Module 9.3: Time Series Analysis with Python
Fall Term 2023

Week 7:

Cointegration

HSLU Lucerne University of Applied Sciences and Arts

Outline in Weeks

1. Introduction; Descriptive Modeling
2. Returns; Autocorrelation; Stationarity
3. ARMA Models
4. Unit Roots; ARIMA Models
5. Volatility Modeling
6. Value at Risk
7. Cointegration

Outline

Cointegration and Common Trends

Suppose we have two time series Y_t and X_t, which are both $I(1)$, and we analyze a regression model of the form

$$Y_t = \beta_1 + \beta_2 X_t + U_t.$$

Here U_t has mean zero but may display autocorrelation. Two cases:

- $U_t \sim I(1)$: if U_t displays no mean-reversion, then Y_t does not revert to the explained part $\beta_1 + \beta_2 X_t$. Even if $\beta_2 = 0$, its t-statistic and R^2 will often seem significant (*spurious regressions*). To avoid this, one should estimate a model in differences, i.e., $\Delta Y_t = a_1 + a_2 \Delta X_t + \Delta U_t$.
- $U_t \sim I(0)$: now Y_t and X_t have a *common* stochastic trend, such that the linear combination $Y_t - \beta_2 X_t$ does not have a trend. This is called *cointegration*.

Example

- Consider the model

$$Y_t = \beta_1 + \beta_2 X_t + U_{1,t}$$
$$X_t = X_{t-1} + U_{2,t}$$

  where $\beta_2 \neq 0$, $U_{1,t}, U_{2,t} \overset{\text{iid}}{\sim} (0, \sigma^2)$ independently of each other.
- X_t is a random walk and thus nonstationary. Y_t contains X_t and is thus also nonstationary. But

$$Y_t - \beta_2 X_t = \beta_1 + U_{1,t}$$

  is stationary: the RHS is white noise plus a constant.
- $(1, -\beta_2)$ is called the *cointegrating vector*.

Cointegration and Common Trends, contd.

- The concept is easily extended to more than two series: if $X_{2t}, \ldots, X_{kt}$ are all $I(1)$ variables, and
$$Y_t = \beta_1 + \beta_2 X_{2t} + \ldots + \beta_k X_{kt} + U_t,$$
then this is a spurious regression if $U_t \sim I(1)$ (and $\beta_i = 0$), and a cointegrating relation if U_t is stationary.
- In other words, cointegration between k integrated series means that there exists a *linear combination*[1] of them which is stationary.
- Examples of possibly cointegrated time series:
  - exchange rates and relative prices (*purchasing power parity*);
  - spot and futures prices of assets or exchange rates;
  - short- and long-term interest rates (*term structure models*);
  - stock prices and dividends (*present value relations*).

[1] i.e., a weighted sum

Outline

Testing for Cointegration

- For cointegrated series, one should exploit the long-run equilibrium relationship between variables for estimation rather than differencing. Differencing would *remove* that structure.
- Engle and Granger proposed the following procedure:
  - Conduct individual unit root tests to ensure all series are $I(1)$.
  - Estimate the regression model

    $$Y_t = \beta_1 + \beta_2 X_{2t} + \ldots + \beta_k X_{kt} + U_t$$

    by ordinary least-squares. Estimates are (super)consistent, but standard errors are wrong because series are $I(1)$.
  - Apply an ADF unit root test (with constant) to the residuals $\hat{u}_t$ from this regression. This yields a test for $H_0 : U_t \sim I(1)$ (spurious regression) against $H_1 : U_t \sim I(0)$ (cointegration). The critical values depend on k. E.g., for $k = 2$ (X_{2t} and an intercept), the 5% c.v. is -3.41.
  - If H_0 is rejected, estimate an *error correction model*.

Engle-Granger critical values

Number of series (w/o constant)	2	3	4	5	6
Critical value	-3.41	-3.80	-4.16	-4.49	-4.74

Error Correction Models

- Cointegration between Y_t and X_t implies that deviations $(Y_{t-1} - \beta_1 - \beta_2 X_{t-1})$ from the equilibrium level should be (partially) corrected in the next period, by Y_t, X_t, or both.
- This leads to a *vector error correction model* (VECM), which in the simplest form is

$$\begin{aligned} \Delta Y_t &= c_1 + \alpha_1 (Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + e_{1t}, \\ \Delta X_t &= c_2 + \alpha_2 (Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + e_{2t}, \end{aligned}$$

  where e_{1t} and e_{2t} are two white noise errors (possibly correlated), and where we expect $\alpha_1 < 0$ and/or $\alpha_2 \beta_2 > 0$.
- We might need to add lags of ΔY_t and/or ΔX_t on RHS to combat autocorrelation.
- The *Granger representation theorem* states that cointegration implies an error correction model (possibly with more lags), and vice versa; see exercises.

Engle-Granger Procedure

- The VECM

$$\begin{aligned}\Delta Y_t &= c_1 + \alpha_1(Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + e_{1t},\\ \Delta X_t &= c_2 + \alpha_2(Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + e_{2t}.\end{aligned}$$

  is estimated by replacing $Y_{t-1} - \beta_1 - \beta_2 X_{t-1}$ by OLS residual $\hat{u}_{t-1} = Y_{t-1} - \hat{\beta}_1 - \hat{\beta}_2 X_{t-1}$, and estimating α_1 and α_2 by OLS.
- Note that $\hat{u}$ is stationary, so this is a valid regression!
- If $\alpha_2 = 0$, then all correction is done by Y_t, and not by X_t. In that case it makes sense to treat X_t as exogenous and Y_t as endogenous, and consider the "single-equation" error correction model

$$\Delta Y_t = c + \alpha_1(Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + e_t.$$

- In general, both Y_t and X_t are endogenous.

Outline

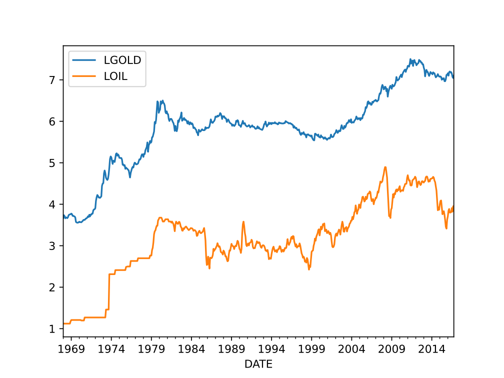
Example

- Until 1971, as part of the Bretton-Woods system of fixed exchange rates, the US dollar was convertible to gold, i.e., it was possible for foreign central banks to redeem US dollars for gold at a fixed rate of 35$ per troy ounce, so that the price of gold was fixed.
- In 1971, US president Nixon unilaterally cancelled the direct convertibility, ultimately ending the Bretton-Woods agreement.
- Gold became a floating asset, and its price increased sharply; in other words, the US$ was massively devalued.

Example continued

- We want to analyze the hypothesis that the increasing price (in US$) of oil is not a consequence of an increased demand for (or a reduced supply of) oil, but rather of a continued devaluation of the US$.
- We have at our disposal monthly data from April 1968 to January 2017 (586 observations) on the following variables:
  - gold_t, the spot price of one troy ounce of gold in US$;
  - oil_t, the spot price of one barrel of WTI crude oil in US$.
- Idea: if the relative price of oil expressed in units of gold $\mathrm{oil}_t/\mathrm{gold}_t$ is stationary, then this implies that $\log(\mathrm{oil}_t) - \log(\mathrm{gold}_t)$ is stationary, so that $\log(\mathrm{oil}_t)$ and $\log(\mathrm{gold}_t)$ must be cointegrated if the individual series are integrated.

$\log(\text{gold}_t)$ and $\log(\text{oil}_t)$



Example continued

Step 1 Test that the variables are integrated (ADF test with constant and trend)

```
ADF, p, crits, res = tsa.stattools.adfuller(df["LOIL"], regression='ct', autolag='AIC', store=True)
print("ADF = ", ADF, "\np   = ", p)
```

```
ADF =  -2.5524224546607397
p   =  0.3022670842559252
```

```
ADF, p, crits, res = tsa.stattools.adfuller(df["LGOLD"], regression='ct', autolag='AIC', store=True)
print("ADF = ", ADF, "\np   = ", p)
```

```
ADF =  -2.559158224027972
p   =  0.29904292389348935
```

Neither test rejects, so the series are I(1).

Example continued

Step 2 Estimate long-run relationship $\mathrm{loil}_t = \beta_1 + \beta_2 \mathrm{lgold}_t + U_t$

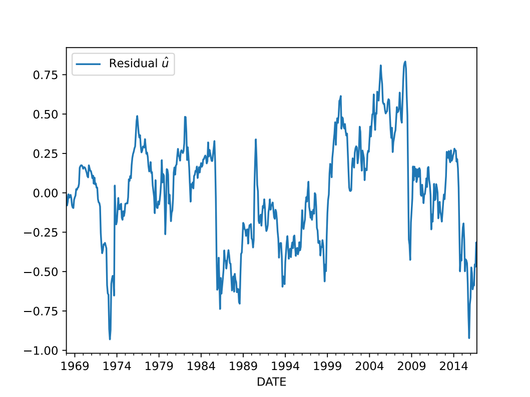
Estimated long run relationship

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                   LOIL   R-squared:                       0.873
Model:                            OLS   Adj. R-squared:                  0.873
Method:                 Least Squares   F-statistic:                     4015.
Date:                Wed, 11 Oct 2023   Prob (F-statistic):         6.99e-264
Time:                        15:03:24   Log-Likelihood:                -183.59
No. Observations:                 586   AIC:                             371.2
Df Residuals:                     584   BIC:                             379.9
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept     -2.2590      0.086    -26.167      0.000      -2.429      -2.089
LGOLD          0.9262      0.015     63.364      0.000       0.898       0.955
==============================================================================
Omnibus:                        4.994   Durbin-Watson:                   0.074
Prob(Omnibus):                  0.082   Jarque-Bera (JB):                4.313
Skew:                          -0.133   Prob(JB):                        0.116
Kurtosis:                       2.675   Cond. No.                         38.3
==============================================================================
```

The cointegrating vector is $(1, -\beta_2) = (1, -0.926)$ (if the Engle-Granger test rejects).
Careful: standard errors are wrong, because variables are $I(1)$.

Example continued

Residuals (=equilibrium error) $\hat{u}_t$



Example continued

Step 3 Apply ADF test (with intercept) to `u`.

```
ADF, p, crits, res = tsa.stattools.adfuller(u, regression='c', autolag='AIC', store=True)
print("ADF = ", ADF)
```

```
ADF =  -3.7247713389879844
```

Careful: we need to use the Engle-Granger 5% critical value of -3.41, **not** the one given by `adfuller` (nor the p-value). Conclusion: test rejects the null of “no cointegration”.

Example continued

Step 4a Estimate the VECM equation for `loil`. Note: I threw in a lag of the dependent variable, as there was autocorrelation without it (cf. selected lag length in ADF test).

VECM equation for `loil`

```
                            OLS Regression Results
==============================================================================
Dep. Variable:          df.LOIL.diff()   R-squared:                       0.078
Model:                             OLS   Adj. R-squared:                  0.074
Method:                  Least Squares   F-statistic:                     24.46
Date:                 Wed, 11 Oct 2023   Prob (F-statistic):           6.31e-11
Time:                         16:25:01   Log-Likelihood:                 654.56
No. Observations:                  584   AIC:                            -1303.
Df Residuals:                      581   BIC:                            -1290.
Df Model:                            2
Covariance Type:             nonrobust
==============================================================================
                         coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept              0.0036      0.003      1.111      0.267      -0.003       0.010
u.shift()             -0.0350      0.010     -3.518      0.000      -0.054      -0.015
df.LOIL.diff().shift()  0.2565      0.040      6.399      0.000       0.178       0.335
==============================================================================
Omnibus:                       325.732   Durbin-Watson:                   2.007
Prob(Omnibus):                   0.000   Jarque-Bera (JB):            11453.205
Skew:                            1.828   Prob(JB):                         0.00
Kurtosis:                       24.385   Cond. No.                         12.3
==============================================================================
```

Example continued

Step 4b Estimate the VECM equation for `lgold`.

VECM equation for `lgold`

```
                            OLS Regression Results
==============================================================================
Dep. Variable:         df.LGOLD.diff()   R-squared:                       0.003
Model:                             OLS   Adj. R-squared:                  0.001
Method:                  Least Squares   F-statistic:                     1.616
Date:                 Wed, 11 Oct 2023   Prob (F-statistic):              0.204
Time:                         16:29:45   Log-Likelihood:                 850.41
No. Observations:                  585   AIC:                            -1697.
Df Residuals:                      583   BIC:                            -1688.
Df Model:                            1
Covariance Type:             nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept      0.0059      0.002      2.504      0.013       0.001       0.010
u.shift()      0.0090      0.007      1.271      0.204      -0.005       0.023
==============================================================================
Omnibus:                        69.639   Durbin-Watson:                   1.933
Prob(Omnibus):                   0.000   Jarque-Bera (JB):              356.493
Skew:                            0.368   Prob(JB):                     3.88e-78
Kurtosis:                        6.753   Cond. No.                         3.02
==============================================================================
```

The adjustment coefficient α_2 in the equation for `d(lgold)` is insignificant ($p = 0.204$). So all the adjustment is done by `loil` $\rightarrow$ single-equation ECM.

Example continued

- The final model is the single-equation ECM

$$\Delta \mathrm{loil}_t = 0.0036 - 0.035(\mathrm{loil}_{t-1} - 0.926\mathrm{lgold}_{t-1} + 2.26) + 0.25\Delta \mathrm{loil}_{t-1} + e_{1t}.$$

  In our earlier notation,

$$\Delta Y_t = c + \alpha_1(Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + \gamma \Delta Y_{t-1} + e_{1t},$$

  with $c = 0.0036$, $\alpha_1 = -0.035 < 0$ as desired, $\beta_1 = -2.26$, $\beta_2 = 0.926$, and $\gamma = 0.25$.
- Interpretation: there is an equilibrium relationship between loil and lgold. In case of a disequilibrium, loil adjusts towards the equilibrium. The adjustment amounts to 3.5% per period.

Outline

Learning Goals

Students

- Understand the concept of cointegration,
- are able to test for cointegration using the Engle-Granger procedure,
- and are able to estimate an error correction model.

Homework

- Exercise 7
- Problem 6 from Chapter 8 of Brooks (2019)