ARCH Models in Julia

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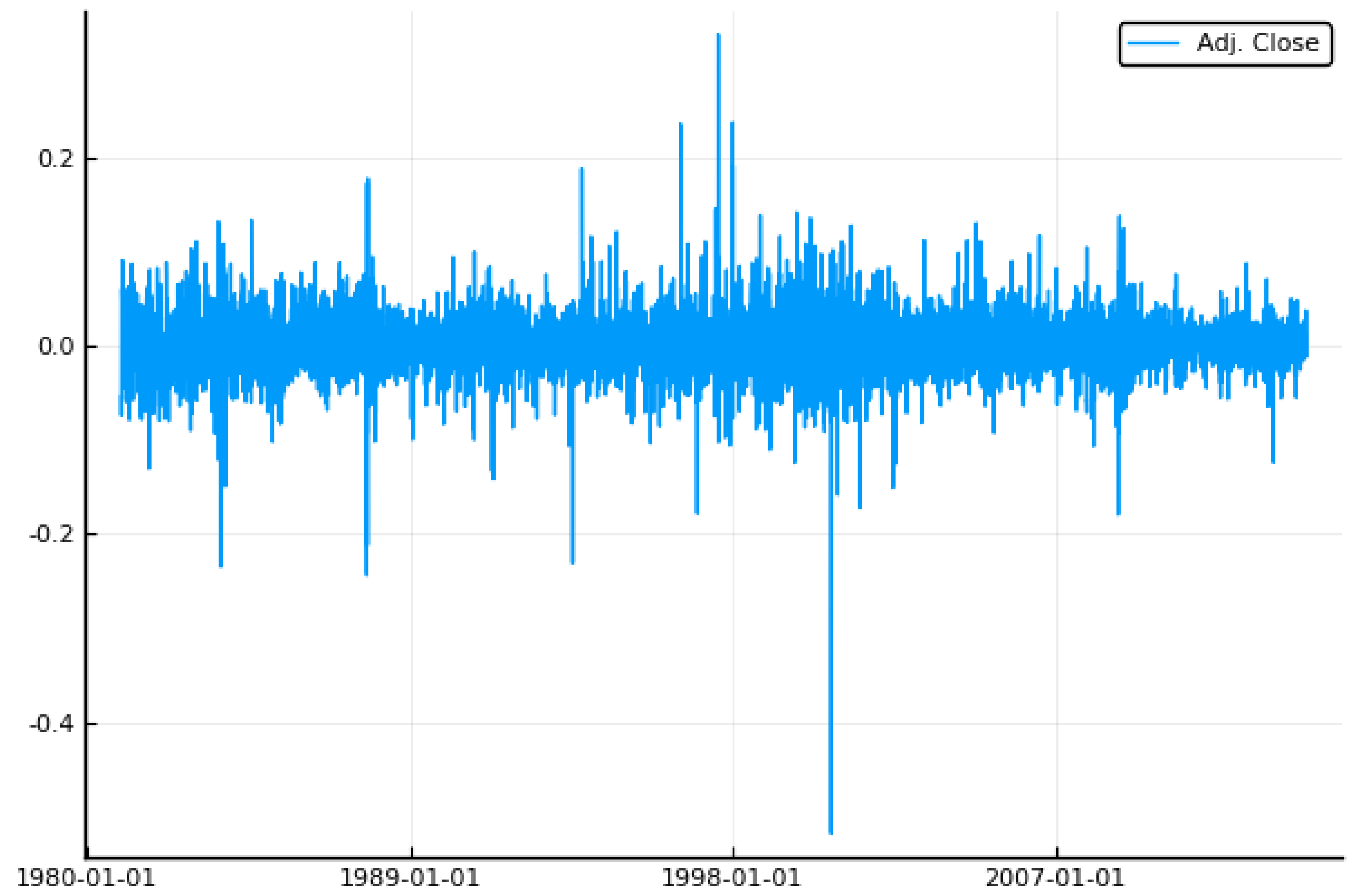
Outline

- Introduction
- Usage
- Benchmarks vs. Matlab

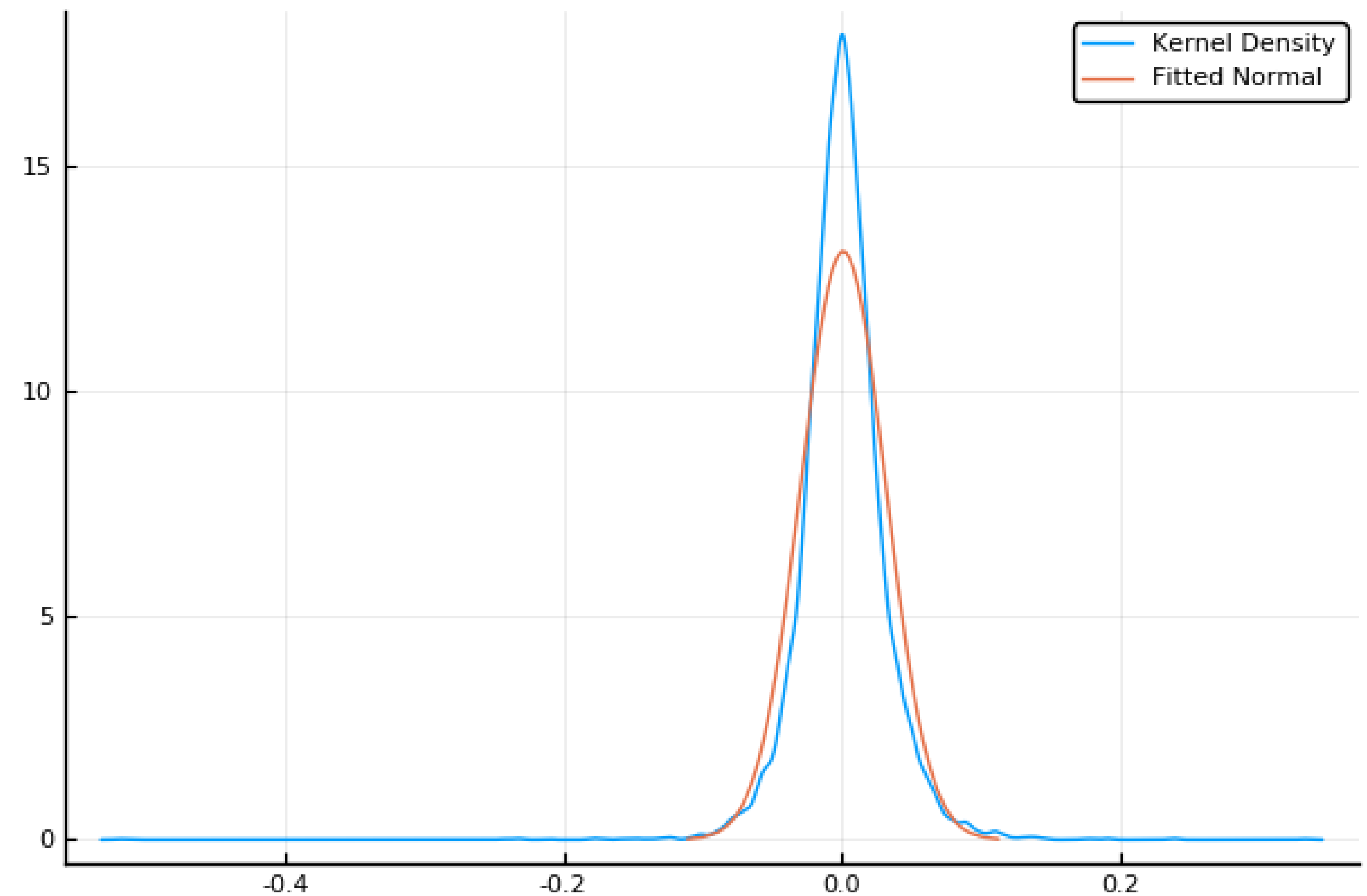
Introduction

- Daily financial returns data exhibit a number of *stylized facts*:
    - Volatility clustering
    - Non-Gaussianity, fat tails
    - Leverage effects: negative returns increase future volatility
- Other types of data (e.g., changes in interest rates) exhibit similar phenomena.
- These effects are important in many areas in finance, in particular in risk management.
- [G]ARCH ([**G**eneralized] **A**utoregressive **C**onditional **V**olatility) models are the most popular for modelling them.

Example: volatility clustering in AAPL returns



Example: fat tails in AAPL return density



(G)ARCH Models

- Basic setup: given a sample of financial returns $\{r_t\}_{t\in\{1,\ldots,T\}}$, decompose r_t as

$$r_t = \mu_t + \sigma_t z_t, \qquad z_t \overset{i.i.d.}{\sim} (0, 1),$$

  where $\mu_t \equiv \mathbb{E}[r_t \mid \mathcal{F}_{t-1}]$ and $\sigma_t^2 \equiv \mathbb{E}[(r_t - \mu_t)^2 \mid \mathcal{F}_{t-1}]$.
- Assume $\mu_t = 0$ for simplicity. Focus is on the *volatility* σ_t. G(ARCH) models make σ_t a function of *past* returns and variances. Examples:

Examples

- ARCH(q) (Engle, Ecta 1982):

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i r_{t-i}^2$$

- GARCH(p, q) (Bollerslev, JoE 1986)

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{i=1}^{q} \alpha_i r_{t-i}^2$$

- EGARCH(o, p, q) (Nelson, Ecta 1991)

$$\log(\sigma_t^2) = \omega + \sum_{i=1}^{o} \gamma_i z_{t-i} + \sum_{i=1}^{p} \beta_i \log(\sigma_{t-i}^2) + \sum_{i=1}^{q} \alpha_i (|z_t| - \mathbb{E}|z_t|)$$

Estimation

- G(ARCH) models are usually estimated by maximum likelihood: with f_z denoting the density of z_t,

$$\max \prod_t f(r_t \mid \mathcal{F}_{t-1}) = \max \prod_t \frac{1}{\sigma_t} f_z(r_t/\sigma_t).$$

- Recursive nature of σ_t means the computation cannot be "vectorized" $\Rightarrow$ loops.
- Julia is very well suited for this. Matlab (and the `rugarch` package for Python) have to implement the likelihood in C.

The ARCH Package

- ARCH.jl is not registered yet; available at https://github.com/s-broda/ARCH.jl
- 0.6 only so far; 0.7 support coming soon.
- Currently supported: simulation and estimation for ARCH, GARCH, and EGARCH models of arbitrary orders, with Gaussian and Student's t errors.
- Designed to be easily extensible with new models, distributions.
- Volatility specifications subtype `VolatilitySpec`. Parametrized on (o, p, q) to facilitate loop unrolling.
- Simulation and estimation return instances of `ARCHModel`, which implements `StatisticalModel` from `StatsBase`.
- Standard errors obtained by AD via `ForwardDiff.jl`.

Usage

In [3]:

```
using ARCH
srand(1); T = 10^4  # sample size
volaspec = GARCH{1, 1}([1., .9, .05])  # [omega, beta, alpha]
am = simulate(volaspec, T; dist=StdTDist(3.))  # returns ARCHModel
fit(GARCH{1, 1}, am.data; dist=StdTDist)  # returns ARCHModel
```

Out[3]:

```
GARCH{1,1} model with Student's t errors, T=10000.


Mean equation parameters:

      Estimate Std.Error  z value Pr(>|z|)
μ    0.0031089  0.028261 0.110007   0.9124

Volatility parameters:

      Estimate Std.Error z value Pr(>|z|)
ω      1.01996   0.16134  6.3218    <1e-9
β₁    0.898131 0.0121042 74.1999   <1e-99
α₁   0.0551944 0.0076214 7.24203   <1e-12

Distribution parameters:

     Estimate Std.Error z value Pr(>|z|)
ν     2.92974  0.096228 30.4458   <1e-99
```

In [4]:
```
# select an EGARCH model without intercept by minimizing AIC; o, p, q < 3
# Uses multiple threads to estimate several (here 2*2*2=8) models
am2 = selectmodel(EGARCH, am.data; meanspec=NoIntercept, criterion=aic, maxlags=2, dist=StdTDist)
```

Out[4]:
```
EGARCH{1,1,1} model with Student's t errors, T=10000.


Volatility parameters:

       Estimate  Std.Error  z value Pr(>|z|)
ω      0.153944  0.0235374  6.54041   <1e-10
γ₁   0.00552637 0.00946358 0.583962   0.5592
β₁     0.955436 0.00737491  129.552   <1e-99
α₁     0.145674  0.0150556  9.67573   <1e-21

Distribution parameters:

     Estimate Std.Error z value Pr(>|z|)
ν     2.91693 0.0953735 30.5842   <1e-99
```

In [5]:
```
# Most of the interface of StatisticalModel is implemented:
# loglikelihood, nobs, fit, fit!, confint, aic, bic, aicc, dof, coef,
# coefnames, coeftable, CoefTable, informationmatrix, islinear, score, vcov:
confint(am2)'
```

Out[5]:
```
2×5 Array{Float64,2}:
 0.107812  -0.0130219  0.940981  0.116165  2.73
 0.200077   0.0240746  0.96989   0.175182  3.10385
```

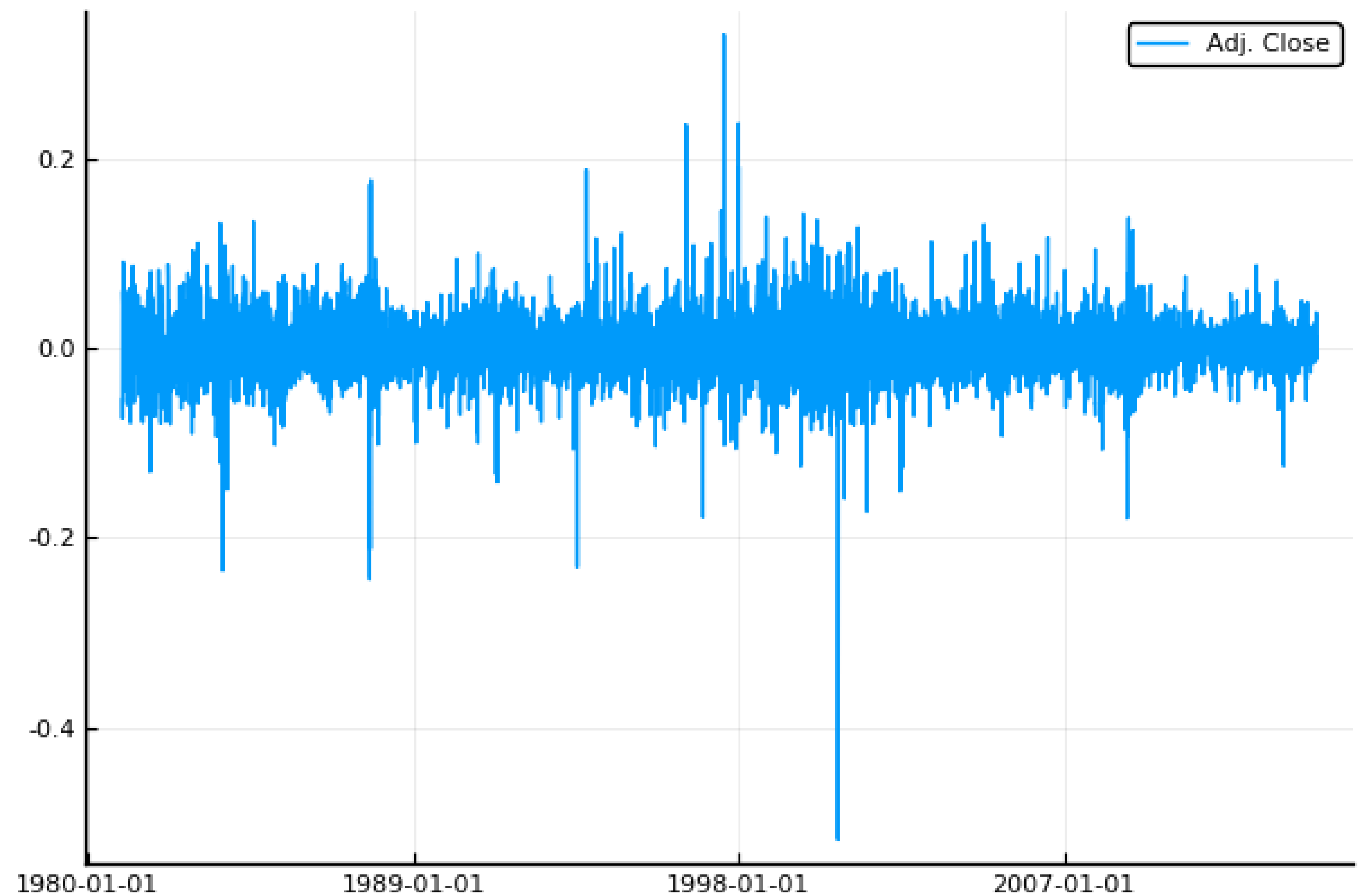
Benchmarks

- Bollerslev and Ghysels (JBES 1996) data is de facto standard in comparing implementations of GARCH models.
- Data consist of daily German mark/British pound exchange rates (1974 observations).

In [6]:

```
@static if !isfile("DMGBP.txt")
    using HTTP
    open("DMGBP.txt", "w") do io
        HTTP.get("http://people.stern.nyu.edu/wgreene/Text/Edition7/TableF20-1.txt", response_stream=io)
    end
end
r = convert.(Float64, readcsv("DMGBP.txt")[2:end]);
@static if !isfile("DMGBP.png")
    using Plots
    plot(r)
    savefig("DMGBP")
end
```

Bollerslev and Ghysels data



GARCH

- Fitting in Julia:

In [10]:
```julia
using BenchmarkTools
@btime fit(GARCH{1, 1}, $r, meanspec=NoIntercept)  # Matlab doesn't use an intercept
```

Out[10]:
```
GARCH{1,1} model with Gaussian errors, T=1974.


Volatility parameters:

      Estimate  Std.Error z value Pr(>|z|)
ω    0.0108661 0.00657449 1.65277   0.0984
β₁    0.804431  0.0730395 11.0136   <1e-27
α₁    0.154597  0.0539319 2.86651   0.0042


  8.068 ms (2617 allocations: 125.69 KiB)
```

- Now Matlab:

In [11]:
```
using MATLAB
mat"version"
```

Out[11]: "9.4.0.813654 (R2018a)"

In [14]:
```
# run this cell a few times to give Matlab a fair chance
mat"tic; estimate(garch(1, 1), $r); toc; 0";
```

```
    GARCH(1,1) Conditional Variance Model (Gaussian Distribution):

                    Value      StandardError    TStatistic     PValue   
                  ________    _____________    __________    __________

    Constant      0.010868      0.0012972        8.3779      5.3896e-17
    GARCH{1}       0.80452       0.016038        50.162               0
    ARCH{1}        0.15433       0.013852        11.141      7.9448e-29

Elapsed time is 0.098643 seconds.
```

- ARCH.jl is faster by a factor of about 5-10, depending on the machine.
- Estimates are quite similar, but standard errors and t-statistics differ.
- So which standard errors are correct? Let's compare with the results from Brooks et. al. (Int. J. Fcst. 2001).

- Brooks et. al. compare implementations of the GARCH(1, 1) model. They use a model with intercept, so let’s re-estimate in Julia (Matlab doesn't seem to allow this):

In [15]:
```
fit(GARCH{1, 1}, r)
```

Out[15]:
```
GARCH{1,1} model with Gaussian errors, T=1974.


Mean equation parameters:

        Estimate  Std.Error   z value Pr(>|z|)
μ    -0.00616637 0.00920163 -0.670139   0.5028

Volatility parameters:

      Estimate  Std.Error z value Pr(>|z|)
ω    0.0107606 0.00649493 1.65677   0.0976
β₁    0.805875  0.0725003 11.1155   <1e-27
α₁    0.153411  0.0536586 2.85903   0.0042
```

- Brooks et. al. give the estimates (t-**stats**) $\mu = -0.00619(\mathbf{-0.67})$, $\omega = 0.0108(\mathbf{1.66})$, $\beta_1 = 0.806(\mathbf{11.11})$, $\alpha_1 = 0.153(\mathbf{2.86})$. Pretty close!

EGARCH

- Julia:

In [16]:
```
@btime fit(EGARCH{1, 1, 1}, $r, meanspec=NoIntercept)
```

Out[16]:
```
EGARCH{1,1,1} model with Gaussian errors, T=1974.


Volatility parameters:

      Estimate Std.Error  z value Pr(>|z|)
ω    -0.128026 0.0518431 -2.46948   0.0135
γ₁   -0.032216 0.0255372 -1.26153   0.2071
β₁    0.911947 0.0331381  27.5196   <1e-99
α₁    0.333243  0.070109  4.75321    <1e-5


  33.940 ms (3365 allocations: 172.89 KiB)
```

- Matlab:

In [17]:
```
mat"tic; estimate(egarch(1, 1), $r); toc; 0"  # Matlab sets o=q
```

Out[17]:
```
0.0
```

```
    EGARCH(1,1) Conditional Variance Model (Gaussian Distribution):

                      Value      StandardError    TStatistic      PValue
                    _________    _____________    __________    __________

    Constant          -0.1283       0.015788       -8.1267      4.4118e-16
    GARCH{1}          0.91186      0.0084535        107.87               0
    ARCH{1}           0.33317       0.021769        15.305      7.1324e-53
    Leverage{1}     -0.032252       0.012564        -2.567        0.010258

Elapsed time is 0.159833 seconds.
```

- Brooks et. al. give no benchmark results. But again, Julia is faster by a factor of about 5-10.

TODO

- 0.7 compatibility
- docs
- forecasting
- more models, distributions
- Value at Risk
- backtesting
- MGARCH

References

- Bollerslev, T (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* **31**, 307–327.
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