2.6 Bestimmen Sie die lokalen Extrema der Funktionen und kontrollieren Sie ihre Ergebnisse mit *Python*.

(a) $f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$. (b) $f(x,y) = (4x^2+y^2)\cdot e^{-x^2-4y^2}$,

a) $f(x,y) = xy + \frac{1}{x} + \frac{1}{y} = xy + x^{-1} + y^{-1}$

$f_x(x,y) = y + (-x)^{-2} = y - \frac{1}{x^2}$ Fall A

$f_y(x,y) = x + (-y)^{-2} = x - \frac{1}{y^2}$ Fall B

$f_{xx}(x,y) = 2x^{-3} = \frac{2}{x^3}$ $\rightarrow f_{xx}(1,1) = \frac{2}{1^3} = 2$

$f_{yy}(x,y) = 2y^{-3} = \frac{2}{y^3}$ $f_{yy}(1,1) = \frac{2}{1^3} = 2$

$f_{xy}(x,y) = 1$ $f_{xy}(1,1) = 1$

$f_{yx}(1,1) = 1$

Fall A und Fall B

$y - \frac{1}{x^2} = 0 \rightarrow y = \frac{1}{x^2}$

$x - \frac{1}{y^2} = 0$

$x - \frac{1}{\frac{1}{x^2}} = 0 \rightarrow x - x^2 = 0 \rightarrow x = x^2 \rightarrow x = 1$ $K_1(1|1)$

$y - \frac{1}{1^2} = 0 \rightarrow y - 1 = 0 \rightarrow y = 1$

Hessenmatrix

$f_{xx} < 0 \rightarrow$ Max

$f_{xx} > 0 \rightarrow$ Min

$$H_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$\hat{=}$ $\text{Det}_{H_f} = 2\cdot 2 - 1\cdot 1 = 3 \rightarrow 3 > 0$

Hessenmatrix posit $\rightarrow$ Minimum

$$f(x,y) = (4x^2 + y^2) \cdot e^{-x^2 - 4y^2}$$

$$f_x(x,y) = 8x \cdot e^{-x^3 - 4y^3} + (4x^2+y^2) \cdot e^{-x^2 - 4y^2} \cdot (-2x)$$

$$= \underbrace{e^{-x^2 - 4y^2}}_{e>0} \left(8x + (4x^2+y^2) \cdot (-2x)\right)$$

$$f_y(x,y) = 2y \cdot e^{-x^2 - 4y^2} + (4x^2+y^2) \cdot e^{-x^2 - 4y^2} \cdot (-8y)$$

$$= \underbrace{e^{-x^2 - 4y^2}}_{e>0} \left(2y + (4x^2+y^2) \cdot (-8y)\right)$$

Fall A $(f_x(x,y))$

$x = 0 \rightarrow 8 \cdot 0 + (4 \cdot 0 + y^2) \cdot 0 = 0$

Fall B $(f_x(x,y))$

$$8x + (4x^2+y^2) \cdot (-2x) = 0$$

Fall C $(f_y(x,y))$

$y = 0 \rightarrow 2 \cdot 0 + (4x^2 + 0^2) \cdot 0 = 0$

Fall D $(f_y(x,y))$

$$2y + (4x^2+y^2) \cdot (-8y) = 0$$

Fall A und Fall C

$x = 0$

$y = 0$

$K_1(0|0)$

Fall A und Fall D

$x = 0$

$2y + (4x^2 + y^2) \cdot (-8y) = 0$

$2y + (4 \cdot 0 + y^2) \cdot (-8y) = 0$

$2y - 8y^3 = 0 \quad | : y$

$2 - 8y^2 = 0$

$2 = 8y^2 \quad | : 8$

$\frac{2}{8} = y^2 \quad | \sqrt{\ }$

$0{,}5 = \sqrt{\frac{2}{8}} = y_1$

$-0{,}5 = \sqrt{\frac{2}{8}} = y_2$

$2 \cdot (-\sqrt{\frac{2}{8}}) + (\cancel{0} + (-\sqrt{\frac{2}{8}}) \cdot$

$K_2(0|0{,}5)$

$K_3(0|-0{,}5)$

Fall B und Fall C

$8x + (4x^2 + y^2) \cdot (-2x) = 0$

$y = 0$

$8x + (4x^2 + 0^2) \cdot (-2x) = 0$

$8x + (-8x^3) = 0$

$8x - 8x^3 = 0 \quad | : 8x$

$1 - x^2 = 0$

$1 = x^2 \rightarrow x_1 = \sqrt{1} = 1$

$x_2 = -\sqrt{1} = -1$

$K_4 = (1|0)$

$K_5 = (-1|0)$

Fall B und Fall D

$$8x + (4x^2+y^2)^2 \cdot (-2x) = 0$$

$$8x + (-8x^3 - 2xy^2 \qquad = 0 \quad |:x$$

$$8 - 8x^2 - 2y^2 = 0 \quad |:2$$

$$4 - 4x^2 - y^2 =$$

$$4 - y^2 = 4x^2 \quad |:4 \quad |\sqrt{\ }$$

$$\sqrt{1 - \frac{1}{4}y^2} = x$$

$$2y + (4x^2+y^2) \cdot (-8y) = 0$$

$$2y + (-32x^2y - 8y^3 = 0 \quad |:y$$

$$2 - 32x^2 - 8y^2 = 0$$

$$2 - 32 \cdot \left(\sqrt{1-\frac{1}{4}y^2}\right)^2 - 8y^2 = 0$$

$$2 - 32\left(1 - \frac{1}{4}y^2\right) - 8y^2 = 0$$

$$2 - 32 + \frac{\cancel{32}^{\,8}}{\cancel{4}} y^2 - 8y^2 = 0$$

$$-30 = 0 \quad ↯$$

Check Kandidaten:

$$f_x(x,y) = 8x \cdot e^{-x^3-4y^3} + (4x^2+y^2)\cdot e^{-x^2-4y^2}\cdot(-2x)$$

$$= \underbrace{e^{-x^2-4y^2}}_{e>0}\left(8x + (4x^2+y^2)\cdot(-2x)\right) = e^{-x^2-4y^2}\cdot(8x-8x^3-2xy)$$

$$f_y(x,y) = 2y\cdot e^{-x^2-4y^2} + (4x^2+y^2)\cdot e^{-x^2-4y^2}\cdot(-8y)$$

$$= \underbrace{e^{-x^2-4y^2}}_{e>0}\left(2y+(4x^2+y^2)\cdot(-8y)\right) = e^{-x^2-4y^2}\cdot(2y-32x^2y-8y^3)$$

$$f_{xx}(x,y) = -2x e^{-x^2-4y^2}\cdot(8x-8x^3-2xy) + e^{-x^2-4y^2}\cdot(8-8x^2-2y) =$$

$$f_{yy}(x,y) = -8y^2 e^{-x^2-4y^2}\cdot(2y-32yx^2-8y^3) + e^{-x^2-4y^2}(2-32x^2-24y^2) = e^{-x^2-4y^2}\left(-8y^2\cdot(2y-32x^2-8y^3) + (2-32x^2-24y^2)\right)$$

$$= e^{-x^2-4y^2}\cdot\left(64y^3+256x^2y^2-16y^2-32\cancel{x^2}-24y^2+2\right)$$

$$= e^{-x^2-4y^2}\cdot\left(64y^3+256y^2x^2-40y^2-32\cancel{x}^2+2\right)$$

$$f_{xy} = e^{-x^2-4y^2}\cdot\left(64x^3y^2+16xy^4-64xy^2-4xy\right)$$

K	X	Y	f_{xx}	f_{yy}	f_{xy}	Det	Typ
1	0	0	8	2	0	16	min
2	0	$\frac{1}{2}$	$\frac{7{,}5}{e}$	0	0	0	?
3	0	$-\frac{1}{2}$	$\frac{7{,}5}{e}$	$-\frac{16}{e}$	0	$-\frac{120}{e}$	Max
4	1	0	$-\frac{16}{e}$	-30	0	$\frac{480}{e}$	Min
5	-1	0	$-\frac{16}{e}$	-30	0	$\frac{480}{e}$	Min