STAT 4224 HW #1

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1.

a.

	X=0	X=1	X=2	X=3
Y=0	1/12	1/4	3/20	1/60
Y=1	1/60	3/20	1/4	1/12

b.

$E(Y|X=0) = \frac{1}{2}(1) + \frac{3}{10}(2) + \frac{1}{30}(3) = \frac{6}{5}$ $E(Y|X=1) = \frac{3}{10}(1) + \frac{1}{2}(2) + \frac{1}{6}(3) = \frac{9}{5}$ $E(Y) = (\frac{1}{4} + \frac{3}{20})(1) + (\frac{1}{4} + \frac{3}{20})(2) + (\frac{1}{60} + \frac{1}{12})(3) = \frac{3}{2}$

$E(E(Y|X)) = (\frac{6}{5} + \frac{9}{5})\frac{1}{2} = \frac{3}{2} = E(Y)$

c.

$var(Y|X=0) = \frac{1}{2}(1) + \frac{3}{10}(4) + \frac{1}{30}(9) - (\frac{6}{5})^2 = \frac{14}{25}$

$var(Y|X=1) = \frac{3}{10}(1) + \frac{1}{2}(4) + \frac{1}{6}(9) - (\frac{9}{5})^2 = \frac{14}{25}$

$var(Y) = (\frac{1}{4} + \frac{3}{20})(1) + (\frac{1}{4} + \frac{3}{20})(4) + (\frac{1}{60} + \frac{1}{12})(9) - (\frac{3}{2})^2 = \frac{13}{20}$

$E(var(Y|X)) = (\frac{14}{25})(\frac{1}{2}) + (\frac{14}{25})(\frac{1}{2}) = \frac{14}{25}$

$var(E(Y|X)) = (\frac{36}{25})(\frac{1}{2}) + (\frac{81}{25})(\frac{1}{2}) - (\frac{3}{2})^2 = \frac{9}{100}$

Thus, $E(var(Y|X)) + var(E(Y|X)) = \frac{14}{25} + \frac{9}{100} = \frac{13}{20} = var(Y)$

d.

$p(x=1|y=2) = \frac{\frac{1}{4}}{\frac{3}{20}+\frac{1}{4}} = \frac{5}{8}$

2.

a.

$p(x|y,z) = \frac{p(x,y,z)}{p(y,z)} = \frac{p(x,y,z)}{\int p(x',y,z)dx'}$

$= \frac{f(x,z)g(y,z)}{\int f(x',z)g(y,z)dx'}$

$= \frac{f(x,z)}{\int f(x',z)dx'}$

$= \frac{f(x,z)}{h^*(z)} = f^*(x,z)$

b.

$p(y|x,z) = \frac{p(x,y,z)}{p(x,z)} = \frac{p(x,y,z)}{\int p(x,y',z)dy'}$

$= \frac{f(x,z)g(y,z)}{\int f(x,z)g(y',z)dy'}$

$= \frac{g(y,z)}{\int g(y',z)dy'}$

$= \frac{g(y,z)}{h^*(z)} = g^*(y,z)$

c.

$p(x|z) = \int p(x|y,z)p(y|z)dy = f^*(x,z)\int p(y|z)dy = f^*(x,z)$

$p(y|z) = \int p(y|x,z)p(x|z)dy = g^*(y,z)\int p(x|z)dy = g^*(y,z)$

$\Rightarrow p(x,y|z) = p(x|z)p(y|x,z)$

$= f^*(x,z)g^*(y,z) = p(x|z)p(y|z)$

3.

a.

$p(y_1,...y_n|\theta) = \prod_{i=1}^n p(y_i|\theta) = \prod_{i=1}^n \theta^{y_i}(1-\theta)^{1-y_i} = \theta^{\sum y_i}(1-\theta)^{n-\sum y_i}$

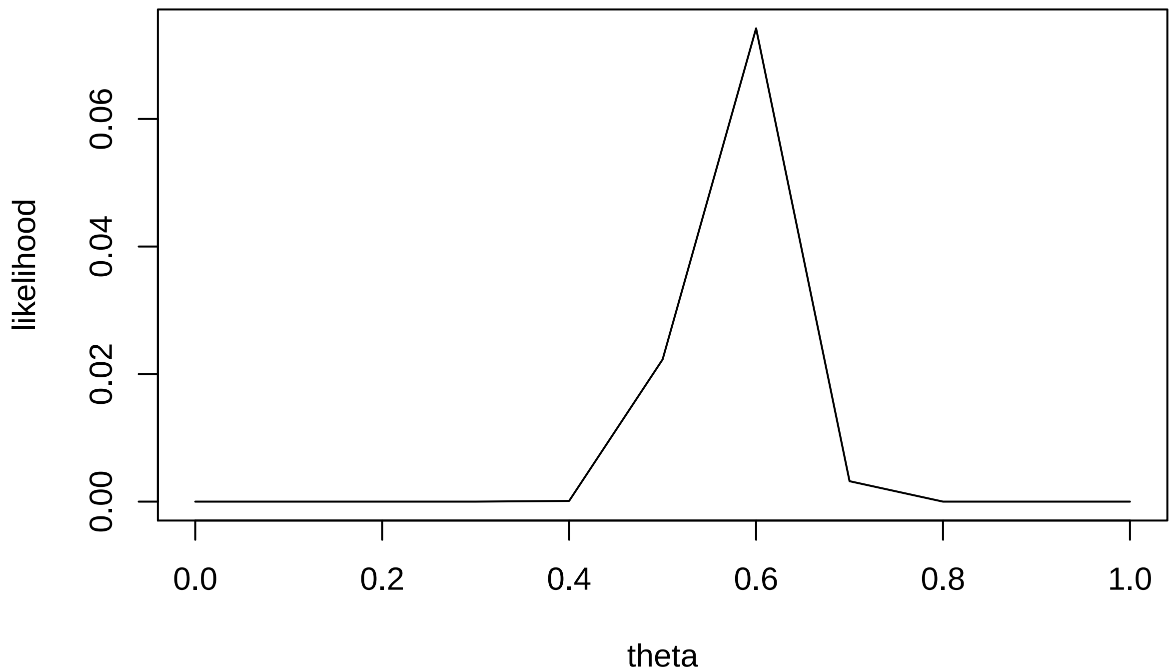
$p(\sum_{i=1}^n y_i = y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$

b.

```
theta<-seq(0,1,.1)
likelihood<-dbinom(58,100,theta)
likelihood
```

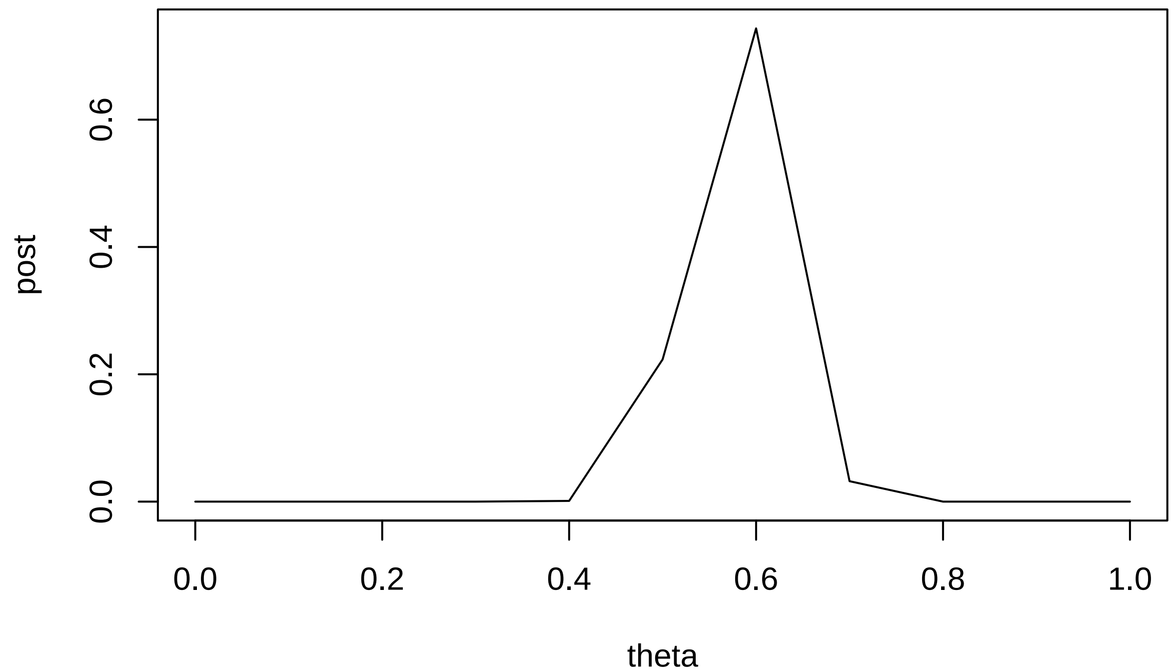
```
##  [1] 0.000000e+00 3.383290e-32 6.929040e-17 4.152449e-09 1.129759e-04
##  [6] 2.229227e-02 7.420719e-02 3.205774e-03 2.976000e-07 6.269305e-17
## [11] 0.000000e+00
```

```
plot(theta,likelihood, type = "l")
```



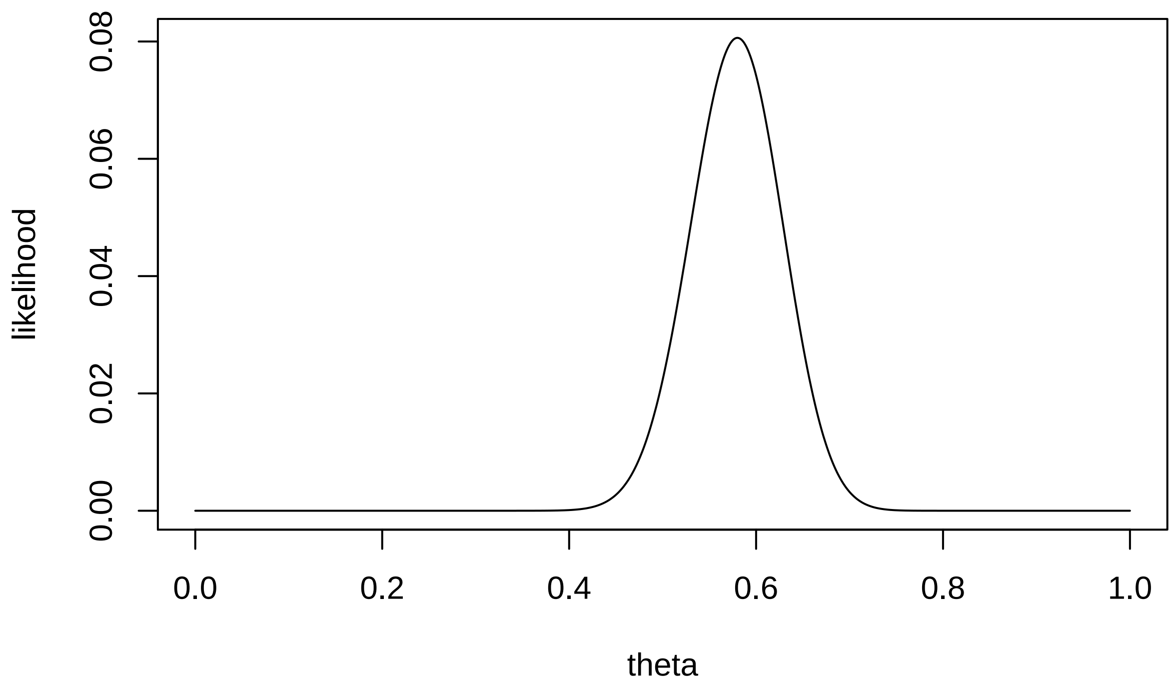
c.

```
likelihood<-dbinom(58,100,theta)
post<-likelihood/sum(likelihood)
plot(theta,post, type = "l")
```



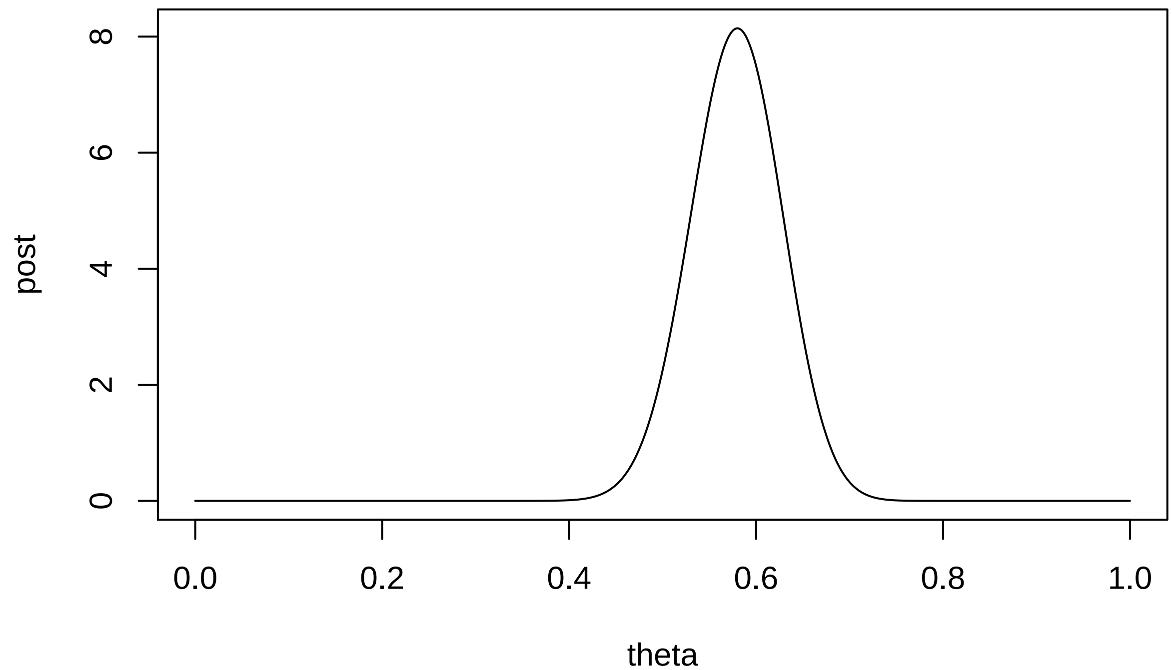
d.

```
theta<-seq(0,1,.001)
likelihood<-dbinom(58,100,theta)
plot(theta,likelihood, type = "l")
```



e.

```
post<-dbeta(theta,58+1,100+1-58)
plot(theta,post,type="l")
```



All the graphs have the same pattern across theta $\in [0, 1]$.

4.

a.

$p(ChildhasHh|Childhasbrowneyes, parentsbothhavebrowneyes)$

$= \frac{p(ChildhasHh, Childhasbrowneyes, parentsbothhavebrowneyes)}{p(Childhasbrowneyes, parentsbothhavebrowneyes)}$

$= \frac{p(childhasHh, parentsbothHh)+p(childhasHh, oneparenthasHh)+p(childhasHh, parentsbothHH)}{p(childhasbrowneyes, parentsbothHh)+p(childhasbrowneyes, oneparenthasHh)+p(childhasbrowneyes, parentsbothHH)}$

$= \frac{\frac{1}{2}4p^2(1-p)^2+\frac{1}{2}4p(1-p)^3+0(1-p)^4}{\frac{3}{4}4p^2(1-p)^2+4p(1-p)^3+(1-p)^4}$

$= \frac{2p}{1+2p}$

b.

i.

$p(JudiisHh|Judyhasbrowneyes, Judy'sparentshavebrowneyes, Judy'spartnerisHh, p)$

$= \frac{\frac{2p}{1+2p}\frac{3}{4}}{\frac{2p}{1+2p}\frac{3}{4}+\frac{1}{1+2p}} = \frac{3p}{3p+2}$

$\hat{p}^2 = .16 \Rightarrow \hat{p} = .4$

$\Rightarrow$ The posterior probability is $\frac{3}{8}$

ii.

The child must be of Hh for the grandchild to have blue eyes. Given $\hat{p} = .4$:

$p(childisHh|...) = \frac{3}{8}\frac{2}{3} + \frac{5}{8}\frac{1}{2} = \frac{9}{16}$

$p(grandchildisHh|...) = \frac{9}{16}(\frac{1}{4}2p(1-p) + \frac{1}{2}p^2) = \frac{9}{80}$