

## Module 9.3: Time Series Analysis

### Fall Term 2023

#### Week 2:

Returns; Autocorrelation; Stationarity

# Outline in Weeks

- 1 Introduction; Descriptive Modelling
- 2 Returns; Autocorrelation; Stationarity
- 3 ARMA Models
- 4 Unit Roots; Regressions between Time Series
- 5 Volatility Modelling
- 6 Value at Risk
- 7 Cointegration

# Outline

- 1 Asset Returns
- 2 Stochastic Processes
- 3 The Efficient Market Hypothesis
- 4 The Autocorrelation Function
- 5 The Random Walk
- 6 Stationary and Integrated Processes
- 7 Epilogue

# Asset Returns

- We consider two definitions of returns:

- 1 *Simple* return between dates  $t - 1$  and  $t$  [or: in period  $t$ ]

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}},$$

where  $P_t$  is the asset price at time  $t$ .

- 2 Continuously compounded return or *log return*

$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right) = \log(1 + R_t).$$

- They are typically very close for daily returns, as

$$r_t = \log(1 + R_t) \approx R_t,$$

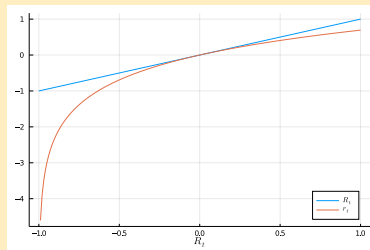
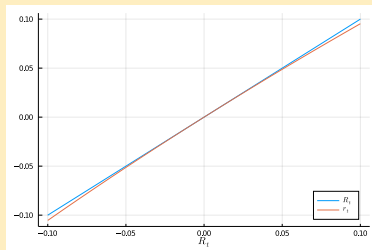
when  $R_t \approx 0$ .

- Log returns and 'simple' returns often are very close, as

$$r_t = \ln(1 + R_t) \approx R_t \text{ when } R_t \approx 0.$$

- Simple returns are bounded below by -1 (100% loss). Log returns live on  $(-\infty, \infty)$ . Easier to model (e.g., normal distribution).

## Simple vs. Log Returns



# Log returns: Intuition

- If a one-period interest rate of  $r$  is compounded  $n$  times, then

$$P_t = (1 + r/n)^n P_{t-1}.$$

- As  $n \rightarrow \infty$ ,  $(1 + r/n)^n \rightarrow e^r$ , so

$$P_t = e^r P_{t-1} \Leftrightarrow r = \log(P_t/P_{t-1}) = \ln P_t - \ln P_{t-1}.$$

# Portfolio Returns

- Advantage of continuously compounded returns: *multi-period return* is sum of single-period returns.
- Advantage of simple returns: *portfolio return* is weighted sum of asset returns.
- Proof: If an investor buys  $n_i$  shares in stock  $i$ , then the value of the portfolio at time  $t - 1$  is  $V_{t-1} = \sum_{i=1}^n n_i P_{i,t-1}$ .
- Ignoring dividends, the payoff is  $V_t = \sum_{i=1}^n n_i P_{i,t}$ , so the return on the portfolio is

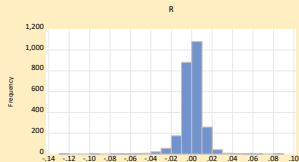
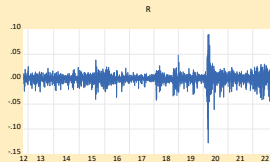
$$\begin{aligned} R_{p,t} &= \frac{V_t - V_{t-1}}{V_{t-1}} = \frac{\sum_{i=1}^n n_i (P_{i,t} - P_{i,t-1})}{V_{t-1}} \\ &= \sum_{i=1}^n \underbrace{\frac{n_i P_{i,t-1}}{V_{t-1}}}_{w_i} \underbrace{\frac{(P_{i,t} - P_{i,t-1})}{P_{i,t-1}}}_{R_{i,t}} = \sum_{i=1}^n w_i R_{i,t}. \end{aligned}$$

# Stylized Facts of Asset Returns

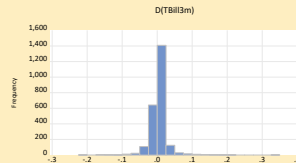
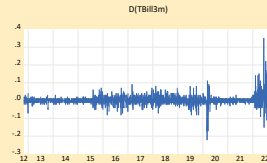
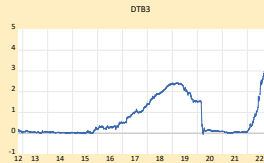
- Prices display (time-varying) *trend*, and variation proportional to price level (motivation for taking logs).
- Returns have constant mean close to zero, and very little autocorrelation.
- Returns display *volatility clustering*: alternating periods of high and low variability.
- Returns have non-Gaussian distribution, *fat tails* (excess kurtosis).
- Interest rates display long swings, very slow *mean-reversion*.
- Interest rate changes have similar characteristics as returns.



## Example: S&P 500 index values and returns, 10/29/2012–10/27/2022



## Example: 3 Month T-Bill rate, 10/29/2012–10/26/2022



# Testing Normality

- Normality can be tested by examining the *skewness* and *kurtosis*.
- Skewness  $SK = m_3 / \sqrt{m_2^3}$  and kurtosis  $K = m_4 / m_2^2$ , where  $m_j$  is the  $j$ -th centralized moment<sup>1</sup>  $m_j = \mathbb{E}[(r_t - \mathbb{E}[r_t])^j]$ .
- A normal distribution has  $SK=0$  and  $K=3$ .
- *Jarque-Bera normality test*:

$$JB = \frac{T}{6} \widehat{SK}^2 + \frac{T}{24} (\widehat{K} - 3)^2,$$

where the skewness and kurtosis of  $r_t$  can be estimated as

$$\widehat{SK} = \hat{m}_3 / \sqrt{\hat{m}_2^3}, \quad \text{and} \quad \widehat{K} = \hat{m}_4 / \hat{m}_2^2, \quad \text{with} \quad \hat{m}_j = \frac{1}{T} \sum_{t=1}^T (r_t - \bar{r})^j.$$

- Under the null hypothesis of normality:  $JB \xrightarrow{d} \chi_2^2$ .

<sup>1</sup>I.e., the second centralized moment  $m^2$  is just the variance, otherwise known as  $\sigma^2$ .

# Outline

- 1 Asset Returns
- 2 Stochastic Processes**
- 3 The Efficient Market Hypothesis
- 4 The Autocorrelation Function
- 5 The Random Walk
- 6 Stationary and Integrated Processes
- 7 Epilogue

# Stochastic Processes

- Time series analysis is concerned with modelling, estimating, analyzing and forecasting returns and other financial and economic variables.
- A *time series*  $\{y_t, t = 1, 2, \dots, T\}$  is a collection of subsequent observations on a particular variable. We view such a time series as a *realization* of a *discrete-time stochastic process*  $\{Y_t, t = 1, 2, \dots\}$ , which is a collection of (dependent) random variables.
- The goal is to determine which process  $\{Y_t\}$  generated the data.
- The distinction between  $\{Y_t\}$  (the process) and  $\{y_t\}$  (the realization) will usually not be emphasized.
- We will not consider continuous-time stochastic processes here (e.g. Brownian motion).

# White Noise

- An important example of a stationary process is the *white noise* process, which has zero mean<sup>2</sup> and zero autocovariances.

$$\begin{aligned}\mathbb{E}[U_t] &= 0, \\ \text{var}(U_t) &= \mathbb{E}[U_t^2] = \sigma^2, \\ \text{cov}(U_t, U_{t-k}) &= \mathbb{E}[U_t U_{t-k}] = 0, \quad k = 1, 2, \dots\end{aligned}$$

- The notation  $U_t$  emphasizes the similarity to regression errors.
- White noise is *unpredictable*.
- It is the building block for other processes (which may be predictable).

---

<sup>2</sup>Brooks allows a white noise process to have a non-zero mean. Usually such a process is called an *uncorrelated* process.

# Outline

- 1 Asset Returns
- 2 Stochastic Processes
- 3 The Efficient Market Hypothesis**
- 4 The Autocorrelation Function
- 5 The Random Walk
- 6 Stationary and Integrated Processes
- 7 Epilogue

# Excursus: The Efficient Market Hypothesis

- The *weak form EMH*<sup>3</sup> posits that past prices and returns cannot predict future returns.
- This implies that no fund manager can consistently outperform the market, at least based on historical prices alone.
- If weak form EMH holds, then returns should be *uncorrelated*. Since the mean return is small for daily data, they should therefore resemble white noise.
- An important application of time series analysis is testing whether the EMH holds.
- The most basic way to do this is to test whether the returns have been generated by a white noise process.

---

<sup>3</sup>Fama (1970). Efficient Capital Markets: A Review of Theory and Empirical Work. *Journal of Finance*, 25(2), pp. 383–417.



# Outline

- 1 Asset Returns
- 2 Stochastic Processes
- 3 The Efficient Market Hypothesis
- 4 The Autocorrelation Function**
- 5 The Random Walk
- 6 Stationary and Integrated Processes
- 7 Epilogue

# Autocorrelation Function

- Recall that if a process  $\{Y_t\}$  is white noise, then it is uncorrelated at all lags (i.e.,  $Y_t$  should be uncorrelated with  $Y_{t-1}$ , with  $Y_{t-2}$ , etc.)
- Formally, its *autocorrelation function (ACF)*  $\tau_s$  is zero for all  $s$ , where the ACF is defined from the *autocovariances*  $\gamma_s$ , as

$$\tau_s = \text{Corr}(Y_t, Y_{t-s}) = \frac{\text{Cov}(Y_t, Y_{t-s})}{\text{Var}(Y_t)} = \frac{\gamma_s}{\gamma_0}, \quad s = 1, 2, \dots$$

# PSA: Population vs. Sample Quantities

- It is important to note that the ACF is a property of a *process*, not of a *sample* (i.e., the observed time series). This makes them *population quantities* or *parameters*.
- The statement on the previous slide says that the random variables  $\{Y_1, Y_2, \dots\}$  generated by a white noise process are uncorrelated.
- Population quantities are *unobserved*. The best we can hope for is to *estimate* them from a sample (a time series).

# PSA: Population vs. Sample Quantities

- To use the normal distribution as an analogy: it has two parameters,  $\mu$  and  $\sigma^2$ . These are *parameters* and thus *unobserved*.
- In a simulation exercise, I can *pretend* to know what  $\mu$  and  $\sigma^2$  are.
- E.g., I can set  $\mu = 0$  and  $\sigma^2 = 4$ , simulate 1000 random numbers  $y_t$ , and give them to you.
- Unlike me, you won't know what  $\mu$  and  $\sigma^2$  are. At best, you can *estimate* them, based on the *sample mean and variance*

$$\bar{y} \equiv \frac{1}{N} \sum_{i=1}^N y_i \quad \text{and} \quad s_y^2 \equiv \frac{1}{N} \sum_{i=1}^N (y_i - \bar{y})^2.$$

- If the sample is large enough, then these will be *close* to  $\mu = \mathbb{E}[Y]$  and  $\sigma^2 = \text{Var}(Y)$  by the law of large numbers (*LLN*).

# The Correlogram

- Applying the analogy to the ACF, the *Sample ACF* or *correlogram* is defined as

$$\hat{\tau}_s = \frac{\hat{\gamma}_s}{\hat{\gamma}_0} = \frac{\sum_{t=s+1}^T (y_t - \bar{y})(y_{t-s} - \bar{y})}{\sum_{t=1}^T (y_t - \bar{y})^2}, \quad \bar{y} = \frac{1}{T} \sum_{t=1}^T y_t.$$

- The correlogram is a *sample quantity*, i.e., I can compute it from a given time series.
- If I want to test if the time series is white noise, I can compare my SACF to the ACF of a white noise process.
- If the two are significantly different, then I can reject the null that the time series was generated by a white noise process.
- See exercises and the spreadsheet `simulations.xlsx`.

# Testing if an Autocorrelation is Zero

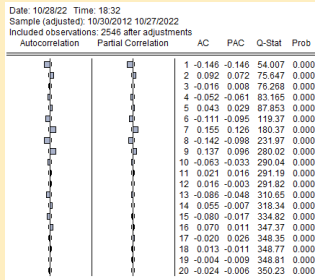
- One can show that under the null that the data were generated by a white noise process, the sample autocorrelations are asymptotically<sup>4</sup> normally distributed with zero mean and variance  $1/T$ .
- This implies that a sample autocorrelation is significantly different from zero if its absolute value is larger than  $1.96/\sqrt{T}$ .
- We can also test whether the first  $m$  autocorrelations are zero jointly: under  $H_0 : \tau_s = 0, s \geq 1$ , the **Ljung-Box**  $Q$ -statistic

$$Q(m) = T(T+2) \sum_{s=1}^m \frac{\hat{\tau}_s^2}{T-s} \xrightarrow{d} \chi^2(m).$$

---

<sup>4</sup>Formally: under  $H_0 : \tau_s = 0, s \geq 1$ ,  $\sqrt{T}\hat{\tau}_s \xrightarrow{d} N(0, 1)$ .

## Example: Correlogram of S&P500 returns



- Thin lines represent critical value  $1.96/\sqrt{T}$ , so autocorrelations at lags 1, 2, 5–10, and 13–17 are significant. Confirmed by the  $Q$ -stats, whose  $p$ -values are all less than 5%.
- Conclusion: some autocorrelation, and hence predictability, in the returns; returns are not white noise.
- Unclear whether predictability is sufficient to exploit with some trading strategy.

# Outline

- 1 Asset Returns
- 2 Stochastic Processes
- 3 The Efficient Market Hypothesis
- 4 The Autocorrelation Function
- 5 The Random Walk**
- 6 Stationary and Integrated Processes
- 7 Epilogue



# From Returns to Asset Prices

- We have seen that the EMH suggests that white noise is a reasonable model for stock returns:

$$r_t = U_t, \quad \text{where } U_t \text{ is white noise (not necessarily normal).}$$

- Recall the definition of log returns:

$$r_t = \log P_t - \log P_{t-1}.$$

- Putting the two together implies that

$$\begin{aligned} \log P_t - \log P_{t-1} &= U_t \Leftrightarrow \\ \log P_t &= \log P_{t-1} + U_t. \end{aligned}$$

- This characterizes the asset price as a *random walk*.

## Definition

A *random walk* is the stochastic process

$$Y_t = Y_{t-1} + U_t,$$

where  $U_t$  is white noise and  $Y_0$  is some fixed starting value.

# Properties of the Random Walk

- The random walk behaves very differently from white noise.
- A quick calculation shows that

$$Y_t = Y_0 + U_1 + U_2 + \cdots + U_t = Y_0 + \sum_{s=1}^t U_s.$$

- From this, it is immediate (see exercises) that

$$\mathbb{E}(Y_t) = Y_0 \quad \text{and} \\ \text{var}(Y_t) = \sigma^2 t.$$

- One can also show that

$$\text{corr}(Y_t, Y_{t-k}) = \sqrt{(t-k)/t}.$$

# Properties of the Random Walk

- In words:
  - The effect of a “shock”  $U_t$  is permanent;  $U_t$  is in all future values  $Y_s, s \geq t$ , whereas for a white noise process,  $U_t$  only affects  $Y_t$ .
  - The variance increases over time, because we add up more and more of the  $U_t$ , all of which are random.
  - The correlogram decreases slowly, approximately linearly (see also `simulation.xlsx`).
- We say that a random walk is not *mean reverting*; one can show that it will (eventually) hit each and every level  $L$ , and its excursions can take arbitrarily long.

# Outline

- 1 Asset Returns
- 2 Stochastic Processes
- 3 The Efficient Market Hypothesis
- 4 The Autocorrelation Function
- 5 The Random Walk
- 6 Stationary and Integrated Processes**
- 7 Epilogue

# Stationary Processes

- Earlier, we rejected the null that the returns on the S&P500 are white noise (although they are close).
- This also implies that the log stock prices are not (exactly) random walks.
- This means that we need to generalize these concepts to allow for other types of stochastic process.
- Specifically, instead of pure white noise, we will consider *stationary processes*.
- Similarly, we generalize the concept of a random walk to *integrated processes*.
- Specific instances of these processes (ARMA and ARIMA models, respectively) will be considered in Weeks 3 and 4.

# Stationarity

## Definition

A process  $\{y_t\}$  is called *weakly stationary* (or second-order, covariance stationary) if the first two moments are time-invariant:

$$\mathbb{E}[Y_t] = \mu, \quad \text{var}(Y_t) = \gamma_0 = \sigma^2, \quad \text{cov}(Y_t, Y_{t-s}) = \gamma_s, \quad t \in \{0, \dots, T\}, s \geq 1.$$

- This means that the mean, variance, and autocovariances (or autocorrelations) do not change over time; i.e., the autocovariance  $\gamma_s$  depends only on the lag  $s$ , not on  $t$ .
- Intuitively, there should not be a significant difference if I calculate the mean, variance, and ACF from the first or second half of the sample.
- White noise is one example of a *stationary* process.
- As we saw, the random walk is not stationary; its variance changes over time.

# Integrated Processes

- Recall that if returns are white noise, then log prices follow a random walk:

$$\log P_t = \log P_{t-1} + U_t$$

- Alternatively, if log prices follow a random walk, then returns are white noise:

$$r_t = \log P_t - \log P_{t-1} = U_t$$

- We write  $\Delta \log P_t$  for  $\log P_t - \log P_{t-1}$ .
- So a process  $Y_t$  is a random walk if  $\Delta Y_t$  is white noise.



# Integrated Processes

- We saw above that for the S&P500, we did not get white noise after differencing, but some other stationary process.
- Such processes are called *integrated*.

## Definition

A process  $Y_t$  is called *integrated* of order 1, or  $I(1)$ , if it is non-stationary itself, but  $\Delta Y_t = Y_t - Y_{t-1}$  is stationary.

- The random walk is the simplest example of an  $I(1)$  process.
- A stationary process is also called  $I(0)$ .
- An  $I(2)$  process would need to be differenced twice to be stationary, but this is rarely necessary in practice.

# Properties of Integrated Processes

- Integrated processes have correlograms that stay close to one, which *die out very slowly*.
- An informal way to check whether the stationarity assumption is reasonable is by inspecting the graph and the correlogram of the series. If the graph displays a tendency to revert to a constant mean, with a more or less constant variance, and the correlogram converges to zero *exponentially fast*, then stationarity may be assumed. A formal test will be introduced later.
- Besides prices, many financial and economic time series (e.g., GDP) do not seem to be stationary, because they display a trending mean, and a variance that increases with the level of the process. The latter phenomenon is usually dealt with by a log-transformation, but then quite often the series is still not stationary.

## Example: ACF of S&P500 returns and log prices

Date: 10/28/22 Time: 18:32

Sample (adjusted): 10/30/2012 10/27/2022

Included observations: 2546 after adjustments

Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob
		1 -0.146	-0.146	54.007	0.000
		2 0.092	0.072	75.647	0.000
		3 -0.016	0.008	76.268	0.000
		4 -0.052	-0.061	83.165	0.000
		5 0.043	0.029	87.853	0.000
		6 -0.111	-0.095	119.37	0.000
		7 0.155	0.126	180.37	0.000
		8 -0.142	-0.098	231.97	0.000
		9 0.137	0.096	280.02	0.000
		10 -0.063	-0.033	290.04	0.000
		11 0.021	0.016	291.19	0.000
		12 0.016	-0.003	291.82	0.000
		13 -0.086	-0.048	310.65	0.000
		14 0.055	-0.007	318.34	0.000
		15 -0.080	-0.017	334.82	0.000
		16 0.070	0.011	347.37	0.000
		17 -0.020	0.026	348.35	0.000
		18 0.013	-0.011	348.77	0.000
		19 -0.004	-0.009	348.81	0.000
		20 -0.024	-0.006	350.23	0.000

Date: 11/03/22 Time: 15:29

Sample: 10/29/2012 10/27/2022

Included observations: 2547

Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob
		1 0.998	0.998	2541.7	0.000
		2 0.997	0.050	5077.1	0.000
		3 0.995	-0.037	7605.3	0.000
		4 0.994	0.017	10127.	0.000
		5 0.992	0.020	12642.	0.000
		6 0.991	-0.004	15151.	0.000
		7 0.990	0.041	17654.	0.000
		8 0.988	-0.075	20150.	0.000
		9 0.987	0.039	22640.	0.000
		10 0.985	-0.028	25123.	0.000
		11 0.984	0.003	27599.	0.000
		12 0.982	0.002	30069.	0.000
		13 0.980	-0.017	32532.	0.000
		14 0.979	0.018	34988.	0.000
		15 0.977	-0.010	37438.	0.000
		16 0.976	0.017	39881.	0.000
		17 0.974	-0.022	42318.	0.000
		18 0.973	-0.000	44748.	0.000
		19 0.971	0.011	47171.	0.000
		20 0.970	0.027	49588.	0.000

## Note: Partial Autocorrelation Function

- The EViews output also shows the (sample) *partial autocorrelation function* ((S)PACF)  $\hat{\tau}_{kk}$ ,  $k = 1, 2, \dots$ , where  $\hat{\tau}_{kk}$  is the OLS estimator of  $\tau_{kk}$  in the regression

$$y_t = \alpha + \tau_{k1}y_{t-1} + \dots + \tau_{kk}y_{t-k} + e_t.$$

*Note:* this is not the model for  $y_t$ , just a regression to estimate  $\tau_{kk}$ !

- The PACF measures the correlation between  $y_t$  and  $y_{t-k}$ , *controlling* for the effect of the intermediate lag. I.e.,  $\tau_{kk}$  only measures the *direct* effect of  $y_{t-k}$  on  $y_t$ .
- For a random walk, it drops to zero after the first lag, because only  $y_{t-1}$  has a direct effect.
- For a stationary process, the ACF and PACF converge to zero at a geometric (exponential) rate as  $k$  increases.
- If the sample ACF and PACF of a time series do not seem to converge at all, or too slowly (linearly), then this is an indication of nonstationarity.

# More Properties of Integrated Series

- *No mean-reversion*. Like the random walk, I(1) processes do not revert to a mean.
- *Persistence of shocks*. Also, the effect of past shocks  $u_{t-i}$  does not die out, whereas for stationary series the effect will decay exponentially. Important for economic policy.
- *Increasing forecast intervals*. For I(0) time series, the long-run 95% forecast interval converges to the unconditional mean  $\pm$  twice the unconditional standard deviation. For an I(1) process the forecast variance does not converge, so forecasts intervals keep increasing.
- *Spurious regressions*. When regressing two integrated time series onto each other, the  $R^2$  and  $t$ -statistic may become very large even if they are totally independent. This is avoided if we regress  $\Delta y_t$  on  $\Delta x_t$ .
- *Asymptotic properties of estimators and tests*. In regressions with I(1) variables, the usual statistical theory breaks down (asymptotic normality of estimators,  $t$ -tests, etc).

# Outline

- 1 Asset Returns
- 2 Stochastic Processes
- 3 The Efficient Market Hypothesis
- 4 The Autocorrelation Function
- 5 The Random Walk
- 6 Stationary and Integrated Processes
- 7 Epilogue**

# Learning Goals

## Students

- know the definitions of simple and log returns,
- know the definition of white noise,
- understand the ACF and PACF and their sample analogs,
- are able to use the correlogram and  $Q$ -statistics to test if a series was generated by a white noise process, and
- are able to distinguish stationary and integrated processes.

# Homework

- Exercise 2
- Questions 9b and 12b from Chapter 6 of Brooks (2019)