Section A

2022 1(a)

A discrete time signal is a signal is a signal defined only at discrete points in time. These signals are obtained by sampling a continuous - time signal at uniform time intervals.

A digital signal is a discrete -time signal with both time and amplitude quantized. It is the result of sampling and quantization.

The frequency of two sinusoidal signal

$f_1 = 10 Hz$

$f_2 = 50 Hz$

$F_s = 40 Hz$

~~To find the aliased frequency~~

The signal $x_1(t) = \cos 2\pi(10)t$

$x_2(t) = \cos 2\pi(50)t$

The corresponding discrete time signal are:

$$x_1(n) = \cos 2\pi \left(\frac{10}{40}\right) n = \cos \frac{\pi}{2} n$$

$$x_2(n) = \cos 2\pi \left(\frac{50}{40}\right) n = \cos \frac{5\pi}{2} n$$

$$\cos \frac{5\pi n}{2} = \cos\left(2\pi + \frac{\pi n}{2}\right) = \cos \frac{\pi n}{2}$$

Hence $x_2(n) = x_1(n)$. The sinusoidal signal signals are identical and consequently in distinguishable. We say that The frequency $F_2 = 50Hz$ is an alias of the frequency $F_1 = 10Hz$ at the sampling rate of 40 Hz.

Aliasing occur when a signal is sampled at a rate lower than twice the highest frequency component,

Here $\frac{F_s}{2} = 20Hz$

$f_1 = 10Hz$ $f_2 = 50Hz$

$\frac{f_s}{2} \ngtr F_2$

2022: 1(b)

Linear Time Invariant Systems are a special and very important class of systems in signal processing and control theory. The properties are

1. Linearity
2. Time Invariance
3. Stability
4. Causality

1. Linearity:

An LTI system obeys the principle of Addictivity and homogeneity.

Addictivity:

If the system's respone to $x_1(t)$ is $y_1(t)$ and $x_2(t)$ is $y_2(t)$

then the response to $x_1(t) + x_2(t)$ is

$$y(t) = y_1(t) + y_2(t)$$

Homogeneity:

If the response to $x(t)$ is $y(t)$ then the response to $a \cdot x(t)$ is $a \cdot y(t)$

2. Time Invariance:

A system is time-invariant if its behavior does not change over time. If the input is delayed by t_0, the output is also delayed by t_0

$$x(t) \rightarrow y(t) \Rightarrow x(t-t_0) \rightarrow y(t-t_0)$$

3. Stability:

If $|x(t)| < M$ for all t, then $|y(t)| < N$ for some constant N.

condition

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad \text{(discrete time)}$$

$$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty \quad \text{(continuous time)}$$

4. causality:

A system is causal if the output at any time depends only on present and past inputs

For an LTI system, this means:

$$h(t) = 0 \quad \text{for } t < 0$$

Given system

$$y(n) = n\,x(n).$$ [Proakis 2.2.5 example]

two input sequence $x_1(n)$ and $x_2(n)$

$$y_1(n) = n\,x_1(n)$$

$$y_2(n) = n\,x_2(n)$$

$$y_3(n) = \mathcal{T}[a_1 x_1(n) + a_2 x_2(n)]$$

$$= n\,[a_1 x_1(n) + a_2 x_2(n)]$$

$$= n \cdot a_1 x_1(n) + n\,a_2 x_2(n)$$

$$a_1 y_1(n) + a_2 y_2(n) = a_1 n\,x_1(n) + a_2 n\,x_2(n)$$

The system Linear. That's why the system is an LTI.

2022 - 1 (c)

$$x(n) = 2^n u(n), \quad 0 \le n \le 6$$

$x(0) = 1 \quad x(1) = 2 \quad x(2) = 4 \quad x(3) = 8$

$x(4) = 16 \quad x(5) = 32 \quad x(6) = 64$

i) $y(n) = x(2n)$

$n=0 \quad y(0) = x(2\times 0) = x(0) = 1$

$y(1) = x(2\cdot 1) = x(2) = 4$

$y(2) = x(2\cdot 2) = x(4) = 16$

$y(3) = x(2\cdot 3) = x(6) = 64$

$y(n) = \{1, 4, 16, 64\}$

ii) $y(n) = x(3-n)$

$y(0) = x(3-0) = x(3) = 8$

$y(1) = x(3-1) = x(2) = 4$

$y(2) = x(3-2) = x(1) = 2$

$y(3) = x(3-3) = x(0) = 1$

$y(n) = \{8, 4, 2, 1\}$

2022-2 (a)

Statement of Sampling Theorem:

If a continuous time signal $x(t)$ has a maximum frequency component f_{max}, then it can be perfectly reconstructed from its samples $x[n] = x(nT_s)$ taken at a uniform sampling rate $f_s = \frac{1}{T_s}$, provided that the sampling rate f_s is greater than the twice maximum frequency component of the signal

$$f_s \geq 2f_{max}$$

The minimum sampling rate $2f_{max}$ is called the Nyquist rate.

In simpler terms, when converting an analog signal into a digital signal (discrete), we must take samples at regular intervals. The sampling theorem tells us the minimum sampling rate needed to perfectly reconstruct the original analog signal from it samples.

If the sampling is done too slowly we get a phenomenon called aliasing where different signals become indistinguishable when sampled.

২০২২ 2 (b)

Aliasing is a distortion that occurs when a continuous signal is sampled at a rate lower than the Nyquist rate. Due to under sampling, different frequency components in the original signal become indistinguisable, resulting a mis-representation of signal.

I have a sinusoidal signal $f = 3KHz$ and sampling rate $f_s = 6KHz$

Nyquist rate $= 2 \times 3KHz = 6KHz$

$f_s \geq 2f$

No aliasing occurs.

sampling rate $= 4KHz$

$f_s < 2f$

aliasing is occur.

Antialiasing filter before sampling a low pass filter that removes frequencies higher than the half sampling rate. It insures that only frequency components that can be correctly sampled remain.

Real world analog signals often contain frequency components higher than the nyquist frequency. If we sample such a signal directly, these high-frequency components will cause aliasing, leading to incorrect or distorted digital representation. ~~a why~~ That's why we need anti-aliasing filter.

2022-3(a)

Difference equations provide a time-domain recursive relationship between the input and output of a LTI discrete time system. They express the current output sample as a linear combination of past output samples and current and past input samples.

Here how they characterize the response:

Input output Relationship:

A difference equation directly defines how the system transforms an input sequence $x[n]$, ~~Given~~ into an output sequence $y[n]$

Implementation:

Difference equations provide a direct blueprint for implementing LTI systems in software or hardware using delays, multipliers and adder.

Analysis:

They are a starting point for analyzing system properties such as the system finding the system's transfer function in the z-domain by taking z transform of the difference equation.

The difference equation for a 3-point moving averaging filter is:

$$y[n] = \frac{1}{3}\left[x[n] + x[n-1] + x[n-2]\right]$$

given input sequence

$$x[k] = [2, 3, 1, 5, 0, 4, 2]$$

$$y[0] = \frac{1}{3}\left[x[0] + x[-1] + x[-2]\right]$$
$$= \frac{1}{3}[2 + 0 + 0] = \frac{2}{3}$$

$$y[1] = \frac{1}{3}\left[x[1] + x[0] + x[-1]\right]$$
$$= \frac{1}{3}[3 + 2 + 0] = \frac{5}{3}$$

$$y[2] = \frac{1}{3}\left[x[2] + x[1] + x[0]\right]$$
$$= \frac{1}{3}[1 + 3 + 2]$$
$$= 2$$

$$y[3] = \frac{1}{3}\{x[3] + x[2] + x[1]\}$$
$$= \frac{1}{3}[5 + 1 + 3]$$
$$= 3$$
$$y[4] = \frac{1}{3}\{x[4] + x[3] + x[2]\}$$
$$= \frac{1}{3}[0 + 5 + 1]$$
$$= 2$$
$$y[5] = \frac{1}{3}\{x[5] + x[4] + x[3]\}$$
$$= \frac{1}{3}[4 + 0 + 5]$$
$$= 3$$
$$y[6] = \frac{1}{3}\{x[6] + x[5] + x[4]\}$$
$$= \frac{1}{3}[2 + 4 + 0]$$
$$= 2$$
$$y[7] = \frac{1}{3}\{x[7] + x[6] + x[5]\}$$
$$= \frac{1}{3}[0 + 2 + 4]$$
$$= 2$$
$$y[8] = \frac{1}{3}\{x[8] + x[7] + x[6]\}$$
$$= \frac{1}{3}[0 + 0 + 2] = \frac{2}{3}$$

The output signal

$$y[n] = \{ \ldots 0, \frac{2}{3}, \frac{5}{3}, 2, 3, 2, 3, 2, 2, \frac{2}{3}, 0, \ldots \}$$

2022-3(c)

we use the Fourier Transform to convert a signal from the time domain to the frewency domain. This help us

i) Analyze the frewency components of signals
ii) simplify the analysis of linear time invariant systems
iii) solve diffential ewation more easily.
iv) understand signal behaviour, filtering, and modulation in the frewency spectrum

Fourier Transform $\delta(t) \cdot u(t)$

$\delta(t)$ is the Dirac delta function

$u(t)$ is the unit step function.

$\delta(t)$ is nonzero only at $t=0$

$u(t) = 1$ for $t \geq 0$ and 0 for $t<0$

$\delta(t) \cdot u(t) = \delta(t)$ for $t=0$

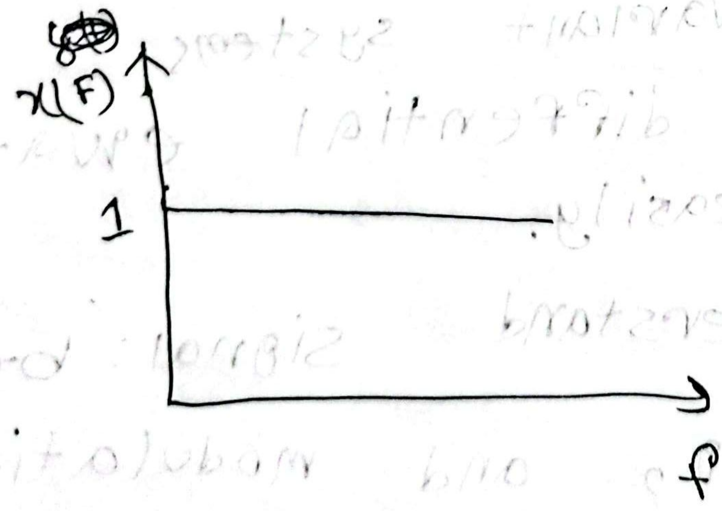
$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi ft}\, dt$$

when $t=0$

$$F(\delta(t)) = e^{-j2\pi f \cdot 0} = e^{0} = 1$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



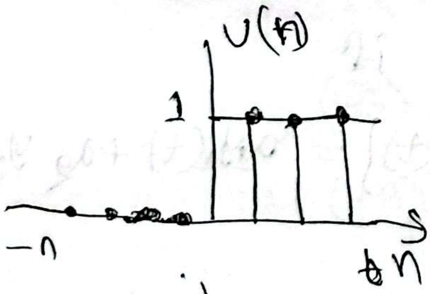
2020 - 3(a)

The unit sample sequence is a discrete time signal with a value of 1 at n = 0 for all other values of n

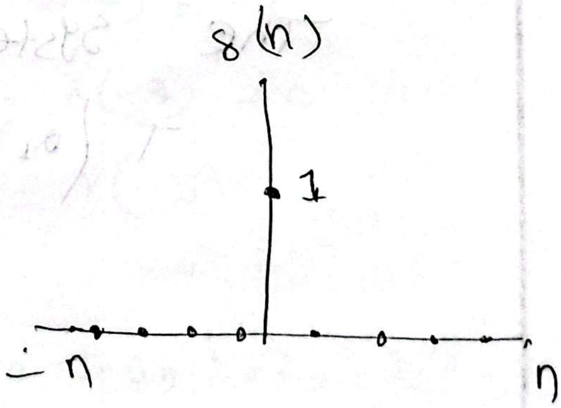
$$\delta[n] = \begin{cases} 1, & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

unit step sequence is a discrete time signal that remains constant at a value of 1 for all time samples greater than or equal to and 0 for all negative time samples.

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



unit step



3. (d) The linear property refers to the behavior of a system, not a signal itself. A system is said to be linear if it satisfies the following two properties:

1. <u>Additivity:</u>

If a system T satisfies

$$T\{x_1(t)\} = y_1(t) \qquad T\{x_2(t)\} = y_2(t)$$

$$T\{x_1(t) + x_2(t)\} = y_1(t) + y_2(t)$$

2. <u>Homogenety (scaling):</u>

$$T\{x(t)\} = y(t)$$

$$T\{a \cdot x(t)\} = a \cdot y(t)$$

The system is linear if

$$T\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 y_1(t) + a_2 y_2(t)$$

2020

3.c) $x(n) = \begin{cases} n & -3 \le n \le 4 \\ 0 & \text{otherwise} \end{cases}$

i) $y(n) = x(n-1)$

$y(0) = x(0-1) = \cancel{x(0) = 0}\ x(-1) = -1$

$y(1) = x(1-1) = x(0) = 0$

$y(2) = x(2-1) = x(1) = 1$

$y(3) = x(3-1) = x(2) = 2$

$y(4) = x(4-1) = x(3) = 3$

$y(5) = x(5-1) = x(4) = 4$

$y(6) = x(6-1) = x(5) = 0$

$y(7) = x(7-1) = x(6) = 0$

$\vdots$

$y(-1) = x(-1-1) = x(-2) = -2$

$y(-2) = x(-2-1) = x(-3) = -3$

$y(-3) = x(-3-1) = x(-4) = 0$

$y(-4) = x(-4-1) = x(-5) = 0$

$y(n) = \{ \ldots 0, -3, -2, -1, 0, 1, 2, 3, 4, 0 \ldots \}$

ii) $y(n) = 3\ x(n+1)$

$y(-5) = 3 \cdot x(-5+1) = 3 \cdot x(-4) = 3 \cdot 0 = 0$

$y(-4) = 3 \cdot x(-4+1) = 3 \cdot x(-3) = 3 \cdot 0 = 0$

$y(-3) = 3 \cdot x(-3+1) = 3 \cdot x(-2) = 3 \cdot -2 = -6$

$y(-2) = 3 \cdot x(-2+1) = 3 \cdot x(-1) = 3 \cdot -1 = -3$

$y(-1) = 3 \cdot x(-1+1) = 3 \cdot x(0) = 3 \cdot 0 = 0$

$y(0) = 3\ x(0+1) = 3 \cdot x(1) = 3 \cdot 1 = 3$

$y(1) = 3 \cdot x(1+1) = 3 \cdot x(2) = 3 \cdot 2 = 6$

$y(2) = 3 \cdot x(2+1) = 3 \cdot x(3) = 3 \cdot 3 = 9$

$y(3) = 3 \cdot x(3+1) = 3 \cdot x(4) = 3 \cdot 4 = 12$

$y(4) = 3 \cdot x(4+1) = 3 \cdot x(5) = 3 \cdot 0 = 0$

$y(n) = \{\ldots 0, -6, -3, 0, 3, 6, 9, 12, 0 \ldots\}$

2020 - 4(a)

An system is a system that satisfies two key Properties:

Linearity:

A system is linear if it satisfies

$T\{x_1(t) + x_2(t)\} = T\{x_1(t)\} + T\{x_2(t)\}$

$T\{a \cdot x(t)\} = a\ T\{x(t)\}$

Together

$$T\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 T\{x_1(t)\} + a_2 T\{x_2(t)\}$$

2. Time Invariance:

A system is time invariant if a time shift in the input causes the same time shift in the output.

$$T\{x(t)\} = y(t)$$

Then any shift t_0:

$$T\{x(t-t_0)\} = y(t-t_0)$$

4. 2020-4(b)

Let $h[n]$ be the impulse response of a discrete time LTI system.

That is if input is $\delta[n]$ then output is

$$T\{\delta[n]\} = h[n]$$

Any discrete time signal

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta(n-k)$$

This is called shifting property of the delta function.

Due to ~~a~~ linearity,

$$T\{x[n]\} = T\left\{\sum_{k=-\infty}^{\infty} x[k] \cdot \delta(n-k)\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \; T\{\delta(n-k)\}$$

Due to time invariance:

$$T\{\delta[n-k]\} = h[n-k]$$

$$y[n] = T\{x[n]\}$$

$$= \sum_{k-\infty}^{\infty} x[k] \cdot h[n-k]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] . h[n-k]$$

This is the convolution sum in discrete time.

~~If~~ $y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$

[continuous time convolution sum]

2020 (4 - c)

c) $x(n) = \{2, 5, 7, 8\}$

$h(n) = \{1, 2, 3, 4\}$

$y(n) = h(n) * x(n)$

length of $y(n) = 4 + 4 - 1 = 7$

```
2   5   7   8
            4   3   2   1
-------------------------
            32  =  y(6)
```

```
2   5   7   8
        4   3   2   1
-------------------------
        28 + 24 = 52 = y(5)
```

~~2 5~~

$h(n)$ \ $x(n)$	2	5	7	8
1	2	5	7	8
2	4	10	14	16
3	6	15	21	24
4	8	20	28	32

$y[n] = \{2, 9, 23, 45, 57, 52, 32\}$

2022- 4.(a)

Importance of ROC in Z Transform:

- The Region of convergence determines for which values of z the z-transform converges
- It is essential to identyfy system stability and causality.
- Two different time-domain signals can have the same z-transform expression but different Roc, which makes the ROC critical for identifying the signal equality.

Given $x(n) = -u(-n-1)$

The unit step function is defined

$u(n) = 1$ for $n \geq 0$

$u(n) = 0$ for $n < 0$

Therefor

$u(-n-1) = 1$ for $-n-1 \geq 0$

$-n \geq 1$

$\therefore n \leq -1$

$u(-n-1) = 0$ for $-n-1 < 0$

$-n < 1$

$n > -1$

$x(n) = -u(-n-1)$ means;

$x(n) = -1$ for $n \leq -1$

$x(n) = 0$ for $n > -1$

This is a left sided sequence.

The z- transform is defined as;

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{-1} (-1)\, z^{-n}$$

Let $m = -n$

when $n = -\infty$ $m = \infty$

$n = -1$ $m = 1$

$$X(z) = -\sum_{m=1}^{\infty} (-1)\, z^{m}$$

$$= -(z^1 + z^2 + z^3 + \cdots)$$

$$= -\frac{z}{1-z}$$

$$= \frac{z}{z-1}$$

ROC: For the geometric series to convergence we need $|z| < 1$

ROC is $|z| < 1$

2022 - 4(b)

$x(n) = \{2, 1, 0, \underline{1}\}$

$h(n) = \{\underline{1}, 2, 3, 1\}$ $\quad y(n) = x(n) * h(n)$

	2	1	0	1
1	2	1	0	1
2	4	2	0	2
3	6	3	0	3
1	2	1	0	1

$y(n) = \{2, 5, 8, 6, 3, 3, \underline{1}\}$

Multiplication of z-domain:

$X(z) = (2z^0 + 1.z^{-1} + 0.z^{-2} + 1z^{-3})$

$= (2 + z^{-1} + z^{-3})$

$$H(z) = 1z^{0} + 2z^{-1} + 3z^{-2} + 1z^{-3}$$

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + z^{-3}$$

$$Y(z) = X(z) \cdot H(z)$$

$$= (2 + z^{-1} + z^{-3})(1 + 2z^{-1} + 3z^{-2} + z^{-3})$$

$$= 2 + 4z^{-1} + 6z^{-2} + 2z^{-3} + z^{-1} + 2z^{-2} + 3z^{-3} + z^{-4} + z^{-3} + 2z^{-4} + 3z^{-5} + z^{-6}$$

$$Y(z) = 2 + 5z^{-1} + 8z^{-2} + 6z^{-3} + 3z^{-4} + 3z^{-5} + z^{-6}$$

The coefficients of $Y(z)$ are $y(n)$.

$\therefore$ The convolution in the time domain is equal to multiplication in z domain.

2022- 4(c)

Causal signal:

A discrete time signal $x(n) = 0$ for $n<0$

This means the signal only exists for non-negative time indices.

$x(n) = (0.5)^n u(n)$

$u(n)$ makes sure that $x(n)$ is zero for $n<0$

ROC for causal signal:

For a causal signal, the z-transform $X(z)$ will have an ROC that is the exterior of a circle. That is the ROC will be of the form $|z| > R$ where R is the magnitude of the outermost pole

$x(n) = (0.5)^n u(n)$

$$X(z) = \sum_{n=0}^{\infty} (0.5)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (0.5 z^{-1})^n$$

~~where~~
$$= 1 + 0.5z^{-1} + (0.5z^{-1})^2 + \cdots$$

$$= \frac{1}{1 - 0.5z^{-1}}$$

ROC:

$$|0.5z^{-1}| < 1$$

$$\frac{|0.5|}{|z|} < 1$$

~~ROC~~ $\boxed{\therefore |z| > 0.5}$

Anti causal signal:

A discrete time signal $x(n)$ is anti-causal if $x(n) = 0$ for $n>0$. This means the signal only exists for non-Positive time indices

$$x(n) = (0.5)^n u(-n)$$

$u(-n)$ makes sure that $x(n)$ is zer for $n>0$

$$u(-n) = 1 \quad \text{for } n \leq 0$$
$$= 0 \quad n > 0$$

ROC for anti causal signals:

For an anti causal signal, the z transform $X(z)$ will have an ROC that is the interior of a circle. That is, the ROC will be of the form $|z| < R$.

$$x(n) = (0.5)^n u(-n)$$

$$X(z) = \sum_{n=-\infty}^{0} (0.5)^n z^{-n}$$

Let $m = -n$

when $n = -\infty$ $m = \infty$

$n = 0$ $m = 0$

$$X(z) = \sum_{m=0}^{\infty} (0.5)^{-m} z^{m}$$

$$= \sum_{m=0}^{\infty} 2^m z^m$$

$$= \sum_{0}^{\infty} (2z)^m$$

$$= 1 + 2z + (2z)^2 + \cdots$$

$$= \frac{1}{1-2z}$$

$|2z| < 1$

$|z| < \frac{1}{2}$

$\therefore |z| < 0.5$