Robot Model and Equations of Motion

November 25, 2023

1 Robot Model: Quadrotor

We have decided to use Quadrotor as our robot. Quadrotor is a type of unmanned aerial vehicle (UAV) or drone that is lifted and propelled by four rotors. Each rotor is mounted at the end of a horizontal arm extending from the central body of the vehicle.

Figure 1: Quadrotor

2 Working of Quadrotors

The quadrotor is a highly non-linear, six degree-of-freedom and under-actuated system. It is under-actuated because inputs are only 4 from 4 motors and total degree-of-freedom is 6. It is controlled by varying the thrust forces of each rotor and balancing the drag torque. A quadrotor has two sets of counter-rotating propellers, therefore neutralizing the effective aerodynamic drag. It has four principal modes of operation(Figure 2):

1. Vertical movement is controlled by simultaneously increasing or decreasing the thrust of all rotors.
2. Yaw moment is created by proportionally varying the speeds of counter-rotating parts to have movement with respect to quadrotor's z-axis.
3. Roll can be controlled by applying differential thrust forces on opposite rotors of the quadrotor to have movement with respect to quadrotor's x-axis.
4. Pitch can be controlled by applying differential thrust forces on opposite rotors of the quadrotor to have movement with respect to quadrotor's y-axis.

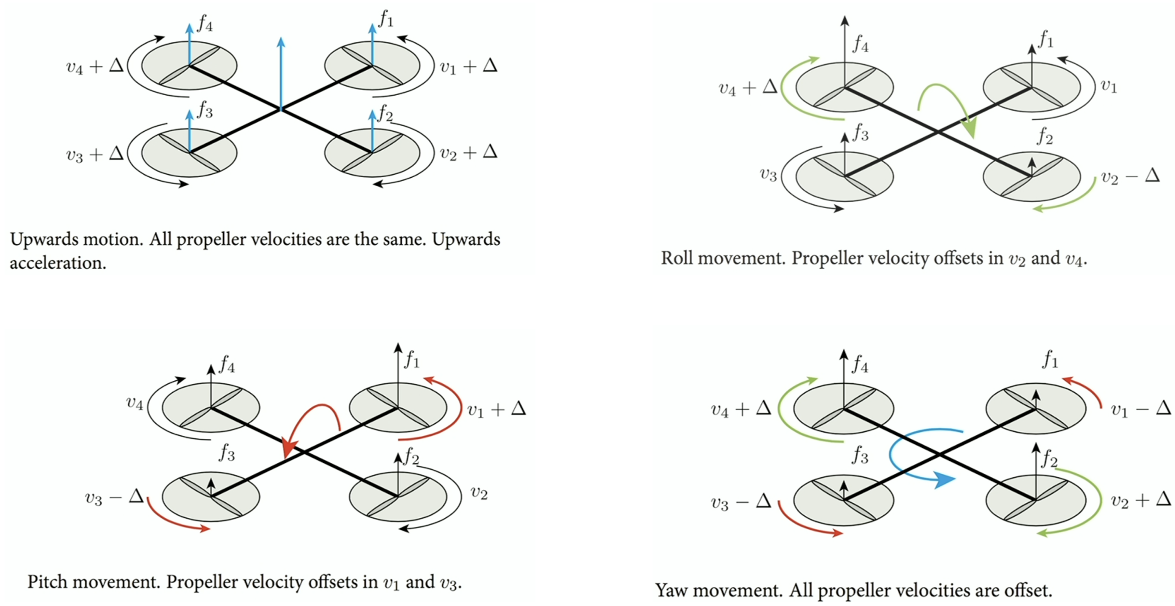

Figure 2: Different modes of operation

3 Quadrotor Model

3.1 Model of a rotor

Each rotor rotates with angular velocity ω and generates a lift force F and moment M. Moment is acting opposite to the directing of rotation. (Figure 3)

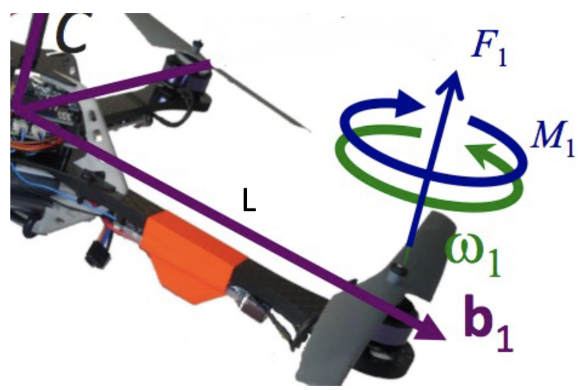

Figure 3: Quadrotor arm

The lift Force F and moment M of ith rotor can be calculated by:
$F_i = k_f * \omega_i^2$ and $k_f = k_T * \rho * D^4$
$M_i = k_m * \omega_i^2$ and $k_m = k_Q * \rho * D^5$

where:
k_T is thrust coefficient
k_Q is torque
rho is fluid density
D is diameter of propeller

3.2 Equations of Motion

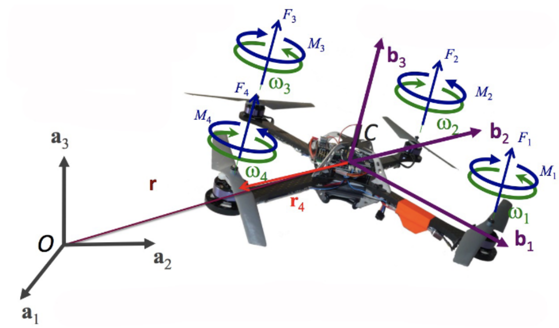

Figure 4: Quadrotor in inertial and body frames

Total thrust and moment is the sum of individual ones in each of the 4 rotors.

Thrust: $F = F_1 + F_2 + F_3 + F_4 - m * g$ Here, F_x are individual lift forces by the propellers and $m * g$ is the one by gravity.

Moment: $M = r_1 * F_1 + r_2 * F_2 + r_3 * F_3 + r_4 * F_4 + M_1 + M_2 + M_3 + M_4$ Here, $r_x * F_x$ are the moments created by forces in quadrotor's centre of gravity and M_x are the individual moments created by the propellers.

3.3 State Variables

u, ϕ, p are the linear velocity along the roll-axis direction, angle rotated and the angular velocity about the roll-axis. w, ψ, r is the linear velocity along the yaw-axis direction, angle rotated and the angular velocity about the yaw-axis. v, θ, q is the linear velocity along the pitch-axis direction, angle rotated and the angular velocity about the pitch-axis. x, y, z are the position of quadrotor along a_1, a_2, a_3 axis of the inertial frame.

3.4 Newton-Euler Equations for Quadrotor

Linear Dynamics Applying Newton's Second Law for system of particles, we get (in inertial frame);
$F = mass * acceleration$
$acceleration(\ddot{r}) = d\dot{r}/dt$, where $\dot{r} = [u, v, w]^T$ (3.3)

Since, w is the yaw-axis in which we calculate thrust, we get;

$$mass * \ddot{r} = \begin{bmatrix} 0 \\ 0 \\ -m * g \end{bmatrix} + R_\psi \phi \theta \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$$

Rotational Dynamics Applying Euler's rotation equations, we get (in body frame);

$M_c =^A dH_c^B/dt =^B dH_c^B/dt +^A \omega^B \times H_c^B$
where, H_c is the angular momentum and ${}^A\omega^B$ is angular velocity of body B in frame A which is given by $p.b_1 + q.b_2 + r.b_3$

General vector form of Euler's equation is; $M_c = I\dot{\omega} + \omega \times (I\omega)$

For Quadrotor, after rearranging the general vector form;

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Let $\gamma = k_M/k_F$, $M_i = \gamma F_i$, we get;

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & L & 0 & -L \\ -L & 0 & L & 0 \\ \gamma & -\gamma & \gamma & -\gamma \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

3.5 Final Joint Equation

Joint Equations using Linear and Rotational dynamics equations, we get;

$$\mathbf{u} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & L & 0 & -L \\ -L & 0 & L & 0 \\ \gamma & -\gamma & \gamma & -\gamma \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

where u is;

$$\begin{bmatrix} & thrust & \\ moment & about & x \\ moment & about & y \\ momemt & about & z \end{bmatrix}$$

4 Control Inputs

Control inputs are in the form of motor speeds for controlling thrust and 3 moments.

$$\begin{bmatrix} T \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} k_F & k_F & k_F & k_F \\ 0 & Lk_F & 0 & -Lk_F \\ -Lk_F & 0 & Lk_F & 0 \\ k_M & -k_M & k_M & -k_M \end{bmatrix} \begin{bmatrix} \omega_1^2 \\ \omega_2^2 \\ \omega_3^2 \\ \omega_4^2 \end{bmatrix}$$