{2}}$

nを整数として、

$$\begin{aligned}i^{\frac{1}{2}} &= \Big(\exp\big(i(\frac{\pi}{2}+2n\pi)\big)\Big)^{\frac{1}{2}}\\ &= \exp\big(\frac{1}{2}i(\frac{\pi}{2}+2n\pi)\big)\\ &= \exp\big(i(\frac{\pi}{4}+n\pi)\big)\\ &= (-1)^{n-1}\frac{1+i}{\sqrt{2}} \quad (n\in\mathbb{Z})\end{aligned}$$

(3)　i^i

nを整数として、

$$\begin{aligned}i^i &= \Big(\exp\big(i(\frac{\pi}{2}+2n\pi)\big)\Big)^{i}\\ &= \exp\big(-(\frac{\pi}{2}+2n\pi)\big) \quad (n\in\mathbb{Z})\end{aligned}$$

(4)　$\sin(i)$

$$\begin{aligned}\sin(i) &= \frac{1}{2i}\big(e^{ii}-e^{-ii}\big)\\ &= \frac{1}{2i}\big(e^{-1}-e^{1}\big)\\ &= \frac{i}{2}\big(e-\frac{1}{e}\big)\end{aligned}$$

(5) $\log{(1+i\sqrt{3})}$

n を整数として、

$$\begin{aligned}\log{(1+i\sqrt{3})} &= \log{\left(2\exp{(i(\frac{\pi}{3}+2n\pi))}\right)} \\ &= \log{2}+i(\frac{\pi}{3}+2n\pi) \quad (n\in\mathbb{Z})\end{aligned}$$

(6) $\frac{2+i}{3-2i}$

$$\begin{aligned}\frac{2+i}{3-2i} &= \frac{(2+i)(3+2i)}{(3-2i)(3+2i)} \\ &= \frac{4+7i}{13}\end{aligned}$$

(7) $\tan{\left(i+\frac{\pi}{3}\right)}$

$$\begin{aligned}\sin{(i)} &= \frac{1}{2i}\left(e^{ii}-e^{-ii}\right) \\ &= \frac{1}{2i}\left(e^{-1}-e^{1}\right) \\ &= \frac{i}{2}\left(e-\frac{1}{e}\right) \\ \cos{(i)} &= \frac{1}{2}\left(e^{ii}+e^{-ii}\right) \\ &= \frac{1}{2}\left(e^{-1}+e^{1}\right) \\ &= \frac{1}{2}\left(e+\frac{1}{e}\right) \\ \tan{(i)} &= \frac{\sin{(i)}}{\cos{(i)}} \\ &= i\frac{e^2-1}{e^2+1} \\ \tan^2{(i)} &= -\frac{(e^2-1)^2}{(e^2+1)^2}\end{aligned}$$

$$
\begin{aligned}
\tan(i+\frac{\pi}{3}) &= \frac{\tan i + \tan\frac{\pi}{3}}{1-\tan i\tan\frac{\pi}{3}} \\
&= \frac{\tan i+\sqrt{3}}{1-\sqrt{3}\tan i} \\
&= \frac{\sqrt{3}(1+\tan^2 i)+4\tan i}{1+3\tan^2 i} \\
&= \frac{\sqrt{3}(1-\frac{(e^2-1)^2}{(e^2+1)^2})+4i\frac{e^2-1}{e^2+1}}{1-3\frac{(e^2-1)^2}{(e^2+1)^2}} \\
&= \frac{\sqrt{3}\big((e^2+1)^2-(e^2-1)^2\big)+4i(e^4-1)}{(e^2+1)^2-3(e^2-1)^2} \\
&= \frac{4\sqrt{3}e^2+4i(e^4-1)}{4e^4-4e^2+4} \\
&= \frac{\sqrt{3}e^2+i(e^4-1)}{e^4-e^2+1}
\end{aligned}
$$

[2]　Laurent 展開 (1)

$z=e^{i\theta}$ とおくと、

$$
\begin{aligned}
J_n(w) &= \frac{1}{2\pi i}\oint_C \frac{f(z,w)}{z^{n+1}}\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\exp\left(\frac{w}{2}(e^{i\theta}-e^{-i\theta})\right)}{\exp\big(i\theta(n+1)\big)}ie^{i\theta}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\exp\big(wi\sin\theta\big)}{\exp\big(i\theta n\big)}\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\big(i(w\sin\theta-n\theta)\big)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(w\sin\theta-n\theta)+i\sin(w\sin\theta-n\theta)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(w\sin\theta-n\theta)\mathrm{d}\theta \quad (\because \sin(w\sin\theta-n\theta)\ \text{は奇関数})
\end{aligned}
$$

以上より、示された。

[3]　Laurent 展開 (2)

(1)

$$
\frac{1}{3z^2-5z-2}=\frac{1}{7}\left(\frac{1}{z-2}-\frac{3}{3z+1}\right)
$$

(2)

(i) $|z| < \frac{1}{3}$

$$\begin{aligned}
\frac{1}{3z^2-5z-2} &= \frac{1}{7}\frac{1}{z-2} - \frac{3}{7}\frac{1}{3z+1} \\
&= -\frac{1}{14}\frac{1}{1-\frac{z}{2}} - \frac{3}{7}\frac{1}{1+3z} \\
&= \frac{1}{14}\sum_{n=0}^{\infty}(\frac{z}{2})^n - \frac{3}{7}\sum_{n=0}^{\infty}(-3z)^n
\end{aligned}$$

(ii) $\frac{1}{3} < |z| < 2$

$$\begin{aligned}
\frac{1}{3z^2-5z-2} &= \frac{1}{7}\frac{1}{z-2} - \frac{3}{7}\frac{1}{3z+1} \\
&= -\frac{1}{14}\frac{1}{1-\frac{z}{2}} - \frac{3}{7}\frac{1}{3z}\frac{1}{1+\frac{1}{3z}} \\
&= -\frac{1}{14}\sum_{n=0}^{\infty}(\frac{z}{2})^n + \frac{3}{7}\sum_{n=0}^{\infty}(-\frac{1}{3z})^{n+1}
\end{aligned}$$

(iii) $|z| > 2$

$$\begin{aligned}
\frac{1}{3z^2-5z-2} &= \frac{1}{7}\frac{1}{z-2} - \frac{3}{7}\frac{1}{3z+1} \\
&= \frac{1}{7}\frac{1}{z}\frac{1}{1-\frac{2}{z}} - \frac{3}{7}\frac{1}{3z}\frac{1}{1+\frac{1}{3z}} \\
&= \frac{1}{14}\sum_{n=0}^{\infty}(\frac{2}{z})^{n+1} + \frac{3}{7}\sum_{n=0}^{\infty}(-\frac{1}{3z})^{n+1}
\end{aligned}$$

(3)

$$\oint_{|z|=1} z^m \mathrm{d}z = \begin{cases} 2\pi i \quad (m=-1) \\ 0 \quad (m=0) \end{cases}$$

なので、(2) における $\frac{1}{3} < |z| < 2$ における Lairent 展開を項別積分すると、

$$\begin{aligned}
\oint_{|z|=1}\frac{\mathrm{d}z}{3z^2-5z^2} &= -\frac{1}{14}\sum_{n=0}^{\infty}\oint_{|z|=1}(\frac{z}{2})^n\mathrm{d}z + \frac{3}{7}\sum_{n=0}^{\infty}\oint_{|z|=1}(-\frac{1}{3z})^{n+1}\mathrm{d}z \\
&= \frac{3}{7}\oint_{|z|=1} -\frac{1}{3z}\mathrm{d}z \\
&= -\frac{1}{7}\oint_{|z|=1} z^{-1}\mathrm{d}z \\
&= -\frac{2\pi}{7}i
\end{aligned}$$

[4]　$\int_0^{2\pi} R(\cos\theta, \sin\theta)\mathrm{d}\theta$ 型の積分

$z = e^{i\theta}$ と置換すると、積分範囲は $|z| = 1$、$\mathrm{d}z = \dfrac{1}{iz}\mathrm{d}\theta$

(1)

$z = e^{i\theta}$ と置換すると、

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta} &= \oint_{|z|=1} \frac{\mathrm{d}z}{iz(a+\frac{z+z^{-1}}{2})} \\ &= \frac{2}{i}\oint_{|z|=1} \frac{\mathrm{d}z}{z^2+2az+1}\end{aligned}$$

$a > 1$ より、積分経路の内側にある極は、$z = -a + \sqrt{a^2-1}$
$z = -a + \sqrt{a^2-1}$ における留数は、

$$\begin{aligned}Res\Big(\frac{1}{z^2+2az+1}, z = -a+\sqrt{a^2-1}\Big) &= \lim_{z\to -a+\sqrt{a^2-1}} \big(z-(-a+\sqrt{a^2-1})\big)\frac{1}{z^2+2az+1} \\ &= \lim_{z\to -a+\sqrt{a^2-1}} \frac{1}{z-(-a-\sqrt{a^2-1})} \\ &= \frac{1}{-a+\sqrt{a^2-1}-(-a-\sqrt{a^2-1})} \\ &= \frac{1}{2\sqrt{a^2-1}}\end{aligned}$$

よって、

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta} &= \frac{2}{i}\oint_{|z|=1} \frac{\mathrm{d}z}{z^2+2az+1} \\ &= \frac{2}{i} \times 2\pi i \times \frac{1}{2\sqrt{a^2-1}} \\ &= \frac{2\pi}{\sqrt{a^2-1}}\end{aligned}$$

(2)

$z = e^{i\theta}$ と置換すると、

$$\begin{aligned}\int_0^{2\pi} \mathrm{d}\theta \frac{\cos 2\theta}{1-2a\cos\theta+a^2} &= \oint_{|z|=1} \mathrm{d}z \frac{\frac{z^2+z^{-2}}{2}}{iz(1-2a\frac{z+z^{-1}}{2}+a^2)} \\ &= -\frac{1}{2ai}\oint_{|z|=1} \mathrm{d}z \frac{z^4+1}{z^2(z^2-(a+\frac{1}{a})z+1)} \\ &= -\frac{1}{2ai}\oint_{|z|=1} \mathrm{d}z \frac{z^4+1}{z^2(z-a)(z-\frac{1}{a})}\end{aligned}$$

$0 < a < 1$ より、積分経路の内側にある極は、$z = 0, a$
2 位の極 $z = 0$ における留数は、

$$
\begin{aligned}
Res\Big(\frac{z^4+1}{z^2(z-a)(z-\frac{1}{a})}, z=0\Big) &= \lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\Big(z^2\frac{z^4+1}{z^2(z-a)(z-\frac{1}{a})}\Big) \\
&= \lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{z^4+1}{z^2-(a+\frac{1}{a})z+1}\Big) \\
&= \lim_{z\to 0}\frac{z^3\big(z^2-(a+\frac{1}{a})z+1\big)-(z^4+1)(2z-a-\frac{1}{a})}{\big(z^2-(a+\frac{1}{a})z+1\big)^2} \\
&= a+\frac{1}{a}
\end{aligned}
$$

1 位の極 $z = a$ における留数は、

$$
\begin{aligned}
Res\Big(\frac{z^4+1}{z^2(z-a)(z-\frac{1}{a})}, z=a\Big) &= \lim_{z\to a}(z-a)\frac{z^4+1}{z^2(z-a)(z-\frac{1}{a})} \\
&= \lim_{z\to a}\frac{z^4+1}{z^2(z-\frac{1}{a})} \\
&= \frac{a^4+1}{a(a^2-1)}
\end{aligned}
$$

よって、

$$
\begin{aligned}
\int_0^{2\pi}\mathrm{d}\theta\frac{\cos 2\theta}{1-2a\cos\theta+a^2} &= -\frac{1}{2ai}\oint_{|z|=1}\mathrm{d}z\frac{z^4+1}{z^2(z-a)(z-\frac{1}{a})}s\frac{1}{2\sqrt{a^2-1}} \\
&= -\frac{1}{2ai}\Big(2\pi i(a+\frac{1}{a})+2\pi i(\frac{a^4+1}{a(a^2-1)})\Big) \\
&= -\frac{\pi}{a}\Big(\frac{a^4-1}{a(a^2-1)}+\frac{a^4+1}{a(a^2-1)}\Big) \\
&= -\frac{2\pi a^2}{a^2-1}
\end{aligned}
$$

[5]　$\int_{-\infty}^{\infty}R(x)\mathrm{d}x$ 型の積分

(1)

$$
I = \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2-2x+2}
$$

とおく。

$$
\int_{C_x}\frac{\mathrm{d}z}{z^2-2z+2} = \int_{-R}^{R}\frac{\mathrm{d}x}{x^2-2x+2} \to I \quad (R\to\infty)
$$

$z=Re^{i\theta}$ とおくと、

$$\begin{aligned}\left|\int_{C_R}\frac{\mathrm{d}z}{z^2-2z+2}\right| &= \left|\int_0^\pi \frac{iRe^{i\theta}\mathrm{d}\theta}{R^2e^{2i\theta}-2Re^{i\theta}+2}\right| \\ &= \int_0^\pi\left|\frac{iRe^{i\theta}\mathrm{d}\theta}{R^2e^{2i\theta}-2Re^{i\theta}+2}\right| \to 0 \quad (R\to\infty) \\ \therefore \quad \int_{C_R}\frac{\mathrm{d}z}{z^2-2z+2} &\to 0 \quad (R\to\infty)\end{aligned}$$

経路 C_x+C_R の内側にある $\frac{1}{z^2-2z+2}$ の極は $z=1+i$ であり、一位の極 $z=1+i$ における留数は、

$$\begin{aligned}Res\Big(\frac{1}{z^2-2z+2}, z=1+i\Big) &= \lim_{z\to 1+i}\big(z-(1+i)\big)\frac{1}{z^2-2z+2} \\ &= \lim_{z\to 1+i}\frac{1}{z-(1-i)} \\ &= \frac{1}{1+i-(1-i)} \\ &= \frac{1}{2i} \\ &= -\frac{1}{2}i\end{aligned}$$

より、

$$\int_{C_x+C_R}\frac{\mathrm{d}z}{z^2-2z+2} = 2\pi i\times\big(-\frac{1}{2}i\big) = \pi \tag{1.1}$$

一方、

$$\int_{C_x+C_R}\frac{\mathrm{d}z}{z^2-2z+2} = \int_{C_x}\frac{\mathrm{d}z}{z^2-2z+2} + \int_{C_R}\frac{\mathrm{d}z}{z^2-2z+2} = I \tag{1.2}$$

(1.1)、(1.2) より、

$$\therefore I = \pi$$

(2)

$$I = \int_{-\infty}^{\infty}\mathrm{d}x\frac{x^2-x+2}{x^4+10x^2+9}$$

とおく。

$$\int_{C_x}\mathrm{d}z\frac{z^2-z+2}{z^4+10z^2+9} = \int_{-R}^{R}\mathrm{d}x\frac{x^2-x+2}{x^4+10x^2+9} \to I \quad (R\to\infty)$$

$z=Re^{i\theta}$ とおくと、

$$\begin{aligned}\left|\int_{C_R}\mathrm{d}z\frac{z^2-z+2}{z^4+10z^2+9}\right| &= \left|\int_0^\pi iRe^{i\theta}\mathrm{d}\theta\frac{R^2e^{2i\theta}-Re^{i\theta}+4}{R^4e^{4i\theta}+10R^2e^{2i\theta}+9}\right| \\ &= \int_0^\pi \mathrm{d}\theta\left|\frac{R^3e^{3i\theta}-R^2e^{2i\theta}+4Re^{i\theta}}{R^4e^{4i\theta}+10R^2e^{2i\theta}+9}\right| \to 0 \quad (R\to\infty) \\ \therefore \quad \int_{C_R}\mathrm{d}z\frac{z^2-z+2}{z^4-10z^2+9} &\to 0 \quad (R\to\infty)\end{aligned}$$

ここで、

$$z^4+10z^2+9=(z+i)(z-i)(z+3i)(z-3i)$$

より、経路 C_x+C_R の内側にある $\frac{z^2-z+2}{z^4-10z^2+9}$ の極は $z=i,3i$ であり、
一位の極 $z=i$ における留数は、

$$\begin{aligned}
Res\Big(\frac{z^2-z+2}{z^4-10z^2+9},z=i\Big)&=\lim_{z\to i}\big(z-i\big)\frac{z^2-z+2}{z^4-10z^2+9}\\
&=\lim_{z\to i}\frac{z^2-z+2}{(z+i)(z+3i)(z-3i)}\\
&=\frac{-1-i+2}{2i\times 4i\times(-2i)}\\
&=\frac{1-i}{16i}\\
&=-\frac{1+i}{16}
\end{aligned}$$

一位の極 $z=3i$ における留数は、

$$\begin{aligned}
Res\Big(\frac{z^2-z+2}{z^4-10z^2+9},z=3i\Big)&=\lim_{z\to 3i}\big(z-3i\big)\frac{z^2-z+2}{z^4-10z^2+9}\\
&=\lim_{z\to 3i}\frac{z^2-z+2}{(z-i)(z+i)(z+3i)}\\
&=\frac{-9-3i+2}{2i\times 4i\times 6i}\\
&=\frac{-7-3i}{-48i}\\
&=\frac{3-7i}{48}
\end{aligned}$$

よって、

$$\int_{C_x+C_R}\mathrm{d}z\frac{z^2-z+2}{z^4+10z^2+9}=2\pi i\times\big(-\frac{1+i}{16}\big)+2\pi i\times\big(\frac{3-7i}{48}\big)=\frac{5\pi}{12}\tag{2.1}$$

一方、

$$\int_{C_x+C_R}\mathrm{d}z\frac{z^2-z+2}{z^4+10z^2+9}=\int_{C_x}\mathrm{d}z\frac{z^2-z+2}{z^4+10z^2+9}+\int_{C_R}\mathrm{d}z\frac{z^2-z+2}{z^4+10z^2+9}=I\tag{2.2}$$

(2.1)、(2.2) より、

$$\therefore I=\frac{5\pi}{12}$$

(3)

$$I=\int_{-\infty}^{\infty}\mathrm{d}x\frac{x^4}{(x^2+2)^2(x^2+3)}$$

とおく。

$$\int_{C_x}\mathrm{d}z\frac{z^4}{(z^2+2)^2(z^2+3)}=\int_{-R}^{R}\mathrm{d}x\frac{x^4}{(x^2+2)^2(x^2+3)}\to I\quad(R\to\infty)$$

$z=Re^{i\theta}$ とおくと、

$$\begin{aligned}\left|\int_{C_R}\mathrm{d}z\frac{z^4}{(z^2+2)^2(z^2+3)}\right| &= \left|\int_0^\pi iRe^{i\theta}\mathrm{d}\theta\frac{R^4e^{4i\theta}}{(R^2e^{2i\theta}+2)^2(R^2e^{2i\theta}+3)}\right| \\ &= \int_0^\pi \mathrm{d}\theta\left|\frac{R^5e^{5i\theta}}{(R^2e^{2i\theta}+2)^2(R^2e^{2i\theta}+3)}\right| \to 0 \quad (R\to\infty) \\ \therefore \quad \int_{C_R}\mathrm{d}z\frac{z^4}{(z^2+2)^2(z^2+3)} &\to 0 \quad (R\to\infty)\end{aligned}$$

経路 C_x+C_R の内側にある $\frac{z^4}{(z^2+2)^2(z^2+3)}$ の極は $z=\sqrt{2}i,\sqrt{3}i$ であり、
二位の極 $z=\sqrt{2}i$ における留数は、

$$\begin{aligned} Res\Big(\frac{z^4}{(z^2+2)^2(z^2+3)}, z=\sqrt{2}i\Big) &= \lim_{z\to\sqrt{2}i}\frac{\mathrm{d}}{\mathrm{d}z}\Big((z-\sqrt{2}i)^2\frac{z^4}{(z^2+2)^2(z^2+3)}\Big) \\ &= \lim_{z\to\sqrt{2}i}\frac{\mathrm{d}}{\mathrm{d}z}\Big(\frac{z^4}{(z+\sqrt{2}i)^2(z^2+3)}\Big) \\ &= \lim_{z\to\sqrt{2}i}\frac{z^3(z+\sqrt{2}i)^2(z^2+3)-z^4 2(z+\sqrt{2}i)(z^2+3)-z^4(z+\sqrt{2}i)^2 2z}{(z+\sqrt{2}i)^4(z^2+3)^2} \\ &= \frac{-2\sqrt{2}i\times(2\sqrt{2}i)^2\times(-2+3)-16\times 2\sqrt{2}i\times(-2+3)-8\times(2\sqrt{2}i)^2\times 2\sqrt{2}i}{(2\sqrt{2}i)^4\times(-2+3)^2} \\ &= \frac{7\sqrt{2}}{4}i \end{aligned}$$

一位の極 $z=\sqrt{3}i$ における留数は、

$$\begin{aligned} Res\Big(\frac{z^4}{(z^2+2)^2(z^2+3)}, z=\sqrt{3}i\Big) &= \lim_{z\to\sqrt{3}i}(z-\sqrt{3}i)\frac{z^4}{(z^2+2)^2(z^2+3)} \\ &= \lim_{z\to\sqrt{3}i}\frac{z^4}{(z^2+2)^2(z+\sqrt{3}i)} \\ &= \frac{9}{(-3+2)^2\times 2\sqrt{3}i} \\ &= -\frac{3\sqrt{3}i}{2} \end{aligned}$$

よって、

$$\int_{C_x+C_R}\mathrm{d}z\frac{z^4}{(z^2+2)^2(z^2+3)} = 2\pi i\times\big(\frac{7\sqrt{2}}{4}i\big)+2\pi i\times\big(-\frac{3\sqrt{3}i}{2}\big) = \Big(3\sqrt{3}-\frac{7\sqrt{2}}{2}\Big)\pi \tag{3.1}$$

一方、

$$\int_{C_x+C_R}\mathrm{d}z\frac{z^4}{(z^2+2)^2(z^2+3)} = \int_{C_x}\mathrm{d}z\frac{z^2-z+2}{z^4+10z^2+9}+\int_{C_R}\mathrm{d}z\frac{z^2-z+2}{z^4+10z^2+9} = I \tag{3.2}$$

(3.1)、(3.2) より、

$$\therefore I = \Big(3\sqrt{3}-\frac{7\sqrt{2}}{2}\Big)\pi$$