

Module 9.3: Time Series Analysis

Fall Term 2022

Week 6:

Value at Risk

Outline in Weeks

- ➊ Introduction; Descriptive Modelling
- ➋ Returns; Autocorrelation; Stationarity
- ➌ ARMA Models
- ➍ Unit Roots; Regressions between Time Series
- ➎ Volatility Modelling
- ➏ Value at Risk
- ➐ Cointegration

Outline

- 1 Value at Risk (VaR)
- 2 VaR Methods: Historical simulation
- 3 VaR Methods: Normal distribution
- 4 VaR Methods: Standardized t distribution
- 5 Expected Shortfall
- 6 Multi-Period VaR
- 7 Backtesting Value at Risk
- 8 Epilogue

Value at Risk

- Consider a portfolio with value $V_{PF,t}$ and daily returns $R_{PF,t+1}$.
- Define the one-day Loss on the portfolio as

$$Loss_{t+1} = V_{PF,t} - V_{PF,t+1}.$$

- The one-day, $100p\%$, dollar **Value at Risk** ($\$VaR_{t+1}^p$) gives the largest loss on the portfolio that we can expect to incur in the next day with level of confidence $100(1-p)\%$.
- Mathematically it is given by

$$\Pr(Loss_{t+1} \leq \$VaR_{t+1}^p) = 1 - p,$$

or equivalently

$$\Pr(Loss_{t+1} > \$VaR_{t+1}^p) = p.$$

Value at Risk

- Usually easier to express the VaR as a percentage of the portfolio value:

$$VaR_{t+1}^p = \frac{\$VaR_{t+1}^p}{V_{PF,t}}.$$

- Hence

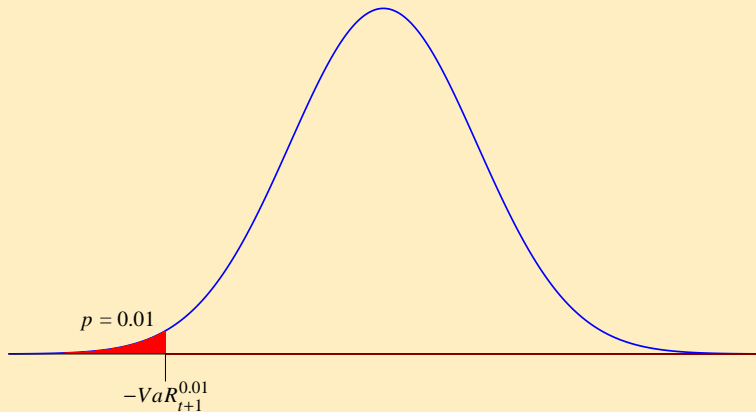
$$\Pr(R_{PF,t+1} < -VaR_{t+1}^p) = p,$$

as

$$R_{PF,t+1} = -\frac{\$Loss_{t+1}}{V_{PF,t}}.$$

- Thus VaR_{t+1}^p is minus the 100 p th *percentile* of the return distribution. Usually $p = 0.01$.
- Definition can be naturally extended to K -day VaR, from the distribution of the K -day returns $R_{PF,t+1:t+K}$.

Probability density function of daily returns



Value at Risk

- Value at Risk was proposed as the standard measure of portfolio risk by the Basel Committee of the Bank of International Settlements in 1996.
- The BC imposed that financial institutions should report the Value at Risk on their positions, such that regulators could check the adequacy of the economic capital as a buffer against market risk.
- Banks were allowed to use their own, internal models for the computation of VaR, but the adequacy of these models should be “backtested” using specific criteria.
- A candidate for a standard model is RiskMetrics (developed by J.P.Morgan).
- VaR is scheduled to be replaced by the expected shortfall (ES) with the rollout of Basel 3. The ES is based on the VaR, however.

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VaR Methods: Historical simulation

Historical simulation assumes that the distribution of tomorrow's portfolio returns is well approximated by the empirical distribution (histogram) of the past m observations

$$\{R_{PF,t}, R_{PF,t-1}, \dots, R_{PF,t+1-m}\}.$$

This is as if we draw, with replacement, from the last m returns and use this to simulate the next day's return distribution.

- The estimator of VaR is given by minus the 100 p th percentile of the sequence of past portfolio returns, that is:
 - sort the returns $\{R_{PF,t}, R_{PF,t-1}, \dots, R_{PF,t+1-m}\}$ in ascending order;
 - define R_{t+1}^p as the number such that 100 p % of the observations are smaller than R_{t+1}^p ;
 - the estimator for VaR is given by

$$\widehat{VaR}_{t+1}^p = -R_{t+1}^p.$$

- R_{t+1}^p can be computed using EViews' @quantile function.

VaR Methods: Historical simulation

Problems / limitations of historical simulation:

- Last year(s) of data not necessarily representative for the next few days (e.g. because of volatility clustering).
- Similar problems as historical volatility (choice of m).
- A large m is required to compute 1% VaR with any degree of precision.
- By focussing on left tails, extreme positive returns are ignored.

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VaR Methods: Normal distribution

- Another simple approach is to assume $R_{t+1} = R_{PF,t+1} \sim N(\mu, \sigma^2)$ and to estimate μ and σ^2 using historical data.
- Denoting the inverse distribution function (quantile function) of the Normal, as Φ_p^{-1} , The VaR becomes

$$VaR_{t+1}^p = -\mu - \sigma \Phi_p^{-1}.$$

For example, $\Phi_{.01}^{-1} = -2.326$. For daily data we can take $\mu = 0$.

VaR Methods: Normal distribution

- The normal model can be easily extended to a *conditionally* normal model. Assume $R_{t+1} \sim N(0, \sigma_{t+1}^2)$ where σ_{t+1}^2 may be estimated by:
 - EWMA / RiskMetrics;
 - univariate GARCH;
 - multivariate GARCH.

The VaR then becomes $VaR_{t+1}^p = -\sigma_{t+1} \Phi_p^{-1}$.

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VaR Methods: Standardized t distribution

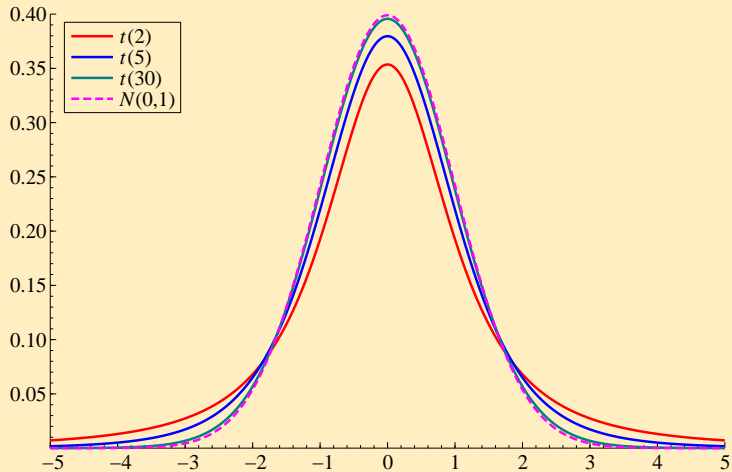
- The VaR methods described on the previous slides require that financial returns are normally distributed.
- This can be tested by the Jarque-Bera test and is usually rejected.
- Solution: use Student's $t(d)$ distribution, where d.o.f. $d > 0$ need not be integer.
- d is just a shape parameter. Small values correspond to fat tails. As $d \rightarrow \infty$, we approach the $N(0, 1)$ distribution.
- For $d > 2$, the variance of a $t(d)$ random variable x is $d/(d - 2)$; the distribution of

$$z = \frac{x}{\sqrt{\text{var}(x)}} = \sqrt{\frac{d-2}{d}} x$$

is called *standardized* $t(d)$, denoted $\tilde{t}(d)$.

- For $d > 4$ the excess kurtosis is $6/(d - 4)$. The distributions are symmetric around 0 (hence mean and skewness are 0).

Student's t densities



VaR Methods: Standardized *t* distribution

- The GARCH model $R_{t+1} = \sigma_{t+1}z_{t+1}$, $\sigma_{t+1}^2 = \omega + \alpha R_t^2 + \beta \sigma_t^2$, may be extended to $z_t \sim \tilde{t}(d)$, where d is an extra parameter that can be estimated by maximum likelihood.
- In practice this GARCH-*t* model often gives a substantially better fit than the Gaussian model. The main problem is that the standardized residuals usually have an asymmetric distribution, with a longer left tail than right tail.

Estimation of GARCH- t in EViews:

Dependent Variable: R
 Method: ML ARCH - Student's t distribution (BFGS / Marquardt steps)
 Date: 12/01/22 Time: 17:14
 Sample (adjusted): 10/30/2012 10/27/2022
 Included observations: 2546 after adjustments
 Convergence achieved after 27 iterations
 Coefficient covariance computed using outer product of gradients
 Presample variance: backcast (parameter = 0.7)
 GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000904	0.000126	7.176877	0.0000
Variance Equation				
C	2.92E-06	6.17E-07	4.743725	0.0000
RESID(-1)^2	0.221296	0.026849	8.242232	0.0000
GARCH(-1)	0.773192	0.023233	33.27982	0.0000
T-DIST. DOF	5.275193	0.568869	9.273121	0.0000
R-squared	-0.002211	Mean dependent var		0.000390
Adjusted R-squared	-0.002211	S.D. dependent var		0.010946
S.E. of regression	0.010958	Akaike info criterion		-6.810961
Sum squared resid	0.305579	Schwarz criterion		-6.799488
Log likelihood	8675.354	Hannan-Quinn criter.		-6.806800
Durbin-Watson stat	2.285926			

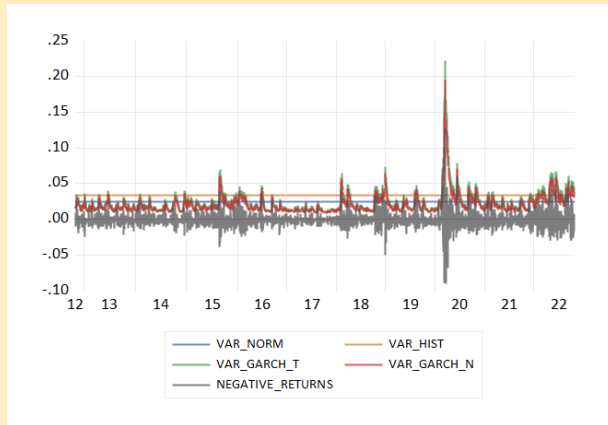
VaR Methods: Standardized *t* distribution

- Let $\tilde{t}_p^{-1}(d)$ be 100*p*% quantile of the standardized *t* distribution $\tilde{t}(d)$ and $t_p^{-1}(d)$ the percentile 100*p*% of the *t* distribution $t(d)$.
- The implied VaR now is

$$VaR_{t+1}^p = -\sigma_{t+1} \tilde{t}_p^{-1}(d) = -\sigma_{t+1} \sqrt{\frac{d-2}{d}} t_p^{-1}(d),$$

where, e.g., $\tilde{t}_{.01}^{-1}(6) = -2.566$.

Example: minus S&P500 returns, with 1% VaR based on (full-sample) historical simulation, normal distribution, and a GARCH(1, 1) with Normal and t errors



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Expected Shortfall

Limitations of Value at Risk:

- VaR is not informative about the magnitude of the losses if they exceed the VaR. Two distributions could have the same 1% VaR, but with different left tails.
- VaR is not *subadditive*: it is not guaranteed that

$$VaR_{t+1}^p(X + Y) \leq VaR_{t+1}^p(X) + VaR_{t+1}^p(Y).$$

This means that VaR is not a “coherent” risk measure.

Expected Shortfall

- The 1% VaR will be replaced by the 2.5% *expected shortfall* (ES, a.k.a. CVaR), which addresses these problems, on January 1st, 2023:

$$ES_{t+1}^p = -E_t [R_{t+1} | R_{t+1} < -VaR_{t+1}^p] .$$

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Multi-Period VaR

- For the GARCH- $N(0, 1)$ and GARCH- $\tilde{t}(d)$ models, the one-day VaR and ES can be determined analytically (when the estimation is based on daily data).
- However, in practice one often needs risk measures for multi-period returns:

$$R_{t+1:t+K} = \sum_{k=1}^K R_{t+k}.$$

For example, a horizon of two weeks ($K = 10$ trading days) is common.

Multi-Period VaR

- Problem: even if the distribution of the one-period return is known (e.g., normal), that of $R_{t+1:t+K}$ is not (because the variance is not deterministic).
- *Monte Carlo simulation* is a possible solution: we let the computer generate a large number of scenarios of K daily returns, and compute from this the conditional distribution of the K -day return, and hence the K -day VaR and ES.
- Quick-and dirty practitioner solution: scale the one-day VaR with \sqrt{K} (*square root of time rule*). Only correct under normality.

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Backtesting Value at Risk

- The Basel Committee requires that methods to evaluate VaR be backtested (<http://www.bis.org/publ/bcbssc223.pdf>).
- They recommend constructing the 1% VaR over the last 250 trading days (≈ 1 year), and counting the number of times losses exceed the day's VaR figure (termed *exceptions* or *violations*).
- A method is said to lie in the:
 - *Green zone*, in case of 0–4 exceptions;
 - *Yellow zone*, in case of 5–9 exceptions;
 - *Red zone*, in case of 10 exceptions or more.
- The capital charge for the bank changes according to the zone.

Backtesting Value at Risk

How can we test if a VaR method is accurate?

- Define the *hit sequence*

$$I_{t+1} = \begin{cases} 1, & \text{if } R_{t+1} < -VaR_{t+1}^p, \\ 0, & \text{if } R_{t+1} > -VaR_{t+1}^p. \end{cases}$$

- Consider a test period that covers $t + 1 \in \{1, \dots, T\}$, then the number of exceptions is given by $T_1 = \sum_{t=1}^T I_t$.
- The proportion of exceptions is given by $\hat{\pi} = T_1/T$ which is an estimator of $\Pr(R_{t+1} < -VaR_{t+1}^p)$.
- Recall that if the model that generated VaR_{t+1}^p is correctly specified, then

$$\Pr(R_{t+1} < -VaR_{t+1}^p) = p,$$

independent of any information at time t .

Backtesting Value at Risk

- Hence, under the null hypothesis of correct specification, the hit sequence $\{I_{t+1}\}$ are independent Bernoulli random variables, and so $T_1 = \sum_{t=1}^T I_t$ has a Binomial(T, p) distribution.
- We can test this hypothesis (e.g., with $p = 0.01$) based on the t -statistics

$$t_0 = \frac{\hat{\pi} - p}{\sqrt{p(1-p)/T}} \quad \text{or} \quad t = \frac{\hat{\pi} - p}{\sqrt{\hat{\pi}(1-\hat{\pi})/T}}.$$

- Under H_0 their asymptotic distribution is $N(0, 1)$.
- The second t -statistic is equal (up to degrees-of-freedom correction) to OLS-based t -statistic in regression of $I_{t+1} - p$ on a constant.

Backtesting Value at Risk

- The previous test only checks *unconditional* coverage, i.e., $\Pr(I_{t+1} = 1) = p$ *on average*. However, misspecification often is due to the fact that the hits I_{t+1} are not independent over time.
- If exceptions are clustered, then if today there was an exception a risk manager can infer that the probability of occurring another exception tomorrow is higher than p . Hence, there is misspecification.
- We would like to test if the VaR violations are *independent* over time, the null hypothesis is

$$H_0 : \Pr(I_{t+1} = 1 | I_t = 1) = \Pr(I_{t+1} = 1 | I_t = 0),$$

which implies $\Pr(I_{t+1} = 0 | I_t = 0) = \Pr(I_{t+1} = 0 | I_t = 1)$.

Backtesting Value at Risk

- Also of interest is to test if the VaR violations are independent over time and if the number of violations is correct (*conditional coverage*)

$$H_0 : \Pr(I_{t+1} = 1 | I_t = 1) = \Pr(I_{t+1} = 1 | I_t = 0) = p.$$

- A simple approach to test these hypotheses is to consider the linear regression model

$$I_{t+1} - p = b_0 + b_1 I_t + e_{t+1}$$

- The *conditional coverage* hypothesis is equivalent to $H_0 : b_0 = 0$ and $b_1 = 0$ which can be tested using a F -test.
- The *independence* hypothesis is equivalent to $H_0 : b_1 = 0$ which can be tested using a t -statistic.

Backtesting Value at Risk

Results for S&P500 returns, 4 different methods

	Hist	Norm	GARCHn	GARCHt
$\hat{\pi} \quad (\times 100)$	0.98	2.04	2.59	1.65
$t(\pi = 0.01)$	-0.09	3.72	5.06	2.57
\hat{b}_1	0.03	0.08	0.00	0.03
$t(b_1 = 0)$	1.54	3.91	0.23	1.60
$F(b_0 = b_1 = 0)$	1.18	14.60	12.80	4.59

The critical values for the t and F tests are, respectively, ± 1.96 and 3.00.

Note: this result is highly unusual. Usually, the GARCHt model fares best.

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Learning Goals

Students

- know the definitions of VaR and Expected Shortfall,
- understand the limitations of the VaR,
- are able to construct VaR forecasts based on various methods,
- and are able to backtest VaR forecasts.

Homework

- Exercise 6