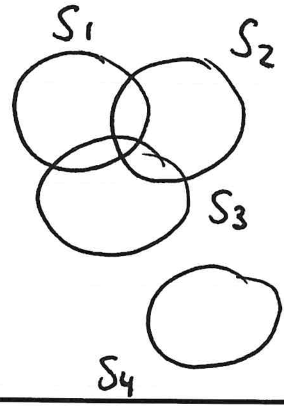
This is known as a "diagonal argument", due to Cantor (1870s). Note that it in fact shows that $(0,1)$ is uncountable.

<u>Corollary 9.</u> There are uncountably many transcendental numbers.

<u>Proof</u> If $\mathbb{R}\setminus\mathbb{A}$ were countable, then since $\mathbb{A}$ is countable, $\mathbb{R} = \mathbb{R}\setminus\mathbb{A} \cup \mathbb{A}$ would be countable. ※. $\square$

<u>Theorem 10</u> $\mathcal{P}(\mathbb{N})$ is uncountable.

<u>Proof 1</u> If $\mathcal{P}(\mathbb{N})$ were countable, we could list the subsets of $\mathbb{N}$ as $S_1, S_2, S_3, \ldots$.



Let $S = \{n \in \mathbb{N} : n \notin S_n\}$.

Then S is not on our list since $\forall n \in \mathbb{N}$, $S \neq S_n$ (as S and S_n differ in their membership of the element n). $\times\!\!\!\times$

Hence $\mathcal{P}(\mathbb{N})$ is uncountable. $\square$

Note that this is again a "diagonal argument".

<u>Proof 2</u> Note that there is an injection from $(0,1)$ into $\mathcal{P}(\mathbb{N})$: write $x \in (0,1)$ in binary $0.x_1x_2x_3\ldots$ with $x_i \in \{0,1\}$ (not ending in an infinite string of 1s) and set $f(x) = \{n : x_n = 1\}$,

Eg. $0.11101000\ldots \longmapsto \{1,2,3,5\}$.

This is an injection. $\square$

In fact, Proof 1 of Theorem 10 shows the following.

<u>Theorem 11</u> For any set X, there is no bijection between X and $\mathcal{P}(X)$.

<u>Proof</u> Given any function $f: X \to \mathcal{P}(X)$, we shall show that f is not a surjection. Indeed, let

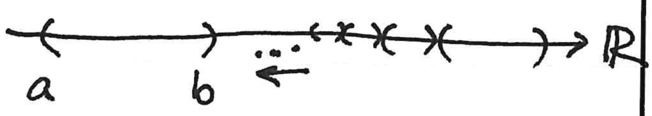
$S = \{x \in X : x \notin f(x)\}$. Then S does not belong to the image of f, since $\forall x \in X$, S and $f(x)$ differ in the element x, and thus $S \neq f(x)$. $\square$

<u>Remarks</u> (1) This is reminiscent of Russell's Paradox.

(2) In fact, it gives another proof that there is no universal set. For suppose we had such a universal set V, then we would have $\mathcal{P}(V) \subseteq V$, in which case there would certainly be a surjection from V to $\mathcal{P}(V)$.

<u>Example</u> Let $\{A_i : i \in I\}$ be a family of open intervals of $\mathbb{R}$ which are pairwise disjoint. Must the family be countable?

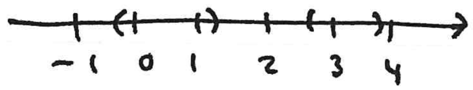
<u>Warning</u>: There is no "next" interval after (a, b).



The family $\{A_i : i \in I\}$ is nevertheless countable.

<u>Proof 1</u> Each interval A_i contains a rational, and $\mathbb{Q}$ is countable, so since the intervals are disjoint, we have an injection from I into $\mathbb{Q}$. Hence the family $\{A_i : i \in I\}$ is countable. $\square$

<u>Proof 2</u> The set $\{i \in I : A_i \text{ has length} \geq 1\}$ is countable as it injects into $\mathbb{Z}$.



Similarly, the set $\{i \in I : A_i \text{ has length} \geq \frac{1}{2}\}$ is countable as it injects into $\frac{1}{2}\mathbb{Z}$.

More generally, for each $n \in \mathbb{N}$, $\{i \in I : A_i \text{ has length} \geq \frac{1}{n}\}$ is countable.

Now $\{A_i : i \in I\}$ is countable as it is a countable union of countable sets. $\square$

<u>Summary</u> To show that X is uncountable

(1) run a diagonal argument on X;

(2) inject your favourite uncountable set into X.

To show that X is countable

(1) list it (may fiddly);

(2) inject it into $\mathbb{N}$;

(3) use "countable unions of countable sets are countable";

(4) if "in/near" $\mathbb{R}$, consider $\mathbb{Q}$.