Solution to Exercise 1

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See Jupyter notebook.

Solution to Exercise 2

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1. (a) The first thing to observe is the difference between the population quantities (parameters) μ and σ, and their estimates $\bar{y}$ and s_y, which are sample quantities. The latter will generally be close to the former because of the law of large numbers, but not the same. The estimates also change every time new random numbers are drawn. The time series plot shows that the observations are randomly scattered around 0. The autocorrelations in the correlogram will mostly be insignificant, i.e., statistically indistinguishable from zero. Formally, the hypothesis being tested is $H_0 : \tau_s = 0$ vs. $H_a : \tau_s \neq 0$. Even if H_0 is true here, there is still a 5% probability that any given autocorrelation will be significant; this is called the type-1 error: the probability of rejecting the null even though it is true. A similar statement concerns the Q-statistics, which test whether the first m autocorrelations are all equal to zero[1]. **Important**: make sure to look at the formulas behind the cells and make sure you understand how they work; specifically, the correlogram, the Q-stats, and their respective critical values. You don't need to understand how the random numbers u_t themselves are generated (they use a trick called inverse transform sampling).

   (b) The time series plot looks very different from that in the other sheet, because a random walk is not mean reverting. Also, the correlogram and the Q-stats are now highly significant, so that we (correctly) reject the null that the data were generated from a white noise process. In fact, the slow and almost linear decay of the correlogram suggests (correctly) that the data were generated by an integrated process. **Important**: make sure you understand how the simulated random walk y_t is constructed, by always taking "yesterday's" value y_{t-1} and adding u_t to it.

2. See Jupyter notebook.

3. (a) By repeatedly plugging in,

$$\begin{aligned} Y_t &= Y_{t-1} + U_t \\ &= Y_{t-2} + U_{t-1} + U_t \\ &= Y_{t-3} + U_{t-2} + U_{t-1} + U_t \\ &\vdots \\ &= Y_0 + \sum_{s=1}^{t} U_s \end{aligned}$$

   as claimed.

[1]Formally, $Q(m)$ can be used to test $H_0 : \tau_1 = \tau_2 = \ldots = \tau_m = 0$.

(b) The result from the previous question implies that

$$\begin{aligned}\mathbb{E}\left[Y_t\right] &= \mathbb{E}\left[Y_0 + \sum_{s=1}^{t} U_s\right] \\ &= Y_0 + \sum_{s=1}^{t} \mathbb{E}[U_s] = Y_0.\end{aligned}$$

Here, we have used that the expectation of a sum is the sum of the expectations, together with the fact that Y_0 is assumed to be a constant, and that $\mathbb{E}[U_t] = 0$ because U_t is white noise. For the variance, recall that in general, $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y) + 2\,\mathrm{cov}(X,Y)$. But since the U_t are all independent, we have that $\mathrm{var}(U_s + U_t) = \mathrm{var}(U_s) + \mathrm{var}(U_t) + 0 = 2\sigma^2$. Thus

$$\begin{aligned}\mathrm{var}\left[Y_t\right] &= \mathrm{var}\left[Y_0 + \sum_{s=1}^{t} U_s\right] \\ &= \mathrm{var}\left[\sum_{s=1}^{t} U_s\right] \\ &= \sum_{s=1}^{t} \mathrm{var}(U_s) \\ &= \sum_{s=1}^{t} \sigma^2 \\ &= t \cdot \sigma^2.\end{aligned}$$

Solution to Exercise 3

Simon A. Broda

1. (a) We clearly see that unless $|\phi_1|$ approaches 1, the process is stationary; the time series plot looks mean-reverting, and the sample autocorrelations decay exponentially as they should. We also see that $\bar{y}$ is close to $\mathbb{E}[Y_t] = \alpha/(1-\phi_1)$, and that s_y^2 is close to $\text{var}[Y_t] = \sigma^2/(1-\phi_1^2)$.

   (b) If $\phi_1 = 1$, we have a random walk, and α becomes the drift: $\mathbb{E}[Y_t] = Y_0 + \alpha \cdot t$.

   (c) See Jupyter notebook.

2. See Jupyter notebook.

3. (a) By repeatedly plugging in,

$$\begin{aligned} Y_t &= \alpha + Y_{t-1} + U_t \\ &= \alpha + (\alpha + Y_{t-2} + U_{t-1}) + U_t \\ &\vdots \\ &= Y_0 + \alpha \cdot t + \sum_{s=1}^{t} U_s, \end{aligned}$$

   so that

$$\mathbb{E}[Y_t] = Y_0 + \alpha \cdot t,$$

   because white noise has expectation zero. The derivation of the variance is the same as for the case without drift from last week and thus omitted here.

   (b) The previous question shows that the random walk with drift is not stationary, because its mean and variance change over time. For it to be I(1), its first difference ΔY_t should be stationary. We immediately se that $\Delta Y_t = Y_t - Y_{t-1} = (\alpha + Y_{t-1} + U_t) - Y_{t-1} = \alpha + U_t$. This is just white noise plus a constant, which is stationary.

   (c) Since $\{U_t\}$ is white noise, U_t is uncorrelated with Y_{t-1}, so

$$\begin{aligned} \text{var}(Y_t) &= \text{var}\left(\alpha + \phi_1 Y_{t-1} + U_t\right) \\ &= \phi_1^2 \text{var}(Y_{t-1}) + \text{var}(U_t) + 2\phi_1 \text{cov}(Y_{t-1}, U_t) \qquad = \phi_1^2 \text{var}(Y_t) + \sigma^2, \end{aligned}$$

   where the final equality holds because Y_t is stationary, which implies that $\text{var}(Y_t) = \text{var}(Y_{t-1})$. Thus, if and only if $|\phi_1| < 1$,

$$\text{var}(Y_t) = \frac{\sigma^2}{1-\phi_1^2}.$$

   Note that $\text{var}(Y_t) > \text{var}(Y_{t-1})$ if $|\phi_1| \geq 1$, i.e., the variance grows without bounds in that case.

(d) **Optional**: For the MA(1) process

$$Y_t = \alpha + U_t + \theta_1 U_{t-1},$$

we have that

$$\begin{aligned}\mathbb{E}[Y_t] &= \mathbb{E}[\alpha + U_t + \theta_1 U_{t-1}] \\ &= \alpha + \mathbb{E}[U_t] + \theta_1 \mathbb{E}[U_{t-1}] \\ &= \alpha.\end{aligned}$$

For the variance,

$$\begin{aligned}\gamma_0 &= \operatorname{var}(Y_t) = \operatorname{var}(\alpha + U_t + \theta_1 U_{t-1}) \\ &= \operatorname{var}(U_t + \theta_1 U_{t-1}) \\ &= \operatorname{var}(U_t) + \theta_1^2 \operatorname{var}(U_{t-1}) + 2\theta_1 \operatorname{cov}(U_t, U_{t-1}) \\ &= \sigma^2 + \theta_1^2 \sigma^2 + 0 \\ &= \sigma^2(1 + \theta_1^2).\end{aligned}$$

For the first autocovariance,

$$\begin{aligned}\gamma_1 &= \operatorname{cov}(Y_t, Y_{t-1}) \\ &= \operatorname{cov}(\alpha + U_t + \theta_1 U_{t-1}, \alpha + U_{t-1} + \theta_1 U_{t-2}) \qquad (\dagger) \\ &= \operatorname{cov}(\theta_1 U_{t-1}, U_{t-1})\end{aligned}$$

because white noise is uncorrelated. Hence

$$\begin{aligned}\gamma_1 &= \theta_1 \operatorname{cov}(U_{t-1}, U_{t-1}) \\ &= \theta_1 \operatorname{var}(U_{t-1}) \\ &= \theta_1 \sigma^2.\end{aligned}$$

Higher order autocorrelations will be zero, because there will no common U_t terms in (†). Plugging these into the definition of the ACF, we have

$$\tau_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta_1 \sigma^2}{\sigma^2(1 + \theta_1^2)} = \frac{\theta_1}{1 + \theta_1^2}.$$

(e) **Optional**: The ACF is obtained by repeatedly substituting $Y_{t-i} = \phi_1 Y_{t-i-1} + \alpha + U_{t-i}$:

$$\begin{aligned}Y_t &= \phi_1 Y_{t-1} + \alpha + U_t \\ &= \phi_1^2 Y_{t-2} + \phi_1(\alpha + U_{t-1}) + \alpha + U_t \\ &= \phi_1^3 Y_{t-3} + \phi_1^2(\alpha + U_{t-2}) + \phi_1(\alpha + U_{t-1}) + \alpha + U_t \\ &\vdots \\ &= \phi_1^k Y_{t-k} + \sum_{i=0}^{k-1} \phi_1^i \alpha + \sum_{i=0}^{k-1} \phi_1^i U_{t-i}. \qquad (1)\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma_k &= \operatorname{cov}(Y_t, Y_{t-k}) = \phi_1^k \operatorname{cov}(Y_{t-k}, Y_{t-k}) + \sum_{i=0}^{k-1} \phi_1^i \operatorname{cov}(U_{t-i}, Y_{t-k}) \\ &= \phi_1^k \operatorname{var}(Y_{t-k}),\end{aligned}$$

so that

$$\tau_k = \frac{\gamma_k}{\gamma_0} = \phi_1^k.$$

Solution to Exercise 4

Simon A. Broda

1. If we set α and β_1 to the same value, e.g., 1, then both series trend up by 1 each period. The correlograms look quite similar, but the time series plots don't really, even if we crank up the variance of the trend-stationary series. This is because the latter is stationary after de-trending (subtracting off $\beta_1 \cdot t$, whereas the random walk with drift remains nonstationary if we subtract $\alpha \cdot t$ (it becomes a random walk without drift).

2. See Jupyter notebook.

3. (a) For $Y_{1,t}$, we have

$$\begin{aligned}
\mathbb{E}[\Delta Y_{1,t}] &= \mathbb{E}[Y_{1,t} - Y_{1,t-1}] \\
&= \mathbb{E}[\delta t + U_{1,t} - (\delta(t-1) + U_{1,t-1})] \\
&= \delta + \mathbb{E}[U_{1,t} - U_{1,t-1})] \\
&= \delta.
\end{aligned}$$

For $Y_{2,t}$,

$$\begin{aligned}
\mathbb{E}[\Delta Y_{2,t}] &= \mathbb{E}[Y_{2,t} - Y_{2,t-1}] \\
&= \mathbb{E}[\delta + Y_{2,t-1} + U_{2,t} - Y_{2,t-1}] \\
&= \mathbb{E}[\delta + U_{2,t}] \\
&= \delta.
\end{aligned}$$

(b) Consider the AR(2) process

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + U_t.$$

We would like to test the null that $\phi_1 + \phi_2 = 1$ (unit root) vs. $\phi_1 + \phi_2 < 1$ (stationarity). This can be done by rearranging the equation as follows:

$$\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + U_t && | - Y_{t-1} \\
Y_t - Y_{t-1} &= (\phi_1 - 1) Y_{t-1} + \phi_2 Y_{t-2} + U_t && | \pm \phi_2 Y_{t-1} \\
Y_t - Y_{t-1} &= (\phi_1 - 1) Y_{t-1} + \phi_2 Y_{t-1} - \phi_2 Y_{t-1} + \phi_2 Y_{t-2} + U_t \\
Y_t - Y_{t-1} &= (\phi_1 + \phi_2 - 1) Y_{t-1} - \phi_2 \Delta Y_{t-1} + U_t \\
\Delta Y_t &= \psi Y_{t-1} + \alpha_1 \Delta Y_{t-1} + U_t,
\end{aligned}$$

where $\psi := (\phi_1 + \phi_2 - 1)$ and $\alpha_1 := -\phi_2$. Thus testing $\phi_1 + \phi_2 = 1$ vs. $\phi_1 + \phi_2 =< 1$ is equivalent to testing $\psi = 0$ vs. $\psi < 0$ in a regression of ΔY_t onto Y_{t-1}, augmented by one lag of ΔY_{t-1}.

Solution to Exercise 5

Simon A. Broda

1. See Jupyter notebook.
2. (a) Splitting out the first term of the sum immediately yields

$$
\begin{aligned}
\widehat{\sigma}^2_{t+1,EWMA} &= (1-\lambda)\sum_{j=0}^{\infty}\lambda^j r^2_{t-j} \\
\widehat{\sigma}^2_{t+1,EWMA} &= (1-\lambda)\lambda^0 r^2_{t-0} + (1-\lambda)\sum_{j=1}^{\infty}\lambda^j r^2_{t-j} \\
\widehat{\sigma}^2_{t+1,EWMA} &= (1-\lambda) r^2_t + (1-\lambda)\sum_{j=0}^{\infty}\lambda^{j+1} r^2_{t-1-j} \\
\widehat{\sigma}^2_{t+1,EWMA} &= (1-\lambda) r^2_t + \lambda(1-\lambda)\sum_{j=0}^{\infty}\lambda^j r^2_{t-1-j} \\
&= (1-\lambda) r^2_t + \lambda\widehat{\sigma}^2_{t,EWMA}.
\end{aligned}
$$

Solution to Exercise 6

Simon A. Broda

See Jupyter notebook.

Solution to Exercise 7

Simon A. Broda

1. (a) If the relative price of oil expressed in units of gold, $\text{oil}_t/\text{gold}_t$, is stationary, then this implies that $\log(\text{oil}_t/\text{gold}_t) = \log(\text{oil}_t) - \log(\text{gold}_t)$ is also stationary, so $\log(\text{oil}_t)$ and $\log(\text{gold}_t)$ must be cointegrated with cointegrating vector (1, -1) if the individual series are integrated.

   (b) See Jupyter notebook.

2. (a) X_t is a random walk, hence $I(1)$. So no, it is not stationary.

   (b) Y_t depends on X_t if $\beta_2 \neq 0$, so it cannot be stationary.

   (c) Yes, because there exists a linear combination of them that is stationary:

   $$Y_t - \beta_2 X_t = \beta_1 + U_{1,t}.$$

   The cointegrating vector is $(1, -\beta_2)$.

   (d) The goal is to find two equations, one with ΔY_t on the LHS, and one with ΔX_t. Both should have the equilibrium error $Y_{t-1} - \beta_1 - \beta_2 X_{t-1}$ on the RHS.
   For Y_t, we find

   $$\begin{aligned}
   Y_t &= \beta_1 + \beta_2 X_t + U_{1,t} && | - Y_{t-1} \\
   \Delta Y_t &= -Y_{t-1} + \beta_1 + \beta_2 X_t + U_{1,t} && | \pm \beta_2 X_{t-1} \\
   \Delta Y_t &= -(Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + \beta_2 \Delta X_t + U_{1,t} \\
   \Delta Y_t &= \alpha_1 (Y_{t-1} - \beta_1 - \beta_2 X_{t-1}) + \beta_2 \Delta X_t + U_{1,t},
   \end{aligned}$$

   where $\alpha_1 = -1$. For X_t,

   $$\begin{aligned}
   X_t &= X_{t-1} + U_{2,t} && | - X_{t-1} \\
   \Delta X_t &= U_{2,t} \\
   \Delta X_t &= 0(Y_{t-1} - aX_{t-1}) + U_{2,t} \\
   \Delta X_t &= \alpha_2 (Y_{t-1} - aX_{t-1}) + U_{2,t}
   \end{aligned}$$

   where $\alpha_2 = 0$. This means that we can treat this as a single-equation ECM.