Calculating the Directivity Factor γ of Transducers from Limited Polar Diagram Information[1]

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The calculation of the directivity factor γ of loudspeakers and microphones (including gun microphones) from polar diagrams measured in two planes only is described. Tables of "weights" for measurements taken at intervals of 10°, 15°, 20°, 22½°, 30°, and 45° are given. Some of the theory is also outlined which uses numerical integration on the surface of the sphere using a product Fejér/Clenshaw–Curtis rule. Rule-of-thumb estimates are given for the maximum directivity factors that can be estimated by this procedure for various intervals.

INTRODUCTION: In a series of notes, Davis [1], [2] and Wilson [3], [4] gave methods of determining the directivity factor γ of loudspeakers from polar diagrams measured in two planes, vertical and horizontal, through the nominal axis of the loudspeaker. Their methods involved integrating the total energy response of the polar diagram over all directions by adding together the energy gain in various directions, each being multiplied by a suitable coefficient or "weight." Unfortunately, their weights were determined by the area of a "slice" of the surface of the sphere of directions. Thus their methods are analogs on the sphere of midpoint or trapezium rules for ordinary numerical integration, and these methods are well known to be of poor accuracy [5].

We give a theoretically optimal method of determining the directivity factor γ of a loudspeaker or microphone from the polar diagrams taken in the vertical and horizontal planes through the nominal axis of the transducer. Since the theory involves some mathematical difficulties, we divide this paper into 2 parts. The first consists of a description of the practical use of our method of determining γ, with tables for data measured at intervals of 10°, 15°, 20°, 22½°, 30°, and 45°, and some "rules of thumb" to indicate when the estimate for γ is likely to be unreliable. The second part consists of the theory which involves a method of numerical integration similar to that proposed by Fejér [6] in 1933, and essentially the same as the method of Clenshaw and Curtis [7], [5, ch. 2 pp. 83-87].

PRACTICAL DETERMINATION OF γ

Suppose that we are given as data two polar diagrams (polar energy responses) $f(\theta)$ and $g(\theta')$, where θ is the angle from the forward axis in the horizontal plane and θ' the angle from the forward axis in the vertical plane. (If these are given in decibels as $F(\theta)$ and $G(\theta')$, then $f(\theta) = 10^{0.1F(\theta)} = \text{anti-log}_{10}\, 0.1F(\theta)$, and similarly for $g(\theta')$).

[1] We use the notation γ rather than Q or R_θ (see [3]) since the latter two can be misleading (R_θ depends on a function of *two* angles, θ and ϕ). γ has been used for "directivity factor" on several previous occasions in this *Journal* (see [8], [12]).

Suppose further that we have measured $f(\theta)$ and $g(\theta')$ at intervals of $180°/n$. The problem is to estimate the directivity factor γ of the exact polar diagram $h(\theta,\phi)$, where θ is the angle from the axis, and ϕ the angle around the axis. The mathematical definition of γ is

$$\gamma = \frac{h(\theta,\phi)_{\max}}{\frac{1}{4\pi}\iint h(\theta,\phi)\sin\theta\, d\theta\, d\phi} \tag{1}$$

where the integration is over all directions (θ,ϕ). This is simply the ratio of maximum energy gain to the average of the energy gains over all directions.

We may compute γ from our data as follows. Writing $(1/4\pi)\iint h(\theta,\phi)\sin\theta\, d\theta\, d\phi$ as $I[h]$, then

$$I[h] \cong w_0 f(0) + \tfrac{1}{4}\sum_{i=1}^{n-1} w_i\left[f\left(\frac{180°}{n}i\right) + f\left(-\frac{180°}{n}i\right) + g\left(\frac{180°}{n}i\right) + g\left(-\frac{180°}{n}i\right)\right] + w_n f(180°) \tag{2}$$

where the weights w_i will be given shortly. Note that $f(0) = g(0)$ and $f(180°) = g(180°)$. Then we have that

$$\gamma = h(\theta,\phi)_{\max}/I[h]. \tag{3}$$

Thus, as long as we know $h(\theta,\phi)_{\max}$, this completes our determination of γ. $h(\theta,\phi)_{\max}$ is often equal to $f(0)$, but for asymmetric polar diagrams it may be determined by inspection of the polar diagrams.

The weights w_i are given in Table I for $n = 4, 6, 8, 9, 12$, and 18, i.e., for $180°/n = 45°, 30°, 22\tfrac{1}{2}°, 20°, 15°$, and $10°$. Theoretically, as indicated in the following section, they are given by the formulas

$$w_0 = w_n = (1/2n){\sum_{r=0,2,4,\ldots}^{n}}'' -\frac{1}{r^2-1}$$
$$w_i = (1/n){\sum_{r=0,2,4,\ldots}^{n}}'' -\frac{1}{r^2-1}\cos\left(\frac{180°\, ri}{n}\right) \quad \text{for } i \neq 0, n \tag{4}$$

where the sums are over all even r up to n, and where the double prime indicates that the terms with $r = 0$ and (in the case that n is even) $r = n$ are halved.

If polar diagrams in additional planes through the forward axis are available (e.g., four planes at 45° from one another), then formula (2) is modified only in that each term of the $\sum_{i=1}^{n-1}$ term becomes an average over the, say, eight half-planes involved, rather than over the four half-polar-diagrams in the case already described.

Accuracy

Theoretically a formula averaging over only two planes in the manner indicated is unreliable for polar energy responses containing spherical harmonic components of order greater than 3 (see next section). The maximum directivity factor that can be achieved with such third-order energy polar diagrams is $\gamma = 6$, and so any estimate of γ in excess of, or comparable to, 6 (say, $\gamma \geq 4$) obtained by this method must be regarded as unreliable.

However, in many cases, such as gun microphones or paraboloidal reflector microphones, we have reason to suppose that there is a reasonable degree of symmetry about the axis of the polar diagram, i.e., that there are no spherical harmonic components of the polar diagram of the form $Y_l^m(\theta,\phi)$ (where ϕ is measured around the axis and θ from it) with large "azimuthal" frequency m (say, no $m > 3$). (See any text on spherical harmonics, e.g., [11], for the definition of Y_l^m). In this case the limiting factor in the measurement of γ will be the number $2n$ of points around each plane polar diagram used in the determination. In these circumstances, it turns out (see next section) that polar energy responses of spherical harmonic order up to n can be measured, with a highest possible directivity factor of around $(\tfrac{1}{2}n+1)^2$. Thus for $180°/n = 10°$, and assuming good (but not necessarily perfect) axial symmetry, directivity factors up to about $(\tfrac{1}{2}\times 18+1)^2 = 100$ can be measured, using data spaced apart by 10° intervals.

The weights given by Table I for 10° intervals do not differ very greatly from those given in [1]–[4], certainly not enough to affect results significantly from a practical viewpoint. The reasons for using the values given here may be summarized as follows.

1) The tables given here are the optimal weights from a theoretical viewpoint. Even though errors in the tables of [1]–[4] may be small in comparison with other sources of

Table I.

	Angle (degrees)	Weights w_i
$n = 4$	0 or 180	0.03333 3333
	45 or 135	0.26666 6667
	90	0.40000 0000
$n = 6$	0 or 180	0.01428 5714
	30 or 150	0.12698 4127
	60 or 120	0.22857 1429
	90	0.26031 7460
$n = 8$	0 or 180	0.00793 6508
	22½ or 157½	0.07310 9325
	45 or 135	0.13968 2540
	67½ or 112½	0.18085 8929
	90	0.19682 5397
$n = 9$	0 or 180	0.00617 2840
	20 or 160	0.05828 3728
	40 or 140	0.11264 2162
	60 or 120	0.15097 0018
	80 or 100	0.17193 1253
$n = 12$	0 or 180	0.00349 6503
	15 or 165	0.03302 8712
	30 or 150	0.06577 1266
	45 or 135	0.09238 1692
	60 or 120	0.11348 6512
	75 or 105	0.12633 7847
	90	0.13099 4931
$n = 18$	0 or 180	0.00154 7988
	10 or 170	0.01478 5498
	20 or 160	0.02998 0282
	30 or 150	0.04356 3973
	40 or 140	0.05613 7598
	50 or 130	0.06681 8462
	60 or 120	0.07559 9942
	70 or 110	0.08198 2300
	80 or 100	0.08596 0196
	90	0.08724 7523

error, it is advisable to minimize those sources of error that we can control.

2) The method given in this paper includes (see next section) an analysis of when the result is exact, and describes the source of errors in other cases.

3) The justification of the tables of weights in [1]–[4] relied on the assumption that the successive sample values of the polar diagrams changed slowly with angle. The present method only demands that the angular frequency components of the polar energy diagram have frequencies that are less than half the sampling frequency. This means that useful calculations can be made either when the polar diagram consists of many lobes of comparable magnitude, or when the sampling interval is quite a bit larger than 10°.

It is known [13] that many different methods of estimating the directivity factor (varying from crude methods of estimating the beamwidth to a full sphere integration) give results that are adequately close to the correct value, provided that the polar diagram consists of one main lobe plus other lobes of much smaller magnitude. The present method does not depend on this assumption, and so is also suitable for cases such as bidirectional transducers and transducers with multiple interference nulls within the forward polar diagram.

The spherical harmonic theory of the next section can sometimes give a useful guide as to when the method of this paper will give adequate results. Consider the transducer as lying in a snug-fitting cylinder whose axis coincides with the nominal transducer axis. Let the length of the cylinder be l and its diameter d. Then, as a rule of thumb, the polar energy diagram measured with a sound of wavelength λ will include possibly significant spherical harmonic components whose Fourier frequency around the axis are of orders up to $2\pi d/\lambda$, and whose plane polar diagrams measured through the nominal axis include Fourier components of order up to $2\pi l/\lambda$. Thus ideally, to ensure reasonable accuracy we should have $2\pi d/\lambda \leq 3$ and $2\pi l/\lambda \leq n$, where $n+1$ is the number of sample points used in each half-plane polar diagram.

These rule-of-thumb inequalities often may be grossly violated in practice with little harm, even though one can construct mathematically polar diagrams that give inaccurate estimates for γ when these conditions are grossly violated. Cases where the first condition matters little include transducers with almost perfect axial symmetry (e.g., paraboloidal reflector microphones). A case where the second condition can often be ignored is when the polar diagram is so ragged that it may be regarded as largely random in nature. In other cases, the highest order spherical harmonic components may in fact be of low amplitude. In these latter cases, a visual examination of the polar diagram will often suggest the Fourier frequency n of the highest order significant components, and an interval size of less than $180°/n$ may safely be used. It is, on the other hand, not possible to determine from the data in only two orthogonal planes whether two planes are enough. Ideally, polar diagrams in four or more planes through the axis should be used, but the two-plane method has the advantage of using easily available data.

Two sources of error have not been discussed at all in this paper, but should be recognized. First, if the value $h(\theta,\phi)_{\max}$ does not lie on one of the measured polar diagrams, then our estimate for γ has the same proportionate error due to this cause as the estimate used for $h_{\max}$. We have not investigated the best way of estimating $h_{\max}$ from polar diagrams in two planes only. Second, if we measure transducers that are extended in space or that have large γ, then measurements made too close to the transducer (in the near field) will not give the same polar diagram as that measured at spatial infinity. For transducers not involving velocity transducer elements, the near field for a sound of wavelength λ consists roughly of points whose distance from the transducer is less than about $\frac{1}{4}\pi d^2/\lambda$, where d is the maximum spatial extent (say in meters) of the transducer. It will be seen that we are often forced to make measurements in the near field, and it is not easy to say whether this introduces significant errors in the estimation of γ. We expect this error to be worse for the case (e.g., loudspeakers, reflector microphones) where the maximum spatial extent is orthogonal to the nominal axis rather than along it.

THEORY

We wish to determine $I[h]$ over the surface of a sphere numerically. Numerical integration over the surface of the sphere is now well understood (see McLaren [9] and Stroud [10]), but numerical analysts are primarily concerned with choosing accurate numerical rules involving the smallest possible number of points on the sphere, rather than with choosing these points conveniently for experimental measurement. For example, ideally we should use the 72-point precision 14 (i.e., it is accurate for up to 14th-order spherical harmonics) rule described by McLaren [9] and Stroud [10, ch. 8 p. 302], but this involves using some experimentally very awkward directions in space around our transducer, and these data are somewhat unlikely to be available in published information.

Given information in just the two planes, and then only at equal angular intervals, we shall have to use a product rule [10, ch.2] for numerical integration on the sphere. Denote a direction by its direction cosines (x, y, z) with $x^2 + y^2 + z^2 = 1$. Note that $z = \cos\theta$, $x = \cos\phi \sin\theta$, and $y = \sin\theta \sin\phi$, where θ is the angle from the axis, and ϕ is that around it. Then

$$I[h] = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} h(\cos\phi \sqrt{(1-z^2)}\,, \sin\phi \sqrt{(1-z^2)}, z)\, d\phi\, dz \tag{5}$$

from Eq. (1) by the change of variable $z = \cos\theta$, where $h(x, y, z)$ denotes the polar diagram as a function of direction cosines.

Now consider, for a moment, integrals of functions in one of the variables ϕ and z only. Let $j(\phi)$ be a function of the angle ϕ. Then it is well known (and pretty obvious) that

$$\frac{1}{2\pi} \int_{0}^{2\pi} j(\phi)\, d\phi \cong \frac{1}{m} \sum_{i=1}^{m} j(\frac{360°}{m} i), \tag{6}$$

where the integration rule (6) is exact for functions $j(\phi)$ with no Fourier components above the $(m-1)$th harmonic.

Moreover, for a function $k(z)$ we may always put

$$½\int_{-1}^{1} k(z)\,dz \cong \sum_{i=0}^{n} w_i k(z_i)\,, \tag{7}$$

where the weights w_i and points z_i may be chosen such that Eq. (7) is exact for all polynomials $k(z)$ of degree $\leq n$.

Now $h(\cos\phi\,\sqrt{(1-z^2)}, \sin\phi\,\sqrt{(1-z^2)}, z)$ as a function of ϕ certainly has no Fourier components above nth harmonic if $h(x, y, z)$ has no spherical harmonic component above nth spherical harmonic, and

$$\frac{1}{2\pi}\int_0^{2\pi} h(\cos\phi\,\sqrt{(1-z^2)},\ \sin\phi\,\sqrt{(1-z^2)},\ z)\,d\phi$$

is clearly a polynomial function of the nth degree in z, since $h(x, y, z)$ is also. Putting Eqs. (6) and (7) together, we thus have

$$I[h] \cong \sum_{i=0}^{n} w_i\left[\frac{1}{m}\sum_{j=1}^{m} h\left(\cos\left(\frac{360°}{m}j\right)\sqrt{(1-z_i^2)},\right.\right.$$
$$\left.\left.\sin\left(\frac{360°}{m}j\right)\sqrt{(1-z_i^2)},\ z_i\right)\right]\,, \tag{8}$$

which is exact provided that h has no spherical harmonic components of order exceeding n, and that h as a function of ϕ has no Fourier components of greater than $(m-1)$th harmonic. The formula (8) is termed a product rule for the surface of the sphere (see [10, ch. 2, pp. 40-43]).

In our case, we require $m = 4$, so as to use points in two perpendicular planes disposed around the nominal axis of the transducer. Thus our rule necessitates that h as a function of ϕ have no Fourier components of greater than third harmonic.

We also require in our case, because of our preassigned choice of measurement points at equal angles around a polar diagram, that

$$z_i = \cos\left(\frac{180°}{n}i\right). \tag{9}$$

The choice of the w_i in Eq. (7) that leads to the most accurate integration rule (in the sense of integrating exactly polynomials $k(z)$ of as high a degree as possible) with the particular points (9) is known as the "interpolatory rule" for the points (9). This interpolatory rule is in fact that described by Clenshaw and Curtis [7], [5, ch. 2 pp. 83-87], although they do not give the w_i explicitly in the form (4). Formulas very similar to Eq. (4) are given by Fejér [6] for interpolatory rules for the points $z_i = \cos([180°/(n+1)]i)$ with $i = 1,2,\ldots,n$, and for $z_i = \cos((180°/n)(i-½))$ with $i = 1,2,\ldots,n$.

Since an explicit derivation for Eq. (4) does not seem to be published, we give a quick proof here, but see also [7], [5, pp. 83-87]. If Eq. (7) obeying Eq. (9) is to be accurate for all polynomials $k(z)$ of degree $\leq n$, then for all $0 \leq m \leq n$,

$$½\int_{-1}^{1} T_m(z)\,dz = \sum_{i=0}^{n} w_i T_m(\cos\theta_i) \tag{10}$$

where $\theta_i = 180°\,i/n$ and $T_m(z)$ is the mth degree Chebyshev polynomial defined by

$$T_m(\cos\theta) = \cos m\theta.$$

Thus,

$$½\int_{-1}^{1} T_m(z)\,dz = ½\int_0^{\pi} \cos m\theta\,\sin\theta\,d\theta$$
$$= ¼\int_0^{\pi} \sin(m+1)\theta - \sin(m-1)\theta\,d\theta$$
$$= 0 \quad \text{if } m \text{ odd}$$
$$= \frac{-1}{m^2-1} \text{ if } m \text{ even.}$$

Thus

$$\sum_{i=0}^{n} w_i \cos m\theta_i = 0 \quad \text{if } m \text{ odd} \tag{11}$$
$$= \frac{-1}{m^2-1} \quad \text{if } m \text{ even.}$$

Note that the left-hand side of (11) is the Fourier cosine transform of w_i, and use the fact that

$$\frac{1}{n}\sum_{i=0}^{n}{}''\cos r\theta_i \cos m\theta_i = \begin{cases} 0 & \text{if } r \neq m \\ 1 & \text{if } r = m = 0 \text{ or } r = m = n \\ ½ & \text{if } r = m \text{ and } 0 < r < n \end{cases}$$

where the double prime indicates to halve the $i = 0$ and $i = n$ terms to get

$$w_0 = w_n = (1/2n)\sum_{r=0,2,4,\ldots}^{n}{}'' \frac{-1}{r^2-1}$$

$$w_i = (1/n)\sum_{r=0,2,4,\ldots}^{n}{}'' \frac{-1}{r^2-1}\cos r\theta_i \quad \text{when } 0 < i < n$$

which proves Eq. (4).

To summarize, we have shown that the method of determining $I[h]$ given in the preceding section is exact provided that h is not of order greater than n and has no Fourier component of order greater than 3 about the nominal axis at which the two planes of measurement meet. This is clearly the best we can do with the points at our disposal, since we cannot better Eq. (6) for m points on the circumference of a circle, and the w_i have been chosen to be the best possible.

Finally, we give some degree of justification for the rule of thumb for estimating the highest γ that can be reliably measured as follows. The "polar diagrams" that we have considered are all polar energy responses, and not polar amplitude responses. Since energy is the square of amplitude, an amplitude polar diagram of spherical harmonic order $½n$ will give an energy polar diagram of order n. By the directivity factor theorem of [8], the maximum γ that can be obtained from an amplitude polar diagram of order $½n$ is $\gamma = (½n+1)^2$. Thus the rule (2) with Eq. (4), which is reliable only for polar energy responses of order $\leq n$, is certainly unreliable for γ exceeding $(½n+1)^2$.

Strictly, the above argument only applies when n is even, but in practice, it is a useful guide also for odd n. Take, for example, the case of energy polar diagrams of order $\leqslant 3$ (we have already shown that γ can be computed exactly in this case). An axially symmetric polar diagram of this type may be written $h(x, y, z) = az^3 + bz^2 + cz + d$. Because h is a polar energy response, it must be positive for $-1 \leqslant z \leqslant 1$, so that $\gamma = h_{\max}/I[h]$ is maximized for a function of the form $h(x, y, z) = (z+1+\beta)(z+1-\alpha)^2$, where $\beta \geqslant 0$ and $0 \leqslant \alpha < 2$, if we put the maximum of h at $z = 1$. The usual methods of finding the maximum for γ shows that this is achieved for $\beta = 0$ and $\alpha = 1$, for which values $\gamma = 6$. Thus the highest γ for third/order polar energy responses is $\gamma = 6$.

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