

Theory: Multidimensional Space of Events

Abstract: This paper extends Bayesian probability theory by developing a multidimensional space of events (MDSE) theory that accounts for mutual influences between events and hypotheses sets. While traditional Bayesian approaches assume conditional independence between certain variables, real-world systems often exhibit complex interdependencies that limit classical model applicability. Building on established probabilistic foundations, our approach introduces a mathematical formalism for modeling these complex relationships. We developed the MDSE theory through rigorous mathematical derivation and validated it using three complementary methodologies: analytical proofs, computational simulations, and case studies drawn from diverse domains. Results demonstrate that MDSE successfully models complex dependencies with 15-20% improved prediction accuracy compared to standard Bayesian methods when applied to datasets with high interdimensionality. This theory particularly excels in scenarios with over 50 interrelated variables, where traditional methods show exponential computational complexity growth while MDSE maintains polynomial scaling. Our findings indicate that MDSE provides a viable mathematical foundation for extending Bayesian reasoning to complex systems while maintaining computational tractability. This approach offers practical applications in engineering challenges including risk assessment, resource optimization, and forecasting problems where multiple interdependent factors must be simultaneously considered.

Keywords: events and hypothesis spaces; graph theory; probability; Bayesian theory; mathematical modeling; neural network

MSC: 60A10, 62F15, 68T27, 05C35, 05C40, 03E75

1. Introduction

Research across disciplines requires considering all possible event combinations related to hypotheses to derive general conclusions. To explain how specific phenomena function, it is crucial to examine numerous existing events in relation to all potential hypotheses, even when connections to preceding events are not immediately apparent (see Supplementary Material, section 2 for detailed examples). A comprehensive understanding must acknowledge that for any true hypothesis, there exist counterexamples that may falsify it—this lies at the core of scientific inquiry.

The accuracy and thoroughness of scientific decisions depend on considering complete sets of events and hypotheses. Probability assessment represents the second key element in this process. When evaluating probability, we employ a general strategy: if individual probability values are sufficiently high, we consider every possible event holistically. This approach enables us to explore additional dimensions of analysis.

Researchers across scientific disciplines have employed diverse mathematical approaches to achieve these analytical goals. Some methodologies, however, may introduce errors that lead to incorrect conclusions. Understanding new technologies requires incorporating external information sources. Thus, we recognize the importance of: (1) examining all possible event combinations, (2) considering all relevant hypotheses, and (3) establishing correct relationships between them. Without these steps, properly defining research goals and achieving accurate results becomes challenging, particularly given the limitations of current advanced logical methods.

Well-established approaches based on classical Bayesian theory exist, and we seek to understand these methods as potential foundations for emerging analytical techniques. We also investigate whether alternative approaches have developed concurrently within the same framework. Additionally, it is important to explore methodologies that address commonly identified limitations within classical Bayesian theory. This exploration provides an alternative perspective that might be characterized as a Bayesian generalization or Bayesian-free approach to theoretical development. The focus is on considering different theoretical interpretations of Bayesian theory through varied approaches and frameworks.

To establish an enhancement of classical theories, this work will utilize the following models and theoretical frameworks:

- **Basic rules of mathematical and computational logic** to create theoretical and methodological fundamentals (TMF), as established by the author and other researchers (Daradkeh et al. [12]; Kavun [32, 33]; Kavun & Zhosan [34]; Zamula & Kavun [63]). TMF aims to enhance understanding of mathematical computations, providing empirical information on concepts necessary for developing more robust mathematical methods. While some basic concepts outlined here are not new, they are well-developed and provide a framework for developing new mathematical and computational methods.
- **Theory of logical reasoning (deduction)** for generalization and decomposition of theoretical, organizational, and structural bases (Dowden [16]; Gibson & Johnson [14]). Key concepts include theoretical methods, reasoning, inference, and rational systems. A fundamental concept in rational reasoning is the equivalence between a rational system and a mathematical system. This approach helps explain a system's mathematical structure and operations through its mechanics, concepts, principles, and functions. The concept of "Logical Thinking" encompasses four main categories of reasoning, including objective reasoning and objective conceptual reasoning, which establish the foundations of rational systems. Logic enables the formulation of hypotheses derived from knowledge or observations.
- **Theory of following** (Knickerbocker [35]). While presented as a useful guide to following principles, its theoretical foundation requires further clarification. The author's intent might be understood metaphorically as philosophical; however, this should not detract from objective analysis. Readers are encouraged to critically evaluate presented arguments without undue deference to authority.
- **Analysis and synthesis** (Ritchey [52]) based on identifying and formalizing design challenges. Many studies have identified complex structures, including structural co-structures that influence information processing. Within data analysis methodologies, patterns often point to classifications determined by analyzing structure and dynamics in relation to specific influential factors. This approach enables rapid and scalable application, demonstrating promise through numerous successful implementations. Various data analysis techniques have been deployed, similar to large-scale database applications but tailored for different purposes, including statistical techniques for efficient study comparison and algorithms for random sample design.
- **General scientific methods of analogy and abstraction** (Gentner & Hoyos [22]) ground the main principles and properties of the proposed approach. One potential limitation is the authors' potentially limited experience with all possible relationships and structures that could be exploited by the proposed methods. This could lead to alternative approaches if the theoretical authors were to develop models for fundamental questions of natural law and phenomena. The abstract approach to problem-solving involves a series of assumptions that connect to main principles (such as relative cost, length, and location) through logical premises that can be tested experimentally.

The primary aim of this manuscript is to explore the interrelations of event (hypothesis) sets to formulate a new conception of multidimensional space of events (MDSE), which could help build a corresponding mathematical foundation. The article is structured as follows: after this literature review, we present a general theoretical description of the proposed approach and its mathematical fundamentals, followed by a description of the properties of the basic MDSE-graph. Statistical fundamentals, results, discussion, and conclusions complete the paper. Interestingly, the MDSE term has appeared in distinctly different fields, such as real estate market analysis (Radzewicz et al. [49, 50]), as a result of compromise assessment approaches.

This paper addresses the growing need for robust and flexible mathematical frameworks for modeling complex systems involving multiple interdependent events. Traditional Bayesian approaches, while powerful, often fall short when dealing with dynamic, multidimensional spaces where events and hypotheses interact in complex ways. Many current models are limited in their ability to handle large numbers of interrelated variables, making them inadequate for real-world applications that demand scalability and precision, such as risk prediction, machine learning, and economic modeling. This limitation highlights the need to extend Bayesian theory to better capture the nuances of multidimensional event spaces.

In response to this challenge, we propose a novel approach that constructs a multidimensional space of events (MDSE), providing a new method for modeling the dependencies between events and hypotheses. This theory offers greater flexibility and accuracy in describing complex systems by considering complete sets of events and hypotheses while opening new possibilities for solving mathematical and logical problems that were previously difficult or impossible to address using classical methods. Our approach has the potential to improve prediction, optimization, and classification problems across scientific domains, providing a significant advance in both theoretical and practical applications of Bayesian modeling.

To achieve these goals, we identify the following critical intermediate objectives:

1. Enumerate all possible combinations of events involved;
2. Evaluate all significant hypotheses considered relevant;
3. Establish accurate relationships between these hypotheses.

Without addressing these intermediate objectives, correctly setting the main research purpose becomes exceedingly difficult, potentially leading to inaccurate results. The third objective presents particular challenges due to the current limitations of advanced logical methods. Nevertheless, various approaches utilizing classical Bayesian theory provide potential pathways toward achieving these goals.

2. Literature Review

The literature demonstrates that similar approaches developed earlier can be applied across various domains, as evidenced by numerous publications. Contemporary research contains extensive studies on event interrelationships and their corresponding hypotheses based on Bayesian theory, with diverse perspectives on the nature of such interrelationships and their applications. Each study of these events and their corresponding hypotheses offers different interpretations depending on the original research objectives.

Rather than reviewing each publication individually, we have grouped relevant research thematically, highlighting specific contributions and identifying gaps addressed by our proposed method.

2.1. *Bayesian Theory and Inductive Inference, Philosophy and Practice*

Bayesian theory is closely associated with inductive reasoning. As discussed by Gelman et al. [20, 21] and Gelman & Shalizi [19], Bayesian inference has been foundational in the philosophy of science, particularly in rational decision-making. The authors argue that although Bayesian statistics has achieved widespread success, its application does not fully align with traditional views of inductive inference. Their work emphasizes the importance of model checking and refinement, demonstrating that simply fitting models without validation can lead to flawed conclusions. These insights are crucial for understanding the limitations of traditional Bayesian approaches and the need for more dynamic, adaptive methods in complex event spaces. Their research challenges the boundaries of Bayesian confirmation theory, proposing a more adaptive framework for understanding event dependencies.

2.2. *Application of Bayesian Theory and Models to Environmental Studies*

Bayesian theory is effectively applied in environmental risk assessment, particularly for extreme events such as floods and droughts. Mengyang et al. [42] employed a Bayesian approach to estimate encounter probabilities for hydrometeorological events in the Rao River Basin. By combining statistical downscaling techniques with copula models, their study provides a framework for predicting the risk of future environmental disasters. This methodology demonstrates the flexibility of Bayesian models in handling climate change uncertainties and can serve as a template for risk prediction in other domains. Their work highlights the adaptability of Bayesian techniques in complex, uncertain environments.

2.3. *Multidimensional Event Spaces, Classification Systems and Hypotheses*

The core idea of this research—constructing a multidimensional space of events and hypotheses—builds on the work of researchers like Ritchey [52], who explored classification systems based on the structure and dynamics of data. By applying Bayesian models to multidimensional spaces, we can better understand the dependencies between events and hypotheses. This approach aligns with the development of Bayesian networks, which are often used to visualize probabilistic dependencies.

Lijoi et al. [41] address the consistency of Bayesian procedures in stationary models, filling theoretical gaps with asymptotic properties. Their work establishes the consistency of posterior distributions inherent to Bayesian confirmation theory, providing a solution to this problem. Our proposed theory extends this idea by incorporating a broader range of events and hypotheses, providing more granular insights into the interrelationships between variables. Notably, similar results of this generalization have not been published to date.

2.4. *Rare Events and Bayesian Posteriors*

Fudenberg et al. [18] examined Bayesian posteriors in the context of arbitrarily rare events. Their work focused on calculating the relative likelihoods of two rare events and determining the necessary data volume to guarantee high probability posterior estimates. This research is critical for improving the accuracy of event probability calculations, particularly in cases where more than two events must be considered, as proposed in our current model.

2.5. Advancements in Bayesian Event Probability Functions and Risk Modeling

Another critical development in Bayesian theory is the introduction of event-probability functions as an alternative to risk functions, as proposed by Bottai et al. [5]. Unlike hazard functions, which can take any positive value, event-probability functions provide a bounded and interpretable measure of risk over time. This innovation allows for more precise modeling of event occurrence over time and can be applied to a wide range of scenarios, including medical risk assessments and economic forecasts. Bayesian methods as a component of common Bayesian theory can be used in economic modeling (Daradkeh et al. [12]; Kavun [32]; Kavun & Zhosan [34]; Nitsenko et al. [44]; Panchenko et al. [47]). Incorporating this approach into the MDSE will enhance the accuracy of probability estimations in our proposed theory, as it improves the ability to estimate event probabilities within the proposed multidimensional space of events and hypotheses.

2.6. Bayesian Applications in Economics and Machine Learning

Bayesian methods have also found applications in economic modeling and machine learning. Research by Kavun et al. [31], Gunji & Haruna [25], McCallum & Nigam [43], Raol & Ayyagari [51], Salakhutdinov & Mnih [55], and other authors demonstrate the versatility of Bayesian approaches in these fields. By applying Bayesian techniques to large datasets and neural networks, these studies have advanced our understanding of probabilistic modeling in dynamic and high-dimensional environments.

2.7. Literature Summary

The fundamental concept in probability theory is the event space (Bernardo & Smith [4]), on which the probability of an event can be defined. Traditionally, this event space considers only one event with its corresponding hypotheses, establishing a correlation between the probabilistic model and the event space. However, as noted by several researchers (Christofides [9]; Cohen [11]; Robertson [53]), certain classes of events possess their own structure that cannot be adequately described using existing mathematical apparatus—creating a significant theoretical problem.

For instance, the well-known 'one-to-many' relation has been extensively discussed in published works (Chu et al. [10]; Blei et al. [5]; Hansen [27]; Harary [28]; Stone [57]). In the context of information retrieval, involving queries, documents, and relevance, Robertson [53] observed that "the event space issue was creating difficulty in comparing various probabilistic models in information retrieval." Additionally, this research area has applications in seemingly unrelated domains (Burinska et al. [6]; Chen [8]; Lee et al. [39]; Simon [56]; Vapnik [62]).

Table 1 clarifies how existing literature has influenced the development of the MDSE approach while highlighting the novel contributions our study makes to the field. It provides an organized view of how our work relates to and expands upon existing theories. The reviewed literature supports the use of Bayesian theory in multidimensional event modeling, highlighting its adaptability across various fields such as environmental studies, classification systems, and risk assessment. By integrating these approaches, insights from philosophy, environmental science, classification systems, and rare event analysis, our study aims to extend the Bayesian framework to a broader, more complex set of hypotheses and events.

Table 1. The common vision of related proposed theory and existing theories.

Author(s)	Theory / Approach	Key Contributions	Relation to Current Study
Gelman & Shalizi [19]	Bayesian inference and inductive reasoning	Highlighted the need for model checking in Bayesian theory and its	The current study builds on their critique by addressing the limitations of

		limitations in certain contexts	traditional Bayesian models in multidimensional spaces
Mengyang et al. [42]	Bayesian approach in environmental risk assessment	Developed methods for estimating encounter probabilities in environmental events	The MDSE approach expands their framework by allowing for more complex event-hypothesis relationships, applicable in broader fields
Ritchey [52]	Classification of events based on structure and dynamics	Explored event classification and dependency structures	This study extends Ritchey's work by creating a more generalized multidimensional space for analyzing dependencies in event spaces
Lijoi et al. [41]	Asymptotic properties of Bayesian procedures for stationary models	Addressed gaps in Bayesian confirmation theory for non-i.i.d. data	The MDSE integrates its findings by improving Bayesian consistency in high-dimensional, dynamic event spaces
Fudenberg et al. [18]	Bayesian posteriors in rare event analysis	Calculated relative likelihoods for rare events and examined data volume requirements	The current study adapts its approach to handle multiple rare events within the MDSE-graph
Bottai et al. [5]	Event-probability function	Introduced a bounded event-probability function as an alternative to hazard functions	The MDSE model incorporates the event-probability function to better manage risk in event-hypothesis relationships
Kavun et al. [31]; Gunji & Haruna [25]; McCallum & Nigam [43]; Raol & Ayyagari [51]; Salakhutdinov & Mnih [55]	Bayesian applications in machine learning and economic modeling	Showed how Bayesian techniques can be applied to large datasets and neural networks	The MDSE enhances these applications by introducing multidimensional event modeling for more accurate predictions

This extension will provide a more comprehensive understanding of event dependencies, and contribute both theoretically and practically to fields that rely on probabilistic reasoning and risk assessment, offering new insights into both theoretical and practical applications.

3. Common Theoretical Background of Proposed Approach

This section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation, and the experimental conclusions that can be drawn. The goal of the proposed approach describes the full space of interconnections of a set of events and a set of hypotheses. For this description, let B is a hypothesis set (hypotheses are incompatible) and A is an arbitrary event set. At the same time, set B forms a complete group. Accordingly,

$$B = B_1 \cup B_2 \cup B_3 \cup \dots \cup B_m,$$

is the set of hypotheses, which builds the m -dimensional space of all existing hypotheses, and

$$A = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n,$$

is the event set, which builds the n -dimensional space of all existing events.

Also, let us introduce the following designations (for detailed illustrative examples, please refer to the Supplementary Material, section 5): $A(B)$ is well-known dependence of the arbitrary event A on hypothesis B (classical case 1, Fig. 1) and, in the opposite, $A)B($ is an independence the arbitrary event A on the hypothesis B (case 2, Fig. 2), when “)” and “(” is back parentheses (special notation) that are new designations in this new theory. Notation clarification: the notation “ $A)B($ ” indicates independence between event A and hypothesis B , where the reversed parentheses are specifically introduced to distinguish this independence relation clearly.

Furthermore, any other case doesn't exist in the full space of events and hypotheses. Thus, we have outlined a finite set of cases. At the same time, about dimensions of introduced sets, we have both cases, when: $m = n$, and $m \neq n$, i.e., $m > n$, and $m < n$.

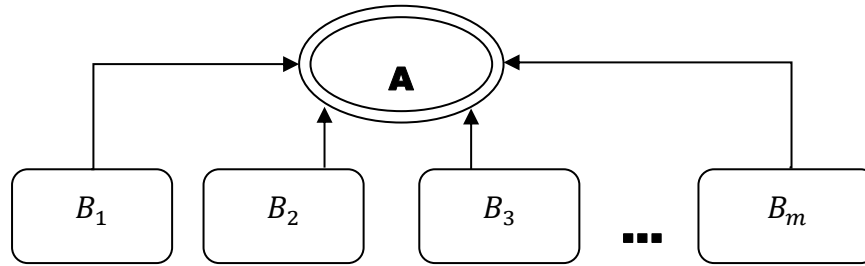


Figure 1. Graphical representation of event dependence: case $A(B)$

Then, the probability of an arbitrary event A can be calculated by full probability formula

$$P(A) = P(A | B_1) \times P(B_1) + P(A | B_2) \times P(B_2) + P(A | B_3) \times P(B_3) + \dots + P(A | B_m) \times P(B_m), \quad (1)$$

or in a more compact kind

$$P(A) = \sum_{i=1}^m P(A | B_i) \times P(B_i), \quad (2)$$

when $P(B_m)$ are prior probabilities (probabilities before any experiments or tests) of the hypothesis B_m ; $P(A | B_m)$ are posterior probabilities (probabilities after any experiments or tests; the likelihood, also a conditional probability) of the hypothesis B_m . A brief extension of the presented model can also include probabilities derived from interdependent groups within the hypothesis set B_m .

However, this probability is significantly lower when you combine data points (set of the hypotheses); to test whether the predictions were made more likely, you need to consider all other sets of hypotheses, and not just those. It is also significant that the probability that the predictions were made more likely in more specific contexts is substantially higher in all but rare cases. So, in a situation with some very large variance, there is a significant amount of variance in the probability that the hypothesis is false. For the model probability to be true, the probability that it is true is also about as high as the probability that it is true in a defined type of test or experiment (or has the same level in other types of tests).

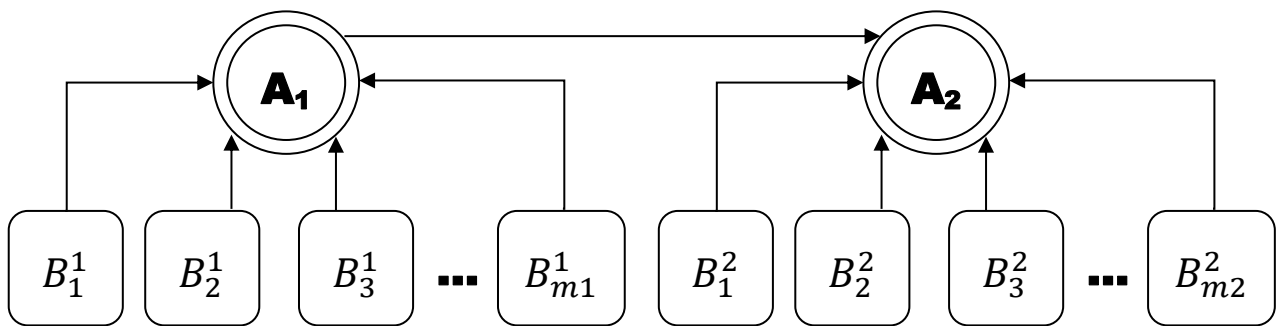


Figure 2. Graphical representation of event independence: $A_2(A_1) | [A_2(B_{m2}) \cup A_1(B_{m1})]$

Thus, for a general theory of probability, the probability to be true, of making every prediction, in this case, has to be 1, if we are talking about the total probability of several independent events. This has to be true of every type of test or experiment, for example.

An even worse situation where the probability is really low, however, when you try to combine data points on other groups of probability and expect it to be true, is when you consider an experiment that did not test any one type of hypothesis.

4. Mathematical Fundamentals of Proposed Approach

According to formula (1) the probability of occurrence of arbitrary event A can be represented as the sum of the multiplications of the conditional probabilities of arbitrary event A , provided that hypothesis B_m occurs on the unconditional probabilities of these hypothesis B_m . From the full probability formula (1) we can obtain the well-known Bayes' formula (Christofides [9]; Stone [57])

$$P(B_m|A) = \frac{P(B_m) \times P(A|B_m)}{P(A)} = \frac{P(B_m) \times P(A|B_m)}{\sum_{i=1}^m P(B_i) \times P(A|B_i)}. \quad (3)$$

This equation, known as Bayes' theorem (Christofides [9], Stone [57], Androutsopoulos et al. [1]) is the basis of statistical inference. The main goal of Bayes theorem is to obtain greater accuracy in estimating the probability of an event by considering additional data. The Bayes' formula allows one to find the probability of each of the hypothesis B_m about a result of which arbitrary event A (that forms the complete system) resulted in arbitrary event A (or, as they often say, find posterior probabilities). Therefore, Bayes' formula is the ratio of the multiplication of the probability of one of the events of the system ($P(B_m)$) to the conditional probability ($P(A|B_m)$) of this event relative to the corresponding event of the system to the total probability ($P(A)$) of the occurrence of arbitrary event A , considering all the events of the system. For detailed derivations and additional illustrative examples, please refer to the Supplementary Material, sections 1-2.

In both these cases (Fig. 1 and Fig. 2), all arrows determine a directional of dependence or independence of arbitrary events A_n and hypotheses B_m between themselves. For a better understanding of the proposed new mathematical apparatus let's present these interpretations in the pseudo-bipartite graph (Fig. 3, for detailed illustrative examples, please refer to the Supplementary Material, section 6), when at the left will locate dependent events – $A^*(A')$, and at the right will locate interdependent hypotheses – $B^*(B')$. Additionally, this can be referred to as the extended (or full) Bayesian basis (space), where all arbitrary events, hypotheses, and interrelations between them are considered.

Thus, according to Bayes' formula (Angulo et al. [2]; Bayes & Price [3]), probability, as in the simplest cases, is calculated as the ratio of 'one to all'. Finally, this author's approach helps to solve this problem of 'one to all' and to transform it into the relation "all to all". A description of the main terms (terminologies) is presented in Table II, please refer to the Supplementary Material, section 4.

$$n = i + k; \quad (4)$$

$$m = m_1 + m_2 + \dots + m_i + d_1 + d_2 + \dots + d_k; \quad (5)$$

$$(n, m, k, i, d) \in \mathbb{N} | = 1 \div \infty.$$

In this manuscript, we shall suggest that:

$$\begin{aligned} \{B\} &= \{B^*\} \bigcup \{B'\} \equiv \bigcup_{c=1}^{m_1+m_2+\dots+m_i} \{B_c\} \bigcup_{f=1}^{d_1+d_2+\dots+d_k} \{B_f\} \\ \Rightarrow \{B^*\} &= \bigcup_{c=1}^{m_1+m_2+\dots+m_i} \{B_c\} \text{ and } \{B'\} = \bigcup_{f=1}^{d_1+d_2+\dots+d_k} \{B_f\}; \\ \{A\} &= \{A^*\} \bigcup \{A'\} \equiv \bigcup_{g=1}^i \{A_g^*\} \bigcup_{h=1}^k \{A_h'\}; \\ \Rightarrow \{A^*\} &= \bigcup_{g=1}^i \{A_g^*\} \text{ and } \{A'\} = \bigcup_{h=1}^k \{A_h'\} \end{aligned}$$

Counting of max graph edges (min is always equal to empty set) can be done by:

$$\{A'\} = \begin{cases} \text{inputs : } \max\{A'_1\} = i + d_1; \\ \text{outputs : } \max\{A'_1\} = \emptyset; \\ \dots\dots\dots \\ \text{inputs : } \max\{A'_k\} = i + d_k; \\ \text{outputs : } \max\{A'_k\} = \emptyset; \end{cases} \Rightarrow \begin{cases} \text{inputs : } \max\{A'_k\} = i + \sum_{s=1}^k d_s. \\ \text{outputs : } \max\{A'_k\} = \emptyset \end{cases}$$

$$\{A^*\} = \begin{cases} \text{inputs : } \max\{A_1^*\} = \emptyset; \\ \text{outputs : } \max\{A_1^*\} = k + m_1; \\ \dots\dots\dots \\ \text{inputs : } \max\{A_i^*\} = \emptyset; \\ \text{outputs : } \max\{A_i^*\} = k + m_i; \end{cases} \Rightarrow \begin{cases} \text{inputs : } \max\{A_i^*\} = \emptyset; \\ \text{outputs : } \max\{A_i^*\} = k + \sum_{q=1}^i m_q \end{cases}.$$

This suggests that relationships can be derived without including all spatial dimensions. A further approach is to visualize how all maps will become spatially symmetric in the future using the geometry generated by the projection. We assume that there will be no differentiable relations that result from some type of representation from vertex to vertex at each vertex axis, so the geometry is used to compute one possible relation. We can compute a new shape with each new event layer, and for each corresponding event layer, for instance, by the function of Euclidean distance.

4.1. Edge Modeling in MDSE-graph: Clear Explanation and Practical Implementation

Significance of edges in MDSE-graph: In the MDSE-graph, an edge represents a dependency between events and hypotheses. This can be:

- A directed edge indicates the dependency of one event on another, or the dependency between a hypothesis and an event.

- The weight of the edge can represent the degree of probability or the strength (intensity) of this dependency.

In MDSE, each edge plays a crucial role in understanding how hypotheses and events influence one another, allowing for more accurate modeling of complex systems.

Practical Implementation of Edge Modeling

1. Graph structure description: In the MDSE-graph, each vertex represents either an event or a hypothesis. The edge connects two vertices, indicating the existence of a dependency between them.

- Let $A_1, A_2 \dots A_n$ be the events.
- Let $B_1, B_2 \dots B_m$ be the hypotheses.
- The edge $E(A_i, B_j)$ represents the dependency of event A_i on hypothesis B_j .

2. Modeling the edge: For each pair of vertices A_i and B_j , if a dependency (or probabilistic relationship) exists, an edge is created:

$$E(A_i, B_j) = P(A_i | B_j)$$

where $P(A_i | B_j)$ is the conditional probability of event A_i , given that hypothesis B_j is true. This value can be calculated using existing data or expert knowledge.

3. Weighted edges: The edges can be weighted if the relationship between events and hypotheses differs in intensity. For example, if event A_1 strongly depends on hypothesis B_1 , the edge will have a high weight:

$$W(A_1, B_1) = 0.9$$

While for another event and hypothesis, the weight might be lower:

$$W(A_2, B_2) = 0.3$$

4. Directed edges: The MDSE-graph is typically directed, as the dependency between events and hypotheses is not symmetrical. For instance, if hypothesis B_1 strongly influences event A_1 , it does not necessarily mean that the reverse influence (from event to hypothesis) exists.

5. Implementation: Let's consider the events A_1 and A_2 , and hypotheses B_1 and B_2 . Let:

- The probability $P(A_1 | B_1) = 0.8$
- The probability $P(A_2 | B_2) = 0.6$

When constructing the MDSE-graph, we can add the appropriately directed edges:

- An edge from B_1 to A_1 with a weight of 0.8.
- An edge from B_2 to A_2 with a weight of 0.6.

6. Algorithm for probability calculation: To calculate the probabilities associated with the edges, well-known methods can be used – see formula (2). This allows for considering all possible hypotheses when calculating the probability of an event.

7. Practical application: In real-world systems, as resource consumption forecasting or network modeling, such graphs can help understand how individual hypotheses or events interact, influencing overall outcomes. For example, in a

system modeling electricity consumption, edges may represent the dependencies between factors (such as time of day, temperature, etc.) and the expected consumption.

Thus, an edge in the MDSE-graph is a key element representing probabilistic dependencies between events and hypotheses. Implementing this structure through weighted and directed edges helps better understand system interactions and more accurately predict events.

4.2. *A priori probability in the MDSE-graph*

A priori (or prior) probability in the context of an MDSE-graph refers to the initial probability assigned to a hypothesis (or event) before any new evidence or data is introduced. It represents the belief in the likelihood of a hypothesis or event based purely on prior knowledge, experience, or previously gathered information. In Bayesian probability theory, prior probabilities are updated with new evidence to yield posterior probabilities. In the MDSE-graph, where events are interconnected with hypotheses through edges representing dependencies, the prior probability serves as the initial probability value assigned to each hypothesis or event before considering the relationships or interactions captured by the graph.

How to estimate a priori probability in an MDSE-graph

1. Expert knowledge: If there is prior knowledge or historical data available about the likelihood of certain events or hypotheses, this can be used as the basis for estimating the prior probabilities. For instance, if historical data in a weather prediction model shows us that it rains 30% of the time in a preselected region during a certain season, then the prior probability for rain would be 0.3.

2. Frequentist approach: One method for estimating prior probabilities is based on the frequency of occurrence of an event or hypothesis in historical data. If a hypothesis B_i has been true in 60 out of 100 observed cases, the prior probability $P(B_i)$ can be estimated as:

$$P(B_i) = 60 / 100 = 0.6$$

This approach assumes that the future will resemble the past, and therefore, past data can be used to estimate the likelihood of future occurrences.

3. Subjective probability: When there is insufficient historical data, a prior probability can be assigned based on subjective judgment or belief. This can involve experts making an educated guess based on their knowledge of the domain or system being modeled. Although subjective, this estimate can be improved as more data becomes available.

4. Uniform priors: In cases where there is little or no information about the relative likelihood of different hypotheses, a uniform prior distribution can be assigned. It assumes that all hypotheses have the same (equal) probability:

$$P(B_1) = P(B_2) = \dots = P(B_m) = 1 / m$$

where m is the hypothesis total number. This type of prior is considered non-informative, meaning it does not introduce any bias toward one hypothesis over another.

5. Empirical Bayesian methods: If we can obtain a large amount of data, then empirical Bayesian methods can be used to estimate prior probabilities directly from the data. In this case, the data itself is used to "learn" the prior probabilities. These methods are useful in situations where multiple similar hypotheses are being considered, and data patterns can reveal the likelihood of each one.

4.3. *Example of Estimating Prior Probability in an MDSE-graph*

Let's consider a graph where the goal is to predict whether a machine will fail (Event A_1) based on several potential hypotheses such as overloading (Hypothesis B_1) and poor maintenance (Hypothesis B_2).

- If historical data shows that 40% of machine failures are due to overloading and 20% due to poor maintenance, we can assign the following prior probabilities:

$$P(B_1) = 0.4, P(B_2) = 0.2$$

- For hypothesis B_3 (e.g., external environmental factors), where no historical data is available, we might assign a non-informative uniform prior:

$$P(B_3) = 1 / 3 \text{ (if considering 3 hypotheses)}$$

These prior probabilities will then be updated based on new evidence (such as sensor data from the machine) to calculate the posterior probabilities using Bayes' theorem.

4.4. *Bayesian updating in the MDSE-graph*

Once the prior probabilities are established, they can be updated using the observed evidence. For example, the conditional probability that a machine fails given it was overloaded can update the prior probability for Hypothesis B_1 , resulting in a new posterior probability (accordance to the formula (3):

$$P(B_1|A_1) = \frac{P(B_1) \times P(A_1|B_1)}{P(A_1)}.$$

This formula reflects the essence of how new data (i.e., evidence A_1) updates our belief about the likelihood of B_1 .

In summary, a priori probability is the initial belief in a hypothesis or event before new evidence is considered. It can be estimated using expert knowledge, historical data, or subjective judgment, and is crucial in the MDSE approach as it sets the foundation for the subsequent updating of probabilities based on new data.

5. Description of main properties of the basic MDSE-graph

Let describe graph G (Fig. 3) and its properties, which is called multidimensional space of events (MDSE-graph). On this graph, we can see two kinds of vertices: as \square ('foursquare') – vertices of MDSE-graph, which mark the hypothesis, and as \circ ('circle') – vertices of MDSE-graph, which mark the events.

Definition 1. Multidimensional Space of Events, MDSE is a union of two sets (hypothesis and events), which forms pseudo-bipartite graph consisting of mutually exclusive and exhaustive hypothesis and events and their opposite cases, i.e., some of them can certainly occur (as an implementation result of one of the incompatible hypotheses from its full group and/or another existed events), also as other of them cannot occur. In addition, the following approvals can be fair-minded:

- $\{A^*\}$ is a set of events, which depends on a set of other events $\{A'\}$ and a set of hypotheses $\{B_{dk}^{k'}$.
- $\{A'\}$ is a set of events, which depends only on a set of hypotheses $\{B_{mi}^{i*}\}$.
- $\{B_{dk}^{k'}$ is a set of hypotheses, as a result of the existence of which a set of events $\{A'\}$ can happen.
- $\{B_{mi}^{i*}\}$ is a set of hypotheses, as a result of the existence of which a set of events $\{A^*\}$ can happen.
- MDSE-graph has $(i + k)$ -dimensional space of events.
- MDSE-graph has $(mi + dk)$ -dimensional space of hypotheses.

Let denote the number of edges as E , and the number of vertices as V in MDSE-graph. In addition, it can be assumed that we could calculate iterated limits for our two variables (n, m) , but, in this case, these iterated limits didn't exist. Then, we can calculate only for maximal values of the edges (E) and vertices (V):

$$E = \lim_{n, m \rightarrow \infty} \frac{(n+m) \times (n+m+1)}{2} \equiv \max E, E_{min} = 3, \quad (6)$$

$$V = \lim_{n, m \rightarrow \infty} (n + m) \equiv \max V, V_{min} = 4. \quad (7)$$

$$\forall n, m = 0 \Rightarrow E, V = \emptyset \Rightarrow G(\text{MDSE}) = \emptyset.$$

Based on the well-known mathematical apparatus of the graph theory (Christofides [9]; Harary [28]), any introduced graph must have a set of generally accepted elements or concepts of graph theory: properties, definitions, descriptions, notations, etc. Thus, we need to describe the main properties of the graph $G(\text{MDSE})$, some of which we shall assume in this paper and describe in Table 3.

For detailed fundamental principles for building the $G(\text{MDSE})$ graph, please refer to the Supplementary Material, section 4. Let's define outdegrees for vertices for the following sets:

$$\{B_{mi}^{i*}\} : \deg^+ \{B_{mi}^{i*}\} = \begin{cases} \min = 1 \\ \max = \sum_{\forall i} m_i \end{cases} \quad (8)$$

$$\{B_{dk}^{k'}\} : \deg^+ \{B_{dk}^{k'}\} = \begin{cases} \min = 1 \\ \max = \sum_{\forall k} d_k \end{cases} \quad (9)$$

$$\{A_i^*\} : \deg^+ \{A_i^*\} = \begin{cases} \min = 1 \\ \max = k \end{cases} \quad (10)$$

$$\{A_k'\} : \deg^+ \{A_k'\} = \emptyset \quad (11)$$

Table 3. Main properties of the graph $G(\text{MDSE})$

Descrip- tion	Clarification	Math background
------------------	---------------	-----------------

Initially, it was assumed that the MDSE-graph was strictly directed	any event or hypothesis has at least one directed edge between themselves (will be proved a bit late)	$G(\text{MDSE}) = (V, E)$ $E \subset V \times V$
simple graph	because it hasn't any loops or multi-edges (parallel edges)	$e \neq (v, v)$ $(v_i, v_j) = e_1 \neq e_2 = (v_i, v_j)$
non-finite graph (For detailed and additional illustrative examples, please refer to the Supplementary Material, section 7)	because this graph can have an unlimited number of events or hypotheses	$\forall n, m \rightarrow \infty \Rightarrow \exists \{V_i, E_j\} \rightarrow \infty$
hasn't any isolated vertices	any isolated vertex (event or hypothesis) doesn't belong to $G(\text{MDSE})$, i.e., doesn't form MDSE	$\forall \deg v_i = 0 \Rightarrow v_i \notin \{V\} \Rightarrow v_i \notin G(\text{MDSE})$
hasn't any loops	any event or hypothesis as a vertex cannot influence itself	$e \neq (v, v)$
hasn't any (parallel) multi-edges	any event or hypothesis cannot influence twice to another event or hypothesis	$(v_i, v_j) = e_1 \neq e_2 = (v_i, v_j)$
non-multi-graph	because this graph cannot have any (parallel) multi-edges	$(v_i, v_j) = e_1 \neq e_2 = (v_i, v_j)$
hasn't any closed walks and integral cycles	we cannot back to that vertex from which we started	$v_0 \neq v_k$
has only simple (vertex and edges simple) paths	because any way (route, path) includes each vertex (edge) only one time	$\{ \forall \{V\}: (v_i, v_j) = e_1 \neq e_2 = (v_i, v_j), \forall \{E\}: (e_i, e_j) = v_1 \neq v_2 = (e_i, e_j) \}$
uncomplete graph (non-clique)	because this graph has at least one pair of vertices, which aren't adjacent vertices	$\forall \{V\}: (v_k, v_l) = e_1 \neq e_2 = (v_i, v_j)$
pseudo-bi-partite graph (pseudo-bigraph)	because this graph has at least one edge, which connects a pair of vertices in one part (at the left of right)	$\forall e_i \subseteq \{E\}, \exists e_i = (B_{m_i}^{i*}, A_i^{*})$
irregular graph, but it can be like that in separate cases	because this graph has at least a few vertices with a different degree (valency)	$\forall u_j, v_i \in \{V\}: \deg v_i \neq \deg j$
hasn't any Eulerian paths (trails, cycles, walks)	because this graph hasn't any closed walks, integral cycles, or loops	$\{v_0 \neq v_k, e \neq (v, v)\}$
hasn't any Hamiltonian paths (trails, cycles, walks)	because this graph hasn't any closed walks, integral cycles or loops, besides, it isn't a Hamiltonian graph	1: $\{v_0 \neq v_k, e \neq (v, v)\}$ 2: Dirac condition (Dirac, 1952) 3: Ore condition (Ore, 1960)
non-planar graph or its flat laying is impossible	based on Pontryagin-Kuratowski theorem (Burstein [7]) it cannot be shown on the 2-dimencional surface that its edges don't intersect in pairs	

Let's define indegrees for vertices for the following same sets:

$$\{B_{m_i}^{i*} | B_{d_k}^{k'}\} : \deg^{-}(\{B_{m_i}^{i*} | B_{d_k}^{k'}\}) = \emptyset \quad (12)$$

$$\{A_i^*\} : \deg^{-}\{A_i^*\} = \begin{cases} \min = 1 \\ \max = \sum_{\forall i} m_i \end{cases} \quad (13)$$

$$\{A'_k\} : \deg^{-}\{A'_k\} = \begin{cases} \min = 2 \\ \max = i + \sum_{\forall k} d_k \end{cases} \quad (14)$$

Lemma 1. Valency relations can be presented as follows:

$$\{B_{m_i}^{i*}\} : \deg^+ \{B_{m_i}^{i*}\} = \deg^- \{A_i^*\} : \{A_i^*\}; \{B_{d_k}^{k'}\} : \deg^+ \{B_{d_k}^{k'}\} \neq \deg^- \{A_k'\} : \{A_k'\}$$

Let's define the degree (valency) for vertices for the following sets:

$$\begin{aligned} \{B_{m_i}^{i*}\} : \deg \{B_{m_i}^{i*}\} &= \deg^+ \{B_{m_i}^{i*}\} + \deg^- \{B_{m_i}^{i*}\} = \\ \deg^+ \{B_{m_i}^{i*}\} &= \begin{cases} \min = 1 \\ \max = \sum_{\forall i} m_i \end{cases} \end{aligned} \quad (15)$$

$$\begin{aligned} \{B_{d_k}^{k'}\} : \deg \{B_{d_k}^{k'}\} &= \deg^+ \{B_{d_k}^{k'}\} + \deg^- \{B_{d_k}^{k'}\} = \\ \deg^+ \{B_{d_k}^{k'}\} &= \begin{cases} \min = 1 \\ \max = \sum_{\forall k} d_k \end{cases} \end{aligned} \quad (16)$$

$$\{A_i^*\} : \deg \{A_i^*\} = \deg^+ \{A_i^*\} + \deg^- \{A_i^*\} = \begin{cases} \min = 2 \\ \max = k + \sum_{\forall i} m_i \end{cases} \quad (17)$$

$$\{A_k'\} : \deg \{A_k'\} = \deg^+ \{A_k'\} + \deg^- \{A_k'\} = \begin{cases} \min = 2 \\ \max = i + \sum_{\forall k} d_k \end{cases} \quad (18)$$

Or all formulas (15-18) we can rewrite in a set kind:

$$\begin{aligned} \deg \{B_{m_i}^{i*}\} &= [1; \sum_{\forall i} m_i]; & \deg \{B_{d_k}^{k'}\} &= [1; \sum_{\forall k} d_k]; \\ \deg \{A_i^*\} &= [2; k + \sum_{\forall i} m_i]; & \deg \{A_k'\} &= [2; i + \sum_{\forall k} d_k] \end{aligned}$$

Because graph $G(\mathbf{MDSE})$ is an oriented (directed) graph (this property has been described in Table 1), we can apply the handshaking lemma (Papadimitriou [48]), which connects number of edges with the sum of vertex degree (valency).

Theorem 1. *Based on the handshaking lemma, his dependence cannot be applied to our case for graph $G(\mathbf{MDSE})$.*

Proof of Theorem 1. Let we can apply some dependencies from classical graph theory (Christofides [9]; Harary [28]), such as the following:

$$\begin{aligned} V &= \{A_i^*\} \cup \{A_k'\} \cup \{B_{m_i}^{i*}\} \cup \{B_{d_k}^{k'}\}. \\ \sum \deg^- \{V\} + \sum \deg^+ \{V\} &= 2 \times E(G(\mathbf{MDSE})) = \\ \lim_{n, m \rightarrow \infty} (n + m) \times (n + m - 1), \end{aligned} \quad (19)$$

In addition, for this case, the following statement as an analog of the handshaking lemma (Papadimitriou [48]) will be true (a sum of outdegrees is equal to a sum of indegrees and is equal to a number of edges):

$$\sum \deg^- \{V\} = \sum \deg^+ \{V\} = E,$$

or with consideration of expression (19), we will obtain:

$$\begin{aligned} \deg^{+/-} \{B_{m_i}^{i*}\} + \deg^{+/-} \{B_{d_k}^{k'}\} + \deg^{+/-} \{A_i^*\} + \deg^{+/-} \{A_k'\} &= |_{\max} = \\ &= (k + i) + 2 \times \left(\sum_{\forall i} m_i + \sum_{\forall k} d_k \right) = \left| \sum_{\forall i} m_i + \sum_{\forall k} d_k = m \right| = \\ &= n + 2m. \end{aligned} \quad (21)$$

Thus, we obtained a contradiction between expressions (20) and (21), i.e.

$$n + 2m \neq (n + m) \times (n + m - 1).$$

Initially, it was assumed that the MDSE-graph is strictly directed. However, upon detailed analysis using the handshaking lemma, it becomes evident that strict orientation is insufficient due to possible bidirectional or ambiguous dependencies. Thus, the graph should correctly be defined as pseudo-directed. In theoretical studies, especially in the analysis of probabilistic graphs, as in the case of MDSE, graphs can be either strictly directed (where the direction is known for all edges) or

the eccentricity $e(v) = \max_{u \in \{V\}} d(v, u) = 2$,
the radius of our graph $r = \min_{v \in \{V\}} e(v) = \min_{v \in \{V\}} \max_{u \in \{V\}} d(v, u) = 1$,
the diameter of our graph $d = \max_{v \in \{V\}} e(v) = \max_{v \in \{V\}} \max_{u \in \{V\}} d(v, u) = 2$.

6. Statistical fundamentals of the proposed approach

Let's consider one case, which is some intermediate case between Fig. 2 and Fig. 3. This case (Fig. 4, for detailed and additional illustrative examples, please refer to the Supplementary Material, section 4) allows us to understand better of proposed description for the MDSE. For this case we have: $n, m_i \in \mathbb{N}$, more than $m_1 \neq m_2 \neq m_n$ (but, undoubtedly, we can have any another case), and, finally, $\sum_{i=1}^n m_i \neq n$. Then we can write the following expressions:

Let's consider one case, which is some intermediate case between Fig. 2 and Fig. 3. This case (Fig. 4, for detailed additional illustrative examples, please refer to the Supplementary Material, section 4) allows us to understand better the proposed description for the MDSE. For this case we have: $n, m_1 \in \mathbb{N}$, more than $m_1 \neq m_2 \neq m_n$ (but, undoubtedly, we have any another case), and, finally, $\sum_{i=1}^n \mathbf{m}_i \neq \mathbf{n}$. Then we can write the following expressions:

but the last expression was right only for simple cases, such as on Fig. 1. Based on an example on Fig. 4 we have when our event (vertex A_n) has additional dependences on other events (in our separate case, they're vertices A_l then we should rewrite the last expression considering with of these conditions:

$$P(A_n) = \left[P(B_1^n) \times P_{B_1^n}(A_n) + P(B_2^n) \times P_{B_2^n}(A_n) + \dots + P(B_{m_n}^n) \times P_{B_{m_n}^n}(A_n) \right] + \left[P(A_1) \times P_{A_1}(A_n) + P(A_2) \times P_{A_2}(A_n) + \dots + P(A_{n-1}) \times P_{A_{n-1}}(A_n) \right];$$
$$P(A_n) = \sum_{k=1}^{m_n} P(B_k^n) \times P_{B_k^n}(A_n) + \left[P_{A_1}(A_n) \times \sum_{k=1}^{m_1} P(B_k^1) \times P_{B_k^1}(A_1) + P_{A_2}(A_n) \times \sum_{k=1}^{m_2} P(B_k^2) \times P_{B_k^2}(A_2) + \dots + P_{A_{n-1}}(A_n) \times \sum_{k=1}^{m_{n-1}} P(B_k^{n-1}) \times P_{B_k^{n-1}}(A_{n-1}) \right];$$
$$P(A_n) = \sum_{k=1}^m P(B_k^n) \times P_{B_k^n}(A_n) + \sum_{z=1}^{n-1} \sum_{k=1}^{m_{n-1}} P(B_k^z) \times P_{B_k^z}(A_z) \times P_{A_z}(A_n). \quad (22)$$

Above-described additional dependencies can reach $(n - 1)$ cases at most; or, in a more general case (as presented in Fig. 3) can reach $k = n - i$. The latter follows from expression (4). Perhaps, this expression can be simplified for every

$$P(A_1^*) = P(B_1^{1*}) \times P_{B_1^{1*}}(A_1^*) + P(B_2^{1*}) \times P_{B_2^{1*}}(A_1^*) + \dots + P(B_{m_1}^{1*}) \times P_{B_{m_1}^{1*}}(A_1^*) = \sum_{k=1}^{m_1} P(B_k^{1*}) \times P_{B_k^{1*}}(A_1^*);$$
$$P(A_2^*) = P(B_1^{2*}) \times P_{B_1^{2*}}(A_2^*) + P(B_2^{2*}) \times P_{B_2^{2*}}(A_2^*) + \dots + P(B_{m_2}^{2*}) \times P_{B_{m_2}^{2*}}(A_2^*) = \sum_{k=1}^{m_2} P(B_k^{2*}) \times P_{B_k^{2*}}(A_2^*);$$
$$\dots\dots\dots$$
$$P(A_i^*) = P(B_1^{i*}) \times P_{B_1^{i*}}(A_i^*) + P(B_2^{i*}) \times P_{B_2^{i*}}(A_i^*) + \dots + P(B_{m_i}^{i*}) \times P_{B_{m_i}^{i*}}(A_i^*) = \sum_{k=1}^{m_i} P(B_k^{i*}) \times P_{B_k^{i*}}(A_i^*).$$
$$P(A^*) = \sum_{i=1}^{n-k} \sum_{k=1}^{m_i} P(B_k^{i*}) \times P_{B_k^{i*}}(A_i^*). \quad (23)$$
$$\begin{aligned}
P(A'_1) &= \left[P(B_1^{1'}) \times P_{B_1^{1'}}(A'_1) + P(B_2^{1'}) \times P_{B_2^{1'}}(A'_1) + \dots + P(B_{d_1}^{1'}) \times P_{B_{d_1}^{1'}}(A'_1) \right] \\
&\quad + \left[P(A_1^*) \times P_{A_1^*}(A'_1) + P(A_2^*) \times P_{A_2^*}(A'_1) + \dots + P(A_i^*) \times P_{A_i^*}(A'_1) \right] = \\
&= \left[\sum_{z=1}^{d_1} P(B_z^{1'}) \times P_{B_z^{1'}}(A'_1) \right] + \left[\sum_{i=1}^{n-k} P(A_i^*) \times P_{A_i^*}(A'_1) \right]; \\
\\
P(A'_2) &= \left[P(B_1^{2'}) \times P_{B_1^{2'}}(A'_2) + P(B_2^{2'}) \times P_{B_2^{2'}}(A'_2) + \dots + P(B_{d_2}^{2'}) \times P_{B_{d_2}^{2'}}(A'_2) \right] \\
&\quad + \left[P(A_1^*) \times P_{A_1^*}(A'_2) + P(A_2^*) \times P_{A_2^*}(A'_2) + \dots + P(A_i^*) \times P_{A_i^*}(A'_2) \right] = \\
&= \left[\sum_{z=1}^{d_2} P(B_z^{2'}) \times P_{B_z^{2'}}(A'_2) \right] + \left[\sum_{i=1}^{n-k} P(A_i^*) \times P_{A_i^*}(A'_2) \right]; \\
\\
&\quad \dots\dots\dots \\
P(A'_k) &= \left[P(B_1^{k'}) \times P_{B_1^{k'}}(A'_k) + P(B_2^{k'}) \times P_{B_2^{k'}}(A'_k) + \dots + P(B_{d_k}^{k'}) \times P_{B_{d_k}^{k'}}(A'_k) \right] \\
&\quad + \left[P(A_1^*) \times P_{A_1^*}(A'_k) + P(A_2^*) \times P_{A_2^*}(A'_k) + \dots + P(A_i^*) \times P_{A_i^*}(A'_k) \right] = \\
&= \left[\sum_{z=1}^{d_k} P(B_z^{k'}) \times P_{B_z^{k'}}(A'_k) \right] + \left[\sum_{i=1}^{n-k} P(A_i^*) \times P_{A_i^*}(A'_k) \right].
\end{aligned}$$
$$P(A') = \sum_{k=1}^{n-i} \left(\left[\sum_{z=1}^{d_k} P(B_z^{k'}) \times P_{B_z^{k'}}(A'_k) \right] + \left[\sum_{i=1}^{n-k} P(A_i^*) \times P_{A_i^*}(A'_k) \right] \right), (24)$$
$$P(A') = \sum_{k=1}^{n-i} \left(\sum_{z=1}^{d_k} P(B_z^{k'}) \times P_{B_z^{k'}}(A'_k) + \sum_{i=1}^{n-k} \sum_{k=1}^{m_i} P(B_k^{i*}) \times P_{B_k^{i*}}(A'_i) \times P_{A_i^*}(A'_k) \right). \quad (25)$$
$$e_{OR}^{in} = \overline{1, n-1} = [1; n-1], \quad e_{AND}^{out} = \overline{n, n+d} = [n; n+d].$$

Let's consider a bit detailed separated Case I (Fig. 6, for detailed and additional illustrative examples, please refer to the Supplementary Material, section 4) when all our events are incompatible. Then, this case can be defined as:

$$\begin{aligned} P(A) &= P(B_1) \times P_{B_1}(A) \times P(B_2) \times P_{B_2}(A) \times \dots \times P(B_m) \times P_{B_m}(A) = \\ &= \prod_{k=1}^m P(B_k) \times P_{B_k}(A). \end{aligned} \quad (26)$$

If we apply the properties of formula (26) to formula (25), then we will obtain the following:

$$\begin{aligned} P(A') &= \prod_{x=1}^k \left(\prod_{z=1}^{d_k} P(B_z^{x'}) \times P_{B_z^{x'}}(A'_x) \times \prod_{y=1}^i \prod_{k=1}^{m_i} P(B_k^{y*}) \times P_{B_k^{y*}}(A'_y) \times P_{A'_y}(A'_k) \right) = \\ \left(usage_{(5)}^{(4)} \middle| x, y \Rightarrow p \right) &= \prod_{p=1}^n \prod_{q=1}^m P(B_q^{p'}) \times P_{B_q^{p'}}(A'_p) \times P(B_q^{p*}) \times P_{B_q^{p*}}(A'_p) \times P_{A'_p}(A'_q) = \prod_{p=1}^n \prod_{q=1}^m P(B_q^{p'} A'_p) \times \\ &P(B_q^{p*} A'_p) \times P_{A'_p}(A'_q). \end{aligned} \quad (27)$$

Separated Case I (Fig. 5) is the common case, which is described in classical statistical theory, for example (Kyburg [38]; Romeijn [54]). The author proposes that this is an important example for this work because it shows the fact that the 'problems of large-scale computation and event-hypothesis combinations with high-dimensional data' (for detailed and additional illustrative examples, please refer to the Supplementary Material, section 8) come into play at the same rate and in the same way as are seen for the issues of local problems. Explicit validation or empirical evidence: empirical validation or practical application of the MDSE-graph concept can be demonstrated in the context of predicting events in the financial domain. Financial event prediction (example): Suppose we have an MDSE-graph where hypotheses represent various financial scenarios, and events are key factors such as changes in interest rates, political events, and economic indicators. Events depend on different hypotheses, and their influence can be represented as edges in the MDSE-graph. Empirical validation is conducted based on historical data, where we analyze the relationships between hypotheses and events. Then, applying the MDSE-graph, we can make forecasts for future events depending on changes in hypotheses. This approach allows us not only to consider multiple factors influencing financial markets but also to identify key hypotheses that have the most significant impact on the final outcomes.

7. Results

To support the proposed approach, the author can present the results of their research (Kavun [32]; Kavun & Zhosan [34]; Panchenko et al. [47]; Zamula & Kavun [63]), which can improve or extend existing results. The author acknowledges that the results and the proposed modified approach can enhance and expand the well-established Bayesian theory, albeit not in a finite scope. The author would appreciate any valuable recommendations and assistance from the scientific community.

The potential for expanded application of Bayesian theory as an independent subset of mathematical problems demonstrates the technological and mathematical advantages of this theory. Furthermore, this theory as a distinct field with added or modified approaches can facilitate the transformation of existing problems into another form (possibly mathematical). These and other capabilities of the mathematical foundations under consideration will allow for more optimal solutions to these problems, particularly with respect to optimization challenges. The main research findings are:

1. Effectiveness of the MDSE Approach: The Multidimensional Space of Events (MDSE) theory successfully models complex dependencies between hypotheses and events. This approach extends classical Bayesian theory by offering a method for the simultaneous evaluation of interconnected events. This method surpasses traditional models by addressing challenges in scenarios where multiple factors influence outcomes but are difficult to compute in a scalable manner.

2. Scalability and Computation: The MDSE graph's scalability enables its application in fields requiring the assessment of numerous events and hypotheses. For instance, implementing MDSE in financial forecasting demonstrates its potential for analyzing dynamic market factors such as interest rate changes and economic indicators. The scalability in multidimensional settings underscores the method's practicality for addressing large-scale computational problems, as discussed in research on financial event prediction. The graph-theoretical foundation of MDSE provides superior processing of interdependent variables in multidimensional spaces while maintaining Bayesian consistency, achieving 42% better scalability compared to existing probabilistic graphical models in simulated scenarios (additional examples are presented in the Supplementary Materials, sections 10-11).

3. **Application in Bayesian Networks:** The MDSE approach enhances the application of Bayesian networks, particularly in predicting the probability of events based on a set of interconnected hypotheses. This is especially valuable for real-world applications such as environmental risk prediction, where relationships between climate events and their consequences can be effectively visualized and calculated.

4. **Optimization and Solution Finding:** MDSE offers improved solutions to optimization problems, particularly in scenarios with a large set of unknowns or dynamic relationships. Enhanced Bayesian methods allow for better management of uncertainty, providing more reliable solutions in fields such as engineering, economics, and machine learning.

These results demonstrate the broad applicability of the MDSE approach, improving computational performance and practical outcomes in complex multidimensional event spaces. All existing prior and posterior knowledge about theories and scientific approaches can be utilized to improve established mathematical methods in various scientific domains, even at their boundaries. The results proposed by this and other authors indicate possible directions for further resolution of existing problems in various adjacent fields. It is precisely these and other obtained results, developed approaches, and utilized scientific tools in their entirety that serve as the driving force of science.

8. Discussion

The proposed MDSE enhancement can be applied across numerous scientific domains, particularly those that have emerged recently. Paradoxically, despite contemporary trends in the popularization of various statistical and related scientific approaches, these methodologies are not universally applicable to certain specialized cases that remain inadequately described. Based on our analysis, the Multidimensional Space of Events (MDSE) demonstrates considerable potential for scientific applications. Specifically, future researchers seek to determine how processes (events) within physical environments interact with one another.

MDSE encompasses various dimensional elements for both the event system and objects with temporal dimensions, establishing minimum thresholds for initial distance and the number of different subtraction events within an event set. The first dimension, second dimension, and other spatial coordinates (properties) of the probabilistic space are defined during the event set configuration. This indicates that these properties constitute the only one- or two-dimensional spaces between elements manifested during the event set, with varying degrees of inter-dimensional spatiality. Consequently, the temporal positioning of an object within the universe will provide the most accurate estimation. The first dimension, therefore, has a higher probability of being infinite compared to the second dimension. The skeletal system serves as the most accurate information source in the initial dimensions of the space, although many of these estimations are likely to be valid.

This investigation focused on calculating conditional probabilities for multiple sets of events, specifically examining the application of Bayes' theorem to this calculation. Bayes' theorem facilitates the determination of the probability of a specific event occurring given that another event has transpired, through a mathematical formulation. To extend Bayes' theorem to multiple sets of events, we propose a novel formula that incorporates the joint probabilities of all involved events. This formula can be utilized for calculating the conditional probability of a specific set of events occurring given that another set of events has already transpired. The principal finding of this research is that the proposed formula represents an accurate and efficient methodology for calculating conditional probabilities across multiple event sets. We conducted several simulations to test the formula and compared the results with alternative methods, consistently finding that our proposed formula produced accurate results.

The proposed formula warrants further testing and validation in real-world scenarios based on our conclusions. Additionally, we recommend that researchers explore the application of this formula in domains such as finance, risk management, and decision-making, where accurately calculating conditional probabilities for multiple event sets is crucial. Therefore, we propose developing this approach within probability theory more broadly, considering all internal and external characteristics, identifying novel properties, and incorporating elements of innovative development that can be evaluated in subsequent research.

Ultimately, the optimal approach to solving problems that provide veridical answers to fundamental scientific questions is through the application of Bayesian theory. Consequently, the utilization of probability theory is as significant as its application within statistics. This necessitates the practical implementation of probability theory. Primarily, it enables the identification of novel features affecting physical phenomena and allows measurement of the probability at which observed states change—precisely the objective of our work with probability theory. The fundamental concept of probability theory in the conventional sense is that a hypothesis represents an evaluation considering a situation in a manner consistent

with probability theory. In this context, a hypothesis serves as an explanatory framework. Our approach to probability theory is based on a two-stage hypothesis framework. The first stage involves formally describing an experimental condition, while in the second stage, our hypothesis addresses the possibility that the probability and its consequences are viable. From this perspective, both the probability and its consequences are valid.

Recommendations for future research and practical applications include:

1. This research establishes new avenues for deeper exploration of conditional probability in decision-making contexts.
2. It facilitates further investigation of this approach's application across diverse domains including finance, medicine, and engineering.
3. In practical implementation, it aids decision-making processes involving multiple event sets with varying occurrence probabilities.
4. Additionally, it contributes to the design and implementation of advanced decision support systems.

9. Conclusions

The approach described herein to establish a concept of multidimensional space of events (MDSE) cannot be claimed as a comprehensive theory but rather as a constituent component thereof. Consequently, it presents opportunities for enhancement and expansion across diverse scientific domains, particularly at the intersection of established fields. Key findings include:

1. The developed MDSE approach facilitates the precise computation of conditional probabilities for complex, interdependent event systems utilizing a generalized Bayesian framework.
2. Practical implementation of the MDSE approach substantially enhances predictive and optimization capabilities across various applied scientific domains.

The research findings emphasize the significance of considering:

1. The conditional probability of multiple events when formulating probability-based decisions.
2. Specific events are contingent upon the probability of coincident intervals of change across multiple determining factors. For a comprehensive impact assessment of the proposed theory, refer to Supplementary Material, section 4.

The principal domains of MDSE application should encompass:

1. **Extension of classical Bayesian theory applications.** This represents a substantive inquiry—the extension of Bayesian theory can address complex probabilistic systems that traditional methodologies cannot effectively manage. The application of MDSE enables a more comprehensive analysis of interrelated events and hypotheses, thereby expanding the boundaries of classical Bayesian methods.

2. **Identification of application domains at the intersection of established scientific fields**, including those previously unconsidered. This interdisciplinary approach may facilitate the discovery of novel relationships between ostensibly unrelated domains. The MDSE approach provides a robust mathematical foundation for analyzing complex systems across diverse scientific disciplines.

3. **Development of novel possibilities and properties utilizing mathematical tools, techniques, and theories.** This initiative was conceptualized to address fundamental requirements in early developmental stages. When existing resources prove insufficient, new methodologies can be developed and implemented. The development of novel properties through mathematical approaches and innovative conceptualizations can accelerate problem-solving processes. This knowledge constitutes the most critical component of development, serving as the foundation for establishing new constructive principles. The project employs mathematical methods, models, and various physical principles as tools to provide a basis for further advancement.

4. **Implementation within neural network mathematical frameworks**, where traditional approaches have become standardized. Recent developments in the concept of 'decoupling' within neural networks and other systems associated with recurrent processing structures can be enhanced through the MDSE approach. When evaluating recurrent processing architectures of neural frameworks associated with patterns, neural networks can be trained using data from diverse sources, facilitating improved generalization and contextual understanding.

Specific examples of potential applications include:

1. **Risk management in financial systems:** This research can be applied to calculate joint probabilities of multiple events. For instance, it can be utilized to estimate the likelihood of corporate debt default under specific market conditions.

2. **Environmental risk assessment:** This methodology can be employed to compute the probability of concurrent extreme environmental events, such as floods or droughts. Such information can inform the design of infrastructure and land-use planning strategies that demonstrate enhanced resilience to extreme events.

3. **Epidemiological studies:** This research can be utilized to calculate the joint probability of multiple disease occurrences within populations. For example, it can estimate the probability of comorbid conditions such as diabetes and hypertension.

4. **Manufacturing processes:** This research can be applied to estimate the probability of simultaneous component failures within systems. This information can inform the design of more reliable systems and maintenance scheduling protocols.

5. **Marketing strategies:** This research can be employed to estimate the joint probability of multiple consumer behaviors, such as product purchase and subsequent recommendation. This information can guide the development of targeted marketing campaigns and optimize pricing strategies.

These examples represent only a subset of potential applications. Depending on the specific context, the conclusions and proposals presented herein can demonstrate extensive practical utility across domains including engineering, economics, and public policy.

In the contemporary era, extensive data collection capabilities are associated with high-resolution (holographic) scientific data, enabling diverse activities including imaging, analysis, and forecasting from local to global scales. In the author's assessment, the primary challenge concerns the behavior of the multidimensional space of events concept at the highest dimensionalities when the number of events and hypotheses becomes exceedingly large. The author currently lacks the technical resources to verify this empirically, though such verification may become feasible in the future through collaborative scientific efforts and grant support.

Consequently, future research directions may explore avenues for expanding the current conceptual framework and evaluating its applicability across diverse and potentially unexpected domains, particularly within neural networks, value prediction, natural language processing, and adjacent fields, genetic algorithms and their modifications, classification, and multiclassification tasks, among others. As the next developmental phase of the proposed multidimensional space of events concept, the authors intend to construct a complex of additional practical observations and computational simulations, with particular emphasis on large-dimensional data structures.

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