数学２D演習　第7回

工学部電気電子工学科 3 年　03200489　末吉七海

[1]　復習

(1)

$z = x + iy$ とおくと、

$$f(z) = e^z = e^x \cos y + ie^x \sin y$$

$f(z) = u + iv$ とすると、

$$\begin{aligned} \partial_x u = e^x \cos y \quad \partial_y v = e^x \cos y \\ \partial_y u = -e^x \sin y \quad \partial_x v = e^x \sin y \\ \therefore \quad \partial_x u = \partial_y v \quad \partial_y u = -\partial_x v \end{aligned}$$

このように CR 方程式が成り立つので、f(z) は微分可能。
すなわち、f(z) の微分は微分の方向によらないので、

$$\frac{\mathrm{d}e^z}{\mathrm{d}z} = \partial_x u + i\partial_x v = e^x \cos y + ie^x \cos y = e^z$$

以上より題意は示された。

(2)

$z = x + iy$ とおくと、

$$f(z) = \log z = \frac{1}{2}\log(x^2 + y^2) + i \arctan\frac{y}{x}$$

$f(z) = u + iv$ とすると、

$$\partial_x u = \frac{x}{x^2 + y^2} \quad \partial_y u = \frac{y}{x^2 + y^2}$$

$\tan v = \frac{y}{x}$ より、

$$\begin{aligned} \frac{1}{\cos^2 v}\partial_x v = -\frac{y}{x^2} \quad \frac{1}{\cos^2 v}\partial_y v = \frac{1}{x} \\ \therefore \quad (1 + \tan^2 v)\partial_x v = -\frac{y}{x^2} \quad (1 + \tan^2 v)\partial_x v = \frac{1}{x} \\ \therefore \quad (\frac{x^2 + y^2}{x^2})\partial_x v = -\frac{y}{x^2} \quad (\frac{x^2 + y^2}{x^2})\partial_x v = \frac{1}{x} \\ \therefore \quad \partial_x v = -\frac{y}{x^2 + y^2} \quad \partial_y v = \frac{x}{x^2 + y^2} \end{aligned}$$

以上より、

$$\partial_x u = \partial_y v \quad \partial_y u = -\partial_x v$$

このように CR 方程式が成り立つので、f(z) は微分可能。
すなわち、f(z) の微分は微分の方向によらないので、

$$\frac{\mathrm{d}e^z}{\mathrm{d}z} = \partial_x u + i\partial_x v = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2} = \frac{1}{x + iy}$$

以上より題意は示された。

(3)

$\frac{e^z}{z^2}$ をローラン展開すると、

$$\frac{e^z}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{z^{n-2}}{n!}$$

(4)

$\frac{\sin z}{z^3}$ をローラン展開すると、

$$\frac{\sin z}{z^3} = \frac{1}{z^3}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n-2}}{(2n+1)!}$$

(5)

$\frac{1}{z(z-1)^2}$ をローラン展開すると、

$$\begin{aligned}\frac{1}{z(z-1)^2} &= \frac{1}{z} - \frac{1}{z-1} + \frac{1}{(z-1)^2} \\ &= \frac{1}{z} + \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty}(n+1)z^n \\ &= \frac{1}{z} + \sum_{n=0}^{\infty}(n+2)z^n\end{aligned}$$

[2]　$\int_{-\infty}^{\infty} R(x)\exp(ix)\mathrm{d}x$ の積分

(1)

$\frac{e^{iz}}{z^2+a^2}$ の極は、$z = \pm ia$ であり、それぞれにおける留数は、

$$Res\left(\frac{e^{iz}}{z^2+a^2}, ia\right) = \frac{e^{-a}}{2ai}$$
$$Res\left(\frac{e^{iz}}{z^2+a^2}, -ia\right) = \frac{e^{a}}{-ai}$$

(2)

積分経路内にある極は、$z = ia$ であるから、

$$\oint_C \frac{e^{iz}}{z^2+a^2} = 2\pi i Res\left(\frac{e^{iz}}{z^2+a^2}, ia\right) = 2\pi i \frac{e^{-a}}{2ai} = \frac{\pi e^{-a}}{a}$$

(3)

$z = Re^{i\theta}(0 < \theta < pi)$ とおくと、$\mathrm{d}z = iRe^{i\theta}\mathrm{d}\theta$

$$\begin{aligned}
\lim_{R\to\infty}\left|\int_{C_R}\mathrm{d}z\frac{e^{iz}}{z^2+a^2}\right| &= \lim_{R\to\infty}\left|\int_0^\pi \mathrm{d}\theta\, iRe^{i\theta}\frac{e^{iR(\cos\theta+i\sin\theta)}}{R^2e^{2i\theta}+a^2}\right| \\
&= \lim_{R\to\infty}\int_0^\pi \mathrm{d}\theta\left|iRe^{i\theta}\right|\left|\frac{e^{iR\cos\theta-R\sin\theta)}}{R^2e^{2i\theta}+a^2}\right| \\
&= \lim_{R\to\infty}\int_0^\pi \mathrm{d}\theta\, R\left|\frac{e^{-R\sin\theta}}{R^2e^{2i\theta}+a^2}\right| \\
&< \lim_{R\to\infty}\int_0^\pi \mathrm{d}\theta\, R\frac{e^{\sin\theta}e^{-R}}{R^2-a^2} \quad (\because R<a) \\
&\leq \lim_{R\to\infty}\int_0^\pi \mathrm{d}\theta\, R\frac{e^{-R}}{R^2-a^2} \quad (\because \sin\theta \leq 1) \\
&= \frac{R}{R^2-a^2}\pi \\
&= 0
\end{aligned}$$

(4)

$$\begin{aligned}
\int_0^\infty \mathrm{d}x\frac{cosx}{x^2+a} &= \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}x\frac{cosx}{x^2+a^2} \\
&= \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}z\frac{e^{iz}}{z^2+a^2} \\
&= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R \mathrm{d}z\frac{e^{iz}}{z^2+a^2} \\
&= \frac{1}{2}\lim_{R\to\infty}\int_{C_x} \mathrm{d}z\frac{e^{iz}}{z^2+a^2} \\
&= \frac{1}{2}\lim_{R\to\infty}\left(\int_C \mathrm{d}z\frac{e^{iz}}{z^2+a^2} - \int_{C_R}\mathrm{d}z\frac{e^{iz}}{z^2+a^2}\right) \\
&= \frac{\pi e^{-a}}{2a}
\end{aligned}$$

[3]　Heaviside ステップ関数の積分表示

$$S(t)=\frac{1}{2\pi i}\lim_{\epsilon\to+0}\int_{-\infty}^{\infty}\frac{e^{itx}}{x-i\epsilon}\mathrm{d}x$$

(i)　$t>0$

$$\begin{aligned}
S(t)&=\frac{1}{2\pi i}\lim_{\epsilon\to+0}\lim_{R\to\infty}\int_{C_x}\frac{e^{itz}}{z-i\epsilon}\mathrm{d}z\\
&=\frac{1}{2\pi i}\lim_{\epsilon\to+0}\lim_{R\to\infty}\oint_{C}\frac{e^{itz}}{z-i\epsilon}\mathrm{d}z\quad(\because \text{ジョルダンの補題})\\
&=\lim_{\epsilon\to+0}Res\left(\frac{e^{itz}}{z-i\epsilon},i\epsilon\right)\\
&=\lim_{\epsilon\to+0}e^{-\epsilon t}\\
&=1
\end{aligned}$$

(ii)　$t=0$

$z=Re^{i\theta}$ とおくと、

$$\begin{aligned}
S(0)&=\frac{1}{2\pi i}\lim_{\epsilon\to+0}\lim_{R\to\infty}\int_{C_x}\frac{1}{z-i\epsilon}\mathrm{d}z\\
&=\frac{1}{2\pi i}\lim_{\epsilon\to+0}\lim_{R\to\infty}\oint_{C}\frac{1}{z-i\epsilon}\mathrm{d}z-\frac{1}{2\pi i}\lim_{\epsilon\to+0}\lim_{R\to\infty}\int_{C_R}\frac{1}{z-i\epsilon}\mathrm{d}z\\
&=\lim_{\epsilon\to+0}Res\left(\frac{1}{z-i\epsilon},i\epsilon\right)-\frac{1}{2\pi i}\lim_{\epsilon\to+0}\lim_{R\to0}\int_0^{\pi}\frac{1}{Re^{i\theta}-i\epsilon}iRe^{i\theta}\mathrm{d}\theta\\
&=1-\frac{1}{2\pi}\lim_{\epsilon\to+0}\lim_{R\to0}\int_0^{\pi}\frac{1}{1-i\frac{\epsilon}{Re^{i\theta}}}\mathrm{d}\theta\\
&=1-\frac{1}{2\pi}\int_0^{\pi}\lim_{\epsilon\to+0}\lim_{R\to0}\frac{1}{1-i\frac{\epsilon}{Re^{i\theta}}}\mathrm{d}\theta\quad\left(\because \frac{1}{1-i\frac{\epsilon}{Re^{i\theta}}}\text{は 1 に一様収束する}\right)\\
&=1-\frac{1}{2\pi}\int_0^{\pi}\mathrm{d}\theta\quad\left(\because \frac{1}{1-i\frac{\epsilon}{Re^{i\theta}}}\text{は 1 に一様収束する}\right)\\
&=1-\frac{1}{2}\\
&=\frac{1}{2}
\end{aligned}$$

(iii)　$t<0$

$z\to-z$ と変数変換すると、

$$\begin{aligned}
S(t)&=\frac{1}{2\pi i}\lim_{\epsilon\to+0}\lim_{R\to\infty}\int_{C_x}\frac{e^{-itz}}{-z-i\epsilon}\mathrm{d}z\\
&=\frac{1}{2\pi i}\lim_{\epsilon\to+0}\lim_{R\to\infty}\int_{C_x}\frac{e^{-itz}}{-z-i\epsilon}\mathrm{d}z\\
&=\frac{1}{2\pi i}\lim_{\epsilon\to+0}\lim_{R\to\infty}\oint_{C}\frac{e^{-itz}}{z-i\epsilon}\mathrm{d}z\quad(\because \text{ジョルダンの補題})\\
&=0
\end{aligned}$$

[4] 多価関数の積分

(1)

$$
\begin{aligned}
\left|\int_{C_R}\frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z\right| &< \int_0^{\pi}\left|\frac{R^{\frac{1}{2}}e^{\frac{i\theta}{2}}}{R^2e^{2i\theta}+1}iRe^{i\theta}\right|\mathrm{d}\theta \quad (z=Re^{i\theta})\\
&= \int_0^{\pi}\left|\frac{R^{\frac{1}{2}}e^{\frac{i\theta}{2}}}{R^3e^{2i\theta}+1}\right|\left|e^{\frac{i\theta}{2}}\right|\left|iRe^{i\theta}\right|\mathrm{d}\theta\\
&= \int_0^{\pi}\left|\frac{R^{\frac{3}{2}}e^{\frac{i\theta}{2}}}{R^2e^{2i\theta}+1}\right|\mathrm{d}\theta\\
&< \int_0^{\pi}\frac{R^{\frac{3}{2}}}{R^2-1}\mathrm{d}\theta\\
&= 2\pi\frac{R^{\frac{3}{2}}}{R^2-1}\\
&\to 0 \quad (as\, R\to\infty)
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_{C_\epsilon}\frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z\right| &< \int_0^{\pi}\left|\frac{\epsilon^{\frac{1}{2}}e^{\frac{i\theta}{2}}}{\epsilon^2e^{2i\theta}+1}i\epsilon e^{i\theta}\right|\mathrm{d}\theta \quad (z=\epsilon e^{i\theta})\\
&= \int_0^{\pi}\left|\frac{\epsilon^{\frac{3}{2}}e^{\frac{i\theta}{2}}}{\epsilon^2e^{2i\theta}+1}\right|\mathrm{d}\theta\\
&< \int_0^{\pi}\frac{\epsilon^{\frac{3}{2}}}{1-\epsilon^2}\mathrm{d}\theta\\
&= 2\pi\frac{\epsilon^{\frac{3}{2}}}{1-\epsilon^2}\\
&\to 0 \quad (as\, \epsilon\to 0)
\end{aligned}
$$

$$
\begin{aligned}
\oint_C\frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z &= 2\pi iRes\left(\frac{z^{\frac{1}{2}}}{z^2+1}, e^{i\frac{\pi}{2}}\right)+2\pi iRes\left(\frac{z^{\frac{1}{2}}}{z^2+1}, e^{\frac{3\pi}{2}}\right)\\
&= 2\pi i\frac{e^{i\frac{\pi}{4}}}{i+i}+2\pi i\frac{e^{i\frac{3\pi}{4}}}{-i-i}\\
&= 2\pi i\frac{\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}}{2i}+2\pi i\frac{-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}}{-2i}\\
&= \sqrt{2}\pi
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{\infty}\frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z &= \lim_{\delta\to 0}\int_0^{\infty}\frac{\left(xe^0+i\delta\right)^{\frac{1}{2}}}{\left(xe^0+i\delta\right)^2+1}\mathrm{d}x \quad (z=xe^0+i\delta)\\
&= \int_0^{\infty}\frac{x^{\frac{1}{2}}}{x^2+1}\mathrm{d}x \quad (\because \frac{\left(xe^0+i\delta\right)^{\frac{1}{2}}}{\left(xe^0+i\delta\right)^2+1}\text{は一様収束})\\
&= J
\end{aligned}
$$

$$
\begin{aligned}
\int_{\infty}^{0}\frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z &= \lim_{\delta\to 0}\int_{\infty}^{0}\frac{\left(xe^{i2\pi}+i\delta\right)^{\frac{1}{2}}}{\left(xe^{i2\pi}+i\delta\right)^2+1}\mathrm{d}x \quad (z=xe^{i2\pi}+i\delta)\\
&= \int_{\infty}^{0}\frac{x^{\frac{1}{2}}e^{i\pi}}{x^2e^{i4\pi}+1}\mathrm{d}x \quad (\because \frac{\left(xe^{i2\pi}+i\delta\right)^{\frac{1}{2}}}{\left(xe^{i2\pi}+i\delta\right)^2+1}\text{は一様収束})\\
&= \int_{\infty}^{0}-\frac{x^{\frac{1}{2}}}{x^2+1}\mathrm{d}x\\
&= J
\end{aligned}
$$

以上より、

$$
\begin{aligned}
J &= \frac{1}{2}\int_0^\infty \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z + \frac{1}{2}\int_\infty^0 \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z \\
&= \frac{1}{2}\oint_C \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z - \frac{1}{2}\int_{C_R} \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z - \frac{1}{2}\int_{C_\epsilon} \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z \\
&= \frac{\sqrt{2}}{2}\pi
\end{aligned}
$$

(2)

z の偏角が 2π から 4π として同様の議論を行う。

(1) と同様に、

$$
\begin{aligned}
\int_{C_R} \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z = 0 \\
\int_{C_\epsilon} \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z = 0
\end{aligned}
$$

また、

$$
\begin{aligned}
\oint_C \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z &= 2\pi i Res\left(\frac{z^{\frac{1}{2}}}{z^2+1}, e^{i\frac{5\pi}{2}}\right) + 2\pi i Res\left(\frac{z^{\frac{1}{2}}}{z^2+1}, e^{i\frac{7\pi}{2}}\right) \\
&= 2\pi i\frac{e^{i\frac{5\pi}{4}}}{i+i} + 2\pi i\frac{e^{i\frac{7\pi}{4}}}{-i-i} \\
&= 2\pi i\frac{\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}{2i} + 2\pi i\frac{-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}{-2i} \\
&= -\sqrt{2}\pi
\end{aligned}
$$

さらに、

$$
\begin{aligned}
\int_0^\infty \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z &= \lim_{\delta\to 0}\int_0^\infty \frac{\left(xe^{i2\pi}+i\delta\right)^{\frac{1}{2}}}{\left(xe^{i2\pi}+i\delta\right)^2+1}\mathrm{d}x \quad (z = xe^{i2\pi}+i\delta) \\
&= \int_0^\infty \frac{x^{\frac{1}{2}}e^{i\pi}}{x^2e^{i4\pi}+1}\mathrm{d}x \quad \left(\because \frac{\left(xe^{i2\pi}+i\delta\right)^{\frac{1}{2}}}{\left(xe^{i2\pi}+i\delta\right)^2+1}\text{は一様収束}\right) \\
&= \int_0^\infty -\frac{x^{\frac{1}{2}}}{x^2+1}\mathrm{d}x \\
&= -J \\
\int_\infty^0 \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z &= \lim_{\delta\to 0}\int_\infty^0 \frac{\left(xe^{i4\pi}+i\delta\right)^{\frac{1}{2}}}{\left(xe^{i4\pi}+i\delta\right)^2+1}\mathrm{d}x \quad (z = xe^{i4\pi}+i\delta) \\
&= \int_\infty^0 \frac{x^{\frac{1}{2}}e^{2i\pi}}{x^2e^{i8\pi}+1}\mathrm{d}x \quad \left(\because \frac{\left(xe^{i4\pi}+i\delta\right)^{\frac{1}{2}}}{\left(xe^{i4\pi}+i\delta\right)^2+1}\text{は一様収束}\right) \\
&= \int_\infty^0 \frac{x^{\frac{1}{2}}}{x^2+1}\mathrm{d}x \\
&= -J
\end{aligned}
$$

以上より、

$$
\begin{aligned}
J &= -\frac{1}{2}\int_0^\infty \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z - \frac{1}{2}\int_\infty^0 \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z \\
&= -\frac{1}{2}\oint_C \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z + \frac{1}{2}\int_{C_R} \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z + \frac{1}{2}\int_{C_\epsilon} \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z \\
&= \frac{\sqrt{2}}{2}\pi
\end{aligned}
$$

これは確かに (1) の答えと一致している。

(3)

z の偏角が $-\pi$ から π として同様の議論を行う。
(1) と同様に、

$$
\begin{aligned}
\int_{C_R} \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z = 0 \\
\int_{C_\epsilon} \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z = 0
\end{aligned}
$$

また、

$$
\begin{aligned}
\oint_C \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z &= 2\pi i Res\left(\frac{z^{\frac{1}{2}}}{z^2+1}, e^{i\frac{\pi}{2}}\right) + 2\pi i Res\left(\frac{z^{\frac{1}{2}}}{z^2+1}, e^{-i\frac{\pi}{2}}\right) \\
&= 2\pi i \frac{e^{i\frac{\pi}{4}}}{i+i} + 2\pi i \frac{e^{-i\frac{\pi}{4}}}{-i-i} \\
&= 2\pi i \frac{\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}{2i} + 2\pi i \frac{\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}}{-2i} \\
&= i\sqrt{2}\pi
\end{aligned}
$$

さらに、

$$
\begin{aligned}
\int_0^\infty \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z &= \lim_{\delta\to 0}\int_0^\infty \frac{\left(xe^{i\pi}+i\delta\right)^{\frac{1}{2}}}{\left(xe^{i\pi}+i\delta\right)^2+1}\mathrm{d}x \quad (z = xe^{i\pi}+i\delta) \\
&= \int_0^\infty \frac{x^{\frac{1}{2}}e^{\frac{i\pi}{2}}}{x^2e^{i2\pi}+1}\mathrm{d}x \quad \left(\because \frac{\left(xe^{i\pi}+i\delta\right)^{\frac{1}{2}}}{\left(xe^{i\pi}+i\delta\right)^2+1}\text{は一様収束}\right) \\
&= \int_0^\infty i\frac{x^{\frac{1}{2}}}{x^2+1}\mathrm{d}x \\
&= iJ
\end{aligned}
$$

$$
\begin{aligned}
\int_\infty^0 \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z &= \lim_{\delta\to 0}\int_\infty^0 \frac{\left(xe^{-i\pi}+i\delta\right)^{\frac{1}{2}}}{\left(xe^{-i\pi}+i\delta\right)^2+1}\mathrm{d}x \quad (z = xe^{-i\pi}+i\delta) \\
&= \int_\infty^0 \frac{x^{\frac{1}{2}}e^{-\frac{i\pi}{2}}}{x^2e^{i2\pi}+1}\mathrm{d}x \quad \left(\because \frac{\left(xe^{-i\pi}+i\delta\right)^{\frac{1}{2}}}{\left(xe^{-i\pi}+i\delta\right)^2+1}\text{は一様収束}\right) \\
&= \int_\infty^0 -i\frac{x^{\frac{1}{2}}}{x^2+1}\mathrm{d}x \\
&= iJ
\end{aligned}
$$

以上より、

$$
\begin{aligned}
J &= -i\frac{1}{2}\int_0^\infty \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z - i\frac{1}{2}\int_\infty^0 \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z \\
&= -i\frac{1}{2}\oint_C \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z + i\frac{1}{2}\int_{C_R} \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z + i\frac{1}{2}\int_{C_\epsilon} \frac{z^{\frac{1}{2}}}{z^2+1}\mathrm{d}z \\
&= \frac{\sqrt{2}}{2}\pi
\end{aligned}
$$

これは確かに (1)(2) の答えと一致している。

[5] $\int_0^\infty R(x)x^\alpha \mathrm{d}x$ 型の積分

(1) $\int_0^\pi \frac{x^\alpha}{x^2+x+1}\mathrm{d}x$

$J = \int_0^\pi \frac{x^\alpha}{x^2+x+1}\mathrm{d}x$ とおく。

$$
\begin{aligned}
\left|\int_{C_R} \frac{z^\alpha}{z^2+z+1}\mathrm{d}z\right| &< \int_0^\pi \left|\frac{R^\alpha e^{i\alpha\theta}}{R^2e^{2i\theta}+Re^{i\theta}+1} iRe^{i\theta}\right| \mathrm{d}\theta \quad (z = Re^{i\theta}) \\
&< \int_0^\pi \frac{R^{\alpha+1}}{|R^2e^{2i\theta}| - |Re^{i\theta}+1|}\mathrm{d}\theta \\
&< \int_0^\pi \frac{R^{\alpha+1}}{R^2-R-1}\mathrm{d}\theta \\
&= 2\pi\frac{R^{\alpha+1}}{R^2-R-1} \\
&\to 0 \quad (as\, R\to\infty)
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_{C_\epsilon} \frac{z^\alpha}{z^2+z+1}\mathrm{d}z\right| &< \int_0^\pi \left|\frac{\epsilon^\alpha e^{i\alpha\theta}}{\epsilon^2e^{2i\theta}+\epsilon e^{i\theta}+1} i\epsilon e^{i\theta}\right| \mathrm{d}\theta \quad (z = \epsilon e^{i\theta}) \\
&< \int_0^\pi \frac{\epsilon^{\alpha+1}}{1 - |\epsilon^2e^{2i\theta}+\epsilon e^{i\theta}|}\mathrm{d}\theta \\
&< \int_0^\pi \frac{\epsilon^{\alpha+1}}{1-\epsilon^2-\epsilon}\mathrm{d}\theta \\
&= 2\pi\frac{\epsilon^{\alpha+1}}{1-\epsilon^2-\epsilon} \\
&\to 0 \quad (as\, \epsilon\to\infty)
\end{aligned}
$$

$$
\begin{aligned}
\oint_C \frac{z^\alpha}{z^2+z+1}\mathrm{d}z &= 2\pi i Res\left(\frac{z^\alpha}{z^2+z+1}, e^{i\frac{2\pi}{3}}\right) + 2\pi i Res\left(\frac{z^\alpha}{z^2+z+1}, e^{i\frac{4\pi}{3}}\right) \\
&= 2\pi i\frac{e^{i\frac{2\alpha\pi}{3}}}{i\sqrt{3}} + 2\pi i\frac{e^{i\frac{4\alpha\pi}{3}}}{-i\sqrt{3}} \\
&= \frac{2\pi}{\sqrt{3}}(e^{i\frac{2\alpha\pi}{3}} - e^{i\frac{4\alpha\pi}{3}})
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty \frac{z^\alpha}{z^2+z+1}\mathrm{d}z &= \lim_{\delta\to 0}\int_0^\infty \frac{\left(xe^0+i\delta\right)^\alpha}{\left(xe^0+i\delta\right)^2+xe^0+i\delta+1}\mathrm{d}x \quad (z = xe^0+i\delta) \\
&= \int_0^\infty \frac{x^\alpha}{x^2+x+1}\mathrm{d}x \quad \left(\because \frac{\left(xe^0+i\delta\right)^\alpha}{\left(xe^0+i\delta\right)^2+xe^0+i\delta+1}\text{は一様収束}\right) \\
&= J
\end{aligned}
$$

$$
\begin{aligned}
\int_{\infty}^{0} \frac{z^\alpha}{z^2+z+1}\mathrm{d}z &= \lim_{\delta\to 0}\int_{\infty}^{0} \frac{\left(xe^{i2\pi}+i\delta\right)^\alpha}{\left(xe^{i2\pi}+i\delta\right)^2+xe^{i2\pi}+i\delta+1}\mathrm{d}x \quad (z = xe^{i2\pi}+i\delta) \\
&= \int_{\infty}^{0} \frac{\left(xe^{i2\pi}\right)^\alpha}{\left(xe^{i2\pi}\right)^2+xe^{i2\pi}+1}\mathrm{d}x \quad \left(\because \frac{\left(xe^{i2\pi}+i\delta\right)^\alpha}{\left(xe^{i2\pi}+i\delta\right)^2+xe^{i2\pi}+i\delta+1}\text{は一様収束}\right) \\
&= \int_{\infty}^{0} e^{i\alpha 2\pi}\frac{x^\alpha}{x^2+x+1}\mathrm{d}x \\
&= -e^{i\alpha 2\pi} J
\end{aligned}
$$

以上より、

$$
\begin{aligned}
J &= \frac{1}{1-e^{i\alpha\pi}}\left(\int_0^{\infty} \frac{z^\alpha}{z^2+z+1}\mathrm{d}z + \int_{\infty}^{0} \frac{z^\alpha}{z^2+z+1}\mathrm{d}z\mathrm{d}z\right) \\
&= \frac{1}{1-e^{i\alpha\pi}}\left(\oint_C \frac{z^\alpha}{z^2+z+1}\mathrm{d}z + \int_{C_\epsilon} \frac{z^\alpha}{z^2+z+1}\mathrm{d}z + \int_{C_R} \frac{z^\alpha}{z^2+z+1}\mathrm{d}z\right) \\
&= \frac{2\pi}{\sqrt{3}}\frac{e^{i\frac{2\alpha\pi}{3}}-e^{i\frac{4\alpha\pi}{3}}}{1-e^{i\alpha 2\pi}} \\
&= \frac{2\pi}{\sqrt{3}}\frac{e^{-i\frac{\alpha\pi}{3}}-e^{i\frac{\alpha\pi}{3}}}{e^{-i\alpha\pi}-e^{i\alpha\pi}} \\
&= \frac{2\pi}{\sqrt{3}}\frac{\sin\frac{\pi}{3}\alpha}{\sin\alpha\pi}
\end{aligned}
$$

(2) $\int_0^\pi \frac{\ln x}{x^2+1}\mathrm{d}x$

$J = \int_0^\pi \frac{\ln x}{x^2+1}\mathrm{d}x$ とおく。

$$
\begin{aligned}
\left|\int_{C_R} \mathrm{d}z\frac{(\ln z)^2}{z^2+1}\right| &= \left|\int_0^{2\pi} \frac{(\ln Re^{i\theta})^2}{R^2e^{2i\theta}+1}iRe^{i\theta}\mathrm{d}\theta\right| \quad (z = Re^{i\theta}) \\
&< \int_0^{2\pi} \frac{(\ln R+i\theta)^2}{R^2-1}R\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{R\left((\ln R)^2+\theta^2\right)}{R^2-1}\mathrm{d}\theta \\
&= 2\pi\frac{R\left((\ln R)^2+\theta^2\right)}{R^2-1} \\
&\to 0 \quad (as\, R\to\infty)
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_{C_\epsilon} \mathrm{d}z\frac{(\ln z)^2}{z^2+1}\right| &= \left|\int_0^{2\pi} \frac{(\ln \epsilon e^{i\theta})^2}{\epsilon^2e^{2i\theta}+1}i\epsilon e^{i\theta}\mathrm{d}\theta\right| \quad (z = \epsilon e^{i\theta}) \\
&< \int_0^{2\pi} \frac{(\ln \epsilon+i\theta)^2}{1-\epsilon^2}\epsilon\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{\epsilon\left((\ln \epsilon)^2+\theta^2\right)}{1-\epsilon^2}\mathrm{d}\theta \\
&= 2\pi\frac{\epsilon\left((\ln \epsilon)^2+\theta^2\right)}{1-\epsilon^2} \\
&\to 0 \quad (as\, \epsilon\to\infty)
\end{aligned}
$$

$$
\begin{aligned}
\oint_c \mathrm{d}z \frac{(\ln z)^2}{z^2+1} &= 2\pi i Res\left(\frac{(\ln z)^2}{z^2+1}, e^{i\frac{\pi}{2}}\right) + 2\pi i Res\left(\frac{(\ln z)^2}{z^2+1}, e^{i\frac{3\pi}{2}}\right) \\
&= 2\pi i \frac{(\ln (e^{i\frac{\pi}{2}}))^2}{2i} + 2\pi i \frac{(\ln (e^{i\frac{3\pi}{2}}))^2}{-2i} \\
&= 2\pi i \frac{(i\frac{\pi}{2})^2}{2i} + 2\pi i \frac{(i\frac{3\pi}{2})^2}{-2i} \\
&= 2\pi^3 \\
\int_0^\infty \frac{(\ln z)^2}{z^2+1} \mathrm{d}z &= \lim_{\delta \to 0} \int_0^\infty \frac{(\ln (xe^0 + i\delta))^2}{(xe^0+i\delta)^2+1} \mathrm{d}x \quad (z = xe^0 + i\delta) \\
&= \int_0^\infty \frac{(\ln x)^2}{x^2+1} \quad (\because \frac{(\ln (xe^0 + i\delta))^2}{(xe^0+i\delta)^2+1} \text{は一様収束}) \\
\int_\infty^0 \frac{(\ln z)^2}{z^2+1} \mathrm{d}z &= \lim_{\delta \to 0} \int_\infty^0 \frac{(\ln (xe^{i2\pi} + i\delta))^2}{(xe^{i2\pi}+i\delta)^2+1} \mathrm{d}x \quad (z = xe^{i2\pi} + i\delta) \\
&= \int_\infty^0 \frac{(\ln (xe^{i2\pi}))^2}{x^2+1} \ \mathrm{d}x \quad (\because \frac{(\ln (xe^{i2\pi} + i\delta))^2}{(xe^{i2\pi}+i\delta)^2+1} \text{は一様収束}) \\
&= \int_\infty^0 \frac{(\ln x + i2\pi)^2}{x^2+1} \ \mathrm{d}x \\
&= \int_\infty^0 \frac{(\ln x)^2 + i4\pi \ln x - 4\pi^2}{x^2+1} \mathrm{d}x \\
&= -i4\pi J - \int_0^\infty \frac{(\ln x)^2}{x^2+1} + 4\pi^2 \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^2\theta} \cos^2\theta \mathrm{d}\theta \quad (x = \tan\theta) \\
&= -i4\pi J - \int_0^\infty \frac{(\ln x)^2}{x^2+1} + 4\pi^2 \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \quad (x = \tan\theta) \\
&= -i4\pi J - \int_0^\infty \frac{(\ln x)^2}{x^2+1} + 2\pi^3
\end{aligned}
$$

以上より、

$$
\begin{aligned}
J &= \frac{1}{-i4}\left(\int_0^\infty \frac{(\ln z)^2}{z^2+1} \mathrm{d}z + \int_\infty^0 \frac{(\ln z)^2}{z^2+1} \mathrm{d}z - 2\pi^3\right) \\
&= \frac{i}{4}\left(\oint_C \frac{(\ln z)^2}{z^2+1} \mathrm{d}z + \int_{C_\epsilon} \frac{(\ln z)^2}{z^2+1} \mathrm{d}z + \int_{C_R} \frac{(\ln z)^2}{z^2+1} \mathrm{d}z - 2\pi^3\right) \\
&= \frac{i}{4}(2\pi^3 - 2\pi^3) \\
&= 0
\end{aligned}
$$

(3) $\int_0^{\pi} \frac{(\ln x)^2}{x^2+1} \mathrm{d}x$

$J = \int_0^{\pi} \frac{(\ln x)^2}{x^2+1} \mathrm{d}x$ とおく。

$$
\begin{aligned}
\left|\int_{C_R} \mathrm{d}z \frac{(\ln z)^3}{z^2+1}\right| &= \left|\int_0^{2\pi} \frac{(\ln Re^{i\theta})^3}{R^2 e^{2i\theta}+1} iRe^{i\theta} \mathrm{d}\theta\right| \quad (z = Re^{i\theta}) \\
&< \int_0^{2\pi} \frac{(\ln R + i\theta)^3}{R^2-1} R\mathrm{d}\theta \\
&< \int_0^{2\pi} \frac{[(\ln R)^2 + 4\pi^2]^{\frac{3}{2}}}{R^2-1} \mathrm{d}\theta \\
&= 2\pi \frac{R\big((\ln R)^2 + \theta^2\big)^{\frac{3}{2}}}{R^2-1} \\
&\to 0 \quad (as\, R \to \infty)
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_{C_\epsilon} \mathrm{d}z \frac{(\ln z)^2}{z^2+1}\right| &= \left|\int_0^{2\pi} \frac{(\ln \epsilon e^{i\theta})^3}{\epsilon^2 e^{2i\theta}+1} i\epsilon e^{i\theta} \mathrm{d}\theta\right| \quad (z = \epsilon e^{i\theta}) \\
&< \int_0^{2\pi} \frac{(\ln \epsilon + i\theta)^3}{1-\epsilon^2} \epsilon\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{\epsilon\big((\ln \epsilon)^2 + \theta^2\big)^{\frac{3}{2}}}{1-\epsilon^2} \mathrm{d}\theta \\
&= 2\pi \frac{\epsilon\big((\ln \epsilon)^2 + \theta^2\big)^{\frac{3}{2}}}{1-\epsilon^2} \\
&\to 0 \quad (as\, \epsilon \to \infty)
\end{aligned}
$$

$$
\begin{aligned}
\oint_c \mathrm{d}z \frac{(\ln z)^3}{z^2+1} &= 2\pi i Res\left(\frac{(\ln z)^3}{z^2+1}, e^{i\frac{\pi}{2}}\right) + 2\pi i Res\left(\frac{(\ln z)^3}{z^2+1}, e^{i\frac{3\pi}{2}}\right) \\
&= 2\pi i \frac{(\ln(e^{i\frac{\pi}{2}}))^3}{2i} + 2\pi i \frac{(\ln(e^{i\frac{3\pi}{2}}))^3}{-2i} \\
&= 2\pi i \frac{(i\frac{\pi}{2})^3}{2i} + 2\pi i \frac{(i\frac{3\pi}{2})^3}{-2i} \\
&= i\frac{13\pi^4}{4}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{\infty} \frac{(\ln z)^3}{z^2+1} \mathrm{d}z &= \lim_{\delta \to 0} \int_0^{\infty} \frac{(\ln(xe^0 + i\delta))^3}{(xe^0+i\delta)^2+1} \mathrm{d}x \quad (z = xe^0 + i\delta) \\
&= \int_0^{\infty} \frac{(\ln x)^3}{x^2+1} \quad (\because \frac{(\ln(xe^0+i\delta))^3}{(xe^0+i\delta)^2+1} \text{は一様収束})
\end{aligned}
$$

$$
\begin{aligned}
\int_{\infty}^{0} \frac{(\ln z)^3}{z^2+1} \mathrm{d}z &= \lim_{\delta \to 0} \int_{\infty}^{0} \frac{(\ln(xe^{i2\pi} + i\delta))^3}{(xe^{i2\pi}+i\delta)^2+1} \mathrm{d}x \quad (z = xe^{i2\pi} + i\delta) \\
&= \int_{\infty}^{0} \frac{(\ln(xe^{i2\pi}))^3}{x^2+1} \ \mathrm{d}x \quad (\because \frac{(\ln(xe^{i2\pi}+i\delta))^3}{(xe^{i2\pi}+i\delta)^2+1} \text{は一様収束}) \\
&= \int_{\infty}^{0} \frac{(\ln x + i2\pi)^3}{x^2+1} \ \mathrm{d}x \\
&= \int_{\infty}^{0} \frac{(\ln x)^3 + i6\pi(\ln x)^2 - 12\pi^2 \ln x - i8\pi^3}{x^2+1} \mathrm{d}x \\
&= -i6J - \int_0^{\infty} \frac{(\ln x)^3}{x^2+1} + i8\pi^3 \times \frac{\pi}{2} \\
&= -i6\pi J - \int_0^{\infty} \frac{(\ln x)^2}{x^2+1} + i4\pi^4
\end{aligned}
$$

以上より、

$$
\begin{aligned}
J &= \frac{1}{-i6}\left(\int_0^\infty \frac{(\ln z)^2}{z^2+1}\mathrm{d}z + \int_\infty^0 \frac{(\ln z)^2}{z^2+1}\mathrm{d}z - i4\pi^4\right) \\
&= \frac{i}{6}\left(\oint_C \frac{(\ln z)^2}{z^2+1}\mathrm{d}z + \int_{C_\epsilon} \frac{(\ln z)^2}{z^2+1}\mathrm{d}z + \int_{C_R} \frac{(\ln z)^2}{z^2+1}\mathrm{d}z - i4\pi^4\right) \\
&= \frac{i}{4}(i\frac{13\pi^4}{4} - i4\pi^4) \\
&= \frac{\pi^3}{8}
\end{aligned}
$$