Solution to Exercise 5

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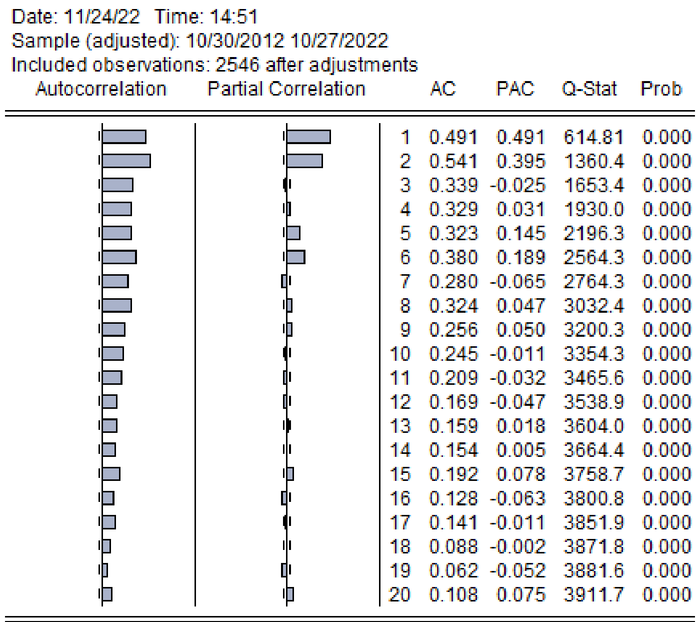
1. (a) The returns are constructed using

```
genr r = dlog(sp500)
```

as usual. We then construct the squared residuals via

```
genr r2 = r^2
```

and generate the correlogram, shown below.



Date: 11/24/22 Time: 14:51
Sample (adjusted): 10/30/2012 10/27/2022
Included observations: 2546 after adjustments

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.491	0.491	614.81	0.000
		2	0.541	0.395	1360.4	0.000
		3	0.339	-0.025	1653.4	0.000
		4	0.329	0.031	1930.0	0.000
		5	0.323	0.145	2196.3	0.000
		6	0.380	0.189	2564.3	0.000
		7	0.280	-0.065	2764.3	0.000
		8	0.324	0.047	3032.4	0.000
		9	0.256	0.050	3200.3	0.000
		10	0.245	-0.011	3354.3	0.000
		11	0.209	-0.032	3465.6	0.000
		12	0.169	-0.047	3538.9	0.000
		13	0.159	0.018	3604.0	0.000
		14	0.154	0.005	3664.4	0.000
		15	0.192	0.078	3758.7	0.000
		16	0.128	-0.063	3800.8	0.000
		17	0.141	-0.011	3851.9	0.000
		18	0.088	-0.002	3871.8	0.000
		19	0.062	-0.052	3881.6	0.000
		20	0.108	0.075	3911.7	0.000

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Alternatively, we can regress the return on an intercept and inspect the correlogram of the squared residuals (under `Residual Diagnostics...`). There is clear evidence of autocorrelation in the squared returns (all Q-stats are significant), indicative of the presence of volatility clustering. Since the SPACF seems to more or less drop to zero after 6 lags, we might try an ARCH(6) model. Usually, a simple GARCH(1, 1) will do better though.

(b) The ARCH-LM test is only offered after a regression has been estimated, so we start by regressing the returns on an intercept. The test is then available under `View`→`Residual Diagnostics`→`Heteroskedasticity Tests`. We choose to include 5 lags (one trading week). The result is shown below.

Heteroskedasticity Test: ARCH

F-statistic	302.2045	Prob. F(5,2535)	0.0000
Obs*R-squared	948.9587	Prob. Chi-Square(5)	0.0000

Test Equation:
Dependent Variable: RESID^2
Method: Least Squares
Date: 11/24/22 Time: 15:11
Sample (adjusted): 11/06/2012 10/27/2022
Included observations: 2541 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	3.14E-05	8.68E-06	3.614959	0.0003
RESID^2(-1)	0.299355	0.019651	15.23388	0.0000
RESID^2(-2)	0.396998	0.020528	19.33915	0.0000
RESID^2(-3)	-0.087729	0.021921	-4.001999	0.0001
RESID^2(-4)	-0.015277	0.020529	-0.744168	0.4568
RESID^2(-5)	0.145365	0.019653	7.396390	0.0000

R-squared	0.373459	Mean dependent var	0.000120
Adjusted R-squared	0.372223	S.D. dependent var	0.000528
S.E. of regression	0.000419	Akaike info criterion	-12.71671
Sum squared resid	0.000444	Schwarz criterion	-12.70292
Log likelihood	16162.59	Hannan-Quinn criter.	-12.71171
F-statistic	302.2045	Durbin-Watson stat	2.054495
Prob(F-statistic)	0.000000		

The null of no heteroskedasticity is clearly rejected; the p-value is essentially zero, and the test observed test statistic $T \cdot R^2_{aux}$ is much larger than the critical value 11.07.

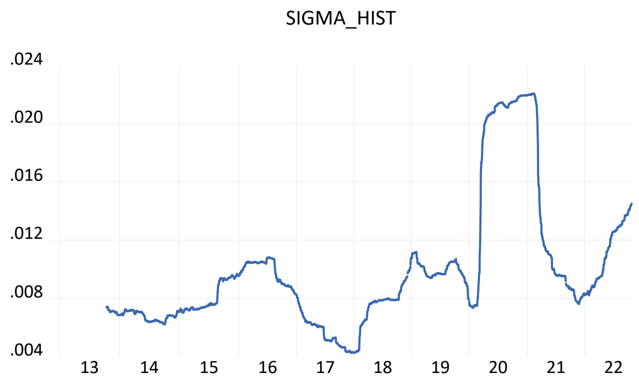
(c) The historical volatility forecasts can be obtained via

```
genr sigma_hist(1) = @sqrt(@movav(r^2, 250))
```

The (1) on the LHS is there because `@movav` includes the observation at time t, and we can only include that in our forecast for time $t+1$. Alternatively, if you don't want to assume a zero mean for the daily returns, you can do

```
genr sigma_hist2(1) = @movstdevp(r, 250)
```

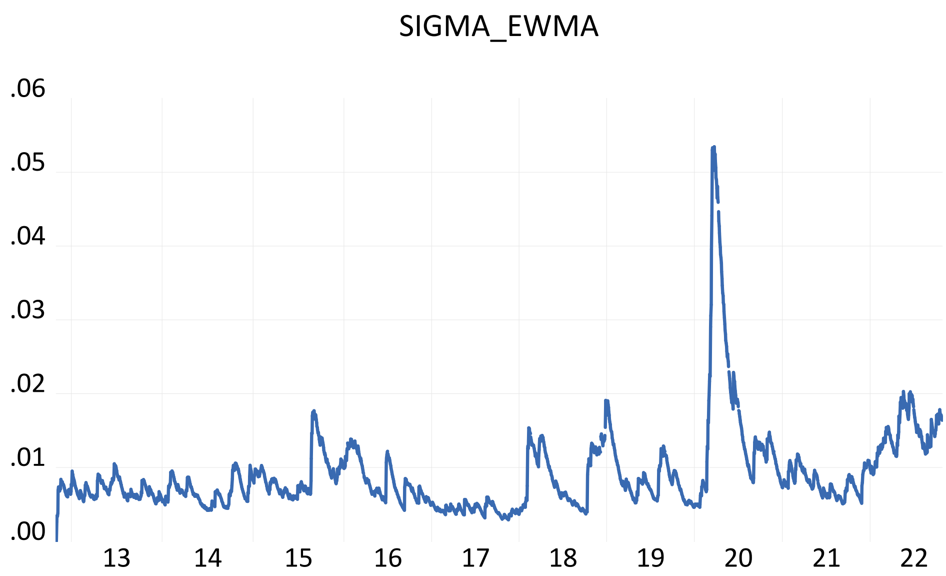
Plotting the historical volatility produces the figure below.



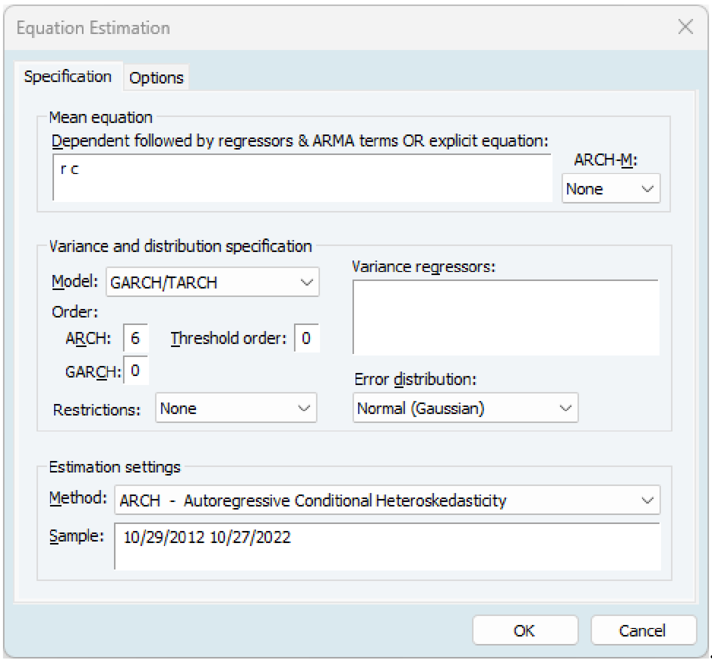
(d) The RiskMetrics volatility with $\lambda = 0.94$ is obtained as follows:

```
scalar lambda = .94
smpl @first+2 @first+2
series sigma_EWMA = r(-1)^2
smpl @first+3 @last
sigma_EWMA = lambda * sigma_EWMA(-1) + (1-lambda) * r(-1)^2
sigma_EWMA = @sqrt(sigma_EWMA)
```

Graphically, it looks as follows.



(e) ARCH models can be estimated by clicking `Quick→Estimate Equation...` and changing the estimation method to ARCH. To fit an ARCH(6), enter the dependent variable at the top, followed by any regressors (this is called the *mean equation*). We only include an intercept, but it's also possible to include ARMA terms if there is any autocorrelation. Then, specify 6 ARCH lags (this is q), zero GARCH lags (this is p), and a zero threshold order (this is for GJR/TARCH):



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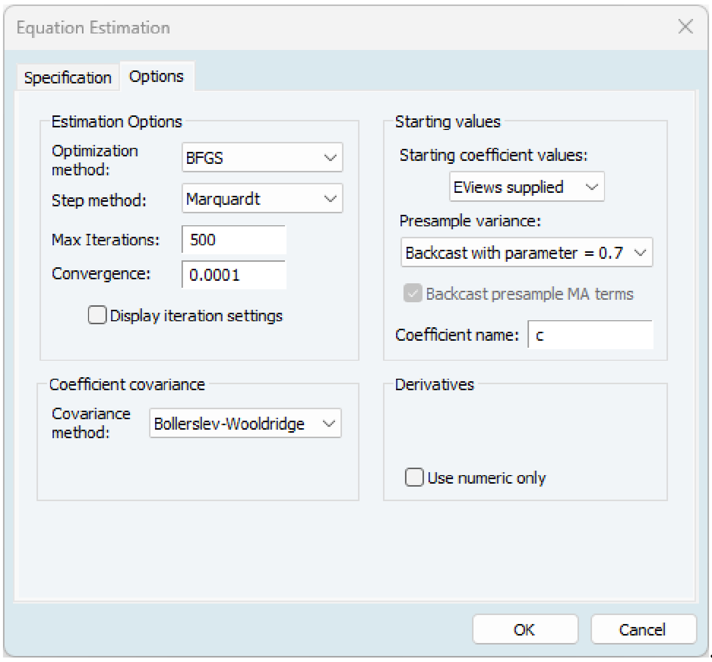
You should also select Bollerslev-Wooldridge standard errors in the `Options` tab, because we will see later that the standardized residuals will not be normally distributed:



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The estimated model is

Dependent Variable: R
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 11/25/22 Time: 18:43
Sample (adjusted): 10/30/2012 10/27/2022
Included observations: 2546 after adjustments
Convergence achieved after 23 iterations
Coefficient covariance computed using Bollerslev-Wooldridge QML
sandwich with expected Hessian
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*RESID(-2)^2 + C(5)*RESID(-3)^2
+ C(6)*RESID(-4)^2 + C(7)*RESID(-5)^2 + C(8)*RESID(-6)^2

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000809	0.000142	5.687025	0.0000
Variance Equation				
C	1.98E-05	2.36E-06	8.385856	0.0000
RESID(-1)^2	0.216152	0.050172	4.308196	0.0000
RESID(-2)^2	0.177032	0.036417	4.861280	0.0000
RESID(-3)^2	0.178068	0.036320	4.902714	0.0000
RESID(-4)^2	0.160963	0.045330	3.550881	0.0004
RESID(-5)^2	0.066833	0.031598	2.115088	0.0344
RESID(-6)^2	0.064229	0.027097	2.370341	0.0178

R-squared	-0.001467	Mean dependent var	0.000390
Adjusted R-squared	-0.001467	S.D. dependent var	0.010946
S.E. of regression	0.010954	Akaike info criterion	-6.734590
Sum squared resid	0.305352	Schwarz criterion	-6.716233
Log likelihood	8581.134	Hannan-Quinn criter.	-6.727932
Durbin-Watson stat	2.287626		

To see if the model is adequate, we can look at the correlogram of the squared standardized residuals (under `View→Residual Diagnostics→ Correlogram Squared Residuals`). It looks as follows.

Date: 11/25/22 Time: 18:47
Sample (adjusted): 10/30/2012 10/27/2022
Included observations: 2546 after adjustments

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob*
		1	0.005	0.005	0.0673	0.795
		2	0.003	0.003	0.0966	0.953
		3	-0.019	-0.019	1.0342	0.793
		4	-0.005	-0.005	1.0924	0.895
		5	-0.033	-0.033	3.8858	0.566
		6	-0.018	-0.018	4.7484	0.576
		7	-0.004	-0.004	4.7962	0.685
		8	-0.003	-0.004	4.8169	0.777
		9	0.015	0.014	5.3654	0.801
		10	0.048	0.046	11.152	0.346
		11	0.004	0.002	11.194	0.427
		12	-0.010	-0.011	11.460	0.490
		13	0.016	0.018	12.111	0.519
		14	-0.015	-0.013	12.654	0.554
		15	0.005	0.009	12.731	0.623
		16	0.014	0.016	13.204	0.658
		17	0.005	0.005	13.275	0.718
		18	0.000	0.001	13.276	0.775
		19	0.005	0.004	13.330	0.821
		20	0.034	0.032	16.233	0.702

*Probabilities may not be valid for this equation specification.

None of the Q-tests reject, so there is no remaining autocorrelation. Alternatively, we can try a GARCH(1, 1) model. This produces

Dependent Variable: R
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 11/25/22 Time: 18:50
Sample (adjusted): 10/30/2012 10/27/2022
Included observations: 2546 after adjustments
Convergence achieved after 18 iterations
Coefficient covariance computed using Bollerslev-Wooldridge QML
sandwich with expected Hessian
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000799	0.000142	5.620743	0.0000
		Variance Equation		
C	4.50E-06	1.14E-06	3.935962	0.0001
RESID(-1)^2	0.223750	0.030237	7.399998	0.0000
GARCH(-1)	0.741309	0.030859	24.02285	0.0000

R-squared	-0.001398	Mean dependent var	0.000390
Adjusted R-squared	-0.001398	S.D. dependent var	0.010946
S.E. of regression	0.010953	Akaike info criterion	-6.742939
Sum squared resid	0.305331	Schwarz criterion	-6.733760
Log likelihood	8587.761	Hannan-Quinn criter.	-6.739609
Durbin-Watson stat	2.287783		

The estimated coefficients are $\hat{\beta} = 0.74$ and $\hat{\alpha} = 0.22$. This is a bit unusual; typically we find $\hat{\beta}$ around 0.9 for daily returns (c.f. $\lambda = 0.94$ for the RiskMetrics model). The fact that the estimated model is close to the stationarity border (recall that stationarity requires $\alpha + \beta < 1$) is typical, though. The correlogram of the squared standardized residuals looks as follows.

Date: 11/25/22 Time: 18:51
Sample (adjusted): 10/30/2012 10/27/2022
Included observations: 2546 after adjustments

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob*
		1	0.001	0.001	0.0011	0.973
		2	-0.000	-0.000	0.0015	0.999
		3	-0.003	-0.003	0.0182	0.999
		4	0.024	0.024	1.4430	0.837
		5	-0.028	-0.028	3.4029	0.638
		6	-0.008	-0.008	3.5668	0.735
		7	-0.021	-0.021	4.6583	0.702
		8	-0.017	-0.017	5.3624	0.718
		9	-0.008	-0.006	5.5064	0.788
		10	0.038	0.037	9.1425	0.519
		11	-0.004	-0.004	9.1864	0.605
		12	-0.017	-0.017	9.8904	0.626
		13	0.000	-0.001	9.8906	0.703
		14	-0.018	-0.020	10.676	0.711
		15	-0.002	-0.001	10.686	0.774
		16	0.003	0.004	10.712	0.827
		17	0.002	0.002	10.720	0.871
		18	-0.007	-0.005	10.850	0.901
		19	-0.007	-0.009	10.990	0.924
		20	0.020	0.017	11.993	0.916

*Probabilities may not be valid for this equation specification.

This looks equally good as the ARCH(6). As usual, we prefer smaller models, so we stick with the GARCH(1, 1)[1]. We can confirm that there is no remaining volatility clustering by running an ARCH-LM test on the standardized residuals (under `View`→`Residual Diagnostics`→`ARCH LM Test`).

Heteroskedasticity Test: ARCH

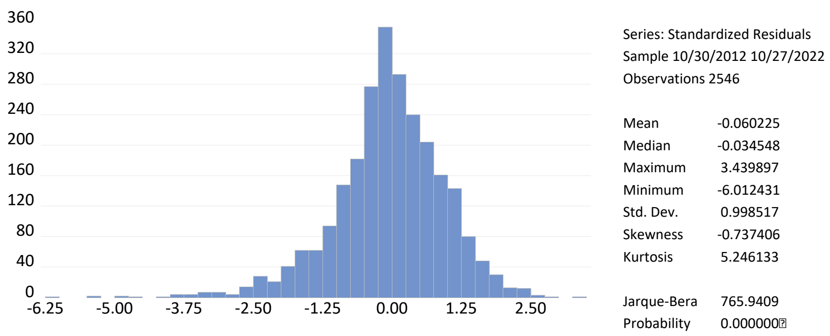
F-statistic	0.676487	Prob. F(5,2535)	0.6413
Obs*R-squared	3.385925	Prob. Chi-Square(5)	0.6407

Test Equation:
Dependent Variable: WGT_RESID^2
Method: Least Squares
Date: 11/25/22 Time: 18:57
Sample (adjusted): 11/06/2012 10/27/2022
Included observations: 2541 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	1.006032	0.060805	16.54509	0.0000
WGT_RESID^2(-1)	0.001390	0.019854	0.069995	0.9442
WGT_RESID^2(-2)	-0.000122	0.019848	-0.006170	0.9951
WGT_RESID^2(-3)	-0.002597	0.019846	-0.130879	0.8959
WGT_RESID^2(-4)	0.023571	0.019845	1.187738	0.2350
WGT_RESID^2(-5)	-0.027765	0.019852	-1.398648	0.1620

R-squared	0.001333	Mean dependent var	1.000523
Adjusted R-squared	-0.000637	S.D. dependent var	2.101669
S.E. of regression	2.102338	Akaike info criterion	4.326336
Sum squared resid	11204.26	Schwarz criterion	4.340126
Log likelihood	-5490.610	Hannan-Quinn criter.	4.331338
F-statistic	0.676487	Durbin-Watson stat	2.000405
Prob(F-statistic)	0.641285		

The test doesn't reject, so it seems that we have successfully modeled the volatility clustering. We can look at a histogram of the standardized residuals by clicking on `View`→`Residual Diagnostics`→`Histogram - Normality Test`. This produces the following plot.



Normality is clearly rejected, another typical finding. This means that we were right to use Bollerslev-Wooldridge standard errors. Alternatively, we could have specified a different error distribution, but we'll reserve that for next week.

EViews doesn't offer a statistical test for the leverage effect, so we'll just go ahead and estimate a TARCH(1, 1, 1) model. The fitted model is

[1]We could also use the BIC to confirm this decision.

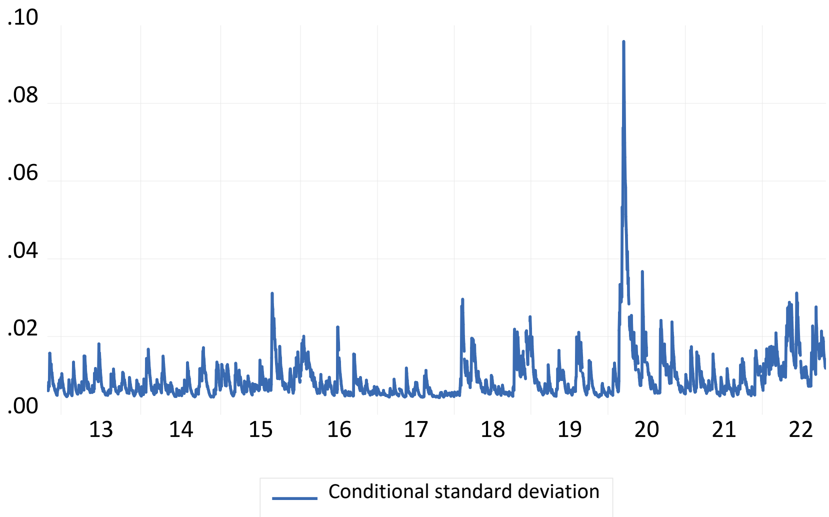
Dependent Variable: R
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 11/25/22 Time: 19:03
Sample (adjusted): 10/30/2012 10/27/2022
Included observations: 2546 after adjustments
Convergence achieved after 26 iterations
Coefficient covariance computed using Bollerslev-Wooldridge QML
sandwich with expected Hessian
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*RESID(-1)^2*(RESID(-1)<0) +
C(5)*GARCH(-1)

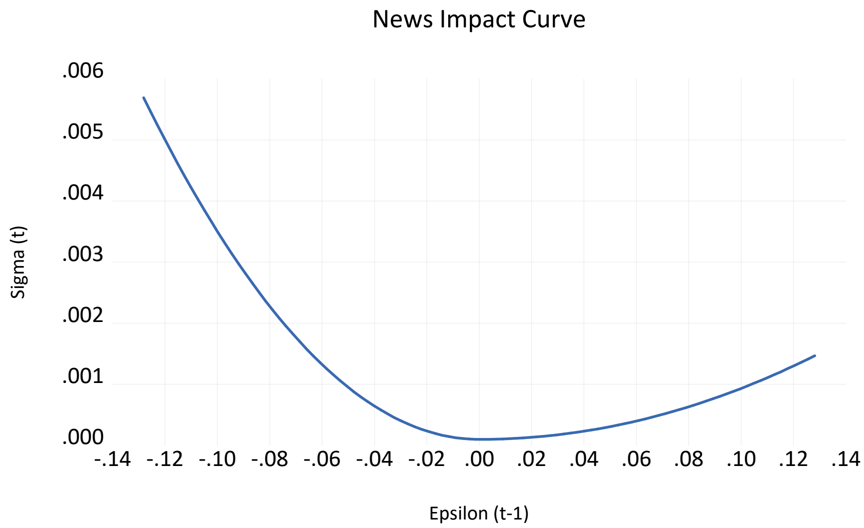
Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000472	0.000132	3.560013	0.0004
Variance Equation				
C	4.18E-06	9.41E-07	4.443218	0.0000
RESID(-1)^2	0.083481	0.072149	1.157069	0.2472
RESID(-1)^2*(RESID(-1)<0)	0.257376	0.063931	4.025838	0.0001
GARCH(-1)	0.755584	0.041655	18.13929	0.0000

R-squared	-0.000056	Mean dependent var	0.000390
Adjusted R-squared	-0.000056	S.D. dependent var	0.010946
S.E. of regression	0.010946	Akaike info criterion	-6.767193
Sum squared resid	0.304922	Schwarz criterion	-6.755719
Log likelihood	8619.636	Hannan-Quinn criter.	-6.763031
Durbin-Watson stat	2.290853		

The asymmetry coefficient $\hat{\gamma} = 0.257$ is clearly significant, so there is clear evidence of leverage. Also note that the ARCH coefficient has become insignificant. This means that volatility shows no significant reaction to good news (positive returns) at all, only to bad news.

(f) The plots can be found under `View→GARCH Graphs` and look as follows.





(g) As usual, we can use EViews for the forecasts, or do it manually. We'll do the latter. To do that, we need the estimated $\widehat{\sigma}_t$ and $\hat{u}_t$. We can get them via `Proc`$\rightarrow$ `Make GARCH Variance Series...` and `Proc`$\rightarrow$ `Make Residual Series....` For the latter, you can specify either the ordinary residuals $\hat{u}_t$ or the standardized residuals $\hat{z}_t$. We need the former; the latter would be needed for forecastin and EGARCH model. We find that the residual for 27/10 is -0.006573, and the corresponding $\widehat{\sigma}_t$ is 0.0001399. This is all we need to produce a forecast, by plugging into our estimated model as follows.

$$\begin{aligned}
\widehat{\sigma}_{t+1}^2 &= \hat{\omega} + \hat{\alpha}\hat{u}_t^2 + \hat{\gamma}\hat{u}_t^2 \mathrm{I}_t + \hat{\beta}\hat{\sigma}_t^2 \\
&= 0.00000418 + 0.083\hat{u}_t^2 + 0.257\hat{u}_t^2 \cdot 1 + 0.756\hat{\sigma}_t^2 \\
&= 0.0^5 418 + 0.083 \cdot (-0.006573)^2 + 0.257 \cdot (-0.006573)^2 + 0.756 \cdot 0.0001399 \\
&= 0.0001246;
\end{aligned}$$

note that the indicator function is 1 because $\hat{u}_t$ is negative.

2. (a) Splitting out the first term of the sum immediately yields

$$\widehat{\sigma}_{t+1,EWMA}^2 = (1-\lambda)\sum_{j=0}^{\infty} \lambda^j r_{t-j}^2$$

$$\widehat{\sigma}_{t+1,EWMA}^2 = (1-\lambda)\lambda^0 r_{t-0}^2 + (1-\lambda)\sum_{j=1}^{\infty} \lambda^j r_{t-j}^2$$

$$\widehat{\sigma}_{t+1,EWMA}^2 = (1-\lambda) r_t^2 + (1-\lambda)\sum_{j=0}^{\infty} \lambda^{j+1} r_{t-1-j}^2$$

$$\begin{aligned}
\widehat{\sigma}_{t+1,EWMA}^2 &= (1-\lambda) r_t^2 + \lambda(1-\lambda)\sum_{j=0}^{\infty} \lambda^j r_{t-1-j}^2 \\
&= (1-\lambda) r_t^2 + \lambda \widehat{\sigma}_{t,EWMA}^2.
\end{aligned}$$

(b) Solving $\widehat{\sigma}^2 = \hat{\omega}/(1 - \hat{\alpha} - \hat{\beta})$ for $\hat{\omega}$, we find $\hat{\omega} = \widehat{\sigma}^2 \cdot (1 - \hat{\alpha} - \hat{\beta})$. Plugging this into the equation

$$\widehat{\sigma}_{t+1}^2 = \hat{\omega} + \hat{\alpha}\hat{u}_t^2 + \hat{\beta}\widehat{\sigma}_t^2$$

yields

$$\begin{aligned}\widehat{\sigma}_{t+1}^2 &= \widehat{\sigma}^2 \cdot (1 - \hat{\alpha} - \hat{\beta}) + \hat{\alpha}\hat{u}_t^2 + \hat{\beta}\widehat{\sigma}_t^2 \\ &= \widehat{\sigma}^2 + \hat{\alpha}(\hat{u}_t^2 - \widehat{\sigma}^2) + \hat{\beta}(\widehat{\sigma}_t^2 - \widehat{\sigma}^2).\end{aligned}$$

(c) The result of the previous question also implies that

$$\widehat{\sigma}_{t+2}^2 = \widehat{\sigma}^2 + \hat{\alpha}(\hat{u}_{t+1}^2 - \widehat{\sigma}^2) + \hat{\beta}(\widehat{\sigma}_{t+1}^2 - \widehat{\sigma}^2).$$

Replacing the unknown $\hat{u}_{t+1}^2$ with its forecast $\widehat{\sigma}_{t+1}^2$, we see that

$$\begin{aligned}\widehat{\sigma}_{t+2}^2 &= \widehat{\sigma}^2 + \hat{\alpha}(\widehat{\sigma}_{t+1}^2 - \widehat{\sigma}^2) + \hat{\beta}(\widehat{\sigma}_{t+1}^2 - \widehat{\sigma}^2) \\ \widehat{\sigma}_{t+2}^2 &= \widehat{\sigma}^2 + (\hat{\alpha} + \hat{\beta})(\widehat{\sigma}_{t+1}^2 - \widehat{\sigma}^2). \qquad (1)\end{aligned}$$

Equation (1) also implies that

$$\widehat{\sigma}_{t+3}^2 = \widehat{\sigma}^2 + (\hat{\alpha} + \hat{\beta})(\widehat{\sigma}_{t+2}^2 - \widehat{\sigma}^2), \qquad (2)$$

and plugging (1) into (2) results in

$$\begin{aligned}\widehat{\sigma}_{t+3}^2 &= \widehat{\sigma}^2 + (\hat{\alpha} + \hat{\beta})((\hat{\alpha} + \hat{\beta})(\widehat{\sigma}_{t+1}^2 - \widehat{\sigma}^2)) \\ &= \widehat{\sigma}^2 + (\hat{\alpha} + \hat{\beta})^2(\widehat{\sigma}_{t+1}^2 - \widehat{\sigma}^2).\end{aligned}$$

Iterating this process produces

$$\widehat{\sigma}_{t+s}^2 = \widehat{\sigma}^2 + (\hat{\alpha} + \hat{\beta})^{s-1}(\widehat{\sigma}_{t+1}^2 - \widehat{\sigma}^2)$$

as claimed.