Arithmetic Coding

CS6025 Data Encoding

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2-3-15

A Problem with Huffman Coding

- The Huffman method is simple, efficient, and produces the best codes for the individual data symbols.
- The Huffman method assigns a code with an integral number of bits to each symbol in the alphabet.
- Information theory shows that a symbol with probability 0.4 should ideally be assigned a 1.32-bit code, since $-\log 2\ 0.4 \approx 1.32$.
- The Huffman method, however, normally assigns such a symbol a code of 1 or 2 bits.

Interval Narrowing

- Arithmetic coding overcomes the problem of assigning integer codes to the individual symbols by assigning one (normally long) code to the entire input file.
- The method starts with a certain interval, it reads the input file symbol by symbol, and it uses the probability of each symbol to narrow the interval.
- The output of arithmetic coding is interpreted as a number in the range [0, 1).

Dividing an Interval for Symbols

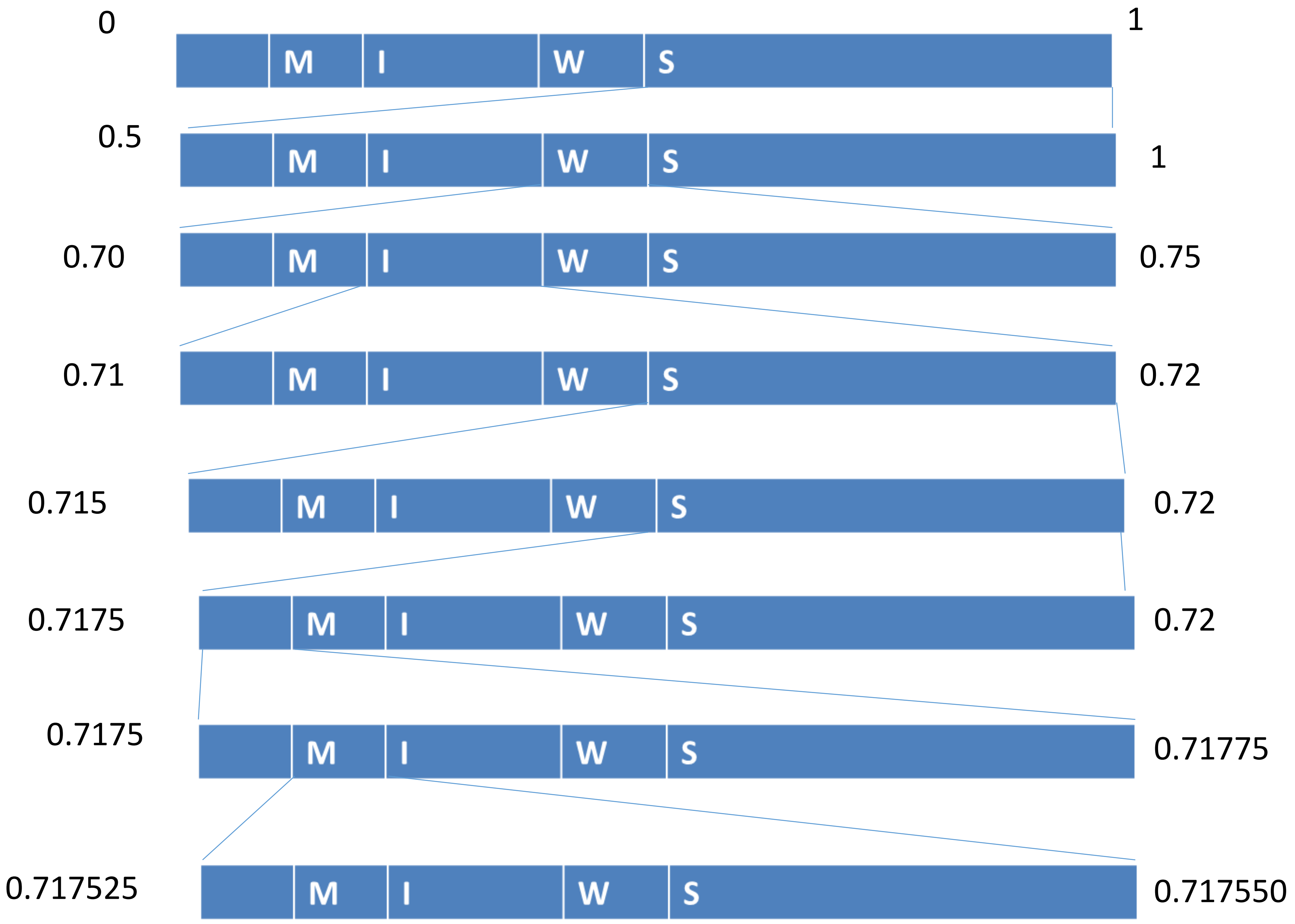
Char	Freq	Prob.	Range	CumFreq
		Total CumFreq=		10
S	5	$5/10 = 0.5$	$[0.5, 1.0)$	5
W	1	$1/10 = 0.1$	$[0.4, 0.5)$	4
I	2	$2/10 = 0.2$	$[0.2, 0.4)$	2
M	1	$1/10 = 0.1$	$[0.1, 0.2)$	1
␣	1	$1/10 = 0.1$	$[0.0, 0.1)$	0

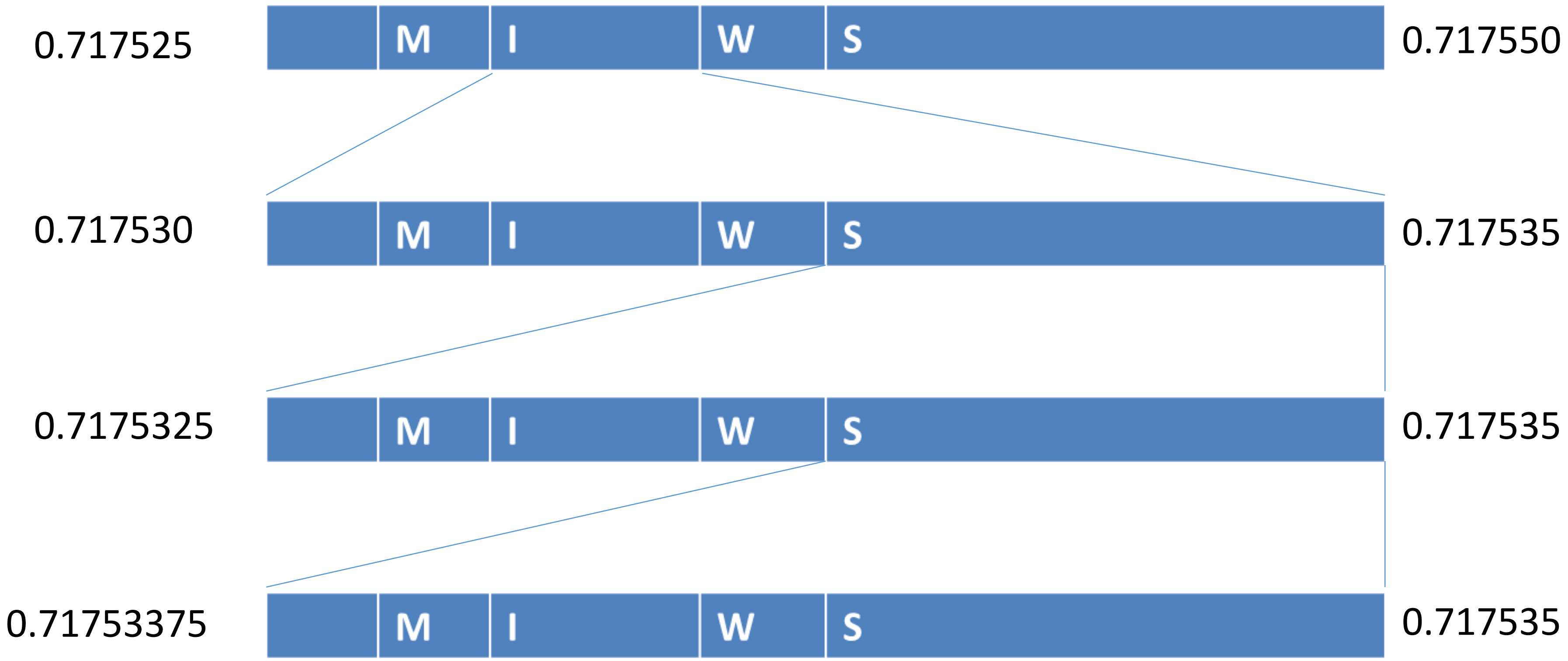
Table 5.42: Frequencies and Probabilities of Five Symbols.

```
NewHigh:=OldLow+Range*HighRange(X);
NewLow:=OldLow+Range*LowRange(X);
```

```
Code:=(Code-LowRange(X))/Range;
```

Char.		The calculation of low and high
S	L	$0.0 + (1.0 - 0.0) \times 0.5 = 0.5$
	H	$0.0 + (1.0 - 0.0) \times 1.0 = 1.0$
W	L	$0.5 + (1.0 - 0.5) \times 0.4 = 0.70$
	H	$0.5 + (1.0 - 0.5) \times 0.5 = 0.75$
I	L	$0.7 + (0.75 - 0.70) \times 0.2 = 0.71$
	H	$0.7 + (0.75 - 0.70) \times 0.4 = 0.72$
S	L	$0.71 + (0.72 - 0.71) \times 0.5 = 0.715$
	H	$0.71 + (0.72 - 0.71) \times 1.0 = 0.72$
S	L	$0.715 + (0.72 - 0.715) \times 0.5 = 0.7175$
	H	$0.715 + (0.72 - 0.715) \times 1.0 = 0.72$
␣	L	$0.7175 + (0.72 - 0.7175) \times 0.0 = 0.7175$
	H	$0.7175 + (0.72 - 0.7175) \times 0.1 = 0.71775$
M	L	$0.7175 + (0.71775 - 0.7175) \times 0.1 = 0.717525$
	H	$0.7175 + (0.71775 - 0.7175) \times 0.2 = 0.717550$
I	L	$0.717525 + (0.71755 - 0.717525) \times 0.2 = 0.717530$
	H	$0.717525 + (0.71755 - 0.717525) \times 0.4 = 0.717535$
S	L	$0.717530 + (0.717535 - 0.717530) \times 0.5 = 0.7175325$
	H	$0.717530 + (0.717535 - 0.717530) \times 1.0 = 0.717535$
S	L	$0.7175325 + (0.717535 - 0.7175325) \times 0.5 = 0.71753375$
	H	$0.7175325 + (0.717535 - 0.7175325) \times 1.0 = 0.717535$


0
1
M
I
W
S
0.5
1
0.70
0.75
0.71
0.72
0.715
0.72
0.7175
0.72
0.7175
0.71775
0.717525
0.717550

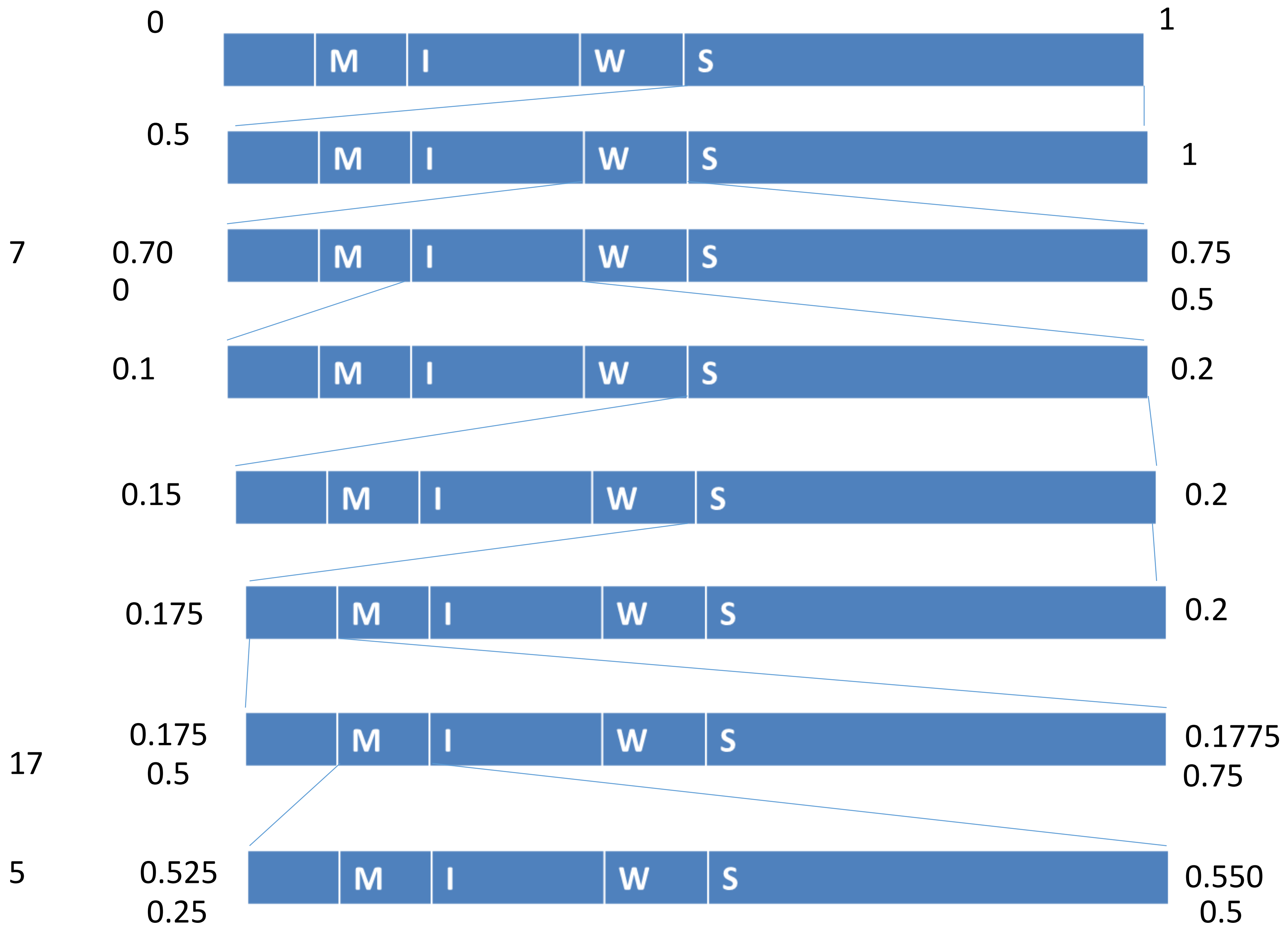

0.717525
M
I
W
S
0.717550
0.717530
M
I
W
S
0.717535
0.7175325
M
I
W
S
0.717535
0.71753375
M
I
W
S
0.717535

Char.	Code−low		Range
S	$0.71753375 - 0.5$	$= 0.21753375$	$/0.5 = 0.4350675$
W	$0.4350675 - 0.4$	$= 0.0350675$	$/0.1 = 0.350675$
I	$0.350675 - 0.2$	$= 0.150675$	$/0.2 = 0.753375$
S	$0.753375 - 0.5$	$= 0.253375$	$/0.5 = 0.50675$
S	$0.50675 - 0.5$	$= 0.00675$	$/0.5 = 0.0135$
␣	$0.0135 - 0$	$= 0.0135$	$/0.1 = 0.135$
M	$0.135 - 0.1$	$= 0.035$	$/0.1 = 0.35$
I	$0.35 - 0.2$	$= 0.15$	$/0.2 = 0.75$
S	$0.75 - 0.5$	$= 0.25$	$/0.5 = 0.5$
S	$0.5 - 0.5$	$= 0$	$/0.5 = 0$

Table 5.45: The Process of Arithmetic Decoding.

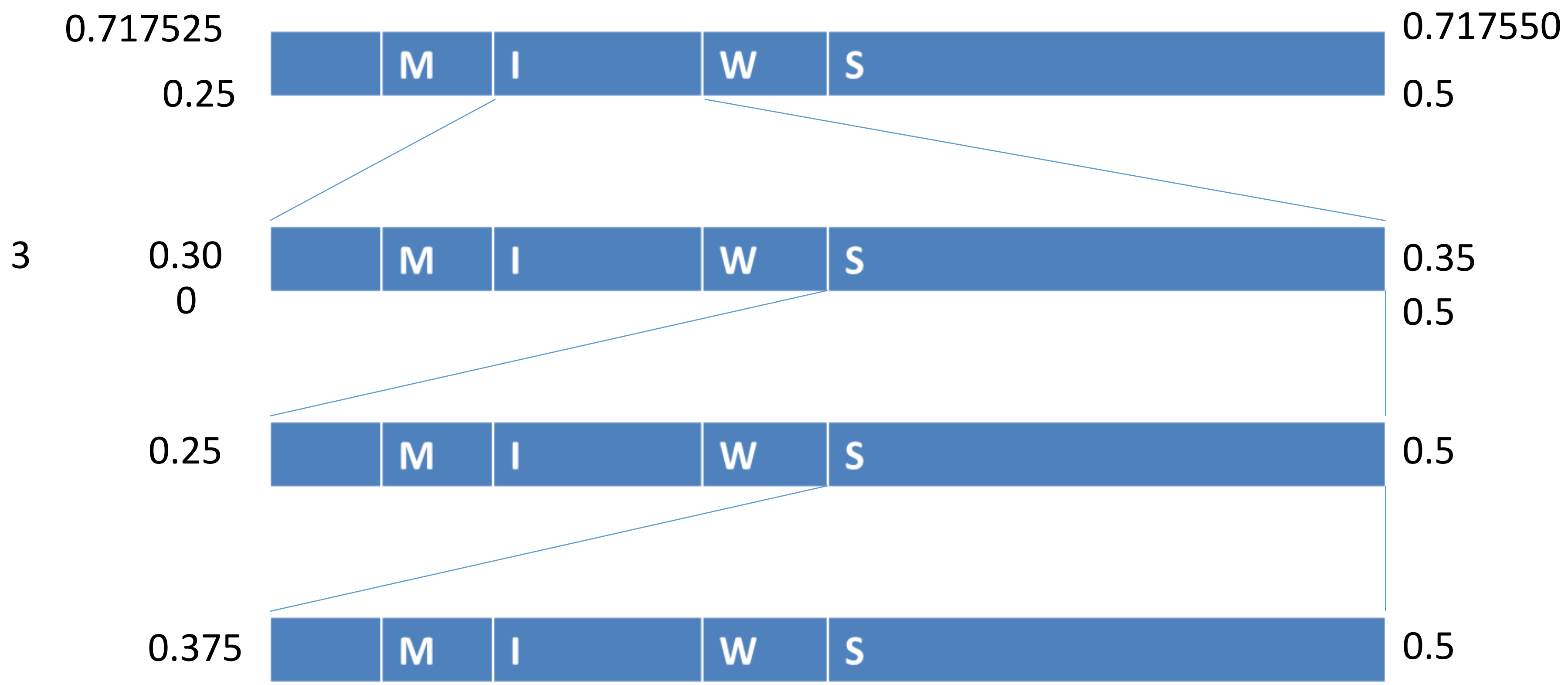
1	2	3	4	5
S	$L = 0+(1 - 0)\times 0.5 = 0.5$	5000		5000
	$H = 0+(1 - 0)\times 1.0 = 1.0$	9999		9999
W	$L = 0.5+(1 - .5)\times 0.4 = 0.7$	7000	7	0000
	$H = 0.5+(1 - .5)\times 0.5 = 0.75$	7499	7	4999
I	$L = 0+(0.5 - 0)\times 0.2 = 0.1$	1000	1	0000
	$H = 0+(0.5 - 0)\times 0.4 = 0.2$	1999	1	9999
S	$L = 0+(1 - 0)\times 0.5 = 0.5$	5000		5000
	$H = 0+(1 - 0)\times 1.0 = 1.0$	9999		9999
S	$L = 0.5+(1 - 0.5)\times 0.5 = 0.75$	7500		7500
	$H = 0.5+(1 - 0.5)\times 1.0 = 1.0$	9999		9999
␣	$L = 0.75+(1 - 0.75)\times 0.0 = 0.75$	7500	7	5000
	$H = 0.75+(1 - 0.75)\times 0.1 = 0.775$	7749	7	7499
M	$L = 0.5+(0.75 - 0.5)\times 0.1 = 0.525$	5250	5	2500
	$H = 0.5+(0.75 - 0.5)\times 0.2 = 0.55$	5499	5	4999
I	$L = 0.25+(0.5 - 0.25)\times 0.2 = 0.3$	3000	3	0000
	$H = 0.25+(0.5 - 0.25)\times 0.4 = 0.35$	3499	3	4999
S	$L = 0+(0.5 - 0)\times 0.5 = .25$	2500		2500
	$H = 0+(0.5 - 0)\times 1.0 = 0.5$	4999		4999
S	$L = 0.25+(0.5 - 0.25)\times 0.5 = 0.375$	3750		3750
	$H = 0.25+(0.5 - 0.25)\times 1.0 = 0.5$	4999		4999

Table 5.51: Encoding SWISS␣MISS by Shifting.



0
1
M I W S
0.5
1
7
0.70
0
0.75
0.5
0.1
0.2
0.15
0.2
0.175
0.2
17
0.175
0.5
0.1775
0.75
5
0.525
0.25
0.550
0.5

7175



0.71753375

0. Initialize `Low`=0000, `High`=9999, and `Code`=7175.

1. `index`= $[(7175-0+1)\times 10-1]/(9999-0+1)=7.1759 \rightarrow 7$. Symbol `S` is selected.
`Low` $=0+(9999-0+1)\times 5/10=5000$. `High` $=0+(9999-0+1)\times 10/10-1=9999$.

2. `index`= $[(7175-5000+1)\times 10-1]/(9999-5000+1)=4.3518 \rightarrow 4$. Symbol `W` is selected.
`Low` $=5000+(9999-5000+1)\times 4/10=7000$. `High` $=5000+(9999-5000+1)\times 5/10-1=7499$.
After the 7 is shifted out, we have `Low`=0000, `High`=4999, and `Code`=1753.

3. `index`= $[(1753-0+1)\times 10-1]/(4999-0+1)=3.5078 \rightarrow 3$. Symbol `I` is selected.
`Low` $=0+(4999-0+1)\times 2/10=1000$. `High` $=0+(4999-0+1)\times 4/10-1=1999$.
After the 1 is shifted out, we have `Low`=0000, `High`=9999, and `Code`=7533.

4. `index`= $[(7533-0+1)\times 10-1]/(9999-0+1)=7.5339 \rightarrow 7$. Symbol `S` is selected.
`Low` $=0+(9999-0+1)\times 5/10=5000$. `High` $=0+(9999-0+1)\times 10/10-1=9999$.

5. $\texttt{index}=[(7533-5000+1)\times 10-1]/(9999-5000+1)=5.0678\rightarrow 5$. Symbol $\texttt{S}$ is selected.
$\texttt{Low}=5000+(9999-5000+1)\times 5/10=7500$. $\texttt{High}=5000+(9999-5000+1)\times 10/10-1=9999$.

6. $\texttt{index}=[(7533-7500+1)\times 10-1]/(9999-7500+1)=0.1356\rightarrow 0$. Symbol ␣ is selected.
$\texttt{Low}=7500+(9999-7500+1)\times 0/10=7500$. $\texttt{High}=7500+(9999-7500+1)\times 1/10-1=7749$.
After the 7 is shifted out, we have $\texttt{Low}$=5000, $\texttt{High}$=7499, and $\texttt{Code}$=5337.

7. $\texttt{index}=[(5337-5000+1)\times 10-1]/(7499-5000+1)=1.3516\rightarrow 1$. Symbol $\texttt{M}$ is selected.
$\texttt{Low}=5000+(7499-5000+1)\times 1/10=5250$. $\texttt{High}=5000+(7499-5000+1)\times 2/10-1=5499$.
After the 5 is shifted out we have $\texttt{Low}$=2500, $\texttt{High}$=4999, and $\texttt{Code}$=3375.

8. `index`$= [(3375 - 2500 + 1) \times 10 - 1]/(4999 - 2500 + 1) = 3.5036 \rightarrow 3$. Symbol `I` is selected.
`Low` $= 2500+(4999-2500+1)\times 2/10 = 3000$. `High` $= 2500+(4999-2500+1)\times 4/10-1 = 3499$.
After the 3 is shifted out we have `Low`=0000, `High`=4999, and `Code`=3750.

9. `index`$= [(3750 - 0 + 1) \times 10 - 1]/(4999 - 0 + 1) = 7.5018 \rightarrow 7$. Symbol `S` is selected.
`Low` $= 0 + (4999 - 0 + 1) \times 5/10 = 2500$. `High` $= 0 + (4999 - 0 + 1) \times 10/10 - 1 = 4999$.

10. `index`$= [(3750 - 2500 + 1) \times 10 - 1]/(4999 - 2500 + 1) = 5.0036 \rightarrow 5$. Symbol `S` is selected.
`Low` $= 2500+(4999-2500+1)\times 5/10 = 3750$. `High` $= 2500+(4999-2500+1)\times 10/10-1 = 4999$.

Symbol Probability Estimation Using the Current Document

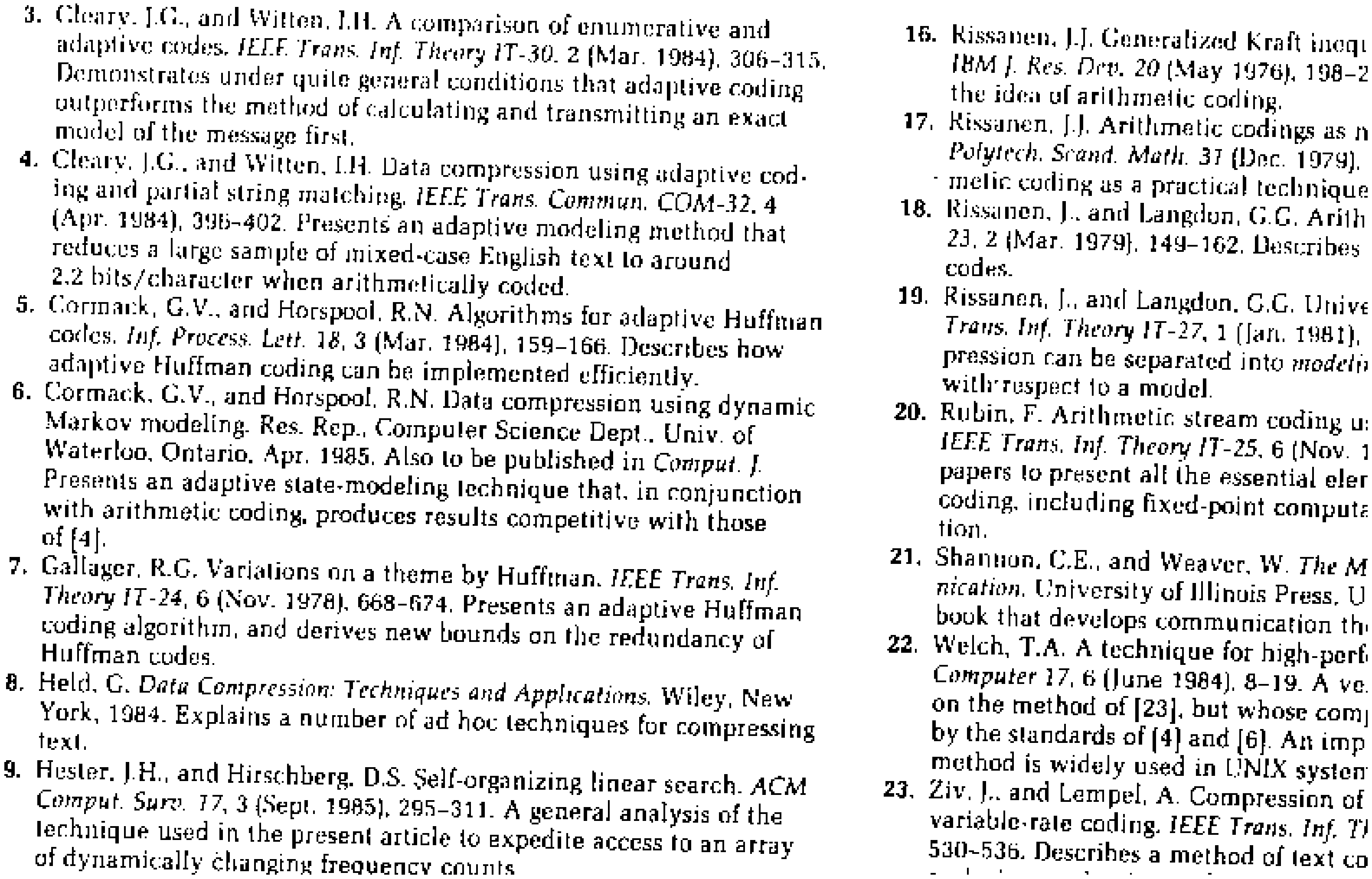
- Arithmetic coding assigns a codeword to each sequence of symbols based on probabilities of symbols.
- Each document may have its own personality.
- Use the processed part of the document to estimate the probabilities of the symbols.
  - Both encoder and decoder can do the same.
  - No probability table needs to be sent.

Symbols in Context

- Part of the processed document may be used as the context to the next symbol.
- Under different context, a symbol may have different probabilities.
- To do adaptive coding, a count for each symbol and each context.
- May further reduce the entropy of data.
- Shorter codewords for documents with distinct personalities or repeated patterns.

Bi-Leval Image

Bi-Level Image for Fax



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4. Cleary, J.G., and Witten, I.H. Data compression using adaptive coding and partial string matching. *IEEE Trans. Commun. COM-32*, 4 (Apr. 1984), 396–402. Presents an adaptive modeling method that reduces a large sample of mixed-case English text to around 2.2 bits/character when arithmetically coded.
5. Cormack, G.V., and Horspool, R.N. Algorithms for adaptive Huffman codes. *Inf. Process. Lett. 18*, 3 (Mar. 1984), 159–166. Describes how adaptive Huffman coding can be implemented efficiently.
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metic coding as a practical technique
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codes.
19. Rissanen, J., and Langdon, G.G. Unive
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22. Welch, T.A. A technique for high-perf
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on the method of [23], but whose com
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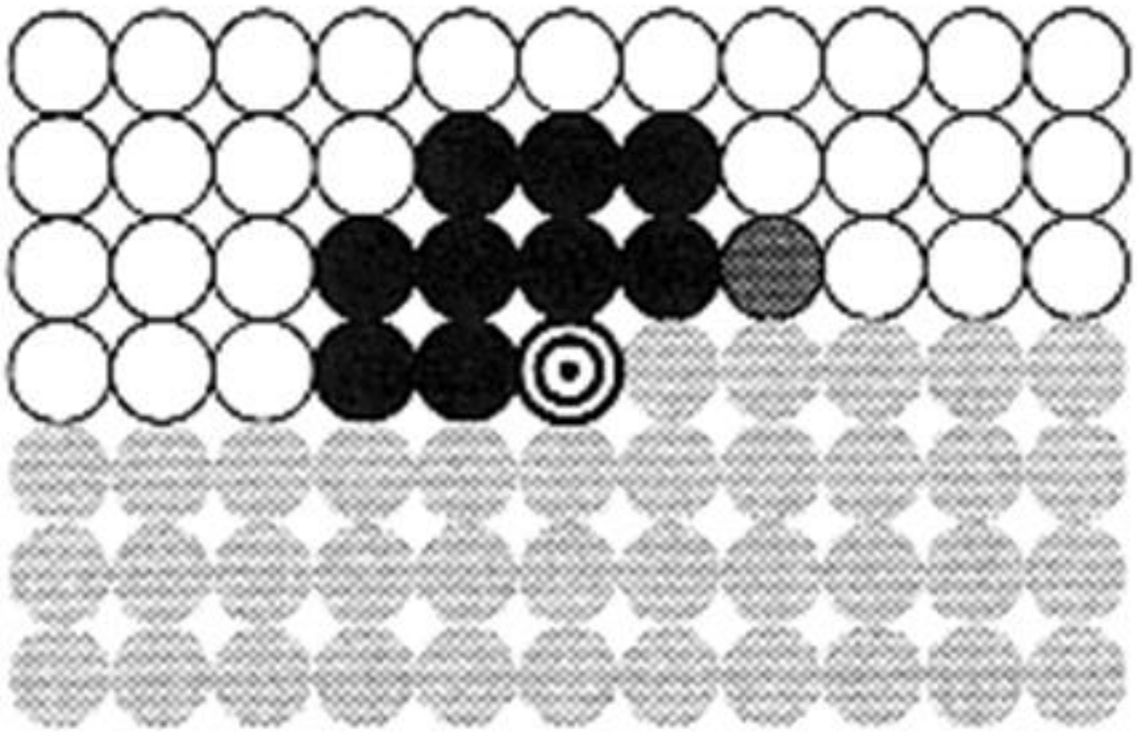
23. Ziv, J., and Lempel, A. Compression of
variable-rate coding. *IEEE Trans. Inf. T*
530–536. Describes a method of text co

PBM Portable Bitmap Format

- A UNIX format (only deals with P4)
- P4\n   // PBM identifier for binary bi-level
- 1024 896\n // width and height
- height * (width / 8) bytes following
- One bit for each pixel
- Left to right
- Top to bottom
- Viewable with Photoshop or H7C.java

Context in JBIG



Adaptive Arithmetic Coding

```
public class H7A{

   static final int maxRange = 65536;  // 2 ** 16
   static final int half = 32768;
   static final int quarter = 16384;
   static final int threequarters = 49152;
   static final int numberOfContexts = 1024;
   int width = 0, height = 0, bytesPerRow = 0;
     // dimension of the image
   boolean[][] bitmap = null;

   int[][] count = new int[numberOfContexts][2];
   int low = 0; int high = maxRange; int follow = 0;
   int buf = 0; int position = 0;

    public static void main(String[] args){
         H7A h7 = new H7A();
         h7.readPBMHeader();
         h7.compress();
    }
```

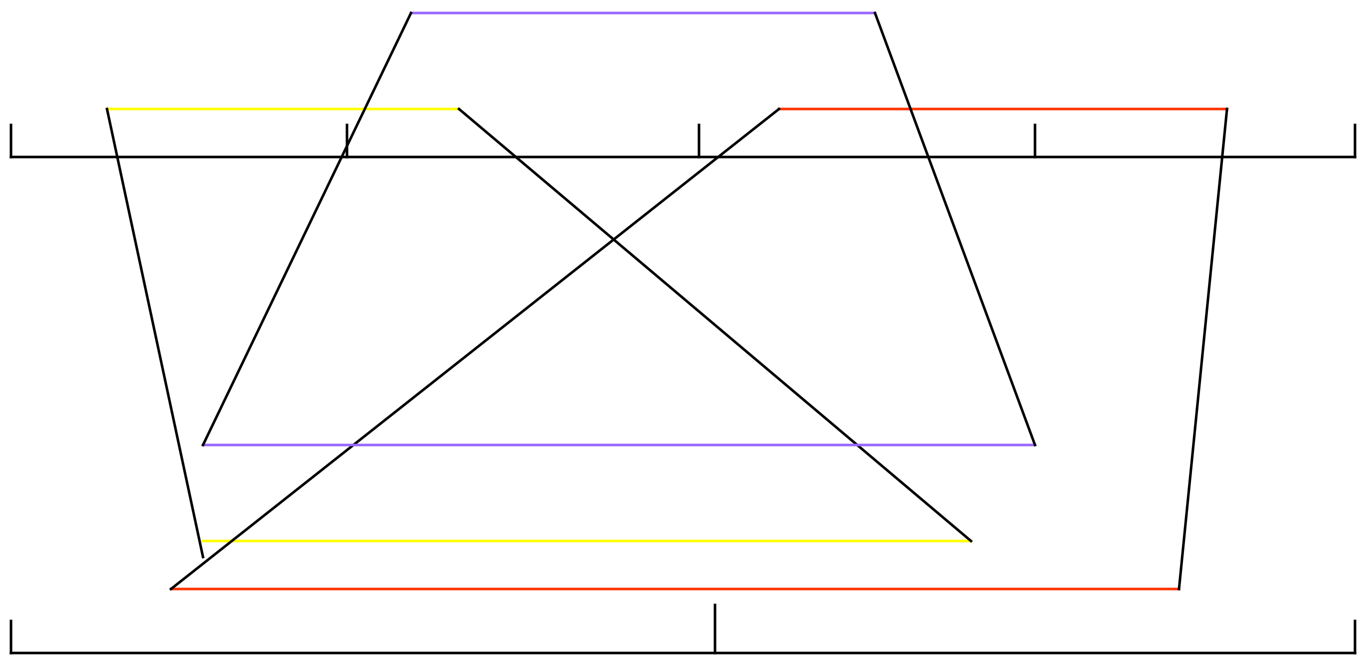
```
void compress(){
   for (int i = 0; i < height; i++){
      int column = 2;
      for (int j = 0; j < bytesPerRow; j++){
         int b = getNextByte();
         for (int test = 0x80; test > 0; test >>= 1){
           int context = getContext(column);
           boolean one = ((test & b) != 0);
           update(context, one);
           incrementCount(context, one);
           bitmap[2][column++] = one;
         }
     }
     for (int j = 2; j < width + 2; j++){
   bitmap[0][j] = bitmap[1][j];  bitmap[1][j] = bitmap[2][j]; }
    }
    if (position > 0){  buf <<= (8 - position);
      System.out.write(buf); }
    System.out.flush();
}
```

getContext

```
int getContext(int column){
  // column >= 2
  int context = 0;
  for (int k = -1; k < 2; k++){
    context <<= 1;
    if (bitmap[0][column + k]) context |= 1;
  }
  for (int k = -2; k < 3; k++){
    context <<= 1;
    if (bitmap[1][column + k]) context |= 1;
  }
  for (int k = -2; k < 0; k++){
    context <<= 1;
    if (bitmap[2][column + k]) context |= 1;
  }
  return context;
}
```

```
void update(int context, boolean one){
   int t = low + count[context][0] * (high - low) /
       (count[context][0] + count[context][1]);
   if (one) low = t; else high = t;
   for (;;){  // double until larger than quarter
      if (high < half){ // most significant bit is 0
        outputBit(0);
        for (int i = 0; i < follow; i++) outputBit(1);
        follow = 0;
        high *= 2;
        low *= 2;
      }else if (low >= half){  // most significant bit is 1
        outputBit(1);
        for (int i = 0; i < follow; i++) outputBit(0);
        follow = 0;
        high = (high * 2) - maxRange;
        low = (low * 2) - maxRange;
      }else if (low > quarter && high <= threequarters){
        follow++; // most significant bit unknown yet
        low = (low * 4 - maxRange) / 2;
        high = (high * 4 - maxRange) / 2;
      }else break;
   }
}
```

Three Cases for Doubling the Current Interval



incrementCount

```
void incrementCount(int context, boolean one){
  int v = one ? 1 : 0;
  if (++count[context][v] >= quarter){  // halve counts
    count[context][0] >>= 1; count[context][1] >>= 1;
    if (count[context][1 - v] == 0) count[context][1 - v] = 1;
  }
}

void outputBit(int bit){
   buf <<= 1;
   if (bit == 1) buf |= 1;
   position++;
   if (position == 8){
     position = 0;
     System.out.write(buf);
     buf = 0;
   }
 }
```

Decoder Program

- The same set of global variables except `follow` is replaced by `codeword.`
- Need a new procedure int inputBit().
- Codeword initialized with the first (most significant) 16 bits of the full codeword.
- Codeword is always between low and high and goes through the same doubling as them followed by adding inputBit().

Adaptive Arithmetic Decoder

```
public class H7B{

   static final int maxRange = 65536;  // 2 ** 16
   static final int half = 32768;
   static final int quarter = 16384;
   static final int threequarters = 49152;
   static final int numberOfContexts = 1024;
   int width = 0, height = 0, bytesPerRow = 0;
   boolean[][] bitmap = null;
   int[][] count = new int[numberOfContexts][2];
   int low = 0; int high = maxRange;
   int inBuf = 0; int inPosition = 0;
   int outBuf = 0; int outPosition = 0;
   int codeword = 0;


  public static void main(String[] args){
      H7B h7 = new H7B();
      h7.readPBMHeader();
      h7.uncompress();
   }
```

```
void uncompress(){
     for (int i = 0; i < 16; i++){
       codeword <<= 1;
       codeword |= inputBit();
     }
     for (int i = 0; i < height; i++){
       for (int j = 0; j < width; j++){
        int context = getContext(j + 2);
        boolean one = update(context);
        incrementCount(context, one);
        bitmap[2][j + 2] = one;
        outputBit(one);
      }
      for (int j = 2; j < width + 2; j++){
        bitmap[0][j] = bitmap[1][j];
        bitmap[1][j] = bitmap[2][j];
      }

    }
    System.out.flush();
 }
```

```
boolean update(int context){
    int t = low + count[context][0] * (high - low) /
      (count[context][0] + count[context][1]);
    boolean ret = codeword >= t;
    if ( ______ ) low = t; else high = t;
    for (;;){ // double until larger than quarter
      if (high < half){
        high *= 2;
        low *= 2;
        codeword *= 2;
        if (inputBit() > 0) codeword |= 1;
      }else if (low >= half){
        high = (high * 2) - maxRange;
        low = (low * 2) - maxRange;
        // code for codeword
      }else if (low > quarter && high <= threequarters){
        low = (low * 4 - maxRange) / 2;
        high = (high * 4 - maxRange) / 2;
        // code for codeword
      }else break;
   }
   return ret;
  }
```

Homework 7: due 2-9-15

- Complete the update() function in the adaptive arithmetic decoding program H7B.java.
- An example PBM file hand.pbm can be used to test the encoding-decoding process.
- Decode test7 into a PBM bi-level image file and.
- Submit the code you write and an image of the uncompressed test7.