# Module 9.3: Time Series Analysis
# Fall Term 2023

**Week 6**:

## Value at Risk

HSLU Lucerne University of Applied Sciences and Arts

# Outline in Weeks

1. Introduction; Descriptive Modelling
2. Returns; Autocorrelation; Stationarity
3. ARMA Models
4. Unit Roots; Regressions between Time Series
5. Volatility Modelling
6. Value at Risk
7. Cointegration

# Outline

# Value at Risk

- Consider a portfolio with value $V_{PF,t}$ and daily returns $R_{PF,t+1}$.
- Define the one-day Loss on the portfolio as

$$\$Loss_{t+1} = V_{PF,t} - V_{PF,t+1}.$$

- The one-day, $100p\%$, dollar *Value at Risk* ($\$VaR^p_{t+1}$) gives the largest loss on the portfolio that we can expect to incur in the next day with level of confidence $100\,(1-p)\,\%$.
- Mathematically it is given by

$$\Pr(\$Loss_{t+1} \leq \$VaR^p_{t+1}) = 1 - p,$$

or equivalently

$$\Pr(\$Loss_{t+1} > \$VaR^p_{t+1}) = p.$$

# Value at Risk

- Usually easier to express the VaR as a percentage of the portfolio value:

$$VaR_{t+1}^{p} = \frac{\$VaR_{t+1}^{p}}{V_{PF,t}}.$$

- Hence

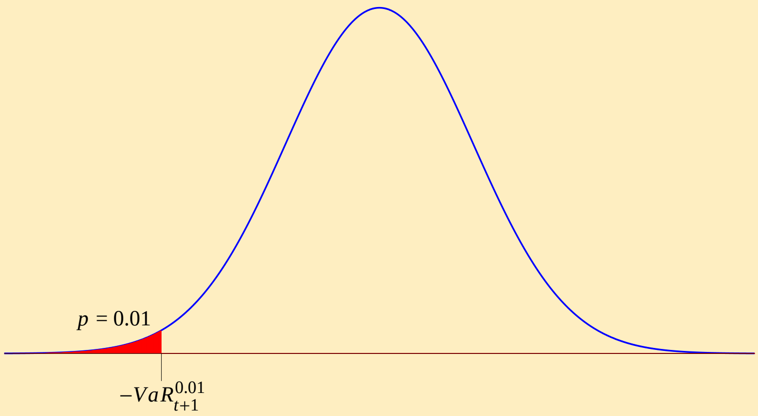
$$\Pr(R_{PF,t+1} < -VaR_{t+1}^{p}) = p,$$

as

$$R_{PF,t+1} = -\frac{\$Loss_{t+1}}{V_{PF,t}}.$$

- Thus $VaR_{t+1}^{p}$ is minus the $100p$th *percentile* of the return distribution. Usually $p = 0.01$.
- Definition can be naturally extended to $K$-day VaR, from the distribution of the $K$-day returns $R_{PF,t+1:t+K}$.

## Probability density function of daily returns



# Value at Risk

- Value at Risk was proposed as the standard measure of portfolio risk by the Basel Committee of the Bank of International Settlements in 1996.
- The BC imposed that financial institutions should report the Value at Risk on their positions, such that regulators could check the adequacy of the economic capital as a buffer against market risk.
- Banks were allowed to use their own, internal models for the computation of VaR, but the adequacy of these models should be "backtested" using specific criteria.
- A candidate for a standard model is RiskMetrics (developed by J.P.Morgan).
- VaR is scheduled to be replaced by the expected shortfall (ES) with the rollout of Basel 3. The ES is based on the VaR, however.

# Outline

# VaR Methods: Historical simulation

Historical simulation assumes that the distribution of tomorrow's portfolio returns is well approximated by the empirical distribution (histogram) of the past $m$ observations $\{R_{PF,t}, R_{PF,t-1}, \ldots, R_{PF,t+1-m}\}$.
This is as if we draw, with replacement, from the last $m$ returns and use this to simulate the next day's return distribution.

- The estimator of VaR is given by minus the 100$p$th percentile of the sequence of past portfolio returns, that is:
  - sort the returns $\{R_{PF,t}, R_{PF,t-1}, \ldots, R_{PF,t+1-m}\}$ is ascending order;
  - define $R^p_{t+1}$ as the number such that 100$p$% of the observations are smaller than $R^p_{t+1}$;
  - the estimator for VaR is given by

$$\widehat{VaR}^p_{t+1} = -R^p_{t+1}.$$

- $R^p_{t+1}$ can be computed using EViews' `@quantile` function.

# VaR Methods: Historical simulation

Problems / limitations of historical simulation:

- Last year(s) of data not necessarily representative for the next few days (e.g. because of volatility clustering).
- Similar problems as historical volatility (choice of $m$).
- A large $m$ is required to compute 1% VaR with any degree of precision.
- By focussing on left tails, extreme positive returns are ignored.

# Outline

# VaR Methods: Normal distribution

- Another simple approach is to assume $R_{t+1} = R_{PF,t+1} \sim N(\mu, \sigma^2)$ and to estimate $\mu$ and $\sigma^2$ using historical data.
- Denoting the inverse distribution function (quantile function) of the Normal, as $\Phi_p^{-1}$, The VaR becomes

$$VaR_{t+1}^{p} = -\mu - \sigma\Phi_p^{-1}.$$

For example, $\Phi_{.01}^{-1} = -2.326$. For daily data we can take $\mu = 0$.

# VaR Methods: Normal distribution

- The normal model can be easily extended to a *conditionally* normal model. Assume $R_{t+1} \sim N(0, \sigma^2_{t+1})$ where $\sigma^2_{t+1}$ may be estimated by:
  - EWMA / RiskMetrics;
  - univariate GARCH;
  - multivariate GARCH.

  The VaR then becomes $VaR^p_{t+1} = -\sigma_{t+1}\Phi^{-1}_p$.

# Outline

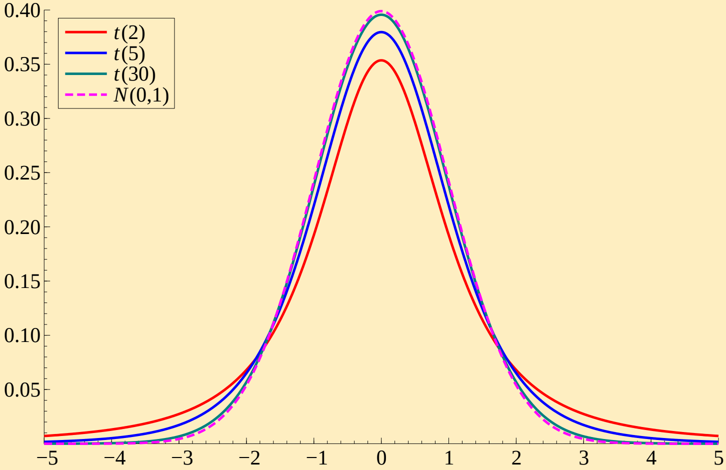
# VaR Methods: Standardized *t* distribution

- The VaR methods described on the previous slides require that financial returns are normally distributed.
- This can be tested by the Jarque-Bera test and is usually rejected.
- Solution: use Student's $t(d)$ distribution, where d.o.f. $d > 0$ need not be integer.
- $d$ is just a shape parameter. Small values correspond to fat tails. As $d \to \infty$, we approach the $N(0, 1)$ distribution.
- For $d > 2$, the variance of a $t(d)$ random variable $x$ is $d/(d-2)$; the distribution of

$$z = \frac{x}{\sqrt{\mathrm{var}(x)}} = \sqrt{\frac{d-2}{d}}x$$

  is called *standardized* $t(d)$, denoted $\tilde{t}(d)$.
- For $d > 4$ the excess kurtosis is $6/(d-4)$. The distributions are symmetric around 0 (hence mean and skewness are 0).

## Student's *t* densities



# VaR Methods: Standardized *t* distribution

- The GARCH model $R_{t+1} = \sigma_{t+1} z_{t+1}$, $\sigma^2_{t+1} = \omega + \alpha R_t^2 + \beta \sigma_t^2$, may be extended to $z_t \sim \tilde{t}(d)$, where $d$ is an extra parameter that can be estimated by maximum likelihood.
- In practice this GARCH-*t* model often gives a substantially better fit than the Gaussian model. The main problem is that the standardized residuals usually have an asymmetric distribution, with a longer left tail than right tail.

## Estimation of GARCH-*t* in EViews:

Dependent Variable: R
Method: ML ARCH - Student's t distribution (BFGS / Marquardt steps)
Date: 12/01/22 Time: 17:14
Sample (adjusted): 10/30/2012 10/27/2022
Included observations: 2546 after adjustments
Convergence achieved after 27 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)

| Variable | Coefficient | Std. Error | z-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.000904 | 0.000126 | 7.176877 | 0.0000 |
| Variance Equation | | | | |
| C | 2.92E-06 | 6.17E-07 | 4.743725 | 0.0000 |
| RESID(-1)^2 | 0.221296 | 0.026849 | 8.242232 | 0.0000 |
| GARCH(-1) | 0.773192 | 0.023233 | 33.27982 | 0.0000 |
| T-DIST. DOF | 5.275193 | 0.568869 | 9.273121 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | -0.002211 | Mean dependent var | 0.000390 |
| Adjusted R-squared | -0.002211 | S.D. dependent var | 0.010946 |
| S.E. of regression | 0.010958 | Akaike info criterion | -6.810961 |
| Sum squared resid | 0.305579 | Schwarz criterion | -6.799488 |
| Log likelihood | 8675.354 | Hannan-Quinn criter. | -6.806800 |
| Durbin-Watson stat | 2.285926 | | |

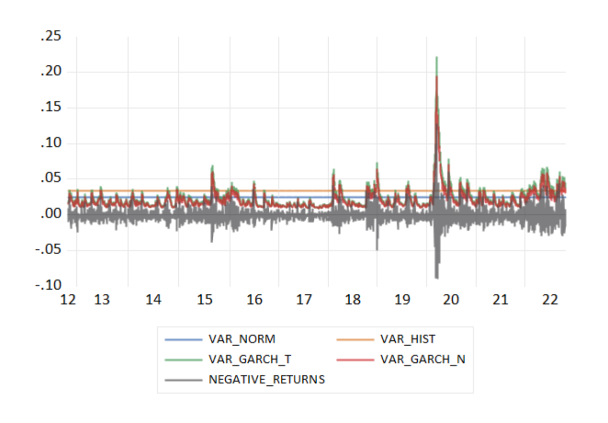
# VaR Methods: Standardized *t* distribution

- Let $\tilde{t}_p^{-1}(d)$ be $100p\%$ quantile of the standardized $t$ distribution $\tilde{t}(d)$ and $t_p^{-1}(d)$ the percentile $100p\%$ of the $t$ distribution $t(d)$.
- The implied VaR now is

$$VaR_{t+1}^{p} = -\sigma_{t+1}\tilde{t}_p^{-1}(d) = -\sigma_{t+1}\sqrt{\frac{d-2}{d}}\,t_p^{-1}(d),$$

where, e.g., $\tilde{t}_{.01}^{-1}(6) = -2.566$.

## Example: minus S&P500 returns, with 1% VaR based on (full-sample) historical simulation, normal distribution, and a GARCH(1, 1) with Normal and *t* errors



# Outline

# Expected Shortfall

Limitations of Value at Risk:

- VaR is not informative about the magnitude of the losses if they exceed the VaR. Two distributions could have the same 1% VaR, but with different left tails.
- VaR is not *subadditive*: it is not guaranteed that

$$VaR^{p}_{t+1}(X+Y) \leq VaR^{p}_{t+1}(X) + VaR^{p}_{t+1}(Y).$$

This means that VaR is not a “coherent” risk measure.

# Expected Shortfall

- The 1% VaR will be replaced by the 2.5% *expected shortfall* (ES, a.k.a. CVaR), which addresses these problems, on January 1st, 2023:

$$ES_{t+1}^{p} = -E_t\left[R_{t+1} | R_{t+1} < -VaR_{t+1}^{p}\right].$$

# Outline

# Multi-Period VaR

- For the GARCH-$N(0,1)$ and GARCH-$\tilde{t}(d)$ models, the one-day VaR and ES can be determined analytically (when the estimation is based on daily data).
- However, in practice one often needs risk measures for multi-period returns:

$$R_{t+1:t+K} = \sum_{k=1}^{K} R_{t+k}.$$

For example, a horizon of two weeks ($K = 10$ trading days) is common.

# Multi-Period VaR

- Problem: even if the distribution of the one-period return is known (e.g., normal), that of $R_{t+1:t+K}$ is not (because the variance is not deterministic).
- *Monte Carlo simulation* is a possible solution: we let the computer generate a large number of scenarios of $K$ daily returns, and compute from this the conditional distribution of the $K$-day return, and hence the $K$-day VaR and ES.
- Quick-and dirty practitioner solution: scale the one-day VaR with $\sqrt{K}$ (*square root of time rule*). Only correct under normality.

# Outline

# Backtesting Value at Risk

- The Basel Committee requires that methods to evaluate VaR be backtested (`http://www.bis.org/publ/bcbsc223.pdf`).
- They recommend constructing the 1% VaR over the last 250 trading days ($\approx$ 1 year), and counting the number of times losses exceed the day's VaR figure (termed *exceptions* or *violations* ).
- A method is said to lie in the:
  - *Green zone*, in case of 0–4 exceptions;
  - *Yellow zone*, in case of 5–9 exceptions;
  - *Red zone*, in case of 10 exceptions or more.
- The capital charge for the bank changes according to the zone.

# Backtesting Value at Risk

How can we test if a VaR method is accurate?

- Define the *hit sequence*

$$I_{t+1} = \begin{cases} 1, & \text{if } R_{t+1} < -VaR_{t+1}^{p}, \\ 0, & \text{if } R_{t+1} > -VaR_{t+1}^{p}. \end{cases}$$

- Consider a test period that covers $t+1 \in \{1, \ldots, T\}$, then the number of exceptions is given by $T_1 = \sum_{t=1}^{T} I_t$.
- The proportion of exceptions is given by $\hat{\pi} = T_1 / T$ which is an estimator of $\Pr\left(R_{t+1} < -VaR_{t+1}^{p}\right)$.
- Recall that if the model that generated $VaR_{t+1}^{p}$ is correctly specified, then

$$\Pr\left(R_{t+1} < -VaR_{t+1}^{p}\right) = p,$$

independent of any information at time $t$.

# Backtesting Value at Risk

- Hence, under the null hypothesis of correct specification, the hit sequence $\{I_{t+1}\}$ are independent Bernoulli random variables, and so $T_1 = \sum_{t=1}^{T} I_t$ has a Binomial($T$,$p$) distribution.
- We can test this hypothesis (e.g., with $p = 0.01$) based on the $t$-statistics

$$t_0 = \frac{\hat{\pi} - p}{\sqrt{p(1-p)/T}} \qquad \text{or} \qquad t = \frac{\hat{\pi} - p}{\sqrt{\hat{\pi}(1-\hat{\pi})/T}}.$$

- Under $H_0$ their asymptotic distribution is $N(0,1)$.
- The second $t$-statistic is equal (up to degrees-of-freedom correction) to OLS-based $t$-statistic in regression of $I_{t+1} - p$ on a constant.

# Backtesting Value at Risk

- The previous test only checks *unconditional* coverage, i.e., $\Pr(I_{t+1} = 1) = p$ *on average*. However, misspecification often is due to the fact that the hits $I_{t+1}$ are not independent over time.
- If exceptions are clustered, then if today there was an exception a risk manager can infer that the probability of occurring another exception tomorrow is higher than $p$. Hence, there is misspecification.
- We would like to test if the VaR violations are *independent* over time, the null hypothesis is

$$H_0 : \Pr(I_{t+1} = 1 | I_t = 1) = \Pr(I_{t+1} = 1 | I_t = 0),$$

which implies $\Pr(I_{t+1} = 0 | I_t = 0) = \Pr(I_{t+1} = 0 | I_t = 1)$.

# Backtesting Value at Risk

- Also of interest is to test if the VaR violations are independent over time and if the number of violations is correct (*conditional coverage*)

$$H_0 : \Pr(I_{t+1} = 1 | I_t = 1) = \Pr(I_{t+1} = 1 | I_t = 0) = p.$$

- A simple approach to test these hypotheses is to consider the linear regression model

$$I_{t+1} - p = b_0 + b_1 I_t + e_{t+1}$$

- The *conditional coverage* hypothesis is equivalent to $H_0 : b_0 = 0$ and $b_1 = 0$ which can be tested using a $F$-test.
- The *independence* hypothesis is equivalent to $H_0 : b_1 = 0$ which can be tested using a $t$-statistic.

# Backtesting Value at Risk

## Results for S&P500 returns, 4 different methods

| | Hist | Norm | GARCHn | GARCHt |
|---|---|---|---|---|
| $\hat{\pi} \quad (\times 100)$ | 0.98 | 2.04 | 2.59 | 1.65 |
| $t(\pi = 0.01)$ | −0.09 | 3.72 | 5.06 | 2.57 |
| $\hat{b}_1$ | 0.03 | 0.08 | 0.00 | 0.03 |
| $t(b_1 = 0)$ | 1.54 | 3.91 | 0.23 | 1.60 |
| $F(b_0 = b_1 = 0)$ | 1.18 | 14.60 | 12.80 | 4.59 |

The critical values for the $t$ and $F$ tests are, respectively, $\pm 1.96$ and 3.00.
*Note*: this result is highly unusual. Usually, the GARCHt model fares best.

# Outline

# Learning Goals

Students

- know the definitions of VaR and Expected Shortfall,
- understand the limitations of the VaR,
- are able to construct VaR forecasts based on various methods,
- and are able to backtest VaR forecasts.

# Homework

- Exercise 6