Solution to Exercise 3

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1. (a) We clearly see that unless $|\phi_1|$ approaches 1, the process is stationary; the time series plot looks mean-reverting, and the sample autocorrelations decay exponentially as they should. We also see that $\bar{y}$ is close to $\mathbb{E}[Y_t] = \alpha/(1-\phi_1)$, and that s_y^2 is close to $\text{var}[Y_t] = \sigma^2/(1-\phi_1^2)$.

   (b) If $\phi_1 = 1$, we have a random walk, and α becomes the drift: $\mathbb{E}[Y_t] = Y_0 + \alpha \cdot t$.

   (c) See Jupyter notebook.

2. See Jupyter notebook.

3. (a) By repeatedly plugging in,

$$\begin{aligned} Y_t &= \alpha + Y_{t-1} + U_t \\ &= \alpha + (\alpha + Y_{t-2} + U_{t-1}) + U_t \\ &\vdots \\ &= Y_0 + \alpha \cdot t + \sum_{s=1}^{t} U_s, \end{aligned}$$

   so that

$$\mathbb{E}[Y_t] = Y_0 + \alpha \cdot t,$$

   because white noise has expectation zero. The derivation of the variance is the same as for the case without drift from last week and thus omitted here.

   (b) The previous question shows that the random walk with drift is not stationary, because its mean and variance change over time. For it to be I(1), its first difference ΔY_t should be stationary. We immediately se that $\Delta Y_t = Y_t - Y_{t-1} = (\alpha + Y_{t-1} + U_t) - Y_{t-1} = \alpha + U_t$. This is just white noise plus a constant, which is stationary.

   (c) Since $\{U_t\}$ is white noise, U_t is uncorrelated with Y_{t-1}, so

$$\begin{aligned} \text{var}(Y_t) &= \text{var}\left(\alpha + \phi_1 Y_{t-1} + U_t\right) \\ &= \phi_1^2 \text{var}(Y_{t-1}) + \text{var}(U_t) + 2\phi_1 \text{cov}(Y_{t-1}, U_t) \qquad = \phi_1^2 \text{var}(Y_t) + \sigma^2, \end{aligned}$$

   where the final equality holds because Y_t is stationary, which implies that $\text{var}(Y_t) = \text{var}(Y_{t-1})$. Thus, if and only if $|\phi_1| < 1$,

$$\text{var}(Y_t) = \frac{\sigma^2}{1-\phi_1^2}.$$

   Note that $\text{var}(Y_t) > \text{var}(Y_{t-1})$ if $|\phi_1| \geq 1$, i.e., the variance grows without bounds in that case.

(d) **Optional**: For the MA(1) process

$$Y_t = \alpha + U_t + \theta_1 U_{t-1},$$

we have that

$$\begin{aligned}\mathbb{E}[Y_t] &= \mathbb{E}[\alpha + U_t + \theta_1 U_{t-1}] \\ &= \alpha + \mathbb{E}[U_t] + \theta_1 \mathbb{E}[U_{t-1}] \\ &= \alpha.\end{aligned}$$

For the variance,

$$\begin{aligned}\gamma_0 = \operatorname{var}(Y_t) &= \operatorname{var}(\alpha + U_t + \theta_1 U_{t-1}) \\ &= \operatorname{var}(U_t + \theta_1 U_{t-1}) \\ &= \operatorname{var}(U_t) + \theta_1^2 \operatorname{var}(U_{t-1}) + 2\theta_1 \operatorname{cov}(U_t, U_{t-1}) \\ &= \sigma^2 + \theta_1^2 \sigma^2 + 0 \\ &= \sigma^2(1 + \theta_1^2).\end{aligned}$$

For the first autocovariance,

$$\begin{aligned}\gamma_1 &= \operatorname{cov}(Y_t, Y_{t-1}) \\ &= \operatorname{cov}(\alpha + U_t + \theta_1 U_{t-1}, \alpha + U_{t-1} + \theta_1 U_{t-2}) \qquad (\dagger) \\ &= \operatorname{cov}(\theta_1 U_{t-1}, U_{t-1})\end{aligned}$$

because white noise is uncorrelated. Hence

$$\begin{aligned}\gamma_1 &= \theta_1 \operatorname{cov}(U_{t-1}, U_{t-1}) \\ &= \theta_1 \operatorname{var}(U_{t-1}) \\ &= \theta_1 \sigma^2.\end{aligned}$$

Higher order autocorrelations will be zero, because there will no common U_t terms in (†). Plugging these into the definition of the ACF, we have

$$\tau_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta_1 \sigma^2}{\sigma^2(1 + \theta_1^2)} = \frac{\theta_1}{1 + \theta_1^2}.$$

(e) **Optional**: The ACF is obtained by repeatedly substituting $Y_{t-i} = \phi_1 Y_{t-i-1} + \alpha + U_{t-i}$:

$$\begin{aligned}Y_t &= \phi_1 Y_{t-1} + \alpha + U_t \\ &= \phi_1^2 Y_{t-2} + \phi_1(\alpha + U_{t-1}) + \alpha + U_t \\ &= \phi_1^3 Y_{t-3} + \phi_1^2(\alpha + U_{t-2}) + \phi_1(\alpha + U_{t-1}) + \alpha + U_t \\ &\vdots \\ &= \phi_1^k Y_{t-k} + \sum_{i=0}^{k-1} \phi_1^i \alpha + \sum_{i=0}^{k-1} \phi_1^i U_{t-i}. \qquad (1)\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma_k = \operatorname{cov}(Y_t, Y_{t-k}) &= \phi_1^k \operatorname{cov}(Y_{t-k}, Y_{t-k}) + \sum_{i=0}^{k-1} \phi_1^i \operatorname{cov}(U_{t-i}, Y_{t-k}) \\ &= \phi_1^k \operatorname{var}(Y_{t-k}),\end{aligned}$$

so that

$$\tau_k = \frac{\gamma_k}{\gamma_0} = \phi_1^k.$$