Module 9.3: Time Series Analysis
Fall Term 2023

Week 7:

Cointegration

HSLU Lucerne University of Applied Sciences and Arts

Outline in Weeks

1. Introduction; Descriptive Modelling
2. Returns; Autocorrelation; Stationarity
3. ARMA Models
4. Unit Roots; Regressions between Time Series
5. Volatility Modelling
6. Value at Risk
7. Cointegration

Outline

Cointegration and Common Trends

Suppose we have two time series Y_t and X_t, which are both $I(1)$, and we analyze a regression model of the form

$$Y_t = \beta_1 + \beta_2 X_t + U_t.$$

Here U_t has mean zero but may display autocorrelation. Two cases:

- $U_t \sim I(1)$: if U_t displays no mean-reversion, then Y_t does not revert to the explained part $\beta_1 + \beta_2 X_t$. Even if $\beta_2 = 0$, its t-statistic and R^2 will often seem significant (*spurious regressions*). To avoid this, one should estimate a model in differences, i.e., $\Delta Y_t = a_1 + a_2 \Delta X_t + \Delta U_t$.
- $U_t \sim I(0)$: now Y_t and X_t have a *common* stochastic trend, such that the linear combination $Y_t - \beta_2 X_t$ does not have a trend. This is called *cointegration*.

Example

- Consider the model

$$
\begin{aligned}
Y_t &= \beta_1 + \beta_2 X_t + U_{1,t} \\
X_t &= X_{t-1} + U_{2,t}
\end{aligned}
$$

  where $\beta_2 \neq 0$, $U_{1,t}, U_{2,t} \overset{\text{iid}}{\sim} (0, \sigma^2)$ independently of each other.
- X_t is a random walk and thus nonstationary. Y_t contains X_t and is thus also nonstationary. But

$$
Y_t - \beta_2 X_t = \beta_1 + U_{1,t}
$$

  is stationary: the RHS is white noise plus a constant.
- $(1, -\beta_2)$ is called the *cointegrating vector*.

Cointegration and Common Trends, contd.

- The concept is easily extended to more than two series: if $X_{2t}, \ldots, X_{kt}$ are all $I(1)$ variables, and
$$Y_t = \beta_1 + \beta_2 X_{2t} + \ldots + \beta_k X_{kt} + U_t,$$
then this is a spurious regression if $U_t \sim I(1)$ (and $\beta_i = 0$), and a cointegrating relation if U_t is stationary.
- In other words, cointegration between k integrated series means that there exists a *linear combination*[1] of them which is stationary.
- Examples of possibly cointegrated time series:
    - exchange rates and relative prices (*purchasing power parity*);
    - spot and futures prices of assets or exchange rates;
    - short- and long-term interest rates (*term structure models*);
    - stock prices and dividends (*present value relations*).

[1] i.e., a weighted sum

Outline

Testing for Cointegration

- For cointegrated series, one should exploit the long-run equilibrium relationship between variables for estimation rather than differencing. Differencing would *remove* that structure.
- Engle and Granger proposed the following procedure:
  - Conduct individual unit root tests to ensure all series are $I(1)$.
  - Estimate the regression model

$$Y_t = \beta_1 + \beta_2 X_{2t} + \ldots + \beta_k X_{kt} + U_t$$

    by ordinary least-squares. Estimates are (super)consistent, but standard errors are wrong because series are $I(1)$.
  - Apply an ADF unit root test (with constant) to the residuals $\hat{u}_t$ from this regression. This yields a test for $H_0 : U_t \sim I(1)$ (spurious regression) against $H_1 : U_t \sim I(0)$ (cointegration). The critical values depend on k. E.g., for $k = 2$ (X_{2t} and an intercept), the 5% c.v. is -3.41.
  - If H_0 is rejected, estimate an *error correction model*.

Engle-Granger critical values

Number of series (w/o constant)	2	3	4	5	6
Critical value	-3.41	-3.80	-4.16	-4.49	-4.74

Error Correction Models

- Cointegration between Y_t and X_t implies that deviations $(Y_{t-1} - \beta_1 - \beta_2 X_{t-1})$ from the equilibrium level should be (partially) corrected in the next period, by Y_t, X_t, or both.
- This leads to a *vector error correction model* (VECM), which in the simplest form is

$$\begin{aligned} \Delta Y_t &= c_1 + \alpha_1 (Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + e_{1t}, \\ \Delta X_t &= c_2 + \alpha_2 (Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + e_{2t}, \end{aligned}$$

  where e_{1t} and e_{2t} are two white noise errors (possibly correlated), and where we expect $\alpha_1 < 0$ and/or $\alpha_2 \beta_2 > 0$.
- We might need to add lags of ΔY_t and/or ΔX_t on RHS to combat autocorrelation.
- The *Granger representation theorem* states that cointegration implies an error correction model (possibly with more lags), and vice versa; see exercises.

Engle-Granger Procedure

- The VECM

$$
\begin{aligned}
\Delta Y_t &= c_1 + \alpha_1 (Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + e_{1t}, \\
\Delta X_t &= c_2 + \alpha_2 (Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + e_{2t}.
\end{aligned}
$$

  is estimated by replacing $Y_{t-1} - \beta_1 - \beta_2 X_{t-1}$ by OLS residual $\hat{u}_{t-1} = Y_{t-1} - \hat{\beta}_1 - \hat{\beta}_2 X_{t-1}$, and estimating α_1 and α_2 by OLS.
- Note that $\hat{u}$ is stationary, so this is a valid regression!
- If $\alpha_2 = 0$, then all correction is done by Y_t, and not by X_t. In that case it makes sense to treat X_t as exogenous and Y_t as endogenous, and consider the "single-equation" error correction model

$$\Delta Y_t = c + \alpha_1 (Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + e_t.$$

- In general, both Y_t and X_t are endogenous.

Outline

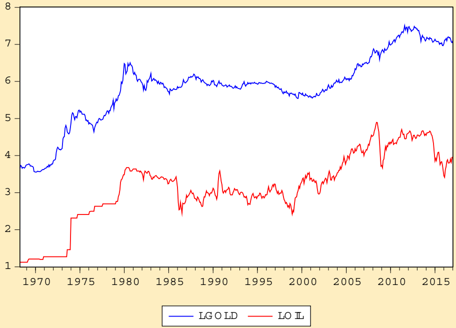
Example

- Until 1971, as part of the Bretton-Woods system of fixed exchange rates, the US dollar was convertible to gold, i.e., it was possible for foreign central banks to redeem US dollars for gold at a fixed rate of 35$ per troy ounce, so that the price of gold was fixed.
- In 1971, US president Nixon unilaterally cancelled the direct convertibility, ultimately ending the Bretton-Woods agreement.
- Gold became a floating asset, and its price increased sharply; in other words, the US$ was massively devalued.

Example continued

- We want to analyze the hypothesis that the increasing price (in US$) of oil is not a consequence of an increased demand for (or a reduced supply of) oil, but rather of a continued devaluation of the US$.
- We have at our disposal monthly data from April 1968 to January 2017 (586 observations) on the following variables:
  - gold_t, the spot price of one troy ounce of gold in US$;
  - oil_t, the spot price of one barrel of WTI crude oil in US$.
- Idea: if the relative price of oil expressed in units of gold $\text{oil}_t/\text{gold}_t$ is stationary, then this implies that $\log(\text{oil}_t) - \log(\text{gold}_t)$ is stationary, so that $\log(\text{oil}_t)$ and $\log(\text{gold}_t)$ must be cointegrated if the individual series are integrated.

$\log(\mathrm{gold}_t)$ and $\log(\mathrm{oil}_t)$



Example continued

Step 0 Take logs: `genr lgold = log(gold), genr loil = log(oil).`

Step 1 Test that the variables are integrated (ADF test with constant and trend)

Null Hypothesis: LOIL has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 1 (Automatic - based on SIC, maxlag=18)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-2.552422	0.3027
Test critical values:	1% level	-3.973847	
	5% level	-3.417533	
	10% level	-3.131184	

*MacKinnon (1996) one-sided p-values.

Null Hypothesis: LGOLD has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 0 (Automatic - based on SIC, maxlag=18)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-1.820648	0.6937
Test critical values:	1% level	-3.973820	
	5% level	-3.417519	
	10% level	-3.131176	

*MacKinnon (1996) one-sided p-values.

Neither test rejects, so the series are I(1).

Example continued

Step 2 Estimate long-run relationship

$$\text{loil}_t = \beta_1 + \beta_2 \text{lgold}_t + U_t$$

and save the residuals (`Proc→Make Residual Series→Ordinary`) as `u`.

Dependent Variable: LOIL
Method: Least Squares
Date: 11/30/20 Time: 16:50
Sample: 1968M04 2017M01
Included observations: 586

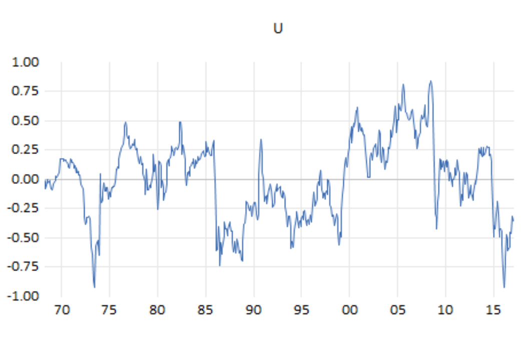
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-2.258963	0.086329	-26.16698	0.0000
LGOLD	0.926232	0.014618	63.36354	0.0000

R-squared	0.873014	Mean dependent var	3.141848
Adjusted R-squared	0.872797	S.D. dependent var	0.929639
S.E. of regression	0.331561	Akaike info criterion	0.633398
Sum squared resid	64.20072	Schwarz criterion	0.648324
Log likelihood	-183.5855	Hannan-Quinn criter.	0.639214
F-statistic	4014.938	Durbin-Watson stat	0.074317
Prob(F-statistic)	0.000000		

The cointegrating vector is $(1, -\beta_2) = (1, -0.926)$ (if the Engle-Granger test rejects).
Careful: standard errors are wrong, because variables are $I(1)$.

Example continued

Residuals (=equilibrium error) $\hat{u}_t$



Example continued

Step 3 Apply ADF test (with intercept) to `u`.

Null Hypothesis: U has a unit root
Exogenous: Constant
Lag Length: 1 (Automatic - based on SIC, maxlag=18)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-3.724771	0.0040
Test critical values:	1% level	-3.441318	
	5% level	-2.866270	
	10% level	-2.569348	

*MacKinnon (1996) one-sided p-values.

Careful: we need to use Engle-Granger 5% critical value of -3.41, not the ones given by EViews. Conclusion: test rejects the null of “no cointegration”.

Example continued

Step 4 Estimate VECM. Note: I threw in a lag of the dependent variable in the equation for `d(loil)`, as there was autocorrelation without it (cf. selected lag length in ADF test).

Dependent Variable: D(LOIL)
Method: Least Squares
Date: 11/30/20 Time: 17:09
Sample (adjusted): 1968M06 2017M01
Included observations: 584 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.003641	0.003278	1.110606	0.2672
U(-1)	-0.034971	0.009942	-3.517509	0.0005
D(LOIL(-1))	0.256547	0.040092	6.399038	0.0000

R-squared	0.077666	Mean dependent var	0.004862
Adjusted R-squared	0.074491	S.D. dependent var	0.082210
S.E. of regression	0.079089	Akaike info criterion	-2.231362
Sum squared resid	3.634195	Schwarz criterion	-2.208914
Log likelihood	654.5578	Hannan-Quinn criter.	-2.222613
F-statistic	24.46179	Durbin-Watson stat	2.006622
Prob(F-statistic)	0.000000		

Dependent Variable: D(LGOLD)
Method: Least Squares
Date: 11/30/20 Time: 17:17
Sample (adjusted): 1968M05 2017M01
Included observations: 585 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.005866	0.002342	2.504458	0.0125
U(-1)	0.008997	0.007077	1.271298	0.2041

R-squared	0.002765	Mean dependent var	0.005871
Adjusted R-squared	0.001054	S.D. dependent var	0.056676
S.E. of regression	0.056646	Akaike info criterion	-2.900562
Sum squared resid	1.870743	Schwarz criterion	-2.885616
Log likelihood	850.4143	Hannan-Quinn criter.	-2.894737
F-statistic	1.616198	Durbin-Watson stat	1.932704
Prob(F-statistic)	0.204130		

The adjustment coefficient α_2 in the equation for `d(lgold)` is insignificant. So all the adjustment is done by `loil` $\rightarrow$ single-equation ECM.

Example continued

- The final model is the single-equation ECM

$$\Delta \mathrm{loil}_t = 0.0036 - 0.035(\mathrm{loil}_{t-1} - 0.926\mathrm{lgold}_{t-1} + 2.26) + 0.25\Delta \mathrm{loil}_{t-1} + e_{1t}$$

  In our earlier notation,

$$\Delta Y_t = c + \alpha_1 (Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + \gamma \Delta Y_{t-1} + e_{1t},$$

  with $c = 0.0036$, $\alpha_1 = -0.035 < 0$ as desired, $\beta_1 = -2.26$, $\beta_2 = 0.926$, and $\gamma = 0.25$.
- Interpretation: there is an equilibrium relationship between loil and lgold. In case of a disequilibrium, loil adjusts towards the equilibrium. The adjustment amounts to 3.5% per period.

Outline

Learning Goals

Students

- Understand the concept of cointegration,
- are able to test for cointegration using both the Engle-Granger procedure and the Johansen test,
- and are able to estimate an error correction model.

Homework

- Exercise 7
- Problem 6 from Chapter 8 of Brooks (2019)