HSLU Lucerne University of Applied Sciences and Arts

# Module Examination MSCBF_RM01_TSA

**Course 9.3: Time Series Analysis** **Autumn Semester** **Mock Exam**

| **Candidate Information** | |
|---|---|
| **Name:** | |
| **First name:** | |

| **Exam Information** | | | |
|---|---|---|---|
| **Duration of exam:** | 90 min | | |
| **Points:** | **Assignment** | **Maximum number of points** | **Number of points awarded** |
| | Question 1 | 27 | |
| | Question 2 | 27 | |
| | Question 3 | 18 | |
| | Question 4 | 18 | |
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| | | | |
| | **Total** | **90** | |
| | **Grade** | | |
| **Aids:** | ▪ Closed book<br>▪ Calculator<br>▪ 8 page summary (4 double sided a4 sheets) | | |
| **Lecturers:** | Dr. Simon A. Broda | | |

| **Exam Instructions** | |
|---|---|
| **Instructions:** | ▪ When performing a statistical test, give the hypotheses, test statistic and its distribution, critical value, observed test statistic, decision (reject/don't reject), and a conclusion in words.<br>▪ For simplicity, **all** tests use the 5% significance level. |

**Question 1** (27 points)

**1.** I have simulated 1000 observations $\{y_t\}$ from an ARMA($p$, $q$) model. Figure 2 on Page 8 shows a plot of the correlogram. Based on it, what do you think $p$ and $q$ are, and why?

6 pts

*Solution*: The PACF becomes insignificant after 2 lags, whereas the ACF decays exponentially. This is indicative of an AR(2) process.

**2.** The 11th sample autocorrelation (not shown in the graph) is $\hat{\tau}_{11} = -0.045$. Test if $\hat{\tau}_{11}$ is significantly different from zero.

6 pts

*Solution*:

- The hypotheses are $H_0 : \tau_{11} = 0$ vs. $H_a : \tau_{11} \neq 0$.
- The test statistic is $\hat{\tau}_{11}$.
- Its distribution is $\mathrm{N}(0, 1/T)$ under $H_0$.
- The critical value is thus $1.96/\sqrt{1000} = 0.062$.
- The observed test statistic is $\hat{\tau}_{11} = -0.045$.
- Since $|-0.045| < 0.062$, we do not reject $H_0$.
- Conclusion: $\tau_{11}$ is not significantly different from zero.

**3.** The output in Figure 3 on Page 8 shows the results of regressing $\Delta y_t$ (`y`) on $y_{t-1}$ (`x1`) and $\Delta y_{t-1}$ (`x2`). Use it to test if the data are integrated.

6 pts

*Solution*:

- The hypotheses are $H_0 : y_t$ is integrated of order 1 vs. $H_a : y_t$ is stationary.
- The test statistic is the estimated coefficient on the lagged level, $y_{t-1}$ (`x1`).
- Its follows a Dickey-Fuller distribution under $H_0$.
- The critical value is -2.86, because a constant has been included.
- The observed test statistic is -6.52.
- Since $-6.52 < -2.86$, we reject $H_0$.
- Conclusion: the data are stationary.

**4.** I have estimated a particular ARMA model. The estimation output is shown in Figure 4 on Page 9. Write down the estimated model in equation form.

3 pts

*Solution*: The estimated AR(2) model is

$$y_t = 0.2158 \cdot (1 - 0.5236 - 0.3595) + 0.5236 \cdot y_{t-1} + 0.3595 \cdot y_{t-2}.$$

Note `archmodel`'s peculiar interpretation of the intercept, `const`: in our notation, it corresponds to $\hat{\alpha}/(1-\hat{\phi}_1-\hat{\phi}_2)$, so that $\hat{\alpha} = \texttt{const} \cdot (1-\hat{\phi}_1-\hat{\phi}_2)$. See the exercises.

**5.** Use the estimated model from the previous question to forecast the value of the series at $t = 1001$. You may need some of the values below. 6 pts

| $t$ | $y_t$ | $u_t$ |
|---|---|---|
| 999 | -0.77 | -4.18 |
| 1000 | -0.75 | -1.41 |

*Solution*: The forecast is

$$\begin{aligned}\widehat{y}_{1001} &= \texttt{const} \cdot (1 - \hat{\phi}_1 - \hat{\phi}_2) + \hat{\phi}_1 \cdot y_{1000} + \hat{\phi}_2 \cdot y_{999} \\ &= 0.2158 \cdot (1 - 0.5236 - 0.3595) + 0.5236 \cdot (-0.75) + 0.3595 \cdot (-0.77) \\ &= -0.644.\end{aligned}$$

**Question 2** (27 points) In this exercise, we analyze the daily returns on Tesla stock between 6/29/2010 and 12/14/2022. The returns are shown in Figure 1.

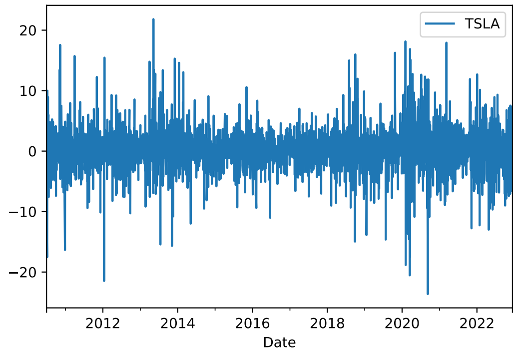



Figure 1: Returns on Tesla stock

The data clearly display volatility clustering, which we want to model using a GARCH model. The output is shown in Figure 5 on Page 9.

**1.** Write the model down as an equation (only the volatility equation). 3 pts

*Solution*:

$$\sigma_{t+1}^2 = 0.14 + 0.0323 \cdot u_t^2 + 0.9564 \cdot \sigma_t^2.$$

**2.** Does your model incorporate a leverage effect? Justify your answer. 6 pts

*Solution*: No. Since $u_t$ (the demeaned return) only enters as its square, the news impact curve is symmetric: volatility reacts equally strongly to a positive and negative shock of the same magnitude.

**3.** Explain what the standardized residuals from a GARCH model are, ideally with an equation. Also, explain what their properties should be if the volatility model is correct. 6 pts

*Solution*: The standardized residuals are defined as $\hat{z}_t = \frac{r_t - \hat{\mu}_t}{\hat{\sigma}_t}$. If the model for $\sigma_t$ is correct, then they should no longer display volatility clustering. This means that there should be no autocorrelation in their squares.

**4.** Regressing the squared standardized residuals on an intercept and 5 of their own lags results in an $R^2$ of 0.00364. Use this to test if the GARCH model has successfully removed the volatility clustering. 6 pts

*Solution*: This is an ARCH-LM test.

- The hypotheses are $H_0$ : no remaining volatility clustering vs. $H_a$ : there is remaining volatility clustering.
- The test statistic is $T \cdot R^2$ in the auxiliary regression.
- Its distribution is $\chi^2(5)$ under $H_0$.
- The critical value is thus 11.07.
- The observed test statistic is $3138 \cdot 0.00364 = 11.42$.

- Since $11.42 > 11.07$, we reject $H_0$ (although barely).
- Conclusion: Some volatility clustering remains.

**5.** Use the model to predict the variance $\sigma^2_{t+1}$ for 12/15/2022, using the following values on 12/14/2022: $\hat{u}_t = -2.72$ and $\sigma^2_t = 15.198$. 6 pts

*Solution*:

$$\begin{aligned}\widehat{\sigma}^2_{t+1} &= 0.14 + 0.0323 \cdot u_t^2 + 0.9564 \cdot \sigma_t^2 \\ &= 0.14 + 0.0323 \cdot (-2.72)^2 + 0.9564 \cdot 15.198 \\ &= 14.91.\end{aligned}$$

**Question 3** (18 points) We now turn our attention to Value at Risk forecasting.

**1.** Use the model from the previous question to predict the 1% Value at Risk for 12/15/2022. Note: If you weren't successful in predicting the variance in the previous question, you can use the value $\widehat{\sigma}^2_{t+1} = 15.00$. 6 pts

*Solution*: The VaR forecast is

$$\begin{aligned} VaR^{0.01}_{t+1} &= -\mu_{t+1} - \sigma_{t+1} \cdot \Phi^{-1}_{0.01} \\ &= -\mu_{t+1} - \sigma_{t+1} \cdot \Phi^{-1}_{0.01} \\ &= -0.1118 - \sqrt{14.91} \cdot (-2.326) \\ &= 8.87. \end{aligned}$$

**2.** In a backtesting exercise, we have created 1% VaR forecasts for the entire sample. From this and the returns, we have created the hit series $\{I_t\}$. 12 pts

(a) (6 pts) Explain how the hit series is defined, and how many times you expect it to equal 1 if the VaR model is correct.

*Solution*: The hit series equals 1 whenever a VaR violation occurs, i.e.,

$$I_{t+1} = \begin{cases} 1, & \text{if } R_{t+1} < -VaR^p_{t+1}, \\ 0, & \text{if } R_{t+1} > -VaR^p_{t+1}. \end{cases}$$

If the model is correct, we would expect a violation 1% of the time. Since $T = 3138$, we would therefore expect around 31 VaR violations.

(b) (6 pts) Figure 6 on page 10 shows the result of regressing $(I_t - 0.01)$ on an intercept and $I_{t-1}$. Use it to test the independence of the VaR violations.

*Solution*:

- The hypotheses are $H_0$ : the VaR violations are independent vs. $H_a$ : they are not.
- The test statistic is the $t$ statistic for the lagged hit series.
- Its distribution is (asymptotically) standard normal under $H_0$.
- The critical value is thus 1.96.
- The observed test statistic is 1.187.
- Since $|1.187| < 1.96$, we do not reject $H_0$.
- Conclusion: there is no evidence of dependence between the VaR violations.

Note: in this case, since I didn't delete the $p$-value, you could alternatively have answered as follows.

- The hypotheses are $H_0$ : the VaR violations are independent vs. $H_a$ : they are not.
- The test statistic is the $t$ statistic for the lagged hit series.
- The observed $p$-value is $0.235 > 0.05$, so we do not reject $H_0$.
- Conclusion: there is no evidence of dependence between the VaR violations.

**Question 4** (18 points) Answer the questions below.

18 pts

**1.**

(a) (3 pts) Spurious regressions can occur between cointegrated variables.

☐ True ☑ False

(b) (3 pts) In a stationary time series, shocks $U_t$ have a transitory effect on the future of the series.

☑ True ☐ False

(c) (3 pts) An ARCH($q$) model for the returns corresponds to an AR($q$) for the squared returns.

☑ True ☐ False

(d) (3 pts) The order $q$ of an MA($q$) model can be determined from the correlogram.

☑ True ☐ False

(e) (3 pts) In the presence of the leverage effect, the news impact curve is steeper to the right of the origin than to the left.

☐ True ☑ False

(f) (3 pts) A VaR model is correctly specified if no VaR violations occur.

☐ True ☑ False

# End of exam.

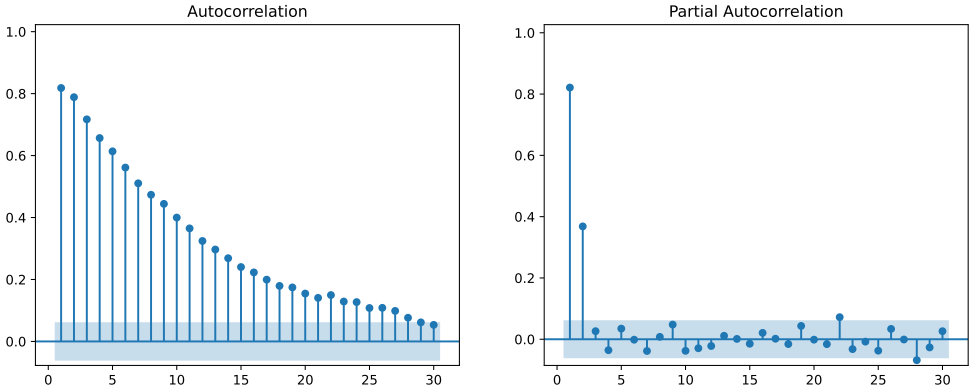



Figure 2: Correlogram of simulated data.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.208
Model:                            OLS   Adj. R-squared:                  0.207
Method:                 Least Squares   F-statistic:                     130.9
Date:                Mon, 16 Oct 2023   Prob (F-statistic):           3.47e-51
Time:                        15:04:00   Log-Likelihood:                -2086.9
No. Observations:                 998   AIC:                             4180.
Df Residuals:                     995   BIC:                             4195.
Df Model:                           2
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
x1            -0.1161      0.018     -6.520      0.000      -0.151      -0.081
x2            -0.3594      0.030    -12.164      0.000      -0.417      -0.301
const          0.0309      0.062      0.497      0.619      -0.091       0.153
==============================================================================
Omnibus:                        0.529   Durbin-Watson:                   2.023
Prob(Omnibus):                  0.768   Jarque-Bera (JB):                0.404
Skew:                           0.003   Prob(JB):                        0.817
Kurtosis:                       3.098   Cond. No.                         3.76
==============================================================================
```

Figure 3: Output for ADF test.

```
                               SARIMAX Results
==============================================================================
Dep. Variable:                      y   No. Observations:                 1000
Model:                 ARIMA(2, 0, 0)   Log Likelihood               -2091.958
Date:                Mon, 16 Oct 2023   AIC                           4191.917
Time:                        15:10:43   BIC                           4211.548
Sample:                             0   HQIC                          4199.378
                               - 1000
Covariance Type:                  opg
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.2158      0.528      0.409      0.683      -0.818       1.250
ar.L1          0.5236      0.032     16.396      0.000       0.461       0.586
ar.L2          0.3595      0.032     11.326      0.000       0.297       0.422
sigma2         3.8369      0.169     22.770      0.000       3.507       4.167
===================================================================================
Ljung-Box (L1) (Q):                   0.13   Jarque-Bera (JB):                 0.36
Prob(Q):                              0.71   Prob(JB):                         0.83
Heteroskedasticity (H):               0.86   Skew:                             0.00
Prob(H) (two-sided):                  0.18   Kurtosis:                         3.09
===================================================================================
```

Figure 4: Estimated ARMA model.

```
                     Constant Mean - GARCH Model Results
==============================================================================
Dep. Variable:                   TSLA   R-squared:                       0.000
Mean Model:             Constant Mean   Adj. R-squared:                  0.000
Vol Model:                      GARCH   Log-Likelihood:               -8267.70
Distribution:                  Normal   AIC:                           16543.4
Method:            Maximum Likelihood   BIC:                           16567.6
                                        No. Observations:                 3138
Date:                Mon, Oct 16 2023   Df Residuals:                     3137
Time:                        15:33:03   Df Model:                            1
                                 Mean Model
===========================================================================
                 coef    std err          t      P>|t|     95.0% Conf. Int.
---------------------------------------------------------------------------
mu             0.1118  5.774e-02      1.936  5.281e-02 [-1.359e-03,  0.225]
                              Volatility Model
============================================================================
                 coef    std err          t      P>|t|      95.0% Conf. Int.
----------------------------------------------------------------------------
omega          0.1400      0.117      1.196      0.232  [-8.938e-02,  0.369]
alpha[1]       0.0323  1.341e-02      2.409  1.599e-02 [6.024e-03,5.859e-02]
beta[1]        0.9564  2.205e-02     43.369      0.000     [  0.913,  1.000]
============================================================================
```

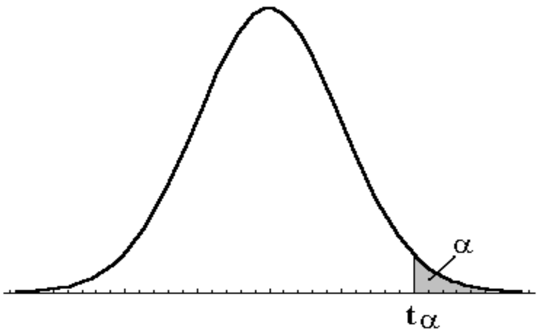
Figure 5: Estimated GARCH model.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:     np.subtract(I, 0.01)   R-squared:                       0.000
Model:                            OLS   Adj. R-squared:                  0.000
Method:                 Least Squares   F-statistic:                     1.409
Date:                Mon, 16 Oct 2023   Prob (F-statistic):              0.235
Time:                        18:54:56   Log-Likelihood:                 1976.8
No. Observations:                3137   AIC:                            -3950.
Df Residuals:                    3135   BIC:                            -3938.
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
b0             0.0065      0.002      2.817      0.005       0.002       0.011
I.shift(1)     0.0212      0.018      1.187      0.235      -0.014       0.056
==============================================================================
Omnibus:                     4069.691   Durbin-Watson:                   2.001
Prob(Omnibus):                  0.000   Jarque-Bera (JB):           412759.104
Skew:                           7.492   Prob(JB):                         0.00
Kurtosis:                      57.160   Cond. No.                         7.76
==============================================================================
```

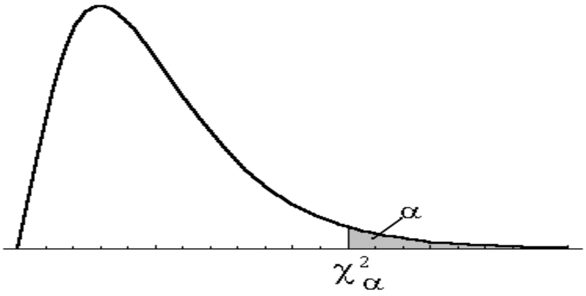
Figure 6: Test regression for the Value at Risk backtest.

**Critical Values of *t***



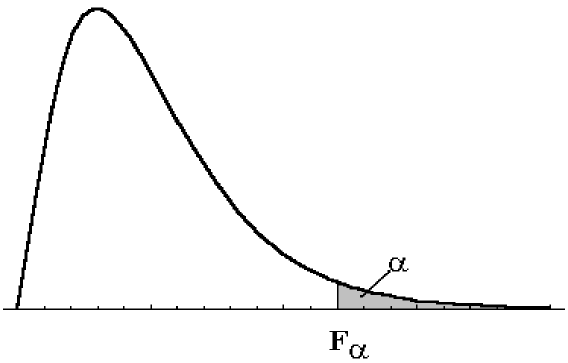

| Degrees of freedom | $t_{.10}$ | $t_{.05}$ | $t_{.025}$ | $t_{.01}$ | $t_{.005}$ | Degrees of freedom | $t_{.10}$ | $t_{.05}$ | $t_{.025}$ | $t_{.01}$ | $t_{.005}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 | **24** | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 |
| **2** | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | **25** | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 |
| **3** | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 | **26** | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 |
| **4** | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 | **27** | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 |
| **5** | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | **28** | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 |
| **6** | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | **29** | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 |
| **7** | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | **30** | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 |
| **8** | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | **35** | 1.306 | 1.690 | 2.030 | 2.438 | 2.724 |
| **9** | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | **40** | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 |
| **10** | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | **45** | 1.301 | 1.679 | 2.014 | 2.412 | 2.690 |
| **11** | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | **50** | 1.299 | 1.676 | 2.009 | 2.403 | 2.678 |
| **12** | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 | **60** | 1.296 | 1.671 | 2.000 | 2.390 | 2.660 |
| **13** | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 | **70** | 1.294 | 1.667 | 1.994 | 2.381 | 2.648 |
| **14** | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 | **80** | 1.292 | 1.664 | 1.990 | 2.374 | 2.639 |
| **15** | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 | **90** | 1.291 | 1.662 | 1.987 | 2.368 | 2.632 |
| **16** | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 | **100** | 1.290 | 1.660 | 1.984 | 2.364 | 2.626 |
| **17** | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 | **120** | 1.289 | 1.658 | 1.980 | 2.358 | 2.617 |
| **18** | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 | **140** | 1.288 | 1.656 | 1.977 | 2.353 | 2.611 |
| **19** | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 | **160** | 1.287 | 1.654 | 1.975 | 2.350 | 2.607 |
| **20** | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 | **180** | 1.286 | 1.653 | 1.973 | 2.347 | 2.603 |
| **21** | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 | **200** | 1.286 | 1.653 | 1.972 | 2.345 | 2.601 |
| **22** | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 | $\infty$ | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 |
| **23** | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 | | | | | | |

**Critical Values of $\chi^2$**




| Degrees of freedom | $\chi^2_{.10}$ | $\chi^2_{.05}$ | $\chi^2_{.025}$ | $\chi^2_{.01}$ | $\chi^2_{.005}$ |
|---|---|---|---|---|---|
| 1 | 2.706 | 3.841 | 5.024 | 6.635 | 7.879 |
| 2 | 4.605 | 5.991 | 7.378 | 9.210 | 10.597 |
| 3 | 6.251 | 7.815 | 9.348 | 11.345 | 12.838 |
| 4 | 7.779 | 9.488 | 11.143 | 13.277 | 14.860 |
| 5 | 9.236 | 11.070 | 12.833 | 15.086 | 16.750 |
| 6 | 10.645 | 12.592 | 14.449 | 16.812 | 18.548 |
| 7 | 12.017 | 14.067 | 16.013 | 18.475 | 20.278 |
| 8 | 13.362 | 15.507 | 17.535 | 20.090 | 21.955 |
| 9 | 14.684 | 16.919 | 19.023 | 21.666 | 23.589 |
| 10 | 15.987 | 18.307 | 20.483 | 23.209 | 25.188 |
| 11 | 17.275 | 19.675 | 21.920 | 24.725 | 26.757 |
| 12 | 18.549 | 21.026 | 23.337 | 26.217 | 28.300 |
| 13 | 19.812 | 22.362 | 24.736 | 27.688 | 29.819 |
| 14 | 21.064 | 23.685 | 26.119 | 29.141 | 31.319 |
| 15 | 22.307 | 24.996 | 27.488 | 30.578 | 32.801 |
| 16 | 23.542 | 26.296 | 28.845 | 32.000 | 34.267 |
| 17 | 24.769 | 27.587 | 30.191 | 33.409 | 35.718 |
| 18 | 25.989 | 28.869 | 31.526 | 34.805 | 37.156 |
| 19 | 27.204 | 30.144 | 32.852 | 36.191 | 38.582 |
| 20 | 28.412 | 31.410 | 34.170 | 37.566 | 39.997 |
| 21 | 29.615 | 32.671 | 35.479 | 38.932 | 41.401 |
| 22 | 30.813 | 33.924 | 36.781 | 40.289 | 42.796 |
| 23 | 32.007 | 35.172 | 38.076 | 41.638 | 44.181 |
| 24 | 33.196 | 36.415 | 39.364 | 42.980 | 45.559 |
| 25 | 34.382 | 37.652 | 40.646 | 44.314 | 46.928 |
| 26 | 35.563 | 38.885 | 41.923 | 45.642 | 48.290 |
| 27 | 36.741 | 40.113 | 43.195 | 46.963 | 49.645 |
| 28 | 37.916 | 41.337 | 44.461 | 48.278 | 50.993 |
| 29 | 39.087 | 42.557 | 45.722 | 49.588 | 52.336 |
| 30 | 40.256 | 43.773 | 46.979 | 50.892 | 53.672 |
| 40 | 51.805 | 55.758 | 59.342 | 63.691 | 66.766 |
| 50 | 63.167 | 67.505 | 71.420 | 76.154 | 79.490 |
| 60 | 74.397 | 79.082 | 83.298 | 88.379 | 91.952 |
| 70 | 85.527 | 90.531 | 95.023 | 100.425 | 104.215 |
| 80 | 96.578 | 101.879 | 106.629 | 112.329 | 116.321 |
| 90 | 107.565 | 113.145 | 118.136 | 124.116 | 128.299 |
| 100 | 118.498 | 124.342 | 129.561 | 135.807 | 140.169 |

**Critical Values of *F*** $\alpha$=.05




NUMERATOR DEGREES OF FREEDOM

| $df_{den}$ \ $df_{num}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 161.4 | 199.5 | 215.7 | 224.6 | 230.2 | 234.0 | 236.8 | 238.9 | 240.5 |
| 2 | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.35 | 19.37 | 19.38 |
| 3 | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 |
| 11 | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | 2.95 | 2.90 |
| 12 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 |
| 13 | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 | 2.77 | 2.71 |
| 14 | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 | 2.70 | 2.65 |
| 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 |
| 16 | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 |
| 17 | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 | 2.55 | 2.49 |
| 18 | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 |
| 19 | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 | 2.48 | 2.42 |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 |
| 21 | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 |
| 22 | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.46 | 2.40 | 2.34 |
| 23 | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.44 | 2.37 | 2.32 |
| 24 | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 |
| 25 | 4.24 | 3.39 | 2.99 | 2.76 | 2.60 | 2.49 | 2.40 | 2.34 | 2.28 |
| 26 | 4.23 | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.39 | 2.32 | 2.27 |
| 27 | 4.21 | 3.35 | 2.96 | 2.73 | 2.57 | 2.46 | 2.37 | 2.31 | 2.25 |
| 28 | 4.20 | 3.34 | 2.95 | 2.71 | 2.56 | 2.45 | 2.36 | 2.29 | 2.24 |
| 29 | 4.18 | 3.33 | 2.93 | 2.70 | 2.55 | 2.43 | 2.35 | 2.28 | 2.22 |
| 30 | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 |
| 40 | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 |
| 60 | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 |
| 120 | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.18 | 2.09 | 2.02 | 1.96 |
| ∞ | 3.84 | 3.00 | 2.60 | 2.37 | 2.21 | 2.10 | 2.01 | 1.94 | 1.88 |

DENOMINATOR DEGREES OF FREEDOM

**Dickey-Fuller critical values for different significance levels $\alpha$**

| 0.01 | 0.025 | 0.05 | 0.10 |
|---|---|---|---|
| | $\tau$ | | |
| -2.58 | -2.23 | -1.95 | -1.62 |
| | $\tau_\mu$ | | |
| -3.43 | -3.12 | -2.86 | -2.57 |
| | $\tau_\tau$ | | |
| -3.96 | -3.66 | -3.41 | -3.12 |

**5% critical values of Engle-Granger test vs. number of regressors k (excluding constant)**

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| -3.41 | -3.80 | -4.16 | -4.49 | -4.74 |