Using the Euler-Lagrange equations to calculate the extremum of functions

How accurately does the spherical and spheroidal model of Earth calculate the minimum distance between two points?

Subject : Mathematics

Calculus of variations

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Contents

1 Introduction

The calculus of variations is a mathematical field which involves the use of variational principles to find particular functions that extremise a certain property[1] of the function. The field was founded by Johann Bernoulli in 1696 when he raised the Brachistochrone curve problem. That was, which path of a particle minimises the time it takes to travel from one point to another when accelerated by gravity[2]. The Euler-Lagrange(EL) equation is a partial differential equation in the calculus of variations with deals with this problem and has other numerous applications in pure and applied maths. In Physics, the EL equation shows that the path taken by a particle obeying Newton's second law, minimises the integral of the Kinetic energy - Potential energy[3]. A physical principle that this essay looks at is the principle of static action. That is, any system in equilibrium minimises its potential energy[4]. In pure maths, the calculus of variations arise in calculating the geodesics of curved surfaces, these are paths along a surface minimising the distance between two points[5].

This essay derives the mathematics needed to use the EL equations and applies to find the shape of a hanging uniform cable and a pure problem computing the geodesic of a sphere analytically and ellipsoid by using Euler's method. Then, the length of this geodesic will be compared to the 'literature value' which will be obtained computationally from the internet using Vincenty's formula. The prerequisites to applying the EL equations are some topics of multivariable calculus.

2 Multi-variable functions

A function that takes in more than a single value is a multi-variable function. Two variable functions can describe surfaces where x and y are independent. For example:

$$z = f(x, y)$$
$$z = x^2 + y^2$$

[1] *Calculus of variations*. Mar. 2021. URL: https://en.wikipedia.org/wiki/Calculus_of_variations.

[2] *Brachistochrone Problem*. URL: https://mathworld.wolfram.com/BrachistochroneProblem.html.

[3] Stephen Lawrence. URL: http://www.physicsinsights.org/lagrange_1.html.

[4] School of Physical Sciences. *The Energy Minimum Principle*. Accessed: 3-11-2020.

[5] *geodesic*. URL: https://dictionary.cambridge.org/dictionary/english/geodesic.

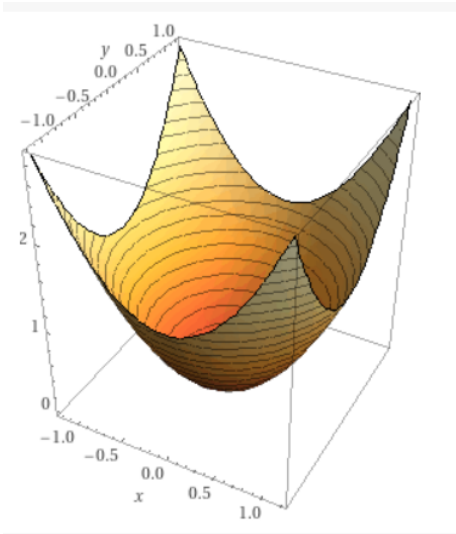

Figure 1: Surface of $z = x^2 + y^2$

2.1 Partial differentiation

The partial derivative of a function calculates the rate at which a function changes with respect to one of its variables where all the other variables are treated as constants.
The derivative is

$$\frac{df}{dx}$$

Partial derivative is

$$\frac{\partial f}{\partial x} \text{ or } f_x(x, y)$$

An example on a three variable function:

$$f(x, y, z) = 5xy + z^2$$
$$\frac{\partial f}{\partial x} = 5y$$
$$\frac{\partial f}{\partial y} = 5x$$
$$\frac{\partial f}{\partial z} = 2z$$

The partial derivative of an n-variable function f where

$$f = f(x_1, x_2, x_3, \ldots, x_n)$$

is

$$\frac{\partial f}{\partial x_1} = \lim_{\Delta x_1 \to 0} \left[\frac{f(x_1 + \Delta x_1, x_2, x_3, \ldots, x_n) - f(x_1, x_2, x_3, \ldots, x_n)}{\Delta x_1} \right] \tag{1}$$

The numerator in the square brackets is a function of only Δx_1 as all the variables $x_2, x_3...$ are taken to be constants.

The nabla operator for an n-variable function returns the partial derivatives of all the independent variables in vector form:

$$f = f(x_1, x_2, x_3, ..., x_n)$$
$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

2.2 Characteristics of a turning point in single variable calculus

For a function of x having an extremum at $x = a$:

$$f'(a) = 0 \tag{2}$$

The third order Taylor expansion of the function f near a is

$$f(x) \approx f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) \tag{3}$$

(3) can be used to approximate the value at a small interval away from x

$$f(x+\Delta x) \approx f(a) + (x+\Delta x-a)f'(a) + \frac{(x+\Delta x-a)^2}{2!}f''(a) + \frac{(x+\Delta x-a)^3}{3!}f'''(a) \tag{4}$$

The difference between (4) and (3) is

$$f(x+\Delta x) - f(x) \approx$$
$$(\Delta x)f'(a) + \frac{((\Delta x)^2 + 2\Delta x(x-a))}{2!}f''(a) + \frac{((\Delta x)^3 + h(x,\Delta x)(x-a))}{3!}f'''(a) \tag{5}$$

where

$$h(x,\Delta x) = \frac{(x+\Delta x-a)^3 - (\Delta x)^3 - (x-a)^3}{x-a}$$

At the extremum $x = a$, the change in $f(x)$ proportional to Δx has the constant of proportionality $f'(a)$ which from (2), is 0. The first order change in the function at the extremum is 0.[6] Other changes to $f(x)$ proportional to the second or higher power of Δx have a constant of proportionality which is not necessarily zero. These are used to determine the extremum of the function.

[6] URL: https://www.feynmanlectures.caltech.edu/II_19.html.

2.3 First derivative test for local extrema

The extremum of multi-variable functions is a result of a combination of maxima and minima with respect to each of the variables. The local extremum of a function is identified using the first derivative test. Analogous to single variable calculus, the function is at an extremum at $p_1, p_2, ..., p_n$ if

$$\nabla f(p_1, p_2, ..., p_n) = \mathbf{0} \tag{6}$$

$\mathbf{0}$ is a zero vector

Proof

Let an n-variable function f be defined as

$$f = f(x_1, x_2, ..., x_n)$$

where $x_1, x_2, ..., x_n$ are independent variables and f has an extremum at the point $(P_1, P_2, ..., P_n)$. A new single variable function g of the variable x_1 can be defined where

$$g(x_1) = f(x_1, P_2, P_3, ..., P_n) \tag{7}$$

The points $P_{2,3,4...n}$ are constants and therefore the function g depends only on x_1.

Due to f having an extremum at $(P_1, P_2, ..., P_n)$, the function $g(x_1)$ also has an extremum at $x_1 = P_1$. By definition of an extremum for a single variable function

$$g'(P_1) = 0$$

The derivative can be rewritten in its limit form and substituting (7) into this yields

$$\begin{aligned} g'(P_1) &= \lim_{\Delta x_1 \to 0} \frac{g(P_1 + \Delta x_1) - g(P_1)}{\Delta x_1} = 0 \\ &= \lim_{\Delta x_1 \to 0} \frac{f(P_1 + \Delta x_1, P_2, P_3, ..., P_n) - f(P_1, P_2, P_3, ..., P_n)}{\Delta x_1} = 0 \end{aligned}$$

This is the definition of the partial derivative from (1) and therefore at the extremum $(P_1, P_2, ..., P_n)$,

$$\frac{\partial f}{\partial x_1} = 0 \tag{8}$$

This proof can be repeated for all the independent variables hence satisfying (6).

2.4 Taylor expansion of a two variable function

For multi-variable function, the n^{th} order partial derivatives of the series must be equal to the n^{th} order partial derivatives of the function. A second order Taylor expansion for an arbitrary function f where

$$f = f(x, y)$$

can be computed.

$$\left.\frac{\partial f}{\partial x}\right|_{x=a, y=b} \equiv f_x(a, b)$$

The second order approximation at the point (a, b) is

$$\begin{aligned} f(a,b) &\approx (x-a)f_x(a,b) + (y-b)f_y(a,b) \\ &+ \frac{(x-a)^2}{2!}f_{xx}(a,b) + \frac{(y-b)^2}{2!}f_{yy}(a,b) \\ &+ \frac{(x-a)(y-b)}{2!}f_{xy}(a,b) + \frac{(x-a)(y-b)}{2!}f_{yx}(a,b) + ... \end{aligned}$$

2.5 Linear approximation of a multi variable function

A linear approximation of a multi-variable function is analogous to using a tangent to approximate a single variable function[7]. Using the Taylor expansion of a multi-variable function, a linear change in a three variable function, Δf, can be made. Linear means proportional to the change Δx, Δy, Δz. Let

$$f = f(x, y, z)$$

Near an arbitrary point (x_0, y_0, z_0), f is approximately

$$f(x,y,z) \approx f(x_0,y_0,z_0) + (x-x_0)\frac{\partial f}{\partial x} + (y-y_0)\frac{\partial f}{\partial y} + (z-z_0)\frac{\partial f}{\partial z} \tag{9}$$

$$f(x+\Delta x, y+\Delta y, z+\Delta z) \approx f(x_0,y_0,z_0) + (x+\Delta x-x_0)\frac{\partial f}{\partial x} + (y+\Delta y-y_0)\frac{\partial f}{\partial y} + (z+\Delta z-z_0)\frac{\partial f}{\partial z} \tag{10}$$

Δf denotes the linear change in f and is the first order difference of (9) and (10)

$$\begin{aligned} \Delta f &= f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z) \\ &\approx \Delta x\frac{\partial f}{\partial x} + \Delta y\frac{\partial f}{\partial y} + \Delta z\frac{\partial f}{\partial z} \end{aligned} \tag{11}$$

[7]Herbert Gross. *RES.18-007 Calculus Revisited: Multivariable Calculus. Fall 2011. Massachusetts Institute of Technology: MIT OpenCourseWare.*

(11) is only a reasonable approximation when the point being evaluated is near the point (x_0, y_0, z_0). When the changes in x, y and z approach 0, the total differential is yielded

$$\lim_{\Delta x,\Delta y,\Delta z\to 0} \Delta f = \Delta x\frac{\partial f}{\partial x} + \Delta y\frac{\partial f}{\partial y} + \Delta z\frac{\partial f}{\partial z} \tag{12}$$

$$\mathrm{d}f = \mathrm{d}x\frac{\partial f}{\partial x} + \mathrm{d}y\frac{\partial f}{\partial y} + \mathrm{d}z\frac{\partial f}{\partial z} \tag{13}$$

In (12), the limit means that higher powers of Δx, Δy, Δz approach zero at a faster rate than Δx, Δy, Δz and can be negated.

2.6 Chain rule for multi-variable functions

Now using

$$f = f(x, y, z)$$

where x, y and z are functions of independent variables a and b

$$x = x(a, b), y = y(a, b), z = z(a, b)$$

The chain rule computes $\frac{\partial f}{\partial a}$ and $\frac{\partial f}{\partial b}$, i.e.

$$\frac{\partial f}{\partial a} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial a}$$

$$\frac{\partial f}{\partial b} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial b} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial b}$$

Proof[8]

From **2.5**, small changes in the variables Δx, Δy, Δz are such that the change in f is

$$\lim_{\Delta x,\Delta y,\Delta z\to 0} \Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z \tag{14}$$

Δx, Δy and Δz are themselves differentials of a function and so

$$\lim_{\Delta a,\Delta b\to 0} \Delta x = \frac{\partial x}{\partial a}\Delta a + \frac{\partial x}{\partial b}\Delta b$$

$$\lim_{\Delta a,\Delta b\to 0} \Delta y = \frac{\partial y}{\partial a}\Delta a + \frac{\partial y}{\partial b}\Delta b$$

$$\lim_{\Delta a,\Delta b\to 0} \Delta z = \frac{\partial z}{\partial a}\Delta a + \frac{\partial z}{\partial b}\Delta b$$

[8] Gross, *RES.18-007 Calculus Revisited: Multivariable Calculus. Fall 2011. Massachusetts Institute of Technology: MIT OpenCourseWare*.

Substituting into (14):

$$\Delta f = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial a}\right)\Delta a + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial b} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial b}\right)\Delta b$$

Dividing by the the differential of a and taking a limit:

$$\begin{aligned}\lim_{\Delta a \to 0} \frac{\Delta f}{\Delta a} = \frac{\partial f}{\partial a} &= \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial a}\right) + \left[\ldots\right]\frac{\Delta b}{\Delta a} \\ &= \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial a}\right)\end{aligned}$$

Similarly for b

$$\lim_{\Delta b \to 0} \frac{\Delta f}{\Delta b} = \frac{\partial f}{\partial b} = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial b} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial b}\right)$$

Note the independent variables:

$$\frac{\Delta b}{\Delta a} = 0 \text{ and } \frac{\Delta a}{\Delta b} = 0$$

2.7 Using the Lagrange multiplier to find the extremum of a function given a constraint

Finding an extremum with some constraint involves incorporating the information of the constraint into the first derivative test of the function. A constraint states a relation that the independent variables must obey is the form of:

$$g(x, y, z) = 0$$

Using this information, the extremum of some function f where the constraint g too is satisfied can be found. Let

$$f = f(x, y, z)$$

$$g(x, y, z) = 0 \to z = h(x, y)$$

Here, the constraint has been rearranged to describe z in terms of x and y. $h(x, y)$ describes this relationship.

$$f(x, y, z) = f(x, y, h(x, y)) = F(x, y)$$

$F(x, y)$ is the function f with the constraint substituted into it. The extremum of F is needed. This is when

$$\nabla F = \mathbf{0}$$

which is

$$
\begin{aligned}
F_x &= f_x + f_z z_x = 0 \\
z_x &= -\frac{f_x}{f_z} \qquad (15)\\
F_y &= f_y + f_z z_y = 0 \\
z_y &= -\frac{f_y}{f_z} \qquad (16)
\end{aligned}
$$

The only unknown's in the previous equations is the partial derivatives of z with respect to x and y. These can be computed using $g(x, y, z)$. As g is expressed explicitly in terms of x, y and z,

$$g(x, y, z) = g(x, y, h(x, y)) = G(x, y)$$

$G = 0$, therefore its partial derivatives are 0.

$$G_x = 0 \text{ and } G_y = 0$$

$$
\begin{aligned}
G_x &= g_x + g_z z_x = 0 \\
z_x &= -\frac{g_x}{g_z} \qquad (17)\\
G_y &= g_y + g_y z_y = 0 \\
z_y &= -\frac{g_y}{g_z} \qquad (18)
\end{aligned}
$$

Equating (15) to (17) and (16) to (18) leads to

$$
\begin{aligned}
-\frac{f_x}{f_z} &= -\frac{g_x}{g_z} \\
-\frac{f_y}{f_z} &= -\frac{g_y}{g_z}
\end{aligned}
$$

For these two equations' conditions to be true, the following must hold

$$f_z = \lambda g_z \qquad (19)$$

$$f_x = \lambda g_x \qquad (20)$$

$$f_y = \lambda g_y \qquad (21)$$

where $\lambda \in \mathbb{R}$.

(19), (20) and (21) become

$$\nabla f = \lambda \nabla g$$

and so

$$\nabla f - \lambda \nabla g = 0 \tag{22}$$

Due to the distributive property of the nabla operator ∇ , (22) becomes

$$\nabla(f - \lambda g) = \mathbf{0} \tag{23}$$

This final form shows that finding the extremum of $f - \lambda g$ is equivalent to finding the extremum of the function f given the constraint g. λ is the Lagrange multiplier[9].

3 Calculus of variations

The calculus of variations looks to find a function such that a property of this function is an extremum. For example to find the function $f(x)$, such that the arc length (see **9.1**) is extremised.

$$s\,(f(x)) = \int_b^a \sqrt{1 + (f')^2}\mathrm{d}x \tag{24}$$

Geometrically, this is the same as minimising the distance between the coordinates $(b,f(b))$ and $(a,f(a))$. The method of solving such a problem uses a variation in the function $f(x)$ denoted by $\eta(x)$. $\eta = \Delta f$ and at the boundaries of $f(x)$, $\eta(x)$ is zero.

$$\eta(a) = 0\,,\, \eta(b) = 0$$

From **2.2** and **2.3**, the first order change in s for the change η evaluated at the extremum is zero. The expression for the change in s, namely δs is

$$\delta s = s(f(x) + \eta(x)) - s(f(x)) \tag{25}$$

$$\sqrt{1 + (f' + \Delta f')^2} \approx \sqrt{1 + (f')^2} + \frac{\partial \sqrt{1 + (f')^2}}{\partial f'}\Delta f' \tag{26}$$

Replacing $\Delta f'$ with η' in (25) and substituting (24),(26) into (25) yields

$$\begin{aligned}\delta s &= \int_b^a \sqrt{1 + (f')^2} + \frac{\partial \sqrt{1 + (f')^2}}{\partial f'}\eta'\mathrm{d}x - \int_b^a \sqrt{1 + (f')^2}\mathrm{d}x \\ &= \int_b^a \frac{\partial \sqrt{1 + (f')^2}}{\partial f'}\eta'\mathrm{d}x \\ &= \int_b^a \frac{f'}{\sqrt{1 + (f')^2}}\eta'\mathrm{d}x = 0 \end{aligned} \tag{27}$$

[9]Gross, *RES.18-007 Calculus Revisited: Multivariable Calculus. Fall 2011. Massachusetts Institute of Technology: MIT OpenCourseWare*.

It can be verified that (27) is satisfied by

$$\frac{f'}{\sqrt{1+(f')^2}} = C \ (C \in \mathbb{R}) \tag{28}$$

Substituting into (27)

$$\begin{aligned}\int_b^a C\eta' \mathrm{d}x &= C(\eta(b) - \eta(a)) \\ &= 0\end{aligned} \tag{29}$$

Solving (28) gives

$$\begin{aligned}
f'(x) &= k\sqrt{1+(f'(x))^2} \\
(f'(x))^2 &= k^2[1+(f'(x))^2] \\
(f'(x))^2(1-k^2) &= k^2 \\
(f'(x))^2 &= \frac{k^2}{(1-k^2)} \\
f'(x) &= \frac{k}{\sqrt{(1-k^2)}} \\
f(x) &= \frac{k}{\sqrt{(1-k^2)}}x + C \\
&= mx + C
\end{aligned}$$

This proves the function minimising the distance between two points is as expected a straight line. Equating the coefficient of η' to a constant, as was stated in (28) is proved by the second lemma of the calculus of variations. The fundamental lemma of the calculus of variations is another similar identity necessary to prove the EL equations. The EL equations generalise the solution for $f(x)$ given $s(f(x))$ where s is of the form

$$S(f(x)) = \int_{x_1}^{x_2} L(f(x), f'(x), x)\mathrm{d}x \tag{30}$$

L is called the Lagrangian. In the example above (24), the Lagrangian was

$$L = \sqrt{(1+(f')^2)}$$

3.1 Fundamental lemma of the calculus of variations

Function f is continuous in the domain $[t_1, t_2]$ and

$$\int_{t_1}^{t_2} f(t)\eta(t)\mathrm{d}t = 0 \tag{31}$$

where η is a variation in the function f such that η is continuous and satisfies the boundary conditions.

$$\eta(t_1) = 0 \text{ and } \eta(t_2) = 0$$

The fundamental lemma of the calculus of variations states that for (31) to be satisfied for all valid η,

$$f(t) = 0$$

Proof by contradiction[10]

To assume the opposite, $g(x)$ needs to be non-zero at some point. $g(c) \neq 0$ where $a \leq c \leq b$. This means that at $x = c$ and in neighbouring region due to the continuity of the function g, there will be a slight bump. h needs to be chosen such that it satisfies the boundary conditions. An appropriate form for h is

$$h(x) = \begin{cases} -(x-a)(x-b) & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$h(x)$ is continuous and positive in the interval $a < x < b$ and is zero at the end points. As $g(x)$ is non- zero in a small region around $x = c$ and $h(x)$ is positive in the whole interval, therefore

$$\int_a^b g(x)h(x)\mathrm{d}x \neq 0$$

This is a contradiction to the claim that $g(x)$ can be non-zero and for the chosen $h(x)$, $g(x)$ must be 0 satisfying (31).

3.2 Euler - Lagrange equations

The EL equations generalise the condition that a function at the extremum must satisfy. The form of the outer most function that is being extremised is the integral of the Lagrangian, L. where L is a multivariable function described in terms of a function, its derivative and the independent variable x. $S(f(x))$ is this outer function

$$S(f(x)) = \int_{x_1}^{x_2} L(f(x), f'(x), x)\mathrm{d}x \tag{32}$$

$$L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$$

[10] nptelhrd. *Mod-01 Lec-36 Calculus of Variations - Three Lemmas and a Theorem.* Mar. 2015. URL: https://www.youtube.com/watch?v=lSsIio4aFuE&t=1362s.

The Lagrangian holds the physical information of the system that needs to be extremised. S can be treated as a single variable function where a small change in the function $f(x)$ is represented by $\eta(x)$. The first order change at the extremum is zero and using the Taylor expansion of the Lagrangian to take the first order change, the EL equations arise

Derivation of EL equations[11]

$$S(f(x)) = \int_{x_1}^{x_2} L(f(x), f'(x), x)\mathrm{d}x$$
$$S(f(x)+\eta(x)) - S(f(x)) = \int_{x_1}^{x_2} L((f(x)+\eta(x), f'(x)+\eta'(x), x) - L((f(x), f'(x), x)\mathrm{d}x$$

where η is a small continuous variation in $f(x)$ and

$$\eta(x_1) = 0 \text{ and } \eta(x_2) = 0$$

The first order change is:

$$L((f(x), f'(x), x) \approx L(f_0, f_0', x) + (f - f_0)\frac{\partial L}{\partial f}\bigg|_{f=f_0} + (f' - f_0')\frac{\partial L}{\partial f'}\bigg|_{f'=f_0'} \tag{33}$$

$$L((f(x)+\eta(x), f'(x)+\eta'(x), x) \approx L(f_0, f_0', x) + (f + \eta - f_0)\frac{\partial L}{\partial f}\bigg|_{f=f_0} + (f' + \eta' - f_0')\frac{\partial L}{\partial f'}\bigg|_{f'=f_0'} \tag{34}$$

$$(34) - (33) \approx \eta\frac{\partial L}{\partial f}\bigg|_{f=f_0} + \eta'\frac{\partial L}{\partial f'}\bigg|_{f'=f_0'}$$
$$S(f(x)+\eta(x)) - S(f(x)) \approx \int_{x_1}^{x_2} \eta\frac{\partial L}{\partial f}\bigg|_{f=f_0} + \eta'\frac{\partial L}{\partial f'}\bigg|_{f'=f_0'} \mathrm{d}x$$
$$\delta S = \int_{x_1}^{x_2} \eta\frac{\partial L}{\partial f}\bigg|_{f=f_0} + \eta'\frac{\partial L}{\partial f'}\bigg|_{f'=f_0'} \mathrm{d}x$$

The small change in the Lagrangian needs to have its η factored out so that it is in the form of (31). η' can be integrated using integration by parts.

$$u = u(x), v = v(x)$$
$$\int_{x_2}^{x_1} uv'\mathrm{d}x \equiv uv\bigg|_{x_2}^{x_1} - \int_{x_2}^{x_1} u'v\mathrm{d}x$$

[11] .

Substituting $u = \frac{\partial L}{\partial f'}$ and $v' = \eta'$

$$\int_{x_1}^{x_2} \eta' \frac{\partial L}{\partial f'}\bigg|_{f'=f_0'} = \left[\eta \frac{\partial L}{\partial f'}\bigg|_{f'=f_0'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\bigg|_{f'=f_0'}\right) \mathrm{d}x$$
$$\left[\eta \frac{\partial L}{\partial f'}\bigg|_{f'=f_0'}\right]_{x_1}^{x_2} = \eta(x_2)\frac{\partial L}{\partial f'}\bigg|_{f'}\bigg|_{x=x_2} - \eta(x_1)\frac{\partial L}{\partial f'}\bigg|_{f'}\bigg|_{x=x_1}$$
$$= 0$$

The η can be factored out

$$\begin{aligned}\delta S &= \int_{x_1}^{x_2} \eta \frac{\partial L}{\partial f}\bigg|_{f=f_0} - \eta \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\bigg|_{f'=f_0'}\right) \mathrm{d}x \\ &= \int_{x_1}^{x_2} \left[\frac{\partial L}{\partial f}\bigg|_{f=f_0} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\bigg|_{f'=f_0'}\right)\right] \eta \, \mathrm{d}x\end{aligned}$$

The first order change in S or just δS is zero at the extremum:

$$\begin{aligned}\delta S &= 0 \\ &= \int_{x_1}^{x_2} \left[\frac{\partial L}{\partial f}\bigg|_{f=f_0} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\bigg|_{f'=f_0'}\right)\right] \eta \, \mathrm{d}x\end{aligned}$$

And then, the Fundamental lemma of the calculus of variations can be used to equate the coefficient of η to zero. This leads to the EL equation.

$$\frac{\partial L}{\partial f}\bigg|_{f=f_0} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\bigg|_{f'=f_0'}\right) = 0 \tag{35}$$

Throughout the derivation, the partial derivatives have been evaluated at the extremum and therefore equating the first order change to zero is appropriate. The EL equation is a necessary condition that the extremising function f_0 will satisfy

A slight simplification of the EL equation will further simplify the calculations to compute Geodesics and Catenary shape.

3.3 Beltrami's identity

When the Lagrangian is not explicitly a function of x, the EL equations can be further simplified. This condition can be written as

$$\frac{\partial L}{\partial x} = 0 \tag{36}$$

This is true as the process of a partial differentiation is keeping all other terms constant and differentiating with respect to the specific variable. Therefore, the partial derivative is 0.
Using the Chain rule, the derivative of the Lagrangian with respect to x can be described in terms of x, $f(x)$ and $f'(x)$

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}x} &= \frac{\partial L}{\partial x} + \frac{\partial L}{\partial f}\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\partial L}{\partial f'}\frac{\mathrm{d}f'}{\mathrm{d}x} \\
\frac{\mathrm{d}f'}{\mathrm{d}x} &= \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \\
\frac{\mathrm{d}L}{\mathrm{d}x} &= \frac{\partial L}{\partial x} + \frac{\partial L}{\partial f}\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\partial L}{\partial f'}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \\
\frac{\mathrm{d}L}{\mathrm{d}x} &= 0 + \frac{\partial L}{\partial f}\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\partial L}{\partial f'}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \qquad (37)
\end{aligned}$$

Combining this with the EL equation will lead to the Beltrami identity[12]. EL equation:

$$\frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\right) = 0$$

Multiplying by $\frac{\mathrm{d}f}{\mathrm{d}x}$ yields

$$\begin{aligned}
\frac{\partial L}{\partial f}\frac{\mathrm{d}f}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\right)\frac{\mathrm{d}f}{\mathrm{d}x} &= 0 \\
\frac{\partial L}{\partial f}\frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\right)\frac{\mathrm{d}f}{\mathrm{d}x}
\end{aligned}$$

substituting into (37) gives

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}x} &= \frac{\partial L}{\partial f}\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\partial L}{\partial f'}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \\
\frac{\mathrm{d}L}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\right)\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\partial L}{\partial f'}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \qquad (38)
\end{aligned}$$

The right hand side can be integrated by making the substitution

$$\frac{\partial L}{\partial f'} = u \text{ and } \frac{\mathrm{d}f}{\mathrm{d}x} = v$$

(38) can be rewritten as

$$\frac{\mathrm{d}L}{\mathrm{d}x} = u'v + uv'$$

Integrating both sides with respect to x and noticing the right-hand side is the derivative of uv yields

$$L = uv + C_0 \ (C \in \mathbb{R})$$

[12] *Beltrami identity*. Feb. 2021. URL: https://en.wikipedia.org/wiki/Beltrami_identity.

Re-substituting the expressions into u and v gives

$$L = \frac{\partial L}{\partial f'}\frac{\mathrm{d}f}{\mathrm{d}x} + C_0$$
$$L - f'\frac{\partial L}{\partial f'} = C_0$$

This is Beltrami's identity

4 Example : Catenary

The shape that a rope or cable takes under its own weight, when its mass is uniformly distributed is called the **Catenary**. One of the underlying theories that will be used is the **Principle of static action**. This states that a system in equilibrium minimises its potential energy[13].

$$\text{Potential energy (PE)} = mgh$$

As the rope is uniform, the change in mass of a small section for a horizontal change $\Delta s \times \rho$, where ρ is a constant density and

$$\begin{aligned}\Delta s &= \sqrt{1 + (f')^2} \\ \Delta m &= \Delta s \times \rho \\ &= \sqrt{1 + (f')^2}\Delta x \times \rho\end{aligned}$$

The formula for the differential of s is explained in **9.1**. The height h is the y value for every point on the curve so

$$h = y$$

A small change in x changes the potential energy by

$$\begin{aligned}\Delta \mathrm{PE} &= \sqrt{1 + (f')^2}\rho g f \times \Delta x \\ \frac{\Delta \mathrm{PE}}{\Delta x} &= \sqrt{1 + (f')^2}\rho g f \\ \lim_{\Delta x \to 0} \frac{\Delta \mathrm{PE}}{\Delta x} = \frac{\mathrm{dPE}}{\mathrm{d}x} &= \sqrt{1 + (f')^2}\rho g f \\ \mathrm{PE} &= \int_{-x_1}^{x_1} \sqrt{1 + (f')^2}\rho g f \mathrm{d}x\end{aligned}$$

[13]Name *. *Samer Adeeb.* URL: `https://sameradeeb-new.srv.ualberta.ca/variational-principles/the-principle-of-minimum-potential-energy-for-conservative-systems-in-equilibrium-2/`.

A constraint on the catneary curve is that its length must be equal to that of the rope being modelled[14]. This is length l:

$$\int_{-x_1}^{x_1} \sqrt{1+(f')^2}\mathrm{d}x = l$$

Note that although this is valid constraint, it will take the form of $g(...) = 0$ and so this constraint becomes

$$\int_{-x_1}^{x_1} \sqrt{1+(f')^2}\mathrm{d}x - l = 0$$

The potential energy must be minimised while satisfying the above constraint. Using the Lagrange multiplier method, the extremum for the following function can be computed, i.e.

$$\int_{-x_1}^{x_1} \sqrt{1+(f')^2}\rho g f \mathrm{d}x - \lambda\left(\int_{-x_1}^{x_1} \sqrt{1+(f')^2}\mathrm{d}x - l\right)$$

Putting this in the form that satisfies the EL equation gives

$$\int_{-x_1}^{x_1} \sqrt{1+(f')^2}\rho g f - \lambda\sqrt{1+(f')^2}\mathrm{d}x + \lambda l$$

$$\lambda l = \int_{-x_1}^{x_1} \frac{\lambda l}{2x_1}\mathrm{d}x$$

$$\int_{-x_1}^{x_1} \sqrt{1+(f')^2}\rho g f - \lambda\sqrt{1+(f')^2} + \frac{\lambda l}{2x_1}\mathrm{d}x$$

The Lagrangian is now

$$L = \sqrt{1+(f')^2}\rho g f - \lambda\sqrt{1+(f')^2} + \frac{\lambda l}{2x_1}$$

The Lagrangian is not explicitly a function of x and therefore, Beltrami's identity is valid

$$\frac{\partial L}{\partial f'} = \frac{f'}{\sqrt{1+(f')^2}}\rho g f - \frac{\lambda f'}{\sqrt{1+(f')^2}}$$

Beltrami's identity:

$$L - f'\frac{\partial L}{\partial f'} = C$$

[14] URL: `https://www.dmf.unicatt.it/~paolini/divulgazione/mateott/catenaria/catenary/catenary.htm`.

Using this, f can be calculated

$$\begin{aligned}
\sqrt{1+(f')^2}\rho g f - \lambda\sqrt{1+(f')^2} + \frac{\lambda l}{2x_1} - \frac{(f')^2}{\sqrt{1+(f')^2}}(\rho g f - \lambda) &= C \\
(1+(f')^2)\rho g f - \lambda(1+(f')^2) + \frac{\lambda l}{2x_1}\sqrt{1+(f')^2} - (f')^2(\rho g f - \lambda) &= C\sqrt{1+(f')^2} \\
\rho g f + (f')^2\rho g f - \lambda - (f')^2\lambda - (f')^2\rho g f + (f')^2\lambda &= \left(C - \frac{\lambda l}{2x_1}\right)\sqrt{1+(f')^2} \\
\rho g f - \lambda &= \left(C - \frac{\lambda l}{2x_1}\right)\sqrt{1+(f')^2} \\
1+(f')^2 &= \frac{(\rho g f - \lambda)^2}{\left(C - \frac{\lambda l}{2x_1}\right)^2} \\
f' &= \sqrt{\left(\frac{\rho g f - \lambda}{C - \frac{\lambda l}{2x_1}}\right)^2 - 1}
\end{aligned}$$

This leads to the separable differential equation

$$\begin{aligned}
f'\frac{1}{\sqrt{\left(\frac{\rho g f - \lambda}{C - \frac{\lambda l}{2x_1}}\right)^2 - 1}} &= 1 \\
\int \frac{1}{\sqrt{\left(\frac{\rho g f - \lambda}{C - \frac{\lambda l}{2x_1}}\right)^2 - 1}}\mathrm{d}f &= x + k
\end{aligned}$$

The left integral now has the form of the derivative of arcosh(x), (see **9.2**). By making the substitution

$$\begin{aligned}
\cosh u &= \frac{\rho g f - \lambda}{C - \frac{\lambda l}{2x_1}} \\
\frac{\mathrm{d}u}{\mathrm{d}f}\sinh u &= \frac{\rho g}{C - \frac{\lambda}{2x_1}} \\
\frac{\mathrm{d}u}{\mathrm{d}f} &= \frac{1}{\sinh u}\frac{\rho g}{C - \frac{\lambda}{2x_1}}
\end{aligned}$$

The Catenary equation is derived

$$\int \frac{1}{\sqrt{\cosh^2 u - 1}} \times \frac{(C - \frac{\lambda}{2x_1}) \sinh u}{\rho g} \mathrm{d}u = x + k$$

$$\int \frac{C - \frac{\lambda}{2x_1}}{\rho g} \mathrm{d}u = x + k$$

$$\frac{C - \frac{\lambda}{2x_1}}{\rho g} u = x + k$$

$$u = \operatorname{arcosh}\left(\frac{\rho g f - \lambda}{C - \frac{\lambda l}{2x_1}}\right)$$

$$\frac{C - \frac{\lambda}{2x_1}}{\rho g} \operatorname{arcosh}\left(\frac{\rho g f - \lambda}{C - \frac{\lambda l}{2x_1}}\right) = x + k$$

Rearranging to make f the subject yields:

$$f = \frac{\alpha}{\rho g} \cosh\left(\frac{\rho g}{\alpha}(x + k)\right) + \frac{\lambda}{\rho g} \tag{39}$$

where

$$\alpha = C - \frac{\lambda l}{2x_1}$$

The minimum point of the cable will be on the y axis meaning $k = 0$. The length of the cable is calculated using the arc length formula. The cable's length, l, is:

$$l = \int_{-x_1}^{x_1} \sqrt{1 + (f')^2} \tag{40}$$

From (40),

$$f'(x) = \sinh\left(\frac{\rho g}{\alpha} x\right)$$

$$\sqrt{1 + (f')^2} = \cosh\left(\frac{\rho g}{\alpha} x\right)$$

$$\int_{-x_1}^{x_1} \cosh\left(\frac{\rho g}{\alpha} x\right) \mathrm{d}x = \frac{2\alpha}{\rho g} \sinh\left(\frac{\rho g}{\alpha} x_1\right) \tag{41}$$

Using (39) and (41), the shape of a freely hanging cable can be calculated given its length and the two points from which it is being held. The shape that a 10 metre uniform cable being held with its two ends 5 metres apart at the same height follows can be calculated

$$\begin{aligned}
\frac{2\alpha}{\rho g}\sinh\left(2.5\times\frac{\rho g}{\alpha}\right) &= 10\\
\frac{\rho g}{\alpha} &= 1.1482(4d.p.)\\
f(2.5) &= 1.1482\cosh\left(\frac{2.5}{1.1482}\right)+\frac{\lambda}{\rho g}=0\\
\frac{\lambda}{\rho g} &= -5.1301\\
f &= 1.1482\cosh\left(\frac{x}{1.1482}\right)-5.1301
\end{aligned}$$

The graph produced is seen in Figure 2.

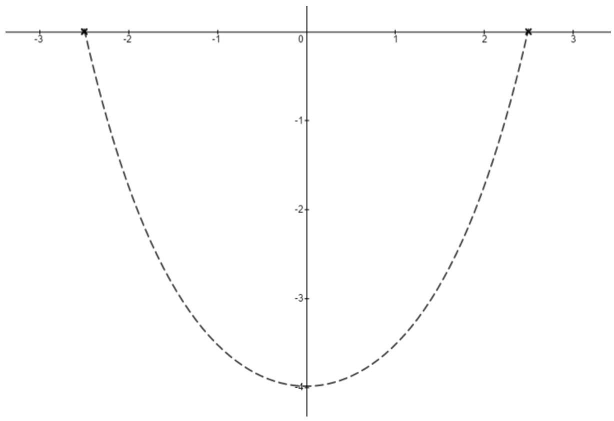

Figure 2: Function of the catenary $y = 1.1482\cosh\left(\frac{x}{1.1482}\right) - 5.1301$ $[-2.5 \leq x \leq 2.5]$

Inverted Catenaries ($-\cosh x$) are the optimum shape for a uniformly dense arch to support its own weight. St Paul's cathedral's dome is a catenoid[15] and the Wembley stadium arch follows an inverted Catenary[16]. The catenary has optimum stability because the rate at which the gradient of a cosh curve increases is proportional to the rate of change of the length of the curve. As the composition of the arch has uniform density and the cross section is constant, the length is proportional to the mass of the cable. This means the weight of the cable is directed perfectly along the curve[17].

[15] *Designs for the Dome*. URL: `https://www.stpauls.co.uk/history-collections/the-collections/architectural-archive/wren-office-drawings/5-designs-for-the-dome-c16871708`.

[16] Paul Shepherd and Marianne. *The catenary goes to Wembley*. Oct. 2017. URL: `https://plus.maths.org/content/catenary-goes-wembley`.

[17] *Maths in a minute: The catenary*. Dec. 2016. URL: `https://plus.maths.org/content/matjhs-minute-catenary`.

5 Geodesics of a curved surface : Sphere

For the geodesic of a sphere[18], spherical coordinates will be used. Spherical coordinates are a 3 dimensional coordinate system which assign a point using three variables r, θ and ϕ.

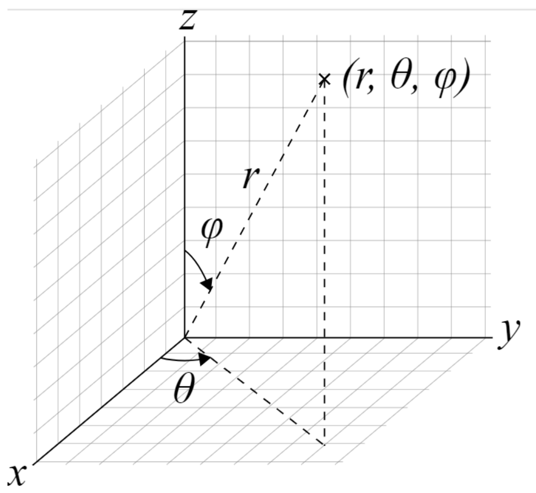

Figure 3: Diagram showing the spherical coordinate system[19]

The transition from Cartesian coordinates to spherical is

$$(x, y, z) \longrightarrow (r, \theta, \phi)$$

$$x = r \sin\phi \cos\theta$$

$$y = r \sin\phi \sin\theta$$

$$z = r \cos\phi$$

$$x^2 + y^2 + z^2 = r^2$$

The arc length in spherical coordinates needs to be computed. In Cartesian coordinates, s is computed by representing a line with a parametric equation. An arbitrary line can be defined by a function $f = f(x, y, z)$ where

$$x = x(t), y = y(t), z = z(t)$$

The 3 dimensional arc length is computed using the Pythagorean theorem. A small change in the

[18]Cambridge University. *Geodesics on the Surface of a Sphere*. Accessed: 3/7/2020.

length, Δs is

$$\begin{aligned}
\Delta s &= \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \\
\frac{ds}{dt} &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \\
s &= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt
\end{aligned} \tag{42}$$

The same process of finding the differential of s in spherical coordinates follows. Using the total differential result from **2.5** and treating r as a constant equal to 1, the following is concluded

$$\begin{aligned}
\Delta x &= \Delta\phi \frac{\partial x}{\partial \phi} + \Delta\theta \frac{\partial x}{\partial \theta} \\
&= \Delta\phi \cos\phi \cos\theta - \Delta\theta \sin\phi \sin\theta \\
(\Delta x)^2 &= (\Delta\phi)^2 \cos^2\phi \cos^2\theta - 2\Delta\phi\Delta\theta \cos\phi \cos\theta \sin\phi \sin\theta + (\Delta\theta)^2 \sin^2\phi \sin^2\theta \\
\Delta y &= \Delta\phi \frac{\partial y}{\partial \phi} + \Delta\theta \frac{\partial y}{\partial \theta} \\
&= \Delta\phi \cos\phi \sin\theta + \Delta\theta \sin\phi \cos\theta \\
(\Delta y)^2 &= (\Delta\phi)^2 \cos^2\phi \sin^2\theta + 2\Delta\phi\Delta\theta \cos\phi \cos\theta \sin\phi \sin\theta + (\Delta\theta)^2 \sin^2\phi \cos^2\theta \\
\Delta z &= \Delta\phi \frac{\partial z}{\partial \phi} \\
&= -\Delta\phi \sin\phi \\
(\Delta z)^2 &= (\Delta\phi)^2 \sin^2\phi \\
(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 &= (\Delta\phi)^2 \cos^2\phi(\cos^2\theta + \sin^2\theta) + (\Delta\theta)^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) + (\Delta\phi)^2 \sin^2\phi \\
&= (\Delta\phi)^2(\cos^2\phi + \sin^2\phi) + (\Delta\theta)^2 \sin^2\phi \\
\Delta s &= \sqrt{(\Delta\phi)^2 + (\Delta\theta)^2 \sin^2\phi} \\
\frac{ds}{d\theta} &= \sqrt{(\phi')^2 + \sin^2\phi}
\end{aligned}$$

$$s = \int \sqrt{(\phi')^2 + \sin^2\phi}\ \mathrm{d}\theta \tag{43}$$

where ϕ' is

$$\frac{d\phi}{d\theta}$$

A curve has one independent variable. This mean that ϕ, although not explicitly defined in terms of θ, is dependent on θ and can be written in terms of θ. Again, ds is the **linear** change in the arc at a point due to a small change in θ and ϕ on a **spherical** surface, as the radius has been taken as a constant 1. As $\frac{\partial L}{\partial \theta} = 0$, (43) qualifies for Beltrami's identity

$$
\begin{aligned}
L &= \sqrt{(\phi')^2 + \sin^2\phi} \\
\frac{\partial L}{\partial \phi'} &= \frac{\phi'}{\sqrt{(\phi')^2 + \sin^2\phi}} \\
L - \phi' \frac{\partial L}{\partial \phi'} &= \sqrt{(\phi')^2 + \sin^2\phi} - \frac{(\phi')^2}{\sqrt{(\phi')^2 + \sin^2\phi}} = C \\
\times\sqrt{(\phi')^2 + \sin^2\phi} & \\
(\phi')^2 + \sin^2\phi - (\phi')^2 &= C\sqrt{(\phi')^2 + \sin^2\phi}
\end{aligned}
$$

The equation can be put into the form, $\phi' f(\phi)$. The form of a separable differential equation

$$
\begin{aligned}
\sin^2\phi &= C\sqrt{(\phi')^2 + \sin^2\phi} \\
\sin^4\phi &= C^2[(\phi')^2 + \sin^2\phi] \\
C^2(\phi')^2 &= \sin^4\phi - C^2\sin^2\phi \\
&= \sin^2\phi(\sin^2\phi - C^2) \\
C\phi' &= \sin\phi\sqrt{\sin^2\phi - C^2} \\
\frac{\phi'}{\sin\phi\sqrt{\sin^2\phi - C^2}} &= \frac{1}{C} \\
\int \frac{1}{\sin\phi\sqrt{\sin^2\phi - C^2}}\mathrm{d}\phi &= \int \frac{1}{C}\mathrm{d}\theta \\
&= \frac{1}{C}\theta + \text{constant}
\end{aligned}
$$

Making the substitution $u = \cot\phi$, the indefinite integral can be calculated

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}\phi} &= -\operatorname{cosec}^2\phi \\
\int \frac{1}{\sin\phi\sqrt{\sin^2\phi - C^2}}\mathrm{d}\phi &= \int \frac{1}{\sin\phi\sqrt{\sin^2\phi - C^2}} \times \frac{-1}{\operatorname{cosec}^2\phi}\mathrm{d}u \\
&= \int \frac{-\sin\phi}{\sqrt{\sin^2\phi - C^2}}\mathrm{d}u \\
&= \int \frac{-1}{\sqrt{1 - C^2\operatorname{cosec}^2\phi}}\mathrm{d}u \\
&= \int \frac{-1}{\sqrt{1 - C^2(u^2+1)}}\mathrm{d}u \quad \because \ (1 + \cot^2\phi = \operatorname{cosec}^2\phi) \\
&= \int \frac{-1}{\sqrt{(1 - C^2) - C^2u^2}}\mathrm{d}u \\
&= \frac{1}{C}\arccos\left(\frac{C}{\sqrt{1 - C^2}}u\right) = \frac{1}{C}\theta + \text{constant} \\
\frac{C}{\sqrt{1 - C^2}}\cot\phi &= \cos(\theta + \theta_0)
\end{aligned}
$$

where θ_0 is a constant.

Understanding this result involves transitioning back from spherical to Cartesian coordinates:

$$
\begin{aligned}
\cot\phi &= \sqrt{\operatorname{cosec}^2\phi - 1} \\
x^2 + y^2 &= \sin^2\theta\phi(\cos^2\theta + \sin^2\theta) = \sin^2\theta \\
\cot\phi &= \sqrt{\frac{1}{x^2 + y^2} - 1} \\
\cos(\theta + \theta_0) &= \cos\theta\cos\theta_0 - \sin\theta\sin\theta_0 \\
\cos\theta &= \frac{x}{\sqrt{x^2 + y^2}} \\
\sin\theta &= \frac{y}{\sqrt{x^2 + y^2}}
\end{aligned}
$$

Substituting these in yields:

$$\frac{C}{\sqrt{1-C^2}}\sqrt{\frac{1}{x^2+y^2}-1} = \frac{x}{\sqrt{x^2+y^2}}\cos\theta_0 - \frac{y}{\sqrt{x^2+y^2}}\sin\theta_0$$

$$\times\sqrt{x^2+y^2} \therefore \frac{C}{\sqrt{1-C^2}}\sqrt{1-x^2-y^2} = x\cos\theta_0 - y\sin\theta_0$$

$$1-x^2-y^2 = z^2$$

$$\frac{C}{\sqrt{1-C^2}}z = x\cos\theta_0 - y\sin\theta_0$$

$$x\cos\theta_0 - y\sin\theta_0 - \frac{C}{\sqrt{1-C^2}}z = 0 \tag{44}$$

This function is describing a plane as the coefficients of x,y,z are constants. Therefore, the minimum path always lies in a plane described by (44). This plane goes through the point $(0, 0, 0)$, meaning the intersection of the plane and spherical surface yields a circle with the largest possible radius for the associated sphere. This is also known as the **great circle**.

6 Geodesic of an oblate ellipsoid

A sphere is the complete revolution of a circle about its diameter. Similarly, an oblate ellipsoid is formed by the complete revolution of an ellipse about its minor axis[20]. An arbitrary point (x, y, z) on an oblate ellipsoid satisfies

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2} = r^2 \tag{45}$$

with spherical coordinates

$$x = a\sin\phi\cos\theta$$

$$y = a\sin\phi\sin\theta$$

$$z = c\cos\phi$$

where $a > c$

The process now is identical to the one for the geodesic of a sphere. The integral defining the distance between two points on the surface is found by taking linear approximations for each differential of x, y and z. Δs is then found using (42). The new integral for the path distance is

$$\int \sqrt{(\phi')^2(a^2\cos^2\phi + c^2\sin^2\phi) + a^2\sin^2\phi}\,\mathrm{d}\theta \tag{46}$$

[20]Eric Weisstein. *Oblate Spheroid*. Accessed: 3-11-2020. URL: `https://mathworld.wolfram.com/OblateSpheroid.html`.

This qualifies for Beltrami's identity where

$$L = \sqrt{(\phi')^2(a^2\cos^2\phi + c^2\sin^2\phi) + a^2\sin^2\phi}$$

$$\begin{aligned}
L - \phi'\frac{\partial L}{\partial \phi'} &= C \\
\frac{\partial L}{\partial \phi'} &= \frac{2\phi'(a^2\cos^2\phi + c^2\sin^2\phi)}{2\sqrt{(\phi')^2(a^2\cos^2\phi + c^2\sin^2\phi) + a^2\sin^2\phi}} \\
L - \phi'\frac{\partial L}{\partial \phi'} &= \sqrt{(\phi')^2(a^2\cos^2\phi + c^2\sin^2\phi) + a^2\sin^2\phi} - \frac{(\phi')^2(a^2\cos^2\phi + c^2\sin^2\phi)}{\sqrt{(\phi')^2(a^2\cos^2\phi + c^2\sin^2\phi) + a^2\sin^2\phi}} = C
\end{aligned}$$

Multiplying by $\sqrt{(\phi')^2(a^2\cos^2\phi + c^2\sin^2\phi) + a^2\sin^2\phi}$ and simplifying yields

$$\begin{aligned}
a^2\sin^2\phi &= C\sqrt{(\phi')^2(a^2\cos^2\phi + c^2\sin^2\phi) + a^2\sin^2\phi} \\
a^4\sin^4\phi &= C^2[(\phi')^2(a^2\cos^2\phi + c^2\sin^2\phi) + a^2\sin^2\phi] \\
a^4\sin^4\phi - C^2a^2\sin^2\phi &= C^2(\phi')^2(a^2\cos^2\phi + c^2\sin^2\phi) \\
1 &= (\phi')^2\frac{C^2(a^2\cos^2\phi + c^2\sin^2\phi)}{a^2\sin^2\phi(a^2\sin^2\phi - C^2)} \\
\phi'\frac{C\sqrt{a^2\cos^2\phi + c^2\sin^2\phi}}{a\sin\phi\sqrt{a^2\sin^2\phi - C^2}} &= 1 \\
\phi'\frac{C\sqrt{a^2 - a^2\sin^2\phi + c^2\sin^2\phi}}{a\sin\phi\sqrt{a^2\sin^2\phi - C^2}} &= 1 \qquad (47)
\end{aligned}$$

The vertical height of an ellipse is $a\sqrt{1-e^2}$ (see **9.3**). The vertical height of this ellipsoid is c.

$$\begin{aligned}
c &= a\sqrt{1-e^2} \\
c^2 &= a^2 - a^2e^2 \\
a^2e^2 &= a^2 - c^2
\end{aligned}$$

Substituting into (47) yields

$$\begin{aligned}
\phi'\frac{C\sqrt{a^2 - a^2e^2\sin^2\phi}}{a\sin\phi\sqrt{a^2\sin^2\phi - C^2}} &= 1 \\
\phi'\frac{\sqrt{1 - e^2\sin^2\phi}}{\sin\phi\sqrt{\frac{a^2}{C^2}\sin^2\phi - 1}} &= 1
\end{aligned}$$

This separable differential equation is solved by

$$\int \frac{\sqrt{1 - e^2\sin^2\phi}}{\sin\phi\sqrt{\frac{a^2}{C^2}\sin^2\phi - 1}}\mathrm{d}\phi = \theta + \text{constant}$$

This is in the form of an elliptic integral and cannot be solved analytically but can be solved numerically. The most common method to compute the geodesic distance is Vincenty's formula[21], which is an iterative method. In order to see the trajectory of the geodesic on an equirectangular projection however, Euler's method with step sizes $h = 0.0001$ will be used. The differential equation being used is

$$\phi' = \frac{\sin\phi\sqrt{\frac{a^2}{C^2}\sin^2\phi - 1}}{\sqrt{1 - e^2\sin^2\phi}}$$

7 Geodesic of a spherical Earth between London and New York

$$90° - \text{Latitude} = \phi \text{ and Longitude} = \theta$$

West and South are negative , North and East are positive

London's latitude will be taken as 51.5074°N and longitude as 0.1287°W and therefore the initial boundary condition of the geodesic is $\theta_0 = -0.1278$, $\phi_0 = 38.4926$. New York's latitude is 40.7128°N and longitude is 74.0060°W and so the final boundary condition is $\theta_1 = -74.0060$, $\phi_1 = 49.2872$. The relation for ϕ and θ on the geodesic obeys.

$$\frac{C}{\sqrt{1 - C^2}}\cot\phi = \cos\left(\theta + \theta_{\text{constant}}\right)$$

Inserting the boundary conditions in this equation yields the values for C and θ_{constant}

$$C = 0.590834 \text{ and } \theta_{\text{constant}} = 23.069592$$

The geodesic equation between these two points is:

$$\phi = \arctan\left(\frac{0.7323238}{\cos(x + 23.069592)}\right)$$
$$\text{Latitude} = 90 - \arctan\left(\frac{0.7323238}{\cos(\text{Longitude} + 23.069592)}\right)$$

Plotting this function on an equirectangular projection (the world map) gives Figure 4.

[21] Nathan Rooy. *Calculate the Distance Between Two GPS Points with Python (Vincenty's Inverse Formula).* URL: https://nathanrooy.github.io/posts/2016-12-18/vincenty-formula-with-python/.

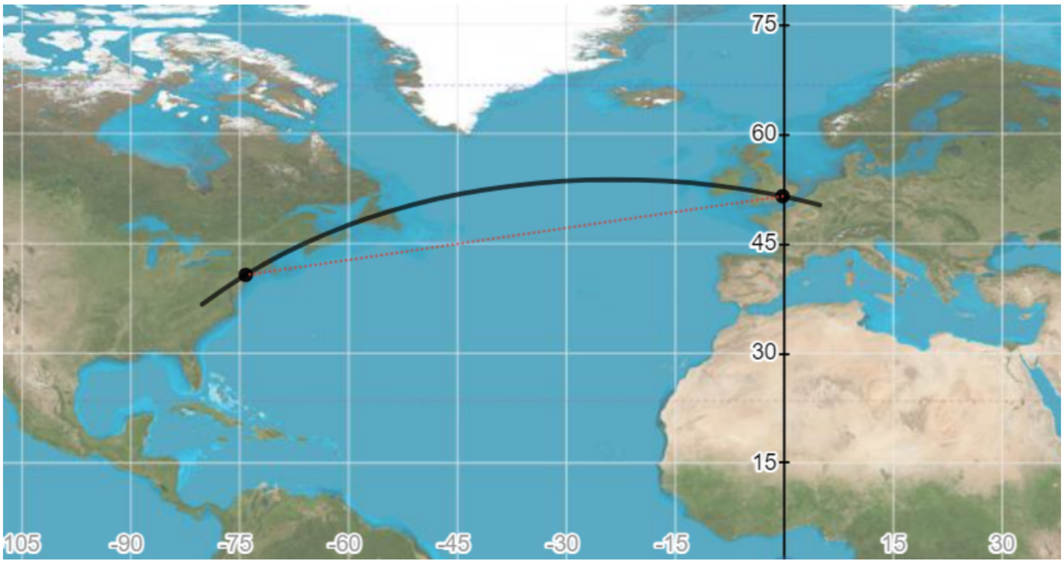

Figure 4: Minimum path from London to New York on a geographical projection

To calculate the minimum distance, the longitude and latitude of the initial and final points are converted to their Cartesian form. Letting the origin be O, initial point have coordinate I and final point F, the great circle distance is

$$= r \times \angle IOF$$

where $\angle IOF$ is the angle between the vectors $\hat{OI}$ and $\hat{OF}$ in radians and r is the volumetric mean radius. The code in Figure 5 returned a distance of 5570.22km, which is 3481.39 miles. So when earth is modelled as a Sphere with its volumetric mean radius[22] being its overall radius, the minimum distance between London and New York is ≈3481 miles. This value will now be compared to the geodesic distance when using a spheroidal model.

[22] David R Williams. *Earth Fact Sheet*. URL: `https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html`.

```
#Constants defined e.g. Volumetric mean radius, locations
r=6371
phi0=38.4926;theta0=-0.1278;phi1=49.2872;theta1=-74.006

#Normalised coordinates of London
x = m.sin(m.radians(phi0))*m.cos(m.radians(theta0))
y = m.sin(m.radians(phi0))*m.sin(m.radians(theta0))
z = m.cos(m.radians(phi0))

#Normalised coordinates of New York
x1 = m.sin(m.radians(phi1))*m.cos(m.radians(theta1))
y1 = m.sin(m.radians(phi1))*m.sin(m.radians(theta1))
z1 = m.cos(m.radians(phi1))

#Put into vector form
vector1 = [x,y,z]
vector2=[x1,y1,z1]

#Dot product
dot_product = np.dot(vector1,vector2)

#arccosine yields the angle as vector modulus is 1
angle = np.arccos(dot_product)

#angle is in radians so r times angle gives arc length
r*angle

5570.222179737957
```

Figure 5: The code used to calculate the Great Circle distance

7.1 Using Euler's method to compute the geodesic of a Spheroid

Due to its rotation , the earth bulges from the equator meaning it resembles an oblate spheroid rather than sphere[23]. Regardless, a comparison of the minimum distances between two points on the earth using the two different models should provide a different minimum distance. To compute Euler's method, the initial conditions are needed to produce the unique geodesic. The differential equation needed is

$$\phi' = \frac{\sin\phi\sqrt{\frac{a^2}{C^2}\sin^2\phi - 1}}{\sqrt{1 - e^2\sin^2\phi}} \tag{48}$$

The derivative is independent of θ, and only dependent on the constant C and and ϕ. This makes sense as a spheroid is a revolution of an ellipse and so the contour of the surface relies strictly on the latitude.

Calculating e for earth (Data taken from[24])

Polar radius: 6356.752km

Equatorial distance: 6378.137km

[23] Charles Q. Choi. *Strange but True: Earth Is Not Round.* Apr. 2007. URL: `https://www.scientificamerican.com/article/earth-is-not-round/`.

[24] Williams, *Earth Fact Sheet*.

From **9.3**,

$$6356.752\text{km} = 6378.137\text{km}\sqrt{1 - e^2}$$

$$e = 0.0818198$$

This gives

$$\phi' = \frac{\sin\phi\sqrt{\frac{6378.137^2}{C^2}\sin^2\phi - 1}}{\sqrt{1 - 0.0818198^2\sin^2\phi}}$$

Here is the function that takes in ϕ and returns ϕ'

```
def PhiPrime(phi):
    #Variables holding the functions so they are easier to refer to in the ODE
    sin = m.sin(m.radians(phi))
    sinsquared = m.sin(m.radians(phi))**2

    #Formula for phi prime is split into its numerator and denominator
    numerator = sin * m.sqrt((a**2)*sinsquared/C0**2 - 1)
    denominator = m.sqrt(1-(e**2 *sinsquared))

    #Positive gradient returned
    return numerator/denominator
```

Figure 6: PhiPrime function which takes in ϕ and returns $\frac{d\phi}{d\theta}$ for a given C

C is unknown and in order to calculate it, ϕ' is needed at the initial position (ϕ_0, θ_0). An estimate is made for ϕ' initially from which the geodesic is calculated, then depending on the width of the geodesic, ϕ' can be adjusted. The initial estimate was taken by calculating ϕ' for the Sphere Geodesic at the starting point.

Finding the ideal gradient that satisfies the endpoints involved splitting the geodesic into two parts. These two parts are due to the turning point of the geodesic. The numerator of the differential equation is

$$\sin\phi\sqrt{\frac{a^2}{C^2}\sin^2\phi - 1}$$

The turning point lies where this is zero. Apart from at the poles, this is $\phi_{\text{max}} = \arcsin\frac{C}{a}$. When the geodesic latitude reaches this maximum point, the gradient switches in direction going from negative to positive (in this case). This gradient switches sign but the magnitudes of ϕ' are unchanged due to (48) being independent of θ. This means the geodesic has a line of symmetry about the $\theta = \theta_{\text{extremum}}$ latitude where θ_{extremum} is the latitude corresponding to the turning point of the geodesic. The geodesic on the oblate Spheroid and symmetry on the oblate Spheroid geodesic is apparent in Figure 7.

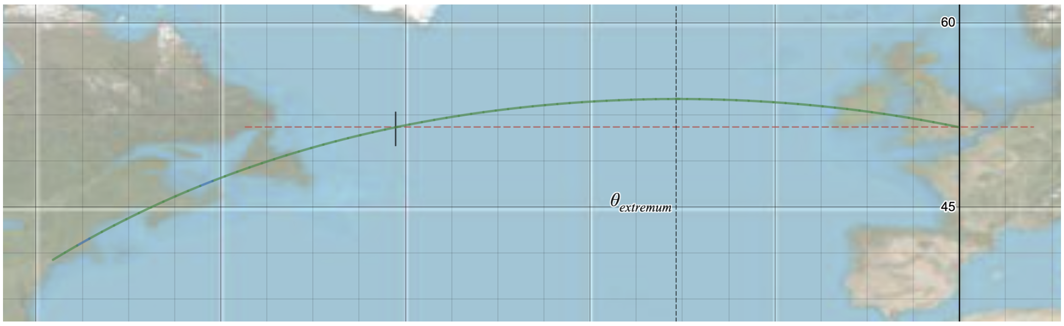

Figure 7: The $\theta_{extremum}$ latitude is the line of symmetry.

C must be chosen such that the ultimate geodesic has the width equal to the longitudinal difference between initial and final point (-0.1278 - - 74.0060 = 73.8782°). Then by using the symmetry of the geodesic, if the geodesic from longitude -0.1278°to $\theta_{extremum}$ is known, then the geodesic from longitude $\theta_{extremum}$ to longitude $2\theta_{extremum}$ + 0.1278°is also known (to the small vertical black line segment). The final part is the geodesic from $2\theta_{extremum}$ + 0.1278°to θ_1. θ_1 is the longitude of the geodesic at the latitude of New York. Both of these values rely on the gradient.

```
NorthwardGeodesicDF['phiprime'][0] = -PhiPrime(38.4926)
NorthwardGeodesicDF['phi'][0]= 38.4926
i=1;c=10**10

#While loop iterates until the turning point of the equirectangular projection is reached
while c>=(m.degrees(np.arcsin(C0/a))+0.00000001):

    #Euler's method being implemented with step size h = 0.0001

    NorthwardGeodesicDF.phi[i]=NorthwardGeodesicDF.phi[i-1] + 0.0001*NorthwardGeodesicDF['phiprime'][i-1]
    c = NorthwardGeodesicDF.phi[i]

    #Negative gradient is needed as going northerly means a decrease in phi
    NorthwardGeodesicDF['phiprime'][i]= -PhiPrime(c)
    i=i+1

SouthwardGeodesicDF['phiprime'][0] = PhiPrime(38.4926)
SouthwardGeodesicDF['phi'][0]= 38.4926
j=1;c=0

#Iterates until phi is equal to 90 - the latitude is that of New York (40.7128°)
while c<49.2872:

    #Euler's method being implemented with step size h = 0.0001
    SouthwardGeodesicDF.phi[j]=SouthwardGeodesicDF.phi[j-1] + 0.0001*SouthwardGeodesicDF['dphi/dtheta'][j-1]
    c = SouthwardGeodesicDF.phi[j]
    SouthwardGeodesicDF['dphi/dtheta'][j]= PhiPrime(c)
    j=j+1
```

Figure 8: Code used to carry out Euler's Method

The use of Euler's method involved over 200, 000 iterations for each part. The geodesic going towards the maximum iterates until it is 0.00000001°in range of the theoretical maximum. The geodesic going south iterates until its latitude has reached that of New York. Then, the longitudes for these two positions are taken and longitudinal width of the geodesic determined.

	thetaN	phi	phiprime
0	-0.1278	38.4926	-0.2073
1	-0.1279	38.49257927	-0.2072989587
2	-0.1280	38.49255854	-0.2072979173

Figure 9: First three rows of the table with the first row containing the boundary conditions

```
#Longitude taken to reach the turning point from London is taken from the longitude at the maximum point
LongitudeN = -(NorthwardGeodesicDF.thetaN[i-2]+0.1278)
#Longitude taken to reach the latitude of New York is taken from the longitude at that latitude
LongitudeS = SouthwardGeodesicDF.thetaS[j-2]+0.1278
#Total longitude covered is calculated. If this is higher than the latitude difference between London
#and New York, the gradient goes down by 0.0001. If it is lower, than goes up by 0.0001
2*LongitudeN + LongitudeS
```

Figure 10: Determining longitudinal width of geodesic using $\theta_{extremum}$ and θ_1

Depending on the longitudinal difference, ϕ' was determined. After the initial approximation of -0.2026 which yielded a geodesic too narrow to cover the longitude and therefore a few decreases later, the optimum ϕ' was -0.2073. This generated a geodesic covering a longitude of 73.8859°, slightly higher than the ideal geodesic but still, this would have a relatively small percentage error (this geodesic lands $\approx$ 830 meters west of the location of New York specified by the coordinates).

A small step size of $h = 0.0001$ was taken. This distance was calculated by converting each position described by each row into its Cartesian coordinate equivalent using the parametric equations on page 19. Then, the Euclidean distance is calculated between adjacent points and summed. For this step size and ϕ', the approximated distance between the two points was 5575.8km or 3484.88 miles.

As $\phi = 90°$- Latitude, a negative ϕ' is is actually northward direction on figure 7.

8 Evaluating Euler's method to approximate Spheroid geodesics and the use of a Sphere to model earth

Distance for Spherical Earth (Great circle formula): 3481.4 miles

Distance for Spheroidal Earth (Euler's method) : 3484.9 miles

Distance for Spheroidal Earth (Vincenty's formula[25]) : 3490.7 miles

[25] URL: https://geodesyapps.ga.gov.au/vincenty-inverse.

The Spherical model for the Earth underestimated the distance between London and New York by over 9 miles (0.266% error). Euler's method was more accurate than this whilst expectedly underestimating the true distance by 5.8 miles (%0.166 error) due to the Euclidean distance being taken between points on the surface of the ellipsoid and the nature of ϕ'. A smaller step size and approximating the curve segments as great circles could decrease this error. Another issue was the resolution limit for ϕ' being only 0.0001. This is the reason the Spheroid's geodesic lands 830 metres west of New York. With faster computation speed, finer results are obtained. Another issue was the turning point of the geodesic, the differential equation begins to zero off as it depended only on latitude while longitude increases steadily at $0.0001°$. To tackle this, when the geodesic latitude was 10^{-8} °below the maximum latitude, the iteration was terminated.

A strength of Euler's method is that it plots latitudes/longitudes as opposed to Vincenty's iterative method which returns only the geodesic length. Despite Euler's method underestimating the lengths heavily and being computationally intensive, it accurately showed that the Spherical model underestimates the geodesic length.

9 Prerequisites explained

9.1 Finding the arc length using integration

Any arc can be treated as small line segments joined to each other. That is, for a small change Δx, the change in the the length of the line segment Δs can be visualised by figure 11:

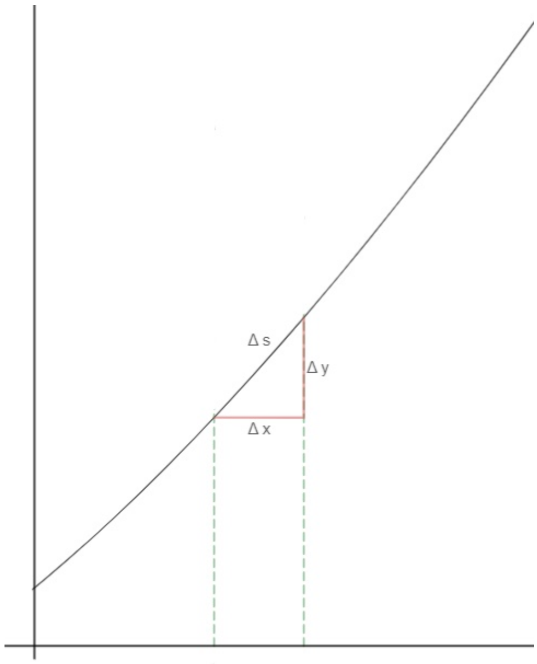

Figure 11: Change in the arc length due to a small change in the x direction

For small Δx, treating Δs as a straight line, the Pythagorean theorem gives

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\frac{\Delta s}{\Delta x} = \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}}$$

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

$$s = \int \sqrt{1 + (y')^2}\mathrm{d}x \tag{49}$$

9.2 Hyperbolic functions

Hyperbolas are parameterised using hyperbolic functions. The hyperbolic sine and hyperbolic cosine functions are defined as:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

The derivatives are

$$\begin{aligned}\frac{\mathrm{d}\cosh(x)}{\mathrm{d}x} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(e^x + e^{-x}) \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(e^x - e^{-x}) \\ &= \sinh(x)\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}\sinh(x)}{\mathrm{d}x} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(e^x - e^{-x}) \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(e^x + e^{-x}) \\ &= \cosh(x)\end{aligned}$$

Osborn's rule[26] neatly derives the hyperbolic identity:

$$\begin{aligned}\cos(x) + i\sin(x) &\equiv e^{ix} \\ \cos(x) - i\sin(x) &\equiv e^{-ix} \\ \cos(x) &\equiv \frac{e^{ix} + e^{-ix}}{2} \\ i\sin(x) &\equiv \frac{e^{ix} - e^{-ix}}{2}\end{aligned}$$

From this:

$$\begin{aligned}\cos(ix) &\equiv \frac{e^{i^2x} + e^{-i^2x}}{2} \equiv \frac{e^{-x} + e^{x}}{2} \equiv \cosh(x) \\ i\sin(ix) &\equiv \frac{e^{i^2x} - e^{-i^2x}}{2} \equiv \frac{e^{-x} - e^{x}}{2} \equiv -\sinh(x) \\ -i\sin(ix) &\equiv \frac{e^{x} - e^{-x}}{2} \equiv \sinh(x)\end{aligned}$$

[26] Brian Gaulter and Mark Gaulter. *Further pure mathematics*. Oxford University Press, 2001.

Hyperbolic function identities are therefore a direct consequence of trigonometric identities:

$$\begin{aligned}
\cos^2(x) + \sin^2(x) &\equiv 1 \\
\cos^2(ix) + \sin^2(ix) &\equiv 1 \\
\cos^2(ix) - (-i\sin(ix))^2 &\equiv 1 \\
\cosh^2(x) - \sinh^2(x) &\equiv 1
\end{aligned}$$

arcosh and arsinh are the inverse function of sinh and cosh. The derivative of arcosh is

$$\begin{aligned}
y &= \operatorname{arcosh}(x) \\
x &= \cosh(y) \\
\frac{\mathrm{d}}{\mathrm{d}x}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(\cosh(y)) \\
1 &= \sinh(y)\frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sinh(y)} \\
\sinh(y) &= \sqrt{\cosh^2(y) - 1} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sqrt{x^2 - 1}}
\end{aligned}$$

9.3 Eccentricity and ellipses

Conics are shapes obtained by the intersection of a plane with a cone. For some point (x, y) on the locus of a conic, The eccentricity, e is the ratio of the distance from (x, y) to the focus (a point) and the shortest distance from (x, y) to a the directrix (a given straight line)[27]. The locus of the conic depends on e.

Conventionally, the focus and initial point on the conic are positioned on the coordinates $(ae, 0)$ and $(a, 0)$ respectively. The directrix is $x = \frac{a}{e}$. These three initial points satisfy the condition of e. The distance between $(ae, 0)$ and $(a, 0)$ is $|ae - a|$ which is e times larger than the direct distance between $x = \frac{a}{e}$ and $(a, 0)$, $|\frac{a}{e} - a|$.

[27] *Eccentricity*. URL: `https://encyclopediaofmath.org/wiki/Eccentricity`.

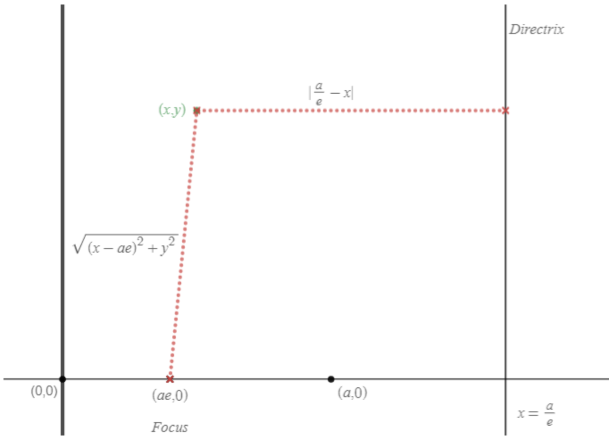

Figure 12: Starting coordinates to map a conic

PD is the distance between the point (x, y) and the directrix and PF is the distance between the locus (x, y) and the focus $(ae, 0)$. Using PF = ePD[28], the locus becomes

$$\sqrt{(x-ae)^2+(y)^2} = e|x-\frac{a}{e}|$$
$$(x-ae)^2+(y)^2 = e^2(x-\frac{a}{e})^2$$
$$x^2+a^2e^2-2aex+y^2 = e^2x^2-2aex+a^2$$
$$(1-e^2)x^2+y^2 = a^2(1-e^2)$$
$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)} = 1 \tag{50}$$

The vertical length of the ellipse through the origin is $a\sqrt{1-e^2}$. The minor axis of an ellipse is the ellipse's shorter line of symmetry.

[28] Brian Gaulter and Mark Gaulter. *Further pure mathematics*. Oxford University Press, 2001.

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