Module 9.3: Time Series Analysis with Python
Fall Term 2023

Week 6:

Value at Risk

Lucerne University of Applied Sciences and Arts

Outline in Weeks

1. Introduction; Descriptive Modeling
2. Returns; Autocorrelation; Stationarity
3. ARMA Models
4. Unit Roots; ARIMA Models
5. Volatility Modeling
6. Value at Risk
7. Cointegration

Outline

Value at Risk

- Consider a portfolio with value $V_{PF,t}$ and daily returns $R_{PF,t+1}$.
- Define the one-day loss on the portfolio as

$$\$Loss_{t+1} = V_{PF,t} - V_{PF,t+1}.$$

- The one-day, $100 \cdot p\%$, dollar *Value at Risk* ($\$VaR^{p}_{t+1}$) is the daily loss which will only be exceeded on the worst $100 \cdot p\%$ of days. Usually, $p = 0.01$.
- Mathematically, it is the value of $\$VaR^{p}_{t+1}$ such that

$$\mathbb{P}(\$Loss_{t+1} > \$VaR^{p}_{t+1}) = p.$$

Value at Risk

- It is usually easier to express the VaR as a percentage of the portfolio value:

$$VaR^p_{t+1} = \frac{\$VaR^p_{t+1}}{V_{PF,t}}.$$

- Hence

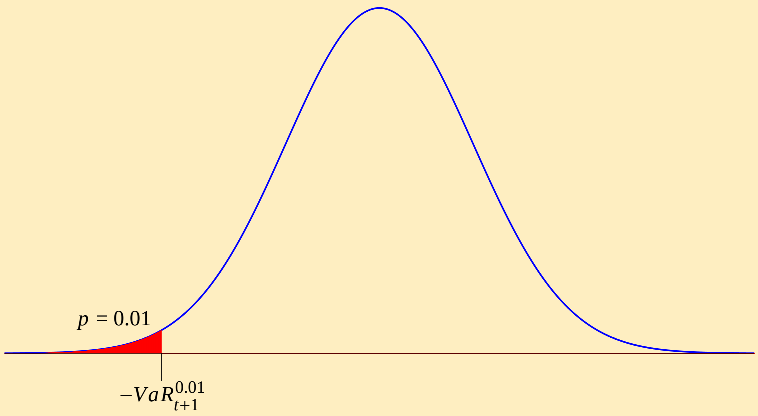
$$\mathbb{P}(R_{PF,t+1} < -VaR^p_{t+1}) = p,$$

  as

$$R_{PF,t+1} = -\frac{\$Loss_{t+1}}{V_{PF,t}}.$$

- Thus VaR^p_{t+1} is minus the $100p$th *percentile* of the return distribution.
- Definition can be naturally extended to K-day VaR, from the distribution of the K-day returns $R_{PF,t+1:t+K}$.

Probability density function of daily returns



Value at Risk

- Value at Risk was proposed as the standard measure of portfolio risk by the Basel Committee of the Bank of International Settlements in 1996.
- The BC imposed that financial institutions should report the Value at Risk on their positions, such that regulators could check the adequacy of the economic capital as a buffer against market risk.
- Banks were allowed to use their own, internal models for the computation of VaR, but the adequacy of these models should be “backtested” using specific criteria.
- A candidate for a standard model is RiskMetrics (developed by J.P.Morgan).
- VaR is being replaced by the expected shortfall (ES) with the rollout of Basel 3. The ES is based on the VaR, however.

Outline

VaR Methods: Historical simulation

Historical simulation assumes that the distribution of tomorrow's portfolio returns is well approximated by the empirical distribution (histogram) of the past m observations $\{R_{PF,t}, R_{PF,t-1}, \ldots, R_{PF,t+1-m}\}$.
This is as if we draw, with replacement, from the last m returns and use this to simulate the next day's return distribution.

- The estimator of VaR is given by minus the 100pth percentile of the sequence of past portfolio returns, that is:
  - sort the returns $\{R_{PF,t}, R_{PF,t-1}, ..., R_{PF,t+1-m}\}$ is ascending order;
  - define R^p_{t+1} as the number such that 100p% of the observations are smaller than R^p_{t+1};
  - the estimator for VaR is given by

$$\widehat{VaR}^p_{t+1} = -R^p_{t+1}.$$

VaR Methods: Historical simulation

Problems / limitations of historical simulation:

- Last year(s) of data not necessarily representative for the next few days (e.g., because of volatility clustering).
- Similar problems as historical volatility (choice of m).
- A large m is required to compute the 1% VaR with any degree of precision, since we are effectively using only 1% of the data to estimate it.

Outline

VaR Methods: Normal distribution

- Another simple approach is to assume $R_{t+1} = R_{PF,t+1} \sim N(\mu, \sigma^2)$ and to estimate μ and σ^2 using historical data.
- Denoting the inverse distribution function (quantile function) of the normal as Φ_p^{-1}, The VaR becomes

$$VaR_{t+1}^{p} = -\mu - \sigma\Phi_p^{-1}.$$

For example, $\Phi_{.01}^{-1} = -2.326$. For daily data one might take $\mu = 0$.

VaR Methods: Normal distribution

- The normal model can be easily extended to a *conditionally* normal model. Assume $R_{t+1} \sim N(\mu_{t+1}, \sigma^2_{t+1})$ where σ^2_{t+1} may be estimated by:
  - rolling windows;
  - EWMA / RiskMetrics;
  - univariate GARCH;
- μ_{t+1} is often just the mean return (e.g., the intercept in the mean equation for a GARCH model)
- The VaR then becomes $VaR^p_{t+1} = -\mu_{t+1} - \sigma_{t+1}\Phi^{-1}_p$.

Outline

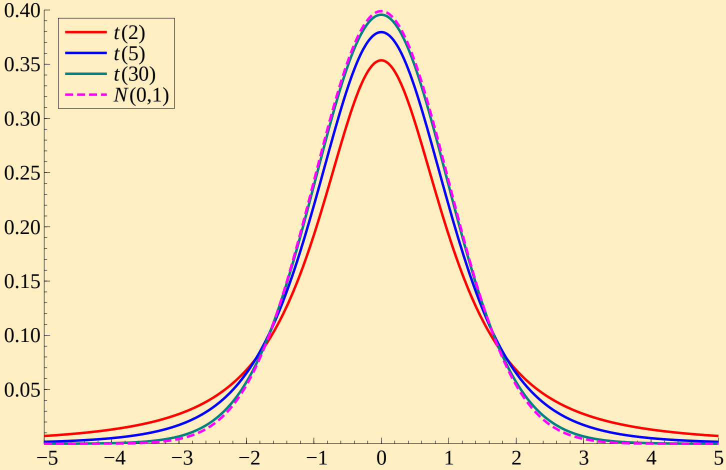
VaR Methods: Standardized t distribution

- The VaR methods described on the previous slides are only applicable if the returns are normally distributed.
- This can be tested by the Jarque-Bera test and is usually rejected.
- Solution: use Student's $t(d)$ distribution, where d.o.f. $d > 0$ need not be integer.
- d is just a shape parameter. Small values correspond to fat tails. As $d \to \infty$, we approach the $N(0, 1)$ distribution.
- For $d > 2$, the variance of a $t(d)$ random variable x is $d/(d-2)$; the distribution of

$$z = \frac{x}{\sqrt{\mathrm{var}(x)}} = \sqrt{\frac{d-2}{d}}x$$

  is called *standardized* $t(d)$, denoted $\tilde{t}(d)$.
- For $d > 4$ the excess kurtosis is $6/(d-4)$. The distributions are symmetric around 0 (hence mean and skewness are 0).

Student's *t* densities



VaR Methods: Standardized t distribution

- The GARCH model $R_{t+1} = \mu_{t+1} + \sigma_{t+1} z_{t+1}$, $\sigma^2_{t+1} = \omega + \alpha R^2_t + \beta \sigma^2_t$, may be extended to $z_t \sim \tilde{t}(d)$, where d is an extra parameter that needs to be estimated.
- In practice this GARCH-t model often gives a substantially better fit than the Gaussian model. The main problem is that the standardized residuals usually have an asymmetric distribution, with a longer left tail than right tail.

Estimation of GARCH-*t* in Python

```
                     Constant Mean - GARCH Model Results
==========================================================================
Dep. Variable:                 log_return   R-squared:                   0.000
Mean Model:                 Constant Mean   Adj. R-squared:              0.000
Vol Model:                          GARCH   Log-Likelihood:           -3036.13
Distribution:      Standardized Student's t   AIC:                     6082.26
Method:                Maximum Likelihood   BIC:                       6111.41
No. Observations:                    2517
Date:                    Tue, Oct 10 2023   Df Residuals:                 2516
Time:                            16:27:43   Df Model:                        1
Mean Model
==========================================================================
coef    std err          t      P>|t|     95.0% Conf. Int.
--------------------------------------------------------------------------
mu             0.0913  1.205e-02      7.578  3.501e-14 [6.768e-02,  0.115]
Volatility Model
==========================================================================
                 coef    std err          t      P>|t|      95.0% Conf. Int.
--------------------------------------------------------------------------
omega          0.0286  6.564e-03      4.353  1.343e-05 [1.571e-02,4.144e-02]
alpha[1]       0.2171  2.891e-02      7.508  6.009e-14     [  0.160,  0.274]
beta[1]        0.7767  2.496e-02     31.114 1.551e-212     [  0.728,  0.826]
Distribution
==========================================================================
                 coef    std err          t      P>|t|  95.0% Conf. Int.
--------------------------------------------------------------------------
nu             5.4748      0.592      9.248  2.293e-20 [  4.314,  6.635]
==========================================================================

Covariance estimator: robust
```

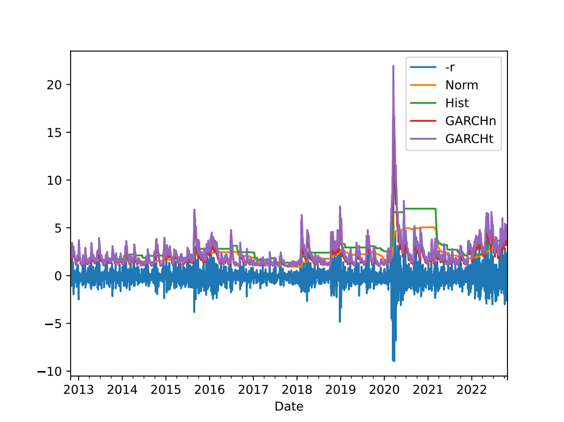
VaR Methods: Standardized t distribution

- Let $\tilde{t}_p^{-1}(d)$ be $100p\%$ quantile of the standardized t distribution $\tilde{t}(d)$ and $t_p^{-1}(d)$ the percentile $100p\%$ of the t distribution $t(d)$.
- The implied VaR now is

$$VaR_{t+1}^p = -\mu_{t+1} - \sigma_{t+1}\tilde{t}_p^{-1}(d) = -\mu_{t+1} - \sigma_{t+1}\sqrt{\frac{d-2}{d}}\,t_p^{-1}(d),$$

where, e.g., $\tilde{t}_{.01}^{-1}(6) = -2.566$.

Example: negative of the S&P500 returns, with 1% VaR based on historical simulation, a rolling normal distribution, and a GARCH(1, 1) with both normal and *t* errors



Outline

Expected Shortfall

Limitations of Value at Risk:

- VaR is not informative about the magnitude of the losses if they exceed the VaR. Two distributions could have the same 1% VaR, but with different left tails.
- VaR is not *subadditive*: it is not guaranteed that

$$VaR^p_{t+1}(X+Y) \leq VaR^p_{t+1}(X) + VaR^p_{t+1}(Y).$$

This means that VaR is not a "coherent" risk measure.

Expected Shortfall

- With the rollout of Basel 3 which started on January 1st, 2023, the 1% VaR is being replaced with the 2.5% *expected shortfall* (ES, a.k.a. CVaR), which addresses these problems.
- It is defined as

$$ES_{t+1}^{p} = -\mathbb{E}\left[R_{t+1} | R_{t+1} < -VaR_{t+1}^{p}\right],$$

  i.e., it represents the average of the losses exceeding the VaR.
- Backtesting the ES is less straightforward than the VaR, and won't be discussed here.
- Note however that a correctly specified ES requires that the VaR be correctly estimated in a first step, so the methods discussed here remain relevant.

Outline

Multi-Period VaR

- As we have seen, the one-day VaR (and ES) can be determined analytically for the GARCH-$N(0,1)$ and GARCH-$\tilde{t}(d)$ models, when estimation is also based on daily data.
- However, in practice one often needs risk measures for multi-period returns:

$$R_{t+1:t+K} = \sum_{k=1}^{K} R_{t+k}.$$

For example, a horizon of two weeks ($K = 10$ trading days) is common.

Multi-Period VaR

- Problem: even if the distribution of the one-period return is known (e.g., normal), that of $R_{t+1:t+K}$ is not (because the variance is not deterministic).
- *Monte Carlo simulation* is a possible solution: we let the computer generate a large number of scenarios of K daily returns, and compute from this the conditional distribution of the K-day return, and hence the K-day VaR and ES.
- Quick-and dirty practitioner solution: scale the one-day VaR with $\sqrt{K}$ (*square root of time rule*). This is strictly speaking only correct under normality.

Outline

Backtesting Value at Risk

- The Basel Committee requires that methods to evaluate VaR be backtested.
- They recommend constructing the 1% VaR over the last 250 trading days ($\approx$ 1 year), and counting the number of times that losses have exceeded the day's VaR figure (termed *exceptions* or *violations*).
- A method is said to lie in the:
  - *Green zone*, in case of 0–4 exceptions;
  - *Yellow zone*, in case of 5–9 exceptions;
  - *Red zone*, in case of 10 exceptions or more.
- The capital charge for the bank changes according to the zone.

Backtesting Value at Risk

How can we test if a VaR method is accurate?

- Define the *hit sequence*

$$I_{t+1} = \begin{cases} 1, & \text{if } R_{t+1} < -VaR^p_{t+1}, \\ 0, & \text{if } R_{t+1} > -VaR^p_{t+1}. \end{cases}$$

- Consider a test period that covers $t+1 \in \{1, \ldots, T\}$, then the number of exceptions is given by $T_1 = \sum_{t=1}^{T} I_t$.
- The proportion of exceptions is given by $\hat{\pi} = T_1 / T$ which is an estimator of $\mathbb{P}\left(R_{t+1} < -VaR^p_{t+1}\right)$.
- Recall that if the model that generated VaR^p_{t+1} is correctly specified, then

$$\mathbb{P}\left(R_{t+1} < -VaR^p_{t+1}\right) = p,$$

independent of any information at time t.

Backtesting Value at Risk

- Hence, under the null hypothesis of correct specification, the hits $\{I_{t+1}\}$ are independent Bernoulli random variables, and so $T_1 = \sum_{t=1}^{T} I_t$ has a Binomial(T, p) distribution.
- We can test this hypothesis (e.g., with $p = 0.01$) based on the t-statistics

$$t_0 = \frac{\hat{\pi} - p}{\sqrt{p(1-p)/T}} \qquad \text{or} \qquad t = \frac{\hat{\pi} - p}{\sqrt{\hat{\pi}(1-\hat{\pi})/T}}.$$

- Under H_0 their asymptotic distribution is $N(0, 1)$.
- The second t-statistic is equal (up to degrees-of-freedom correction) to the OLS-based t-statistic in regression of $I_{t+1} - p$ on a constant; see exercises.

Backtesting Value at Risk

- The previous test only checks *unconditional* coverage, i.e., $\mathbb{P}(I_{t+1} = 1) = p$ *on average*. However, misspecification often is due to the fact that the hits I_{t+1} are not independent over time.
- If exceptions are clustered, then if today there was an exception a risk manager can infer that the probability of occurring another exception tomorrow is higher than p. Hence, there is misspecification.
- We would like to test if the VaR violations are *independent* over time, the null hypothesis is

$$H_0 : \mathbb{P}(I_{t+1} = 1 | I_t = 1) = \mathbb{P}(I_{t+1} = 1 | I_t = 0),$$

  which implies $\mathbb{P}(I_{t+1} = 0 | I_t = 0) = \mathbb{P}(I_{t+1} = 0 | I_t = 1)$.

Backtesting Value at Risk

- Also of interest is to test if the VaR violations are independent over time and if the number of violations is correct (*conditional coverage*)

$$H_0 : \mathbb{P}(I_{t+1} = 1 | I_t = 1) = \mathbb{P}(I_{t+1} = 1 | I_t = 0) = p.$$

- A simple approach to test these hypotheses is to consider the linear regression model

$$I_{t+1} - p = b_0 + b_1 I_t + e_{t+1}$$

- The *conditional coverage* hypothesis is equivalent to $H_0 : b_0 = b_1 = 0$ and can be tested using a F-test.
- The *independence* hypothesis is equivalent to $H_0 : b_1 = 0$ and can be tested using a t-test.

Backtesting Value at Risk

Results for S&P500 returns, 4 different methods

	Norm	Hist	GARCHn	GARCHt
$100 \cdot \hat{\pi}$	3.10	1.67	2.54	1.71
$t(\pi = 0.01)$	6.08	2.62	4.92	2.74
$\hat{b}_1$	0.07	0.10	0.00	0.03
$t(b_1 = 0)$	3.71	5.25	0.30	1.50
$F(b_0 = b_1 = 0)$	25.45	17.24	12.13	4.89

The critical values for the t and F tests are, respectively, ± 1.96 and 3.00.
The GARCHt model fares best, even though correct conditional coverage is still rejected. This is likely driven by the incorrect unconditional coverage, since independence is not rejected. This means that we'd need a different distribution, such as a Skew-t.

Outline

Learning Goals

Students

- know the definitions of VaR and Expected Shortfall,
- understand the limitations of the VaR,
- are able to construct VaR forecasts based on various methods,
- and are able to backtest VaR forecasts.

Homework

- Exercise 6
- *Assignment 1*. Deadline: Sunday after next, 11.59 p.m.