Module 9.3: Time Series Analysis with Python
Fall Term 2023

Week 4:

Unit Roots; ARIMA Models

HSLU Lucerne University of Applied Sciences and Arts

Outline in Weeks

1. Introduction; Descriptive Modeling
2. Returns; Autocorrelation; Stationarity
3. ARMA Models
4. Unit Roots; ARIMA Models
5. Volatility Modeling
6. Value at Risk
7. Cointegration

Outline

Unit Root Testing

- Recall that if a nonstationary series, y_t must be differenced d times before it becomes stationary, then it is said to be *integrated* of order d. Notation: $\mathrm{I}(d)$. When we simply say 'integrated' we mean $\mathrm{I}(1)$. Another synonym is 'the series has a *unit root*'.
- So far, our decision to take differences was based on the correlogram: if autocorrelations decay slowly and approximately linearly, then the series may be integrated and must be differenced.
- This procedure is subjective and unreliable: a *trend-stationary* series, such as $Y_t = \beta_0 + \beta_1 t + U_t$, will also have large and slowly decaying autocorrelations (see `simulation.xlsx`).
- Therefore, it is useful to have a formal testing procedure to distinguish integrated from (trend-) stationary time series. Such tests are called *unit root tests*. The *(Augmented) Dickey-Fuller* test is the most common.

Example: the AR(1) Model

- The simplest example of stationary and integrated time series is the AR(1) model

$$Y_t = \phi_1 Y_{t-1} + U_t,$$

where Y_0 is a constant.

- If $-1 < \phi_1 < 1$, then Y_t is stationary, with mean 0 and variance $\sigma^2/(1-\phi^2)$.
- If $\phi_1 = 1$, then the model becomes a *random walk*, $Y_t = Y_{t-1} + U_t$, with mean Y_0 and variance $\sigma^2 t$.

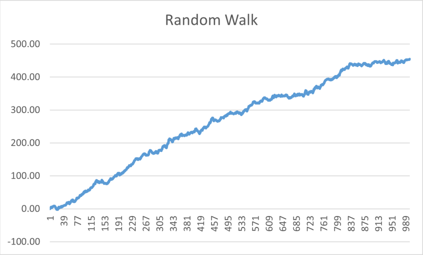
Stochastic vs. Deterministic Trends

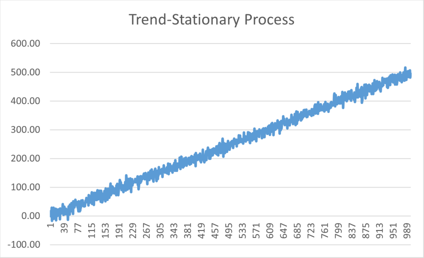
- Consider the two models

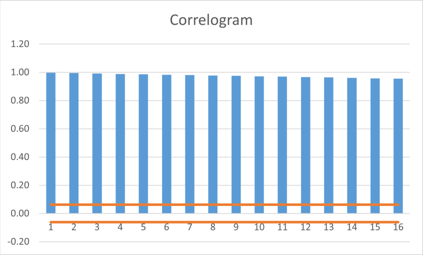
$$Y_{1,t} = \delta t + U_{1,t} \quad \text{and}$$
$$Y_{2,t} = \delta + Y_{2,t-1} + U_{2,t}.$$

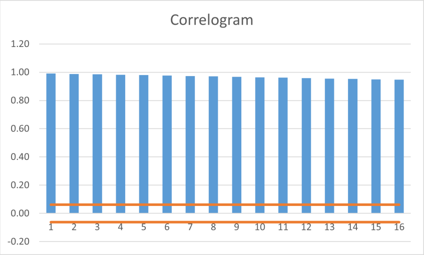
- For both models, $\mathbb{E}[\Delta Y_t] = \delta$, i.e., both series trend upwards by δ each period.
- $Y_{1,t}$ contains a *deterministic trend*: $Y_{1,t} - \delta t = U_{1,t}$ is stationary. In practice, regress y_t on a linear trend like in week 1; the residuals will be stationary.
- $Y_{2,t}$ contains a *stochastic trend*: $Y_{2,t} - \delta t$ is a random walk. It becomes stationary only by differencing, i.e., it is $I(1)$.

Random walk (left) vs. trend stationary process (right).









Why do we care?

1. Inference: OLS estimators have non-standard distribution, so that standard inference is invalid.
2. Forecasting:
   - stationary series display *mean-reversion*; deviations from the mean are corrected in the next period. Shocks U_t have a *transitory*, decreasing effect on future Y_{t+k} (in AR(1) model, the effect is $\phi^k \to 0$);
   - For integrated time series (of order 1): no mean-reversion, shocks U_t have a *persistent* effect on future Y_{t+k}.
3. Spurious regression: regressing two drifting I(1) variables onto each other will spuriously result in significant estimates, because they happen to trend in the same direction.

The Dickey-Fuller Test

- Consider again the AR(1) model

$$Y_t = \phi_1 Y_{t-1} + U_t.$$

- We wish to test the null hypothesis $H_0 : \phi_1 = 1$, against the *one-sided* alternative hypothesis $H_1 : \phi_1 < 1$.
- Under the null, the process is a random walk, and hence integrated. Under the alternative, it is stationary (if $\phi_1 > -1$, which we assume). Therefore, we will be testing $H_0 : Y_t \sim I(1)$ against $H_1 : Y_t \sim I(0)$.
- The *Dickey-Fuller test* is based on the t-statistic for $\phi_1 = 1$ in the AR(1) model, which may be reformulated as the t-statistic for $\psi = 0$ in

$$\Delta y_t = \psi y_{t-1} + u_t,$$

where $\psi := \phi_1 - 1$; note that $\psi < 0$ under the alternative.

The Dickey-Fuller Distribution

- We reject H_0 if the t-statistic is less than the (negative) critical value.
- In classical regressions, the 5% critical value for a one-sided t-test would be -1.645. However, because the regressor Y_{t-1} is non-stationary under the null, a different distribution arises, and the appropriate 5% critical value is $= -1.95$.
- This critical value changes to $= -2.86$ if we add a constant term to the regression:

$$\Delta y_t = \alpha + \psi y_{t-1} + u_t.$$

  This is the relevant test if we want to allow for a non-zero mean $\mathbb{E}[Y_t]$ under the alternative.
- If we want to test a random walk *with drift* against a *trend-stationary* alternative, then the relevant regression is

$$\Delta y_t = \alpha + \delta t + \psi y_{t-1} + u_t,$$

  and the 5% critical value is $= -3.41$.

Choice of Model

- The difference between the three tests is whether or not a constant and a time trend are included in the test regression.
- In practice, the test without constant and trend is almost never applicable.
- The test with an intercept in the regression test is relevant for series such as interest rates and real exchange rates, where we do not expect a linear trend under either the null or the alternative hypothesis.
- Many other economic and financial time series, such as GDP or (log-) asset prices, clearly display an upward trend, in which case a trend should be included.

The Augmented Dickey-Fuller Test

- The *augmented* Dickey-Fuller (ADF) test is an extension of the procedure to the AR(p) model
$$Y_t = \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} + U_t.$$
- This process is integrated under $H_0 : \phi_1 + \ldots + \phi_p = 1$, and stationary under the alternative hypothesis $H_1 : \phi_1 + \ldots + \phi_p < 1$.
- It can be shown (see exercises) that this is equivalent to testing $H_0 : \psi = 0$ against $H_1 : \psi < 0$ in the regression
$$\Delta y_t = \psi y_{t-1} + \alpha_1 \Delta y_{t-1} + \ldots + \alpha_{p^*} \Delta y_{t-p^*} + u_t,$$
where $\psi = \sum_{i=1}^{p} \phi_i - 1$, and $p^* = p - 1$.
- The interpretation of the null and alternative hypothesis, the role of the constant and trend, and the critical values are the same as in the first-order model.

Choosing p

- In practice, p is, of course, *unknown*.
- The number of lags in the test regression must be chosen large enough, such that the residuals have no autocorrelation, but not too large, because this would decrease test power.
- Often the choice is made based on *model selection criteria* (AIC, BIC). The `statsmodels` package has a built-in option to do this automatically.
- The AIC is preferred in this case, because it tends to include more lags than the BIC. This is preferred because our goal in this case is not to find the "correct" model, but to combat autocorrelation in the test regression.
- So effectively, the ADF test is just the DF test (regress Δy_t on y_{t-1}), but with enough lags of ΔY_t thrown in to remove any autocorrelation.

Outline

ARIMA Models

- As discussed last week, the first step in the Box-Jenkins procedure is to make sure that the data are stationary.
- This is usually decided by the ADF test.
- If the test doesn't reject, then one models the first difference ΔY_t as an ARMA process.
- The model for the levels Y_t is then called an *ARIMA(p,d,q)* model: the data are differenced d times, and the result modeled as an ARMA(p,q) process. Usually, $d = 1$.
- To forecast an ARIMA process (with d=1), first predict ΔY_{t+1}, and then let $\hat{Y}_{t+1} = Y_t + \widehat{\Delta Y_{t+1}}$.
- Longer horizon forecast are obtained recursively as usual.

Outline

Learning Goals

Students

- understand the ADF test and can use it to test for unit roots,
- are able to forecast ARIMA models,
- understand the difference between static and dynamic models, and
- know how to test for autocorrelation in a regression

Homework

- Exercise 4
- Questions 2 and 3 from Chapter 8 of Brooks (2019)
- *Assignment 1*. Deadline: Sunday after next, 11.59 p.m.