Investigating discrete Markov processes as a method of predicting the prices of Bitcoin and Litecoin

1 Introduction

Centralized mediums of exchange like commodities, stocks, and government issued currencies have faced heavy criticism in recent times due to several reasons. Financial institutions like commercial banks work as intermediaries between buyers and sellers where they profit through commission, lowering the profits that investors could've made had they directly traded with the buyer[1]. Also, these assets are traded on a central platform like a brokerage where each of the assets is priced at a single value determined by the quoted price; this takes away the seller's freedom to decide the price of their owned asset[2]. Furthermore, placing holdings in commercial banks transfers the risk associated with investing this money to the banks and means less control for the investor. A consequence of this dependence on banks was the Global Recession of 2008, a period of economic downturn due to high-risk investments from US banks which ultimately lead to banks losing funds, people losing confidence in the economy, and an estimated $19 trillion loss from US households[3].

Due to these reasons, Satoshi Nakamoto invented a decentralized 'trustless' currency, Bitcoin, in 2009. Secured using high-level encryption, Bitcoin is a digital currency (or cryptocurrency). The value of a cryptocurrency is not affected by any government's manipulation and there are no intermediaries between the buyers and sellers. The value of a cryptocurrency like any other financial security relies on supply and demand. New Bitcoins are produced by the process of **mining**, which refers to specialized computers solving complex algorithms to calculate a key called a hash which confirms a Bitcoin transaction and lists this transaction on the Blockchain. The Blockchain is a peer to peer network storing a database of completed Bitcoin transactions[4]. Mining is a very cost and energy intensive job and mining companies are incentivized with freshly minted Bitcoins. The rate at which bitcoins are rewarded decreases over time (halves approximately every four years). This decreasing increase in supply aims to ensure the price of Bitcoin also increases[5].

From July 2010 when the price of a bitcoin was $0.0008 to December 2017, when it traded at nearly $20 000 to now at October 2020, when it is trading at over $10,000, it is clear that predicting the change in any financial asset using technical analysis is of decisive importance as accurate predictions can minimize risk and maximize profit.[6]

The **random walk theory** suggests that all the possible changes in different asset prices are analogous to a person randomly walking from one position to another. Where the walker has been before won't determine where they will be next and like this, the previous patterns and prices of an asset will not determine what the price will be tomorrow. There is an independence of future events from past events. However, the current position of a walker does determine which position they will choose next and therefore according to the model, only the present situation will determine the future of a system. That is, the future price of an asset depends only on the present state. A possible position at which the walker can be at is called a **state** of the system.

[1] James Chen. *Financial Intermediary*. Jan. 2020. URL: https://www.investopedia.com/terms/f/financialintermediary.asp.

[2] Will Kenton. *Centralized Market*. Feb. 2020. URL: https://www.investopedia.com/terms/c/centralizedmarket.asp.

[3] Investopedia Staff. *The Great Recession Definition*. Mar. 2020. URL: https://www.investopedia.com/terms/g/great-recession.asp.

[4] Jake Frankenfield. *Bitcoin*. Apr. 2020. URL: https://www.investopedia.com/terms/b/bitcoin.asp.

[5] Alyssa Hertig. *Bitcoin Halving 2020, Explained*. May 2020. URL: https://www.coindesk.com/bitcoin-halving-explainer.

[6] John Edwards. *Bitcoin's Price History*. June 2020. URL: https://www.investopedia.com/articles/forex/121815/bitcoins-price-history.asp.

In this investigation, the random walk theory will be tested on two Cryptocurrencies, **Bitcoin** and **Litecoin** where the transitions of the cryptocurrency between different price ranges will represent the states. Only knowing the present state of a system to predict the immediate future of a state is a feature of the random walk theory and is also known as the Markov property and hence, modelling the price fluctuations of the two currencies as a Markov model is a suitable method to test this theory.

2 Markov Property

To compute this random walk, a **stochastic process** called a **time homogeneous discrete** Markov chain will be used. A stochastic process describes the evolution of a random phenomenon in time.[7]
A discrete time system is one that changes after set intervals of time rather than continuously changing with time.
Time homogeneity is a very useful assumption for this model as it means that the probability distribution of transitioning from a given state k to any other state f is independent of the time but solely dependant on the duration taken to leave some state k and arrive at state f. S_t represents the state at which the system is at time t and f, k represent two potential states of the system. $P(A|B)$ is the standard notation for the probability of event A occurring given that event B has taken place.

$$P(S_{t+m} = f|S_t = k) = P(S_m = f|S_0 = k) \tag{1}$$

(1) shows that the probability of transitions is dependant only on the number of intermediate transitions between the arrival state and starting state, and hence constant for any given time difference m.

The Markov property, also called memorylessness, is a requirement to a Markov process and makes it suitable for this investigation. This is, the future of any system is only dependant on the present state and not influenced by any historical patterns that precede it.[8]

$$P(S_{t+1} = f|S_t = k_0) = P(S_{t+1} = f|S_t = k_0, S_{t-1} = k_1, ..., S_0 = k_t) \tag{2}$$

t is time
S_t denotes the state at which the system is at the time t
k_t is any state that the system is in at time t. e.g. k_0 is the initial state and k_1 is the state of the system one interval later.

(2) shows how if the state is at some state k_0, the probability of it being at state f after 1 time interval is independent of wherever the state had been prior to it its present position. The probabilities of transitioning from any given state to another in a set interval remains constant due to independence from any past transitions (memoryless) and the independence from the time at which the transition is taking place (time homogeneity).

2.1 Matrices and matrix multiplication

A matrix is simply a rectangular array of numbers, where the matrix **M** of dimensions $r \times c$ is a rectangular matrix with **r** rows and **c** columns, this would look as such

$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ r \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \dots & c \end{array} \\ \begin{bmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,c} \\ M_{2,1} & M_{2,2} & \dots & M_{2,c} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r,1} & M_{r,2} & \dots & M_{r,c} \end{bmatrix} \end{array}$$

[7]Fabrice Baudoin. *Stochastic Processes*. URL: https://www.sciencedirect.com/topics/neuroscience/stochastic-processes.
[8]*Markov property*. July 2020. URL: https://en.wikipedia.org/wiki/Markov_property.

Identity matrix (**I**): analogous to 1 in the real number system where a matrix multiplied to **I** is itself. **I** is a matrix of any dimension with its leading diagonal equal to 1 and all other positions equal to 0.
Zero matrix (**0**) : analogous to 0 in real numbers as summing it to any other matrix does not change its value. **0** is a matrix with all its entries equal to 0.

Notational remark : $\mathbf{M}_{\mathrm{i,j}}$ represents the value at the i^{th} row and j^{th} column of the matrix **M**

The matrix multiplication of two matrices is only possible if the number of columns of the first matrix is equal to the number of rows on the second matrix. This is because the row of the first matrix and column of the second matrix undergo a dot product and therefore their lengths must be equal. The multiplication of **A**, an $m \times n$ matrix, and **B**, an $n \times k$ matrix, can be described as such:

$$(\mathbf{AB})_{\mathrm{m,k}} = \sum_{i=1}^{n} \mathbf{A}_{\mathrm{m,i}}\mathbf{B}_{\mathrm{i,k}} \tag{3}$$

Where **m** represents the row number and **k** the column number

e.g

$$\begin{bmatrix} 1 & 3 & 2 \\ 7 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 3 \times 1 + 2 \times 4 \\ 7 \times 2 + 3 \times 1 + 4 \times 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 33 \end{bmatrix}$$

2.2 Scalar multiplication of matrices

If a matrix is multiplied by k where $k \in \mathbb{R}$, the resulting product is the original matrix with all its entries multiplied by k. So

$$k\mathbf{M}_{\mathrm{i,j}} \to k\mathbf{M} \tag{4}$$

This is a scalar multiplication of a matrix. The real number k can be represented in matrix form such that (4) is satisfied. Representing k in the form of a matrix means that it satisfies the conditions for matrix addition and multiplication. The matrix form of k is $k\mathbf{I}$.

$$k \to k\mathbf{I} = \begin{bmatrix} k & 0 & 0\ldots \\ 0 & k & 0\ldots \\ 0 & 0 & k\ldots \\ \vdots & \vdots & \vdots\ddots \end{bmatrix} \tag{5}$$

$k\mathbf{I}$ can be confirmed to satisfy (4) by considering a $m \times n$ matrix **A** and a $r \times m$ matrix **K**, where **K** represents $k\mathbf{I}$. These two matrices satisfy the condition for matrix multiplication and so

$$(\mathbf{KA})_{\mathrm{r,n}} = \sum_{i=1}^{m} \mathbf{K}_{\mathrm{r,i}}\mathbf{A}_{\mathrm{i,n}} \tag{6}$$

Notice from (6), that $\mathbf{K}_{\mathrm{r,i}}$ is non-zero only when i = r and therefore, the summation on the right- side of (6) simplifies to

$$\mathbf{K}_{\mathrm{r,r}}\mathbf{A}_{\mathrm{r,n}}$$

As the r^{th} row and r^{th} column lie on the leading diagonal of **K**, $\mathbf{K}_{\mathrm{r,r}} = k$. And so

$$(\mathbf{KA})_{\mathrm{r,n}} = k\mathbf{A}_{\mathrm{r,n}}$$

2.3 Eigenvectors and eigenvalues

The matrix multiplication of a $m \times n$ matrix with a n variable column vector is a **linear transformation** to that vector. If a vector undergoes a matrix multiplication and its direction stays the same, that vector is known as the eigenvector of that transformation. The associated eigenvalue to that eigenvector is the scale factor by which the vector is stretched or shrunk. A less geometrical interpretation of this is that, when the product of a 1 dimensional matrix and square matrix is a scalar multiple of the 1D matrix, the 1D matrix is the eigenvector of that square matrix. For:

$$\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \lambda \begin{bmatrix} x & y \end{bmatrix}$$

$\begin{bmatrix} x & y \end{bmatrix}$ is the invariant vector or eigenvector of the transformation.
λ is the associated eigenvalue.

A general form of this would be:

$$\mathbf{xT} = \lambda \mathbf{x}$$

where $\mathbf{T}$ is the matrix in the matrix multiplication.
$\mathbf{x}$ is the **eigenvector** associated to the specific **eigenvalue** λ. This can be rewritten in the form using the **distributive** property of matrix multiplication:

$$\mathbf{x}(\mathbf{T} - \lambda) = \mathbf{0}$$

Notice that $\mathbf{T} - \lambda$ makes no sense as $\mathbf{T}$ is vector and λ is a scalar. From (5) it is known that multiplying by $k\mathbf{I}$ has the same effect as multiplying by k. $k\mathbf{I}$ is substituted in place of k, satisfying the conditions of matrix addition which is required next.

$$\mathbf{x}(\mathbf{T} - \lambda\mathbf{I}) = \mathbf{0}$$

An explanation of linear transformations and the distributive property is in the **appendix**.

2.4 Using a transition matrix to compute the Markov chain

A **state** of a system can represent anything. In the later example, one state will represent a 0 - 2% increase in the Bitcoin price. Using matrix notation to define a transition matrix requires assigning a state a number. A transition matrix is where the probabilities of transitioning from some state i to j is are presented on the i^{th} row and j^{th} column of the matrix.[9] That is, a square matrix $\mathbf{T}$ of dimensions n×n with the following property.

$$\mathbf{T}_{\text{i,j}} = P(S_{t+1} = \text{j}|S_t = \text{i}) \tag{7}$$

where S_t represents the state of the system at time t. $S_t = \text{i}$ does not mean the state at time t is the integer i, but means that the state at time t is the state assigned with the number i.

This previous property implies the condition for $\mathbf{T}$ to be a stochastic matrix which is:

$$\sum_{k=1}^{n} \mathbf{T}_{\text{i,k}} = 1$$

That is, each row sum of the matrix is 1. This investigation will use transition matrices of this form.

Using this definition and the time homogeneous characteristic of the Markov chain, I will show that the N^{th} power of a transition matrix provides the probabilities of transitioning between two states in N steps. That is

$$(\mathbf{T}^N)_{\text{i,j}} = P(S_{t+N} = \text{j}|S_t = \text{i}) \tag{8}$$

[9] Stephen Andrilli and David Hecker. *Transition Matrix*. URL: https://www.sciencedirect.com/topics/mathematics/transition-matrix.

The transition matrix, $\mathbf{T}^N$, describes the probabilities of arriving at state j from state i in exactly N transitions. This result can be proven by induction.

Proof

By the definition of the transition matrix in (7), the proposition is true for $N = 1$ as again, by definition:

$$\mathbf{T}^1 = \mathbf{T}$$
$$(\mathbf{T}^1)_{\mathrm{i,j}} = P(S_{t+1} = \mathrm{j}|S_t = \mathrm{i}) \tag{9}$$

This is now being assumed for some integer k:

$$(\mathbf{T}^k)_{\mathrm{i,j}} = P(S_{t+k} = \mathrm{j}|S_t = \mathrm{j}) \tag{10}$$

Now multiplying $\mathbf{T}^k$ and $\mathbf{T}^1$ using (3), this product can be written as:

$$\mathbf{T}^{k+1} = \mathbf{T}^k\mathbf{T}^1 \tag{11}$$
$$(\mathbf{T}^{k+1})_{\mathrm{i,j}} = \sum_{\nu=1}^{n} (\mathbf{T}^k)_{\mathrm{i}\nu} \cdot (\mathbf{T}^1)_{\nu\mathrm{j}} \tag{12}$$

Substituting (9) and (10) in for $(\mathbf{T}^1)_{\nu,\mathrm{j}}$ and $(\mathbf{T}^k)_{\mathrm{i},\nu}$ respectively yields

$$(\mathbf{T}^{k+1})_{\mathrm{i,j}} = \sum_{\nu=1}^{n} P(S_{t+k} = \nu|S_t = \mathrm{i}) \cdot P(S_{t+1} = \mathrm{j}|S_t = \nu) \tag{13}$$

As the process is time homogeneous, the time is allowed to be shifted by any time difference as long as this time difference between the state departure and arrival is a constant. This means:

$$(\mathbf{T}^1)_{\mathrm{i},\nu} = P(S_{t+1} = \mathrm{j}|S_t = \nu) = P(S_{t+k+1} = \mathrm{j}|S_{t+k} = \nu)$$

The constant k has been added to each of the times therefore maintaing the duration of the transition which in this case is 1 time unit. Substituting this value in (13) yields:

$$(\mathbf{T}^{k+1})_{\mathrm{i,j}} = \sum_{\nu=1}^{n} P(S_{t+k} = \nu|S_t = \mathrm{i}) \cdot P(S_{t+k+1} = \mathrm{j}|S_{t+k} = \nu) \tag{14}$$

ν is a variable that iterates from state 1,...,n and is the intermediate state between the process transitioning from state i to j. As the states that ν takes includes every single possible state of the system(exhaustive), the conditional probabilities in (14) can be combined.

$$\sum_{\nu=1}^{n} P(S_{t+k} = \nu|S_t = \mathrm{i}) \cdot P(S_{t+k+1} = \mathrm{j}|S_{t+k} = \nu) = P(S_{t+(\mathbf{k+1})} = \mathrm{j}|S_t = \mathrm{i})$$

And therefore

$$(\mathbf{T}^{k+1})_{\mathrm{i,j}} = P(S_{t+(\mathbf{k+1})} = \mathrm{j}|S_t = \mathrm{i})$$

If (8) were to hold for some value of k, it also holds for $k + 1$, and as it holds for k = 1 by its definition, (8) is proven for all positive integers. It can therefore be stated that:

$$(\mathbf{T}^N)_{\mathrm{i,j}} = P(S_{n+N} = \mathrm{j}|S_n = \mathrm{i})$$

Due to time homogeneity, This can be rewritten as:

$$(\mathbf{T}^N)_{\mathrm{i,j}} = P(S_N = \mathrm{j}|S_0 = \mathrm{i}) \tag{15}$$

2.5 Markov property as an eigenvalue problem

A transition matrix is not enough to calculate the probabilities of being at some state j after N transitions. From (15), the probability of being at state j after N transitions given that the initial state is i is known, however what is the probability of being at state i initially or in probability terms, what is $P(S_0 = \mathrm{i})$?. A row stochastic vector of dimensions $1 \times n$ denoted as $\mathbf{P}_{\mathrm{t}}$ will represent the probabilities of being at some state v at time t in a system with n states. This is

$$(\mathbf{P}_{\mathrm{t}})_{1,v} = P(S_{\mathrm{t}} = v)$$

As it is describing a probability distribution, the sum of its entries is strictly 1. $\mathbf{P}_0$ therefore describes the initial probabilities of starting at state v. This is

$$(\mathbf{P}_0)_{1,v} = P(S_0 = v) \tag{16}$$

I will now show $\mathbf{P}_0\mathbf{T}^N$ is equal to the probability distribution of being at some states at time $t = N$. That is

$$(\mathbf{P}_0\mathbf{T}^N)_{1,v} = P(S_N = v)$$

Using the definition of matrix multiplication in (3), $\mathbf{P}_0\mathbf{T}^N$ can be rewritten

$$\sum_{m=1}^{n} (\mathbf{P}_0)_{1,m} \cdot (\mathbf{T}^N)_{m,v} \tag{17}$$

By substituting (16) and (15) into this sum and then applying standard probability rules, the following can be concluded

$$\begin{aligned}
&= \sum_{m=1}^{n} P(S_0 = m) \cdot P(S_N = v | S_0 = m) \\
&= \sum_{i=m}^{n} P(S_0 = m \cap S_N = v) \\
&= P(S_N = v) \\
&= (\mathbf{P}_N)_{1,v}
\end{aligned}$$

This leads to the useful conclusion:

$$\mathbf{P}_0\mathbf{T}^N = \mathbf{P}_N \tag{18}$$

$\mathbf{P}_N$ is the probability distribution of being at state v, N transitions later. This is useful because if we have the starting probabilities and a transition matrix, then the state of the system at any time can be predicted.

For a large number of transitions, the probability distribution $\mathbf{P}_N$ converges to a stochastic vector called the **steady state vector**. That is:

$$\lim_{N \to \infty} \mathbf{P}_N \to \mathbf{v}$$

where $\mathbf{v}$ represents the steady state vector.
By the definition of the limit, the succeeding probability distributions $\mathbf{T}_{N+1,N+2,N+3....}$ also converge to the vector $\mathbf{v}$:

$$\lim_{N \to \infty} \mathbf{P}_{N+1} \to \mathbf{v}$$

As $N \to \infty$, $\mathbf{v}$ substitutes in for $\mathbf{P}_N$ and $\mathbf{P}_{N+1}$, therefore:

$$\lim_{N \to \infty} \mathbf{P}_N = \lim_{N \to \infty} \mathbf{P}_{N+1} = \mathbf{v}$$

and noticing that, due to the time homogeneity of a Markov Chain, (18) can become $\mathbf{P}_{N+1} = \mathbf{P}_N\mathbf{T}$, a relationship between the steady state vector and transition matrix is established.

$$\mathbf{vT} = \mathbf{v} \tag{19}$$

The substitution shows that $\mathbf{v}$ is the eigenvector of $\mathbf{T}$ with eigenvalue 1. All stochastic matrices have an eigenvalue of 1. This is shown in the **appendix**.

2.6 Markov chain diagrams

Transition states can be better visualized with the diagrams. An example diagram for a two state Markov Chain looks like

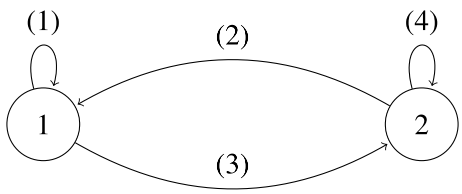

Figure 1: Markov chain diagram for a two state process

$$
\begin{aligned}
(1) &= P(S_{t+1} = 1|P_t = 1) \\
(2) &= P(S_{t+1} = 1|P_t = 2) \\
(3) &= P(S_{t+1} = 2|P_t = 1) \\
(4) &= P(S_{t+1} = 2|P_t = 2)
\end{aligned}
$$

3 Predicting the trading price trend for Bitcoin and Litecoin using a two state Markov model

3.1 Collecting and interpreting the data

In this investigation, the cryptocurrency prices will be taken at set time intervals, the **closing price** of the asset after each day will be taken. The closing price is the final trading price of the cryptocurrency on the particular day.

Two years worth of closing day prices for Bitcoin and Litecoin from Yahoo finance[10] spanning from 25.04.2018 to 25.04.2020 will be used. After downloading the csv file for each cryptocurrency, a Python program was used to extract the necessary information to compute the Markov model. The dataframe from the raw csv file which is downloadable from Yahoo finance is in this form:

	Date	Open	High	Low	Close	Adj Close	Volume
0	2018-04-25	9701.030273	9745.320313	8799.839844	8845.740234	8845.740234	11083100160

Figure 2: Format of the csv Dataframe

To form a Markov model, only the *Close* column from the table is needed.

To use a Markov model, the states that the asset price can be at must be established. Firstly, a very simple two state Markov chain can be used to forecast the price where two states will be **Bear** and **Bull**, where the Bear state means the price has decreased from the preceding day while Bull means the price has increased. These are common terms to represent the downward and upward movement of an asset price

Figure 3 shows the program used for the data abstraction for the two state Markov model.

[10] *Yahoo Finance – stock market live, quotes, business & finance news*. URL: `https://uk.finance.yahoo.com/`.

```python
def BullBearTwoStateMarkovChain(csvfile):

    Data = pd.read_csv(csvfile)
    CPD = Data.Close #Closing Price Data

    States=pd.Series()

    for i in range(len(CPD) - 2):
        if CPD[i+1]>CPD[i]:
            States = States.append(pd.Series([1]))
        else:
            States = States.append(pd.Series([0]))
     #Formatting the Dataframe to acquire the specific columns

    States=States.reset_index()
    States.pop('index')
    States=pd.Series(States[0])
    #Formatting

    StepMatrix=np.zeros((2,2),dtype=int)
    #Empty 2-D matrix declared

    for rowN in range(2):
        for columnN in range(2):
            for time in range(len(States)-2):
                if States[time]==rowN and States[time+1]==columnN:
                    StepMatrix[rowN][columnN]+=1
    return StepMatrix
```

```
In [221]: BullBearTwoStateMarkovChain('BTC-USD.csv')

Out[221]: array([[162, 196],
                 [196, 174]])
```

Figure 3: Matrix output of the function when the Bitcoin file is a parameter

S_t \ S_{t+1}	Bear	Bull	Total
Bear	162	196	358
Bull	196	174	370
			728

Table 1: The number of days within the two year interval, where the price of a Bitcoin transitioned between a bullish and bearish trend

From the number of days in the above table, the probabilities of the transitions occurring can be calculated:

$$P(S_{t+1} = \text{Bear}|S_t = \text{Bear}) = \frac{162}{358}$$
$$P(S_{t+1} = \text{Bull}|S_t = \text{Bear}) = \frac{196}{358}$$
$$P(S_{t+1} = \text{Bear}|S_t = \text{Bull}) = \frac{196}{370}$$
$$P(S_{t+1} = \text{Bull}|S_t = \text{Bull}) = \frac{174}{370}$$

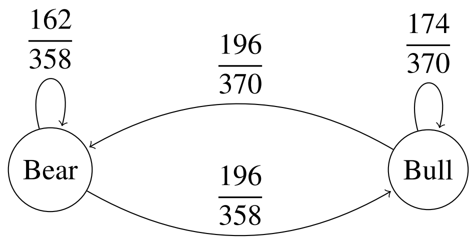

Figure 4: Associated Markov Chain for the two state Bitcoin table

Litecoin:

```
In [222]: BullBearTwoStateMarkovChain('LTC-USD.csv')

Out[222]: array([[185, 195],
                 [195, 153]])
```

Figure 5: Program output for the Litecoin csv file

S_t \ S_{t+1}	Bear	Bull	Total
Bear	185	195	380
Bull	195	153	348
			728

Table 2: The number of days that the **Litecoin** closing price transitioned between each state

The probabilities can be taken from the Litecoin table and yield the following Markov chain .

The next step is creating the matrix equivalent of the Markov chains. The transition matrix for Bitcoin, $\mathbf{T_B}$ and Litecoin, $\mathbf{T_L}$, where L (Low) stands for the Bear state and H (High) stands for the Bull state, looks as such:

$$\mathbf{T_B} = \begin{array}{c} \\ \mathrm{L} \\ \mathrm{H} \end{array}\!\!\begin{array}{c} \begin{array}{cc} \mathrm{L} & \mathrm{H} \end{array} \\ \begin{bmatrix} \frac{162}{358} & \frac{196}{358} \\ \frac{196}{370} & \frac{174}{370} \end{bmatrix} \end{array}, \mathbf{T_L} = \begin{array}{c} \\ \mathrm{L} \\ \mathrm{H} \end{array}\!\!\begin{array}{c} \begin{array}{cc} \mathrm{L} & \mathrm{H} \end{array} \\ \begin{bmatrix} \frac{185}{380} & \frac{195}{380} \\ \frac{195}{348} & \frac{153}{348} \end{bmatrix} \end{array}$$

3.2 Approximating the steady state vectors

The steady state vector will be approached as a larger number of iterations occur. In order to make an approximation, the initial starting probabilities of being at the specific states is needed. $\mathbf{P}_0$ was the notation used early to denote the vector which has the probability distribution of being at some state i at the start.

From the conclusion in (19), the iterative formula being used is:

$$\mathbf{P}_{t+1} = \mathbf{P}_t\mathbf{T} \tag{20}$$

where t in this context, is simply the day which the vector **P** is describing.

For the approximation, an initial distribution for $\mathbf{P}_0$ must be made. Note that the converging steady state vector does not depend on the initial probabilities, but in order to effectively approximate this vector, a reasonable estimate must be made. Assuming that the two currencies behave totally randomly and therefore are bullish and bearish the same number of times, a 0.5, 0.5 probability distribution is a fair approximation for $\mathbf{P}_0$.

$$\mathbf{P}_0 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$$

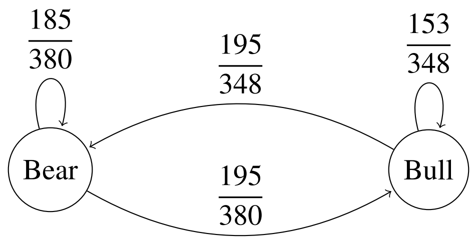

Figure 6: Two state Markov chain for the **Litecoin** closing day price

From here, (20) is directly applied where $\mathbf{T} = \mathbf{T_B}$ and the matrices are multiplied:

$$\begin{aligned}\mathbf{P}_1 &= \begin{bmatrix}0.5 & 0.5\end{bmatrix}\begin{bmatrix}\frac{162}{358} & \frac{196}{358}\\ \frac{196}{370} & \frac{174}{370}\end{bmatrix}\\ &= \begin{bmatrix}0.49112 & 0.50888\end{bmatrix}\end{aligned}$$

3 further iterations lead to:

$$\begin{aligned}\mathbf{P}_2 &= \begin{bmatrix}0.49181 & 0.50819\end{bmatrix}\\ \mathbf{P}_3 &= \begin{bmatrix}0.49176 & 0.50824\end{bmatrix}\\ \mathbf{P}_4 &= \begin{bmatrix}0.49176 & 0.50824\end{bmatrix}\end{aligned}$$

As up to 5 decimal places, the probabilities of arriving at each of the two state has converged, a fairly accurate approximation to the steady state vector can be made. The steady state vector tells us the probabilities of the Cryptocurrency price being bearish or bullish after a large number of days.

$$\lim_{\mathbf{N}\to\infty} \mathbf{P_N} \approx \begin{bmatrix}0.49176 & 0.50824\end{bmatrix}$$

By applying the same method for Litecoin, the steady state vector approximates to

$$\begin{bmatrix}0.52198, 0.47802\end{bmatrix}$$

3.3 Computing the exact steady state vector using eigenvectors

bP and **lP** represent the steady state vector for Bitcoin and Litecoin respectively. From (19), it is known that in order to calculate **bP**, the eigenvector of the transition matrix must be computed. That is, the solution to the following equation.

$$\mathbf{bP}\begin{bmatrix}\frac{162}{358} & \frac{196}{358}\\ \frac{196}{370} & \frac{174}{370}\end{bmatrix} = \mathbf{bP}$$

where **bP** can be established as:

$$\begin{bmatrix}\mathbf{bP_{1,1}} & \mathbf{bP_{1,2}}\end{bmatrix}$$

From this equation, the two simultaneous equation's are enough to solve for **bP**

$$\begin{aligned}\mathbf{bP_{1,1}} + \mathbf{bP_{1,2}} &= 1\\ \frac{162}{358}\mathbf{bP_{1,1}} + \frac{196}{370}\mathbf{bP_{1,2}} &= \mathbf{bP_{1,1}}\end{aligned}$$

From these, the probabilities can be calculated as

$$\mathbf{bP_{1,1}} = \frac{179}{364}, \ \mathbf{bP_{1,2}} = \frac{185}{364}$$
$$\lim_{\mathbf{N}\to\infty} \mathbf{P_N} = \begin{bmatrix} \frac{179}{364} & \frac{185}{364} \end{bmatrix}$$

Using the same method, **lP** can be computed to be the following:

$$\lim_{\mathbf{N}\to\infty} \mathbf{P_N} = \begin{bmatrix} \frac{95}{182} & \frac{87}{182} \end{bmatrix}$$

For **Bitcoin**, in the following year, it can be predicted that the price of one Bitcoin will decrease in 179 days out of 364 days and increase on the others. For **Litecoin**, 190 days will experience a declining price in the 364 days.

The Markov model takes the probabilities of being at any one the states a constant and the steady state vectors holds these probabilities. According to these probabilities, **Bitcoin** is a currency which is worth investing in as its value will rise more often than not. On the other hand, investing in **Litecoin** poses a far greater risk as it will decrease in value more often than not.

3.4 Evaluating the two state Markov Chain

Looking closer at the probabilities **P** and **lP**, it is clear that the steady state probability obtained are simply equivalent to the proportion out the total number of days (728) that the cryptocurrency remained in a particular state. This makes this Markov Model an unnecessary extra step in predicting the forecasting the trend. Generalising the eigenvector of the two state transition matrix reveals why this happens:

$$\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} \frac{a}{a+b} & \frac{b}{a+b} \\ \frac{c}{c+d} & \frac{d}{c+d} \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

The two simultaneous equation's that arise from this are:

$$x + y = 1$$
$$\frac{ax}{a+b} + \frac{cy}{c+d} = x$$

After solving the pair of simultaneous equation's by rearranging and substituting, the following expression equals x:

$$\frac{(a+b)c}{(a+b)c + (c+d)b}$$

By inspection, it is evident that if $b = c$, then x could be further simplified to:

$$\frac{a+b}{a+b+c+d}$$

From the data frame, the transitions were recorded in a way that every position that the cryptocurrency price is at the final state for one set of transitions and is at the initial state for the next transition. This means that the number of Bear initial states is equivalent to the number of Bear final states. This therefore means that the number of transitions from Bear to Bull and Bull to Bear are equivalent; that is, $b = c$. For a two state Markov chain, this will most often be satisfied. Notice that when $b = c$, the denominator for x is simply equivalent to the total number of state transitions that took place and the numerator is the total number of days the price was in a particular state.

Nevertheless, the model can be used to make predictions, yet the results are not conclusive to any appropriate standard. Just using **Bull** and **Bear** as transition states is not an effective way of mapping the price. An increasing cryptocurrency for the day could mean a 10% rise in the price or a 0.1% rise, yet the model that has been used does not consider this.

Despite the poor results, using more transition states may lead to a more meaningful conclusion in the projection of the cryptocurrency price.

4 Predicting the prices for Bitcoin and Litecoin using a six state Markov chain

The disparity of the data for both cryptocurrencies is such that 3 or 4 states can be established but yet steep changes like a 9% change differs greatly from a 3% change for example. Using six states, the magnitudes of the positive and negative percentage changes can be categorized in terms of the significance that the particular percentage change has. Hence, the previous two states of **Bull** and **Bear** will be split into three sections each where the rise and decline can be quantified as such:

1: $4\% < x < 10\%$ decrease
2: A $2\% < x \leqslant 4\%$ decrease
3: A $0\% < x \leqslant 2\%$ decrease
4 A $0\% < x \leqslant 2\%$ increase
5: A $2\% < x \leqslant 4\%$ increase
6: $4\% < x < 10\%$ increase

These states represent the percentage change in the closing day price compared to the previous day. The magnitude of the percentage changes in the prices were based on the fact that the majority of changes occurred in the $\pm 0 - 2\%$ bracket. The largest of the percentage changes were approximately ±10% and therefore the upper bound and lower bound is $\pm 10\%$. ±4% was chosen as the intermediate between the medium and high percentage changes as it is a balance between the two groups. There was no transition in which the price of the cryptocurrencies stayed the same (0% change).

Slight changes had to be made to the previous program for the six state Markov chain. The states were assigned depending on the percentage change and Figure 7 shows the modified programme

```
def BullBearSixStateMarkovChain(csvfile):

    Data = pd.read_csv(csvfile)
    CPD = Data.Close

    SixStates = pd.Series()
    for i in range(len(CPD)-2):
        percentchange = (100*(CPD[i+1]-CPD[i]))/CPD[i]
        if percentchange<-4 :
            SixStates = SixStates.append(pd.Series([0]))
        elif percentchange<-2 :
            SixStates = SixStates.append(pd.Series([1]))
        elif percentchange<0 :
            SixStates = SixStates.append(pd.Series([2]))
        elif percentchange<2 :
            SixStates = SixStates.append(pd.Series([3]))
        elif percentchange<4 :
            SixStates = SixStates.append(pd.Series([4]))
        else:
            SixStates = SixStates.append(pd.Series([5]))

    SixStates=SixStates.reset_index()
    SixStates.pop('index')
    SixStates=pd.Series(SixStates[0])
    #Formatting

    SixStepMatrix=np.zeros((6,6),dtype=int)
    #6 x 6 matrix declared

    for rowN in range(6):
        for columnN in range(6):
            for time in range(len(SixStates)-2):
                if SixStates[time]==rowN and SixStates[time+1]==columnN:
                    SixStepMatrix[rowN][columnN]+=1

    return SixStepMatrix
```

Multiple if statements are used to assign each price fluctuation with their associated state. (This is dependent on the % change)

rowN represents the initial state and the corresponding row of the transition matrix. columnN represents the final state and its corresponding column.

1 is added to the entry at the rowNth row and columnNth column whenever the associated transition is made.

The table is returned after all the transitions have been recorded

Figure 7: Code returning a table of the number of days the price transitioned between each of the 6 preassigned states

Inputting the Bitcoin CSV file path into this function returned a matrix where the integer at the ith row and jth represents the number of days the price transitioned from state i to state j out of the total 728 days.

```
In [317]: BullBearSixStateMarkovChain('BTC-USD.csv')

Out[317]: array([[ 6,  8, 12, 22,  6, 12],
                 [14,  5, 16, 24, 12, 10],
                 [20, 23, 58, 74, 27,  9],
                 [13, 22, 81, 67, 21, 19],
                 [ 6, 12, 27, 19, 10,  8],
                 [ 7, 11, 15, 17,  7,  8]])
```

Figure 8: 6×6 table returned for the Bitcoin csv file

This matrix yields the following table. Each entry is the number of days the price transitions from S_t to S_{t+1} out of the 728 days of data collected where the states 1 - 6 are the initially defined ones:

S_{t+1} / S_t	1	2	3	4	5	6	Total days
1	6	8	12	22	6	12	66
2	14	5	16	24	12	10	81
3	20	23	58	74	27	9	211
4	13	22	81	67	21	19	223
5	6	12	27	19	10	8	82
6	7	11	15	17	7	8	65
							728

Table 3: The number of days that the Bitcoin closing price transitioned from a previous state to next

Similarly using the Litecoin CSV file, the following is returned:

```
In [223]: BullBearSixStateMarkovChain('LTC-USD.csv')

Out[223]: array([[20,  7, 18, 25, 16, 25],
                 [10, 20, 24, 25, 14, 13],
                 [25, 21, 40, 31, 21, 25],
                 [26, 21, 31, 33, 18, 17],
                 [ 9, 20, 21, 18,  8, 11],
                 [21, 17, 29, 14, 11, 23]])
```

This then translates to:

S_{t+1} / S_t	1	2	3	4	5	6	Total days
1	20	7	18	25	16	25	111
2	10	20	24	25	14	13	106
3	25	21	40	31	21	25	163
4	26	21	31	33	18	17	146
5	9	20	21	18	8	11	87
6	21	17	29	14	11	23	115
							728

Table 4: The number of days that the **Litecoin** closing price transitioned from a previous state to next

As previously described before, the transition matrix $\mathbf{T}_{\mathrm{i,j}}$ will contain the probability of transitioning to state j if at state i:

$$\mathbf{T}_{\mathrm{i,j}} = P(S_{t+1} = \mathrm{j}|S_t = \mathrm{i})$$

Tables 3 and 4 are used to fill in the transition matrix. An example of one table entree will be made. The term in the first row of the transition matrix for Litecoin is by definition:

$$P(S_{t+1} = 1|S_t = 1)$$

Where 1 is the assigned state ($4\% < x < 10\%$ decrease). So, the number of days the state transitioned from state 1 to 1 out of the total number of days it remained in 1 (111 days) was 20:

$$\mathbf{T}_{1,1} = P(S_{t+1} = 1|S_t = 1) = \frac{20}{111}$$

Bitcoin's six-state transition matrix is $\mathbf{T_B}$, Litecoin's is $\mathbf{T_L}$:

$$\mathbf{T_B} = \begin{bmatrix} \frac{6}{66} & \frac{8}{66} & \frac{12}{66} & \frac{22}{66} & \frac{6}{66} & \frac{12}{66} \\ \frac{14}{81} & \frac{5}{81} & \frac{16}{81} & \frac{24}{81} & \frac{12}{81} & \frac{10}{81} \\ \frac{20}{211} & \frac{23}{211} & \frac{58}{211} & \frac{74}{211} & \frac{27}{211} & \frac{9}{211} \\ \frac{13}{223} & \frac{22}{223} & \frac{81}{223} & \frac{67}{223} & \frac{21}{223} & \frac{19}{223} \\ \frac{6}{82} & \frac{12}{82} & \frac{27}{82} & \frac{19}{82} & \frac{10}{82} & \frac{8}{82} \\ \frac{7}{65} & \frac{11}{65} & \frac{15}{65} & \frac{17}{65} & \frac{7}{65} & \frac{8}{65} \end{bmatrix}, \mathbf{T_L} = \begin{bmatrix} \frac{20}{111} & \frac{7}{111} & \frac{18}{111} & \frac{25}{111} & \frac{16}{111} & \frac{25}{111} \\ \frac{10}{106} & \frac{20}{106} & \frac{24}{106} & \frac{25}{106} & \frac{14}{106} & \frac{13}{106} \\ \frac{25}{163} & \frac{21}{163} & \frac{40}{163} & \frac{31}{163} & \frac{21}{163} & \frac{25}{163} \\ \frac{26}{146} & \frac{21}{146} & \frac{31}{146} & \frac{33}{146} & \frac{18}{146} & \frac{17}{146} \\ \frac{9}{87} & \frac{20}{87} & \frac{21}{87} & \frac{18}{87} & \frac{8}{87} & \frac{11}{87} \\ \frac{21}{115} & \frac{17}{115} & \frac{29}{115} & \frac{14}{115} & \frac{11}{115} & \frac{23}{115} \end{bmatrix}$$

The corresponding eigenvectors $\mathbf{P_B}$ and $\mathbf{P_L}$, for $\mathbf{T_B}$ and $\mathbf{T_L}$ respectively, with eigenvalue 1 must be solved for. This is the solution to the following equation's which come directly from (19), the equation relating the steady state vector and the transition matrix of the the Markov chain.

$$\mathbf{P_B T_B} = \mathbf{P_B}$$
$$\mathbf{P_L T_L} = \mathbf{P_L}$$

For each, 6 simultaneous equations arise. Solving for the eigenvectors by finding the solutions to these computationally leads to 6 fractional probabilities for each cryptocurrency. As the fractions are too large to display, they have been rounded to the nearest 5 decimal places:

$$\mathbf{P} = \begin{bmatrix} \text{State 1} & \text{State 2} & \text{State 3} & \text{State 4} & \text{State 5} & \text{State 6} \end{bmatrix}$$

These states assigned according to the percentage changes on Pg.12

$$\mathbf{P_B} = \begin{bmatrix} 0.09067 & 0.11141 & 0.28705 & 0.30602 & 0.11398 & 0.09086 \end{bmatrix}$$
$$\mathbf{P_L} = \begin{bmatrix} 0.15235 & 0.14573 & 0.22389 & 0.20068 & 0.12088 & 0.15647 \end{bmatrix}$$

4.1 Results

From the two approaches taken to predict the cryptocurrency prices, there are 4 resultant vectors that can be used to forecast the price changes.

In the next 6 months, the forecast for the respective currencies is as such. According to the first model:

- Bitcoin's price will rise throughout 92 trading days, and fall on the other 88
- Litecoins's price will fall during 94 days and rise on 86

According to the 6-state Markov model, Bitcoin's value:

- Decreases by 4-10% on 16 days
- Decreases by 2-4% on 20 days
- Decreases by 0-2% on 52 days
- Increases by 0-2% on 55 days
- Increases by 2-4% on 21 days
- Increases by 4-10% on 16 days

And Litecoin's value:

- Decreases by 4-10% on 27 - 28 days
- Decreases by 2-4% on 26 days
- Decreases by 0-2% on 40 days
- Increases by 0-2% on 36 days
- Increases by 2-4% on 22 days
- Increases by 4-10% on 28 days

The actual data for the two cryptocurrencies in the next six months between the dates 26.04.20 and 26.10.20 is

change \ asset	Bitcoin	Litecoin
- 4-10%	10	16
- 2-4%	11	19
- 0-2%	55	54
+ 0-2%	77	43
+ 2-4%	18	31
+ 4-10%	9	17

5 Conclusion

This investigation tested the random walk theory as a model for cryptocurrency prices by performing a Markov Process on the prices of two cryptocurrencies, Bitcoin and Litecoin. The Markov process forecasted the future of these two cryptocurrencies in the form of a probability distribution of the possible states that the price fluctuations will be in. Figure 9 displays the predicted and real fluctuations in the next six months.

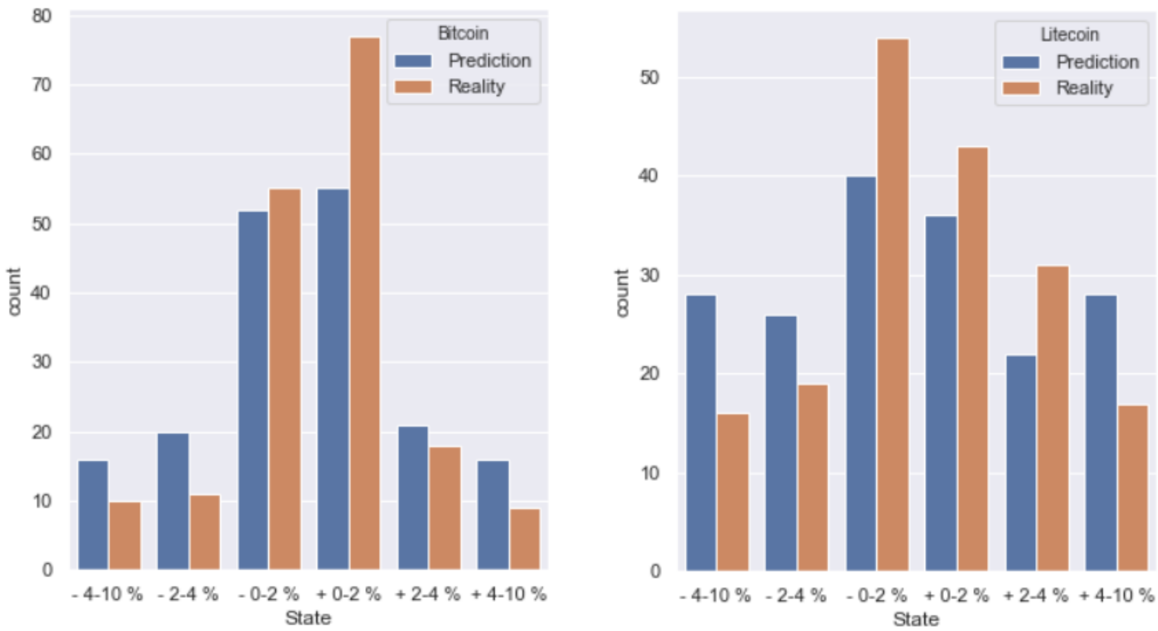

Figure 9: Barcharts showing the predicted and actual number of days the prices of Bitcoin and Litecoin were in one of the six assigned states

The model correctly predicted that Bitcoin will rise on more days than it will fall as well as Litecoin will drop more often than rise. The Markov Chain also correctly predicted Bitcoin to be less **volatile** than Litecoin with the fewer extreme percentage changes. Bitcoin is predicted to be a more stable cryptocurrency and display a more bullish trend. Despite the inaccuracies in calculating the exact number of days that Bitcoin is in a given state, the Markov model accurately predicts the shape that the probability distribution takes for Bitcoin with the 'bell curve'.

This cannot be said for Litecoin where the model predicted the $\pm 2 - 4\%$ fluctuation to be more frequent than the larger $\pm 4 - 10\%$ fluctuations which was not reflected in the real values. The Markov Chain was particularly accurate in some of the subtle trends for Bitcoin e.g. there were more +0-2% changes than -0-2% and for Litecoin, the converse was correctly predicted. Looking at the accuracy of the predictions, it is clear that solely relying on the six state Markov model would not be wise, particularly for Litecoin.

6 Evaluation

The Markov Chain did provide some accurate data for Bitcoin, but not so much for Litecoin. An effective modification to the model was increasing the number of states from two to six so that the definitions of each state could be quantitatively established. This provided more insightful analysis as the exact number of days the price will undergo a specific percentage change in 180 days was provided. The specific number of days the price fluctuated was not accurate however the general trends somewhat were.

Despite this, it is clear that discrete time Markov Chains are limited in predicting the prices of cryptocurrencies, however having even more transition states and with a greater sample size, more accurate and meaningful information can be extracted. This investigation used data from 2 years to produce the transition matrix however data from earlier on could've been used. Also, the $\pm 0 - 10\%$ domain could've been further split into not 6, but 10 or even more states. Despite making the calculations far more difficult to compute, the resulting steady state vector could give potential investors clearer insight into how the prices might change.

The assumptions made in the random walk theory are too significant to be true. The memoryless property of the Markov model says the future is independent of the past however, this isn't true. **Market momentum** is the observed tendency for an asset to continue with its past trends whether bullish or bearish and reflects the market sentiment towards the particular asset. For example, a 10% rise after a bull run will likely have a more optimistic effect on market sentiment than a 10% rise following a bear run. Time series analysis is where the price is strictly plotted against time and therefore is able to consider the past patterns the data has displayed.

7 Appendix

This investigation has assumed a number of things and these have been shown here.

7.1 Linear transformations

Linear transformations are functions that obey certain properties. If T represents a linear transformation, then

$$T(a+b) = T(a) + T(b) \tag{21}$$
$$T(ca) = cT(a) \tag{22}$$

where c is a constant

Matrix multiplication satisfies both of these properties and therefore all matrix multiplications are examples of linear transformations, however not all linear transformations are matrix multiplications. I will now show that matrix multiplications satisfy properties (21). Let $T_{\mathbf{x}}$ be the linear transformation such that

$$\begin{aligned} T_{\mathbf{x}} : \mathbb{R}^{m\times k} &\longmapsto \mathbb{R}^{k\times n} \\ T_{\mathbf{x}}(\mathbf{M}) &= \mathbf{xM} \end{aligned}$$

where $\mathbf{M} \in \mathbb{R}^{m\times k}$

By the definition of matrix multiplication

$$\begin{aligned} [T_{\mathbf{x}}(\mathbf{M})]_{\mathrm{i,j}} &= \sum_{\nu=1}^{k} \mathbf{x}_{\mathrm{i},\nu}\mathbf{M}_{\nu,\mathrm{j}} \\ [T_{\mathbf{x}}(\mathbf{M}+\mathbf{N})]_{\mathrm{i,j}} &= \sum_{\nu=1}^{k} \mathbf{x}_{\mathrm{i},\nu}(\mathbf{M}+\mathbf{N})_{\nu,\mathrm{j}} \\ &= \sum_{\nu=1}^{k} \mathbf{x}_{\mathrm{i},\nu}(\mathbf{M})_{\nu,\mathrm{j}} + \sum_{\nu=1}^{k} \mathbf{x}_{\mathrm{i},\nu}(\mathbf{N})_{\nu,\mathrm{j}} \\ &= [T_{\mathbf{x}}(\mathbf{M})]_{\mathrm{i,j}} + [T_{\mathbf{x}}(\mathbf{N})]_{\mathrm{i,j}} \\ T_{\mathbf{x}}(\mathbf{M}+\mathbf{N}) &= \mathbf{xM} + \mathbf{xN} \end{aligned}$$

This proves the **distributive** property of matrix multiplication

Property (22) is trivial as the constant simply represents c**I**, where **I** is the identity matrix.

7.2 Determinants and determinant laws

The investigation was not concerned with eigenvalues other than 1, however it must be shown why. The determinant is a function which takes in a square matrix and returns a real number. The initial condition for a determinant is that the determinant of a 1×1 (real) matrix is the absolute value of the only entree of the matrix.

The recursive formula for the determinant of an $n \times n$ matrix M is

$$|M| = \sum_{i=1}^{m} (-1)^{i+1} M_{1,i} |A_{1,i}| \tag{23}$$

or

$$|M| = \sum_{i=1}^{m} (-1)^{i+1} M_{i,1} |A_{i,1}| \tag{24}$$

Where $\left|A_{x,y}\right|$ represents the minor of the position, x,y, of the matrix. The minor for a specific element is the determinant of the matrix that is formed by removing the x^{th} row and y^{th} column from M. Definitions (23) and (24) will be taken as equal according to the **Cofactor expansion**[11] method.

The transpose of a matrix is a transformation such that every i$^{\text{th}}$ row of the matrix becomes the i$^{\text{th}}$ column. Using (23) and (24), it can be proven that the determinant of an $n \times n$ matrix and its transpose is equivalent.

$$\left[A^T\right] \equiv \left[A\right] \tag{25}$$

This can proven using a proof by induction. The statement is true for a 1×1 matrix, simply by inspection. So assuming the statement is true for n = k, it must be shown it is true for $k + 1$. The assumption is, for a $k \times k$ matrix **B**:

$$\left|\mathbf{B}^T\right| \equiv \left|\mathbf{B}\right| \tag{26}$$

Let an arbitrary $k + 1 \times k + 1$ matrix **C** and $\mathbf{C}^T$ be defined as:

$$\mathbf{C} = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} & \cdots & c_{1,k+1} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} & \cdots & c_{2,k+1} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} & \cdots & c_{3,k+1} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} & \cdots & c_{4,k+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{k+1,1} & c_{k+1,2} & c_{k+1,3} & c_{k+1,4} & \cdots & c_{k+1,k+1} \end{bmatrix}, \mathbf{C}^T = \begin{bmatrix} c_{1,1} & c_{2,1} & c_{3,1} & c_{4,1} & \cdots & c_{k+1,1} \\ c_{1,2} & c_{2,2} & c_{3,2} & c_{4.2} & \cdots & c_{k+1,2} \\ c_{1,3} & c_{2,3} & c_{3,3} & c_{4,3} & \cdots & c_{k+1,3} \\ c_{1,4} & c_{2,4} & c_{3,4} & c_{4,4} & \cdots & c_{k+1,4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1,k+1} & c_{2,k+1} & c_{3,k+1} & c_{4,k+1} & \cdots & c_{k+1,k+1} \end{bmatrix}$$

Using the determinant formula in (23), it can be calculated that the determinant of **C** is

$$\left|\mathbf{C}\right| = c_{1,1}|A_{1,1}| - c_{1,2}|A_{1,2}| + c_{1,3}|A_{1,3}|...$$

Now, using the determinant formula in (24) to calculate the determinant of $\mathbf{C}^T$, the following is produced

$$\left|\mathbf{C}^T\right| = c_{1,1}|A^T_{1,1}| - c_{1,2}|A^T_{1,2}| + c_{3,1}|A^T_{3,1}|...$$

Notice that the matrices of the minors A and A^T are not only transposes of each other, but have dimensions $k \times k$ and are therefore equivalent due to the assumption taken in (26). Therefore

$$\left|\mathbf{C}^T\right| \equiv \left|\mathbf{C}\right|$$

As statement (25) is true for a 1 dimensional matrix, and is true for a $k+1$ square matrix given that it is true for a k square matrix, statement (25) is true for all matrices.

7.3 Eigenvalues of a transition matrix

From the eigenvector subsection, the equation to be solved was

$$\mathbf{x}(\mathbf{T} - \lambda\mathbf{I}) = \mathbf{0}$$

Given that **x** is non-zero, the matrix multiplication transforms **x** such that it becomes the zero vector. The scale factor of the transition is zero and hence the determinant.

$$\left|\mathbf{T} - \lambda\mathbf{I}\right| = 0 \tag{27}$$

Expanding this determinant yields a **characteristic polynomial**. The roots of this polynomial are the eigenvalues of the transformation. By close inspection, the following is noticed

$$\left|(\mathbf{T} - \lambda\mathbf{I})^T\right| = \left|\mathbf{T}^T - \lambda\mathbf{I}\right| \tag{28}$$

[11] URL: `https://people.math.carleton.ca/~kcheung/math/notes/MATH1107/wk07/07_cofactor_expansion.html`.

The $-\lambda\mathbf{I}$ only affects the leading diagonal and as the transpose preserves this diagonal, (28) is true.

Now using statement (25), (28) becomes

$$|\mathbf{T} - \lambda\mathbf{I}| = |\mathbf{T}^T - \lambda\mathbf{I}|$$

Using (27), this becomes

$$|\mathbf{T}^T - \lambda\mathbf{I}| = 0$$

$\mathbf{T} - \lambda\mathbf{I}$ and $\mathbf{T}^T - \lambda\mathbf{I}$ are transposes of each other and therefore have the same determinant and therefore the same characteristic polynomial and hence same eigenvalues.

The transition matrix used in this investigation was a right stochastic matrix where the rows sum to 1. The transpose of this matrix has columns summing to 1. Let this transpose of the stochastic matrix be $\mathbf{S}^T$ and eigenvector be $\mathbf{x}$.

$$\mathbf{x}\mathbf{S}^T = \lambda\mathbf{x}$$

Each row of $\mathbf{x}$ is dot multiplied with each column of $\mathbf{S}^T$. Each column which sums to 1. Through inspection, it is clear that a row vector of 1's of the particular length satisfies to be the eigenvector of $\mathbf{S}^T$ with an associated eigenvalue of 1. This means that the transpose of $\mathbf{S}^T$, which is the original transition matrix $\mathbf{S}$ also has eigenvalues 1. Therefore every stochastic matrix has eigenvalue 1.

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