3.2 Berechnen Sie die Doppelintegrale

(a) $\int_0^2 \int_1^2 \left(x - 3y^2\right) dy\, dx$

(b) $\int_0^2 \int_0^\pi r \sin^2 \theta d\theta\, dr$

(c) $\int_0^\pi \int_1^2 y \sin(xy)\, dx\, dy$

(d) $\int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx$

Anleitung: Lösen Sie die letzten beiden Doppelintegrale in der angegebenen Reihenfolge, wobei das letzte Doppelintegral durch zweimaliges partielles Integrieren gelöst werden kann.

a) $\int_0^2 \int_1^2 (x-3y^2)\, dy\, dx = \int_0^2 xy - \frac{\cancel{3}y^3}{\cancel{3}}\, dx \Big|_1^2$

$= \int_1^2 2x - 2^3 - (x - 1^3)\, dx = \int_0^2 x - 7\, dx$

$= \frac{x^2}{2} - 7x \Big|_0^2 = \frac{2^2}{2} - 7 \cdot 2 = 2 - 14 = -12$

b) $\int_0^2 \int_0^\pi r \sin^2(\theta)\, d\theta\, dr$

$\int_0^2 \int_0^\pi r \cdot \left(\frac{1 - \cos(\theta)}{2}\right) d\theta\, dr$

$\int_0^1 r \cdot \left(\frac{\theta}{2} - \frac{\sin(\theta)}{4}\right) \Big|_0^\pi$

V	D	I
$1 \cdot t$	r	$\sin^2\theta = \sin\theta \cdot \sin\theta$
	θ	

$\sin^2 x + \cos^2 = 1$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

$\cos^2(x) = \cos^2(x) + \sin^2(x)$

$\sin^2(x) + \cos(2x) + \sin^2(x) = 1$

$2\sin^2(x) = 1 - \cos(2x)$

$$\sin^2(x) = \frac{1-\cos(x)}{2}$$

$$\int_0^2 r \cdot \left[\frac{\pi}{2} - \frac{\sin(\pi)}{4} - \left(0 - \frac{\sin(0)}{4}\right) \right]$$

$$\int_0^2 r \cdot \frac{\pi}{2} \, dr = \frac{r^2 \pi}{4} \Big|_0^2$$

$$= \frac{4\pi}{4} - 0 = \pi$$

c) $\displaystyle\int_0^\pi \int_1^2 y \sin(xy)\, dx\, dy$ $=$

V	D	I
+	y	$\sin(xy)$
−	0	$\frac{1}{y} \cdot (-\cos(x$

$$= \int_0^\pi -\cancel{y}\,\frac{\cos(xy)}{\cancel{y}} \Bigg|_1^2$$

$$= \int_0^\pi -\cos(2y) + \cos(1y)\, dy$$

$$= -\frac{\sin(2y)}{2} + \frac{\sin(1y)}{1} \Bigg|_0^\pi$$

$$= -\frac{\overset{0}{\sin(2\pi)}}{2} + \frac{\overset{0}{\sin(\pi)}}{1} - \left(-\frac{\overset{0}{\sin(2\cdot 0)}}{2} + \frac{\overset{0}{\sin(1\cdot 0)}}{1}\right)$$

$$= 0$$

d) $\displaystyle\int_1^2\int_0^{\pi} y\sin(xy)\,dy\,dx$

V	D	I
+	y	$\sin(xy)$
−	1	$-\frac{\cos(xy)}{x}$
+	0	$-\frac{\sin(xy)}{x^2}$

$$= \int_1^2 -\frac{y\cos(xy)}{x} + \frac{\sin(xy)}{x^2}\Bigg|_0^{\pi}$$

$$= \int_1^2 -\frac{\pi\cos(x\pi)}{x} + \frac{\sin(x\pi)}{x^2} - \left(\frac{\overset{0}{0\cdot\cos(0)}}{x} + \frac{\overset{0}{\sin(0\cdot x)}}{x^2}\right)$$

$$= \int_1^2 -\frac{\pi\cos(x\pi)}{x} + \frac{\sin(x\pi)}{x^2}\,dx$$

$f(x)\cdot g'(x)$
$= f(x)\cdot g(x)$
$- \int f'(x)\cdot g(x)$

$$= \underbrace{\int_1^2 \frac{\sin(x\pi)}{x^2}}_{1} - \pi\underbrace{\int_1^2 \frac{\cos(x\pi)}{x}}_{2}$$

↳ Partiell ~~Int~~ Int.

$$\int f'g = fg - \int f g'$$

$\cos(x\pi)x^{-1}$

$f' = \cos(\pi x)$ $\quad g = \frac{1}{x}$

$f = \frac{\sin(\pi x)}{\pi}$ $\quad g' = -\frac{1}{x^2}$

$$= \int_1^2 \frac{\sin(x\pi)}{x^2} - \pi\left[\overset{fg}{\frac{\sin(\pi x)}{\pi x}} - \int_1^2 -\overset{fg'}{\frac{\sin(\pi x)}{\pi x^2}}\right]$$

$$= \int_1^2 \frac{\sin(x\pi)}{x^2} - \frac{\pi \overset{fg}{\sin(\pi x)}}{\pi x} \overset{*}{+} \pi \int_1^2 -\overset{fg'}{\frac{\sin(\pi x)}{\pi x^2}}$$

$$= \int_1^2 \frac{\sin(x\pi)}{x^2} - \frac{\pi \overset{f}{\sin(\pi x)}}{\pi} \cdot \overset{g}{\left(\frac{1}{x}\right)} + \int_1^2 -\overset{fg'}{\frac{\cancel{\pi}\sin(\pi x)}{\cancel{\pi} x^2}}$$

heben sich auf

$$= - \frac{\pi \sin(\pi x)}{\pi x} \Big|_1^2$$

$$= 0 - 0 = 0$$