Solution to Exercise 3

Simon A. Broda

1. (a) We clearly see that unless $|\phi_1|$ approaches 1, the process is stationary; the time series plot looks mean-reverting, and the sample autocorrelations decay exponentially as they should. We also see that $\bar{y}$ is close to $\mathbb{E}[Y_t] = \alpha/(1 - \phi_1)$, and that s_y^2 is close to $\text{var}[Y_t] = \sigma^2/(1 - \phi_1^2)$.

   (b) If $\phi_1 = 1$, we have a random walk, and α becomes the drift: $\mathbb{E}[Y_t] = Y_0 + \alpha \cdot t$.

   (c) The correlograms of the AR(1), MA(1), and ARMA(1, 1) look respectively as follows.

Date: 11/10/22 Time: 18:49
Sample: 1 1000
Included observations: 1000

	AC	PAC	Q-Stat	Prob
1	0.684	0.684	469.49	0.000
2	0.454	-0.026	676.61	0.000
3	0.285	-0.030	758.29	0.000
4	0.217	0.075	805.59	0.000
5	0.172	0.014	835.50	0.000
6	0.107	-0.054	847.09	0.000
7	0.055	-0.013	850.13	0.000
8	-0.007	-0.058	850.18	0.000
9	-0.051	-0.036	852.83	0.000
10	-0.088	-0.040	860.62	0.000
11	-0.127	-0.063	877.09	0.000
12	-0.113	0.040	889.93	0.000
13	-0.115	-0.039	903.34	0.000
14	-0.079	0.044	909.72	0.000
15	-0.059	0.003	913.28	0.000
16	-0.013	0.058	913.45	0.000
17	-0.013	-0.046	913.62	0.000
18	-0.012	0.001	913.76	0.000
19	-0.005	0.004	913.78	0.000
20	0.021	0.030	914.25	0.000

Date: 11/10/22 Time: 18:50
Sample: 1 1000
Included observations: 1000

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.499	0.499	249.45	0.000
		2	0.062	-0.249	253.29	0.000
		3	0.053	0.197	256.13	0.000
		4	0.041	-0.103	257.81	0.000
		5	-0.001	0.041	257.81	0.000
		6	-0.017	-0.040	258.11	0.000
		7	0.034	0.086	259.26	0.000
		8	0.073	0.010	264.61	0.000
		9	0.065	0.037	268.89	0.000
		10	0.040	-0.009	270.51	0.000
		11	0.049	0.050	272.96	0.000
		12	0.066	0.018	277.42	0.000
		13	0.039	-0.002	278.95	0.000
		14	0.015	0.008	279.20	0.000
		15	0.011	-0.006	279.31	0.000
		16	-0.020	-0.041	279.71	0.000
		17	-0.042	-0.014	281.53	0.000
		18	-0.031	-0.009	282.54	0.000
		19	-0.040	-0.043	284.16	0.000
		20	-0.066	-0.040	288.59	0.000

Date: 11/10/22 Time: 18:51
Sample: 1 1000
Included observations: 1000

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.833	0.833	695.61	0.000
		2	0.555	-0.452	1004.8	0.000
		3	0.361	0.272	1135.7	0.000
		4	0.242	-0.140	1194.8	0.000
		5	0.181	0.146	1227.9	0.000
		6	0.142	-0.107	1248.3	0.000
		7	0.103	0.050	1259.0	0.000
		8	0.071	-0.027	1264.1	0.000
		9	0.049	0.018	1266.5	0.000
		10	0.039	0.012	1268.0	0.000
		11	0.039	0.009	1269.5	0.000
		12	0.045	0.024	1271.5	0.000
		13	0.061	0.046	1275.3	0.000
		14	0.077	-0.008	1281.3	0.000
		15	0.079	0.000	1287.6	0.000
		16	0.071	0.003	1292.8	0.000
		17	0.056	-0.017	1295.9	0.000
		18	0.039	0.005	1297.5	0.000
		19	0.029	0.001	1298.3	0.000
		20	0.027	0.018	1299.1	0.000

For the AR(1), we see that the SACF decays geometrically, while the SPACF drops to zero (more or less) after 1 lag. For the MA(1), we see that the picture is reversed (the fact that the sign of the SPACF alternates does not play a role, as long as its absolute value decays geometrically). For the ARMA(1, 1), both SACF and SPACF decay geometrically, so it's impossible to determine the order of an ARMA(p, q) process (here, $p = 1$ and $q = 1$) from the correlogram.

2. (a) The correlogram looks as shown below.

Date: 11/10/22 Time: 17:06
Sample (adjusted): 1981Q1 2022Q2
Included observations: 166 after adjustments

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.695	0.695	81.608	0.000
		2	0.446	-0.072	115.39	0.000
		3	0.228	-0.105	124.27	0.000
		4	-0.048	-0.286	124.66	0.000
		5	-0.051	0.284	125.11	0.000
		6	-0.068	-0.077	125.91	0.000
		7	-0.120	-0.122	128.45	0.000
		8	-0.160	-0.206	132.96	0.000
		9	-0.149	0.220	136.91	0.000
		10	-0.135	-0.048	140.19	0.000
		11	-0.124	-0.102	142.96	0.000
		12	-0.100	-0.136	144.76	0.000
		13	-0.097	0.135	146.48	0.000
		14	-0.070	0.030	147.38	0.000
		15	-0.045	-0.092	147.76	0.000
		16	-0.035	-0.120	147.99	0.000
		17	-0.031	0.072	148.17	0.000
		18	-0.052	-0.003	148.68	0.000
		19	-0.094	-0.161	150.34	0.000
		20	-0.126	-0.127	153.37	0.000

Geometrically decaying ACF, PACF drops to zero after one lag, even though some later values are significant. Still, a simple AR(1) might suffice. We can estimate it by entering `gdp_growth c ar(1)` under `Quick`→ `Estimate Equation...`. The result is

Dependent Variable: GDP_GROWTH
Method: ARMA Maximum Likelihood (OPG - BHHH)
Date: 11/10/22 Time: 17:10
Sample: 1981Q1 2022Q2
Included observations: 166
Convergence achieved after 18 iterations
Coefficient covariance computed using outer product of gradients

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.017264	0.003884	4.444747	0.0000
AR(1)	0.691619	0.042082	16.43513	0.0000
SIGMASQ	0.000206	8.10E-06	25.38758	0.0000

R-squared	0.483435	Mean dependent var	0.017256
Adjusted R-squared	0.477096	S.D. dependent var	0.020010
S.E. of regression	0.014469	Akaike info criterion	-5.611745
Sum squared resid	0.034126	Schwarz criterion	-5.555504
Log likelihood	468.7748	Hannan-Quinn criter.	-5.588916
F-statistic	76.27284	Durbin-Watson stat	1.892005
Prob(F-statistic)	0.000000		

Inverted AR Roots	.69

Everything is significant, and the estimated model is stationary (AR coefficient is less than 1 in absolute value). The residual correlogram (under `View`→`Residual Diagnostics...` looks like this:

Date: 11/10/22 Time: 17:13
Sample (adjusted): 1981Q1 2022Q2
Q-statistic probabilities adjusted for 1 ARMA term

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.052	0.052	0.4549	
		2	0.042	0.039	0.7559	0.385
		3	0.119	0.115	3.1750	0.204
		4	-0.378	-0.398	27.779	0.000
		5	0.021	0.078	27.854	0.000
		6	0.034	0.052	28.053	0.000
		7	-0.042	0.052	28.357	0.000
		8	-0.091	-0.318	29.819	0.000
		9	-0.043	0.035	30.151	0.000
		10	-0.023	0.047	30.249	0.000
		11	-0.040	0.023	30.543	0.001
		12	0.012	-0.210	30.569	0.001
		13	-0.051	-0.036	31.051	0.002
		14	-0.011	0.073	31.073	0.003
		15	0.011	0.039	31.097	0.005
		16	0.001	-0.137	31.097	0.009
		17	0.030	-0.030	31.266	0.012
		18	0.019	0.094	31.336	0.018
		19	-0.029	-0.012	31.490	0.025
		20	-0.075	-0.215	32.563	0.027

The ACF and PACF at lag 4 are both significant. It's not obvious which model to use for this. One idea is to use an ARMA(1, 4), which we can estimate by entering `gdp_growth c ar(1) ma(1 to 4)` in the estimation window, resulting in the estimation output and residual correlogram below.

Dependent Variable: GDP_GROWTH
Method: ARMA Maximum Likelihood (OPG - BHHH)
Date: 11/10/22 Time: 17:17
Sample: 1981Q1 2022Q2
Included observations: 166
Convergence not achieved after 500 iterations
Coefficient covariance computed using outer product of gradients

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.017535	0.000617	28.43690	0.0000
AR(1)	0.886003	0.051325	17.26249	0.0000
MA(1)	-0.064953	10.31694	-0.006296	0.9950
MA(2)	0.072189	24.96029	0.002892	0.9977
MA(3)	-0.079997	9.575807	-0.008354	0.9933
MA(4)	-0.927237	416.0201	-0.002229	0.9982
SIGMASQ	0.000122	0.003757	0.032452	0.9742

R-squared	0.693653	Mean dependent var	0.017256
Adjusted R-squared	0.682093	S.D. dependent var	0.020010
S.E. of regression	0.011282	Akaike info criterion	-6.019585
Sum squared resid	0.020238	Schwarz criterion	-5.888356
Log likelihood	506.6255	Hannan-Quinn criter.	-5.966318
F-statistic	60.00317	Durbin-Watson stat	2.049757
Prob(F-statistic)	0.000000		

Inverted AR Roots	.89			
Inverted MA Roots	1.00	-.00-1.00i	-.00+1.00i	-.93

Date: 11/10/22 Time: 17:21
Sample (adjusted): 1981Q1 2022Q2
Q-statistic probabilities adjusted for 5 ARMA terms

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.026	-0.026	0.1172	
		2	-0.035	-0.036	0.3237	
		3	0.105	0.104	2.2167	
		4	0.038	0.043	2.4661	
		5	-0.016	-0.007	2.5086	
		6	0.052	0.044	2.9845	0.084
		7	0.025	0.019	3.0950	0.213
		8	-0.066	-0.063	3.8743	0.275
		9	-0.056	-0.069	4.4363	0.350
		10	0.027	0.011	4.5672	0.471
		11	-0.016	-0.006	4.6112	0.595
		12	0.005	0.022	4.6154	0.707
		13	-0.067	-0.070	5.4282	0.711
		14	0.006	0.009	5.4353	0.795
		15	0.003	0.007	5.4367	0.860
		16	-0.011	-0.001	5.4595	0.907
		17	-0.001	-0.005	5.4596	0.941
		18	-0.014	-0.020	5.4985	0.963
		19	-0.019	-0.012	5.5697	0.976
		20	-0.044	-0.044	5.9329	0.981

The residual correlogram looks fine now, but note that all the MA coefficients are insignificant. Maybe we can get away with dropping the first 3, leading to the *subset AR model*

$$Y_t = \alpha + \phi_1 Y_{t-1} + U_t + \theta_4 U_{t-4}.$$

This can be estimated by entering `gdp_growth c ar(1) ma(4)` in the estimation window. The results are shown below.

Dependent Variable: GDP_GROWTH
Method: ARMA Maximum Likelihood (OPG - BHHH)
Date: 11/10/22 Time: 17:26
Sample: 1981Q1 2022Q2
Included observations: 166
Convergence achieved after 32 iterations
Coefficient covariance computed using outer product of gradients

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.017502	0.001291	13.55181	0.0000
AR(1)	0.867376	0.024798	34.97700	0.0000
MA(4)	-0.846949	0.054300	-15.59769	0.0000
SIGMASQ	0.000132	5.58E-06	23.64866	0.0000

R-squared	0.668602	Mean dependent var	0.017256
Adjusted R-squared	0.662465	S.D. dependent var	0.020010
S.E. of regression	0.011625	Akaike info criterion	-6.016511
Sum squared resid	0.021893	Schwarz criterion	-5.941524
Log likelihood	503.3704	Hannan-Quinn criter.	-5.986073
F-statistic	108.9462	Durbin-Watson stat	2.106125
Prob(F-statistic)	0.000000		

Inverted AR Roots	.87			
Inverted MA Roots	.96	.00-.96i	.00+.96i	-.96

Date: 11/10/22 Time: 17:27
Sample (adjusted): 1981Q1 2022Q2
Q-statistic probabilities adjusted for 2 ARMA terms

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.054	-0.054	0.4897	
		2	0.044	0.041	0.8149	
		3	0.060	0.065	1.4274	0.232
		4	0.043	0.048	1.7454	0.418
		5	-0.039	-0.040	2.0137	0.570
		6	0.114	0.103	4.2740	0.370
		7	-0.023	-0.014	4.3679	0.498
		8	-0.058	-0.068	4.9624	0.549
		9	-0.081	-0.098	6.1174	0.526
		10	0.057	0.047	6.7027	0.569
		11	-0.043	-0.012	7.0308	0.634
		12	0.008	0.004	7.0429	0.721
		13	-0.092	-0.092	8.5734	0.661
		14	0.029	0.029	8.7322	0.726
		15	-0.020	0.013	8.8080	0.787
		16	-0.010	-0.022	8.8249	0.842
		17	-0.024	-0.029	8.9355	0.881
		18	0.007	-0.003	8.9435	0.916
		19	-0.038	-0.006	9.2194	0.933
		20	-0.045	-0.062	9.5998	0.944

The correlogram still looks fine, and all coefficients are now significant. The subset model has a smaller BIC (-5.94 vs. -5.89), so is preferred (better tradeoff between fit and parsimony). The AIC seems to prefer the larger model; this is typical.

Instead of the above manual procedure, we can automate the procedure of finding the model by using autoarma: just paste

```
freeze(armatable) gdp_growth.autoarma(diff=0
, select=sic, maxar=4, maxma=4, atable) forec c
```

into the estimation window (all on one line). This produces the table below.

Model Selection Criteria Table
Dependent Variable: GDP_GROWTH
Date: 11/10/22 Time: 17:43
Sample: 1980Q1 2022Q2
Included observations: 166

Model	LogL	AIC	BIC*	HQ
(0,3)(0,0)	502.567881	-5.994794	-5.901059	-5.956746
(1,4)(0,0)	506.625533	-6.019585	-5.888356	-5.966318
(1,3)(0,0)	503.415183	-5.992954	-5.880473	-5.947297
(2,3)(0,0)	504.232016	-5.990747	-5.859519	-5.937481
(2,4)(0,0)	506.761916	-6.009180	-5.859204	-5.948304
(3,3)(0,0)	504.562277	-5.982678	-5.832703	-5.921802
(0,4)(0,0)	497.181159	-5.917845	-5.805364	-5.872188
(4,3)(0,0)	504.564472	-5.970656	-5.801934	-5.902171
(3,4)(0,0)	504.231623	-5.966646	-5.797924	-5.898161
(4,4)(0,0)	500.953990	-5.915108	-5.727639	-5.839013
(3,2)(0,0)	489.173962	-5.809325	-5.678096	-5.756058
(2,2)(0,0)	484.459400	-5.764571	-5.652090	-5.718914
(4,1)(0,0)	486.270962	-5.774349	-5.643121	-5.721082
(2,1)(0,0)	476.307048	-5.678398	-5.584664	-5.640351
(4,2)(0,0)	482.783828	-5.720287	-5.570312	-5.659411
(4,0)(0,0)	477.086618	-5.675742	-5.563261	-5.630085
(1,2)(0,0)	474.444937	-5.655963	-5.562229	-5.617916
(1,0)(0,0)	468.774819	-5.611745	-5.555504	-5.588916
(2,0)(0,0)	469.185600	-5.604646	-5.529658	-5.574208
(1,1)(0,0)	469.105705	-5.603683	-5.528696	-5.573245
(3,1)(0,0)	474.139810	-5.640239	-5.527757	-5.594582
(3,0)(0,0)	470.120246	-5.603858	-5.510124	-5.565811
(0,1)(0,0)	460.483380	-5.511848	-5.455607	-5.489019
(0,2)(0,0)	460.901021	-5.504832	-5.429844	-5.474394
(0,0)(0,0)	414.274265	-4.967160	-4.929666	-4.951941

We see that the BIC selects a MA(3) model[1], but it has a higher BIC than our subset model, because `autoarma` doesn't consider subset models. Estimating this model via

```
gdp_growth c ma(1 to 3)
```

results in the output below.

[1]The AIC selects an ARMA(1, 4)

Dependent Variable: GDP_GROWTH
Method: ARMA Maximum Likelihood (OPG - BHHH)
Date: 11/10/22 Time: 17:47
Sample: 1981Q1 2022Q2
Included observations: 166
Convergence achieved after 106 iterations
Coefficient covariance computed using outer product of gradients

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.017386	0.004378	3.971259	0.0001
MA(1)	0.887055	0.033045	26.84407	0.0000
MA(2)	0.897620	0.063248	14.19207	0.0000
MA(3)	0.857028	0.038752	22.11553	0.0000
SIGMASQ	0.000134	5.66E-06	23.57846	0.0000

R-squared	0.664468	Mean dependent var	0.017256
Adjusted R-squared	0.656132	S.D. dependent var	0.020010
S.E. of regression	0.011734	Akaike info criterion	-5.994794
Sum squared resid	0.022166	Schwarz criterion	-5.901059
Log likelihood	502.5679	Hannan-Quinn criter.	-5.956746
F-statistic	79.70872	Durbin-Watson stat	2.075441
Prob(F-statistic)	0.000000		

Inverted MA Roots	.02-.96i	.02+.96i	-.92

Date: 11/10/22 Time: 17:49
Sample (adjusted): 1981Q1 2022Q2
Q-statistic probabilities adjusted for 3 ARMA terms

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.038	-0.038	0.2491	
		2	-0.060	-0.062	0.8630	
		3	0.026	0.021	0.9778	
		4	-0.032	-0.034	1.1579	0.282
		5	0.000	0.000	1.1579	0.560
		6	0.047	0.042	1.5389	0.673
		7	-0.034	-0.029	1.7367	0.784
		8	-0.107	-0.106	3.7447	0.587
		9	-0.063	-0.079	4.4586	0.615
		10	0.024	0.010	4.5637	0.713
		11	-0.033	-0.038	4.7629	0.783
		12	-0.013	-0.021	4.7950	0.852
		13	-0.070	-0.080	5.6764	0.842
		14	0.008	0.009	5.6875	0.893
		15	0.000	-0.012	5.6875	0.931
		16	-0.015	-0.032	5.7301	0.955
		17	0.009	-0.012	5.7438	0.973
		18	-0.005	-0.011	5.7486	0.984
		19	-0.016	-0.018	5.7945	0.990
		20	-0.044	-0.067	6.1640	0.992

This looks fine too. I prefer to stick with our subset model, because it has a lower BIC.

(b) The estimated parameter c in EViews corresponds to $c = \mathbb{E}[Y_t] = \alpha/(1-\phi_1)$, so we have $\hat{\alpha} = \hat{c}(1-\hat{\phi}_1) = 0.017502 \cdot (1 - 0.867376) = 0.002321$. Thus, our final model is

$$Y_t = 0.002321 + 0.867376 Y_{t-1} + U_t - 0.846949 U_{t-4}.$$

The manual forecast for 2022Q3 is therefore

$$\begin{aligned}\widehat{y}_{t+1} &= 0.002321 + 0.867376 \cdot 0.024037 - 0.846949 \cdot 0.001429 \\ &= 0.021960.\end{aligned}$$

The value for y_{2022Q2}, 0.024037, can be obtained from the spreadsheet view. The value 0.001429 corresponds to $\hat{u}_{2021Q3}$. To find it, go to the estimation output, click on `Proc→Make Residual Series...`, and open the resulting series in spreadsheet view.

The manual forecast for 2022Q4 can be constructed analogously. It requires y_{2022Q3}, which we replace with our forecast from the previous question. Hence

$$\begin{aligned}\widehat{y}_{t+2} &= 0.002321 + 0.867376 \cdot 0.021960 - 0.846949 \cdot (-0.003883)\\ &= 0.024657,\end{aligned}$$

where $-0.003883 = \hat{u}_{2021Q4}$. The same forecasts can be obtained using EViews. First, inside the workfile pane on the left, go to `Proc→ Structure / Resize Current Page...`, and resize the file so that it includes 2022Q3 and 2022Q4. Next, in the pane with the estimation output, click on `Forecast`. Keep the default of a dynamic forecast, and set the forecast sample to 2022Q3:2022Q4. Your forecast will be saved as a new series `gdp_growthf` You can open it in spreadsheet view and confirm that the forecasts are the same as those obtained above,

3. (a) By repeatedly plugging in,

$$\begin{aligned}Y_t &= \alpha + Y_{t-1} + U_t\\ &= \alpha + (\alpha + Y_{t-2} + U_{t-1}) + U_t\\ &\vdots\\ &= Y_0 + \alpha \cdot t + \sum_{s=1}^{t} U_s,\end{aligned}$$

so that

$$\mathbb{E}[Y_t] = Y_0 + \alpha \cdot t,$$

because white noise has expectation zero. The derivation of the variance is the same as for the case without drift from last week and thus omitted here.

(b) The previous question shows that the random walk with drift is not stationary, because its mean and variance change over time. For it to be I(1), its first difference ΔY_t should be stationary. We immediately se that $\Delta Y_t = Y_t - Y_{t-1} = (\alpha + Y_{t-1} + U_t) - Y_{t-1} = \alpha + U_t$. This is just white noise plus a constant, which is stationary.

(c) Since $\{U_t\}$ is white noise, U_t is uncorrelated with Y_{t-1}, so

$$\begin{aligned}\mathrm{var}(Y_t) &= \mathrm{var}\left(\alpha + \phi_1 Y_{t-1} + U_t\right)\\ &= \phi_1^2 \mathrm{var}(Y_{t-1}) + \mathrm{var}(U_t) + 2\phi_1 \mathrm{cov}(Y_{t-1}, U_t) \qquad = \phi_1^2 \mathrm{var}(Y_t) + \sigma^2,\end{aligned}$$

where the final equality holds because Y_t is stationary, which implies that $\mathrm{var}(Y_t) = \mathrm{var}(Y_{t-1})$. Thus, if and only if $|\phi_1| < 1$,

$$\mathrm{var}(Y_t) = \frac{\sigma^2}{1 - \phi_1^2}.$$

Note that $\mathrm{var}(Y_t) > \mathrm{var}(Y_{t-1})$ if $|\phi_1| \geq 1$, i.e., the variance grows without bounds in that case.

(d) For the MA(1) process

$$Y_t = \alpha + U_t + \theta_1 U_{t-1},$$

we have that

$$\begin{aligned}\mathbb{E}[Y_t] &= \mathbb{E}[\alpha + U_t + \theta_1 U_{t-1}]\\ &= \alpha + \mathbb{E}[U_t] + \theta_1 \mathbb{E}[U_{t-1}]\\ &= \alpha.\end{aligned}$$

For the variance,

$$\begin{aligned}\gamma_0 &= \operatorname{var}(Y_t) = \operatorname{var}(\alpha + U_t + \theta_1 U_{t-1})\\ &= \operatorname{var}(U_t + \theta_1 U_{t-1})\\ &= \operatorname{var}(U_t) + \theta_1^2 \operatorname{var}(U_{t-1}) + 2\theta_1 \operatorname{cov}(U_t, U_{t-1})\\ &= \sigma^2 + \theta_1^2\sigma^2 + 0\\ &= \sigma^2(1+\theta_1^2).\end{aligned}$$

For the first autocovariance,

$$\begin{aligned}\gamma_1 &= \operatorname{cov}(Y_t, Y_{t-1})\\ &= \operatorname{cov}(\alpha + U_t + \theta_1 U_{t-1}, \alpha + U_{t-1} + \theta_1 U_{t-2}) && (\dagger)\\ &= \operatorname{cov}(\theta_1 U_{t-1}, U_{t-1})\end{aligned}$$

because white noise is uncorrelated. Hence

$$\begin{aligned}\gamma_1 &= \theta_1 \operatorname{cov}(U_{t-1}, U_{t-1})\\ &= \theta_1 \operatorname{var}(U_{t-1})\\ &= \theta_1\sigma^2.\end{aligned}$$

Higher order autocorrelations will be zero, because there will no common U_t terms in (†). Plugging these into the definition of the ACF, we have

$$\tau_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta_1\sigma^2}{\sigma^2(1+\theta_1^2)} = \frac{\theta_1}{1+\theta_1^2}.$$

(e) **Optional**: The ACF is obtained by repeatedly substituting $Y_{t-i} = \phi_1 Y_{t-i-1} + \alpha + U_{t-i}$:

$$\begin{aligned}Y_t &= \phi_1 Y_{t-1} + \alpha + U_t\\ &= \phi_1^2 Y_{t-2} + \phi_1(\alpha + U_{t-1}) + \alpha + U_t\\ &= \phi_1^3 Y_{t-3} + \phi_1^2(\alpha + U_{t-2}) + \phi_1(\alpha + U_{t-1}) + \alpha + U_t\\ &\vdots\\ &= \phi_1^k Y_{t-k} + \sum_{i=0}^{k-1}\phi_1^i\alpha + \sum_{i=0}^{k-1}\phi_1^i U_{t-i}. && (1)\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma_k &= \mathrm{cov}(Y_t, Y_{t-k}) = \phi_1^k \mathrm{cov}(Y_{t-k}, Y_{t-k}) + \sum_{i=0}^{k-1} \phi_1^i \mathrm{cov}(U_{t-i}, Y_{t-k}) \\ &= \phi_1^k \mathrm{var}(Y_{t-k}),\end{aligned}$$

so that

$$\tau_k = \frac{\gamma_k}{\gamma_0} = \phi_1^k.$$