KNOT GROUPS

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1. Introduction

Knot theory is the mathematical study of knots, simple closed polygonal curves in $\mathbb{R}^3$. This subject has a rich history partly due to the fact that it has applications to various other fields such as chemistry, biology, and physics. The mathematical study of knots was said to be introduced by Gauss in the 1830s. The first attempts at classification of knots types were then made around 40 years later. Lord Kelvin's theory of atoms, which stated that there was a relationship between chemical properties of elements and knotting that occurred in the atoms, motivated Tait, a mathematical physicist, to create a list of all knots that could be drawn with a few number of crossings. However, the formal mathematics addressing this study was not developed yet so proving two knots were different was mostly based on empirical evidence. There was a breakthrough during the early 1900s when mathematicians such as Dehn and Alexander, amongst many others, introduced algebraic methods to classify knots. For instance, they studied knots using knot groups and homological invariants of the knot complement such as the Alexander polynomial. The Alexander polynomial was introduced in the 1920s by Alexander, and this invariant has close connections with the fundamental group; in fact, it can be calculated using the knot group with a tool called Fox calculus. By the 1930s, knot theory was well developed and Reidemeister published his first book about knots.

2. Knot groups

The fundamental group, introduced by Poincaré in 1895 in his paper "Analysis Situs," is a topological invariant that measures loops with fixed basepoints in a space. The knot group of a knot K, denoted $\pi_1(S^3 \setminus K)$, is the fundamental group of the knot complement, and it can be expressed using the Wirtinger presentation, a presentation with relations of the form $wg_iw^{-1} = g_j$ where w is word consisting of g_k's.

Given a knot diagram with c crossings, suppose we traverse an arc from an undercrossing to the next undercrossing and label each of these arcs. We then see that the knot diagram has c arcs. Intuitively, the generators of the knot group represent loops around the c arcs, and each crossing gives rise to a relation of the form $t_i = wt_jw^{-1}$. Therefore, we have

$$\pi_1(S^3 \setminus K) = \langle t_1, \ldots, t_c | r_1, \ldots, r_c \rangle$$

with r_i being $t_i = wt_jw^{-1}$ like described earlier. It was proved that one of the relations is redundant so knot groups can be simplified. The deficiency of a group presentation is the number of generators minus the number of relations, and the Wirtinger presentation can partly help us see that every (tame) knot group has deficiency one.

Since knot groups are knot invariants, we know every knot group associated to the same knot must be isomorphic. Then, an interesting problem concerning knot groups is determining the isomorphism between knot groups coming from different knot projections of the same knot. Given two finite presentations, there is no specific algorithm telling us if these groups are isomorphic. Tietze first decribed the group isomorphism problem in 1908, which asked if there was a way to determine whether two groups were isomorphic. Around fifty years later, Adian and Rabin independently proved the group isomorphism problem is unsolvable, meaning there does not exist an algorithm solving every instance of this problem.

For this project, we focus on the following problem on page 14 from [4].

Problem 1 (L. Moser)**.** Is there a geometric characterization of knots whose groups have one relator?

Knot groups are finitely generated and a simple way to measure the complexity of the group is to look at the number of generators. We list some examples of knot groups below.

Example 1.
Unknot: $\pi_1(S^3 \setminus K) = \mathbb{Z}$
Trefoil: $\pi_1(S^3 \setminus K) = \langle a, c | aca = cac \rangle$
Figure-eight: $\pi_1(S^3 \setminus K) = \langle a, c | c^{-1}a^{-1}cac^{-1} = a^{-1}cac^{-1}a^{-1} \rangle$

Note the knot group of the trefoil, for example, only has two generators even though the trefoil has three crossings since one relation is always redundant. In the late '50s, Papakyriakopoulos proved Dehn's lemma, which states that only the unknot has a knot group generated by one element. Therefore, using the above logic, the simplest class of nontrivial knots is thought to have two generators in the knot group. Problem 1 arises due to the fact that it can be difficult to determine the rank of a group as well as recognize a two-generator knot.

3. Tunnels

According to the update given in Kirby's list, the answer to Problem 1 is conjectured to be tunnel one knots. We first use the following fact presented in [6]. For any knot K in S^3, there is a collection of arcs $\{\gamma_1, \ldots, \gamma_k\}$ such that $(\bigcup \gamma_i) \cap K = \partial (\bigcup \gamma_i)$ and the exterior of $K \cup (\bigcup \gamma_i)$ is homeomorphic to a genus $k+1$ handlebody. Each arc in this system is called the tunnel of K, and the tunnel number of K is the minimum number of disjoint tunnels needed to make the exterior manifold a handlebody. We now briefly explain why the fact stated earlier is true. Consider the projection of K with c crossings, and draw tunnels $\gamma_1, \ldots, \gamma_c$ at each crossing. Then, $N(K \cup \gamma_1 \cup \cdots \cup \gamma_c)$ is isotopic to a standard genus $c+1$ handlebody in S^3 due to a deformation retraction. By the theory of Heegaard splittings, we conclude the complement must also be a genus $c+1$ handlebody. We show an example of how to get a standard genus 4 handlebody with the trefoil.

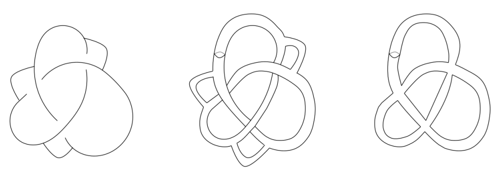

Figure 1. Standard genus 4 handlebody

Dually, Bleiler [1] explained that the exterior of a tunnel n knot can be constructed by attaching n two-handles to a genus $n+1$ handlebody. This space then deformation retracts onto a wedge of $n+1$ circles with n disks attached so the fundmental group of a tunnel n knot is generated by $n+1$ elements and given by n relations. For the case of a tunnel one knot, this means we attach one two-handle to a genus 2 handlebody. A genus 2 handlebody does indeed deformation retract onto a wedge of 2 circles, and the single two-handle contributes to the disk that is attached to the wedge sum. Hence, the rank of the fundamental group of a tunnel one knot is two.

We now describe the tunnel numbers for a few knots.

Example 2.
i. The unknot has tunnel number 0. The thickened unknot is a solid torus, and we know S^3 can be constructed using two solid tori. Thus, the complement of the thickened unknot is also a solid torus, which is a genus 1 handlebody.
ii. In Figure 1, we have three tunnels for the trefoil but it turns out this is not the tunnel number. We can draw less arcs and still obtain a handlebody as shown below.

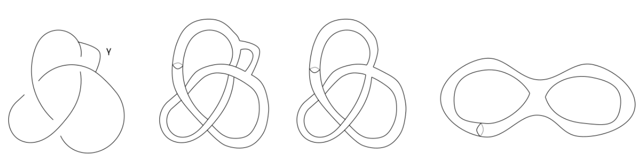

Figure 2.

The arc γ in the figure is a tunnel. The space $N(K \cup \gamma)$ is isotopic to a standard genus 2 handlebody so the exterior of this neighborhood is also a genus 2 handlebody. Therefore, the trefoil knot has tunnel number 1.

We observe that Problem 1 has connections to Heegaard splittings. In fact, it is stated that this problem is a special case of the following question: when does the minimal number of generators for the fundamental group of a 3-manifold equal the minimal genus of a Heegaard splitting? As a very simple example, let's consider S^3. The easiest way to construct this space is by gluing together the boundaries of two 3-balls. This means we have a genus 0 splitting. The fundamental group of S^3 is trivial so we see the question vacuously holds for this case. Since I do not know much about Heegaard splittings, I will not elaborate any further.

4. State of the problem

Before describing some specific updates, we mention two remarks given in [4]. First, we note that the knot group of a two-bridge knot is generated by two elements, one of which is a meridian, and expressed by one relator. In addition, the knot group of a (p, q) torus knot is given by $\langle x, y | x^p = y^q \rangle$. Therefore, two-bridge knots and (p, q) torus knots are two-generator.

We know tunnel one knots are two-generator, but what about the converse? In 1984, Scharlemann conjectured that two-generator knots are tunnel one, and ten years later, Bleiler [1] proved this conjecture for two-generator cable knots, which are knots inside a torus. More specifically, let K_1 be a knot inside an unknotted solid torus. If we knot the solid torus in the shape of a second knot K_2, then this will alter K_1 to a new knot K_3 called a satellite knot. The knot K_2 is called the companion knot. Furthermore, if K_1 is a (p, q) torus knot, then we call the resulting knot K_3 a (p, q) cable. Bleiler proved two-generator cable knots are tunnel one using Morimoto and Sakuma's classification of tunnel one satellite knots given in 1991. He wrote the tunnels could be constructed directly from the knot descriptions. To prove the converse, he used Dehn surgery and some other tools on a given tunnel one (p, q) cable to prove they are two-generator.

In 1997, Bleiler and Jones [2] classified satellite knots with two-generator knot groups satisfying the property that one of the generators is represented by a meridian. We first go over some terms before stating two corollaries given in the paper. An embedded arc α in a handlebody is said to be trivial if there exists another arc β in the boundary of the handlebody whose endpoints agree with α such that $\alpha \cup \beta$ bounds a disk in the handlebody. A knot in a 3-manifold has a (g, b)-decomposition if the knot intersects the closure of each complementary handlebody to a Heegaard surface of genus g in b trivial arcs. For our purpose, we focus on knots with $(1, 1)$-decompositions so the knot intersects the closure of each complementary handlebody to a Heegaard surface of genus 1 in one trivial arc. These knots are referred to as one bridge with respect to a torus. An element of $\pi_1(S^3 \setminus K)$ is meridional if the element can be represented by a simple closed curve in the boundary, and the knot exterior has a (g, b)-presentation if the fundamental group has $g + b$ generators, b of which are meridional. A knot exterior then has a $(1, 1)$-presentation when the fundamental group has 2 generators, one of which is meridional. We now state the corollaries.

Corollary 1 (Corollary 3.2 from [2])**.** A $(1, 1)$-presented satellite knot in S^3 that is one-bridge with respect to a torus is tunnel one.

Corollary 2 (Corollary 3.3 from [2])**.** A two-generator satellite knot with no $(1, 1)$-presentation has tunnel number at least two.

These two corollaries follow from a theorem given in the same paper, which states that the exterior of a $(1, 1)$-presented satellite knot in S^3 decomposes as the union along a torus of a simple link exterior X_A and torus knot exterior X_B. In addition, X_A embeds in S^3 as the complement of a two bridge link and is attached to X_B such that the meridian of the two-bridge link is identified with the Seifert fiber (I googled this but I still do not understand what it is.) of X_B. In particular, the second corollary tells us there are certain two-generator (satellite) knots that are not tunnel one.

Knot groups provide us with an algebraic method to classify knots, and Problem 1 allows us to examine the relationship between this algebraic technique and geometric properties of knots.

References

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