Solution to Exercise 4

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1. If we set α and β_1 to the same value, e.g., 1, then both series trend up by 1 each period. The correlograms look quite similar, but the time series plots don't really, even if we crank up the variance of the trend-stationary series. This is because the latter is stationary after de-trending (subtracting off $\beta_1 \cdot t$, whereas the random walk with drift remains nonstationary if we subtract $\alpha \cdot t$ (it becomes a random walk without drift).

2. See Jupyter notebook.

3. (a) For $Y_{1,t}$, we have

$$\begin{aligned}
\mathbb{E}[\Delta Y_{1,t}] &= \mathbb{E}[Y_{1,t} - Y_{1,t-1}] \\
&= \mathbb{E}[\delta t + U_{1,t} - (\delta(t-1) + U_{1,t-1})] \\
&= \delta + \mathbb{E}[U_{1,t} - U_{1,t-1})] \\
&= \delta.
\end{aligned}$$

For $Y_{2,t}$,

$$\begin{aligned}
\mathbb{E}[\Delta Y_{2,t}] &= \mathbb{E}[Y_{2,t} - Y_{2,t-1}] \\
&= \mathbb{E}[\delta + Y_{2,t-1} + U_{2,t} - Y_{2,t-1}] \\
&= \mathbb{E}[\delta + U_{2,t}] \\
&= \delta.
\end{aligned}$$

(b) Consider the AR(2) process

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + U_t.$$

We would like to test the null that $\phi_1 + \phi_2 = 1$ (unit root) vs. $\phi_1 + \phi_2 < 1$ (stationarity). This can be done by rearranging the equation as follows:

$$\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + U_t && | - Y_{t-1} \\
Y_t - Y_{t-1} &= (\phi_1 - 1) Y_{t-1} + \phi_2 Y_{t-2} + U_t && | \pm \phi_2 Y_{t-1} \\
Y_t - Y_{t-1} &= (\phi_1 - 1) Y_{t-1} + \phi_2 Y_{t-1} - \phi_2 Y_{t-1} + \phi_2 Y_{t-2} + U_t \\
Y_t - Y_{t-1} &= (\phi_1 + \phi_2 - 1) Y_{t-1} - \phi_2 \Delta Y_{t-1} + U_t \\
\Delta Y_t &= \psi Y_{t-1} + \alpha_1 \Delta Y_{t-1} + U_t,
\end{aligned}$$

where $\psi := (\phi_1 + \phi_2 - 1)$ and $\alpha_1 := -\phi_2$. Thus testing $\phi_1 + \phi_2 = 1$ vs. $\phi_1 + \phi_2 =< 1$ is equivalent to testing $\psi = 0$ vs. $\psi < 0$ in a regression of ΔY_t onto Y_{t-1}, augmented by one lag of ΔY_{t-1}.