LATTICES WITH MANY BORCHERDS PRODUCTS

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ABSTRACT. We prove that there are only finitely many isometry classes of even lattices L of signature $(2, n)$ for which the space of cusp forms of weight $1 + n/2$ for the Weil representation of the discriminant group of L is trivial. We compute the list of these lattices. They have the property that every Heegner divisor for the orthogonal group of L can be realized as the divisor of a Borcherds product. We obtain similar classification results in greater generality for finite quadratic modules.

1. INTRODUCTION

Let L be an even lattice of signature $(2, n)$ and write $\mathrm{O}(L)$ for its orthogonal group. In his celebrated paper [Bo1] R. Borcherds constructed a map from vector valued weakly holomorphic elliptic modular forms of weight $1 - n/2$ to meromorphic modular forms for $\mathrm{O}(L)$ whose zeros and poles are supported on Heegner divisors. Since modular forms arising in this way have particular infinite product expansions, they are often called *Borcherds products*. They play important roles in different areas such as Algebraic and Arithmetic Geometry, Number Theory, Lie Theory, Combinatorics, and Mathematical Physics.

By Serre duality, the obstructions for the existence of weakly holomorphic modular forms with prescribed principal part at the cusp at ∞ are given by vector valued cusp forms of dual weight $1 + n/2$ transforming with the Weil representation associated with the discriminant group of L [Bo2]. In particular, if there are no non-trivial cusp forms of this type, then there are no obstructions, and every Heegner divisor is the divisor of a Borcherds product. A lattice with this property is called *simple*.

It was conjectured by the third author that there exist only finitely many isomorphism classes of such simple lattices. Under the assumptions that $n \geq 3$ and that the Witt rank of L is 2, it was proved by M. Bundschuh that there is an upper bound on the determinant of a simple lattice [Bu]. Unfortunately, this bound is very large and therefore not feasible to obtain any classification results. The argument of [Bu] is based on volume estimates for Heegner divisors and the singular weight bound for holomorphic modular forms for $\mathrm{O}(L)$.

The purpose of the present paper is twofold. First, we show that for any $n \geq 1$ (without any additional assumption on the Witt rank) there exist only finitely many isomorphism classes of even simple lattices of signature $(2, n)$, see Theorem 4.5 and Corollary 4.7. Second, we develop an efficient algorithm to determine all of these lattices. It turns out that there are exactly 362 isomorphism classes. Table 1 shows how many of those occur in the different

signatures. The corresponding genus symbols (see Section 2) of these lattices are listed in Tables 7, 8, 9[1].

TABLE 1.   Number of simple lattices of signature $(2, n)$.

n	1	2	3	4	5	6	7	8	9	10	$11 \leq n \leq 17$	18	$19 \leq n \leq 25$	26	$n > 26$
#	256	67	15	5	3	4	2	3	3	2	0	1	0	1	0

It is also interesting to record the Witt ranks of these lattices. For $n = 1$, we have 26 anisotropic lattices. The corresponding modular varieties are Shimura curves while the remaining 230 modular varieties for $n = 1$ are modular curves. For $n = 2$, there are 24 of Witt rank 1 and 43 of Witt rank 2 but no anisotropic lattices. Finally, if $n \geq 3$, all simple lattices have Witt rank 2.

Along the way, we obtain several results on modular forms associated with finite quadratic modules which are of independent interest and which we now briefly describe. A finite quadratic module is a pair consisting of a finite abelian group A together with a $\mathbb{Q}/\mathbb{Z}$-valued non-degenerate quadratic form Q on A, see [Ni], [Sk2]. Important examples of finite quadratic modules are obtained from lattices. If L is an even lattice with dual lattice L', then the quadratic form on L induces a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form on the discriminant group L'/L.

Recall that there is a Weil representation ρ_A of the the metaplectic extension $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[A]$ of a finite quadratic module A, see Section 3. If $k \in \frac{1}{2}\mathbb{Z}$, we write $S_{k,A}$ for the space of cusp forms of weight k and representation ρ_A for the group $\mathrm{Mp}_2(\mathbb{Z})$. We say that a finite quadratic module A is k-simple if $S_{k,A} = \{0\}$. With this terminology, an even lattice L is simple if and only if L'/L is $(1 + n/2)$-simple.

The dimension of the space $S_{k,A}$ can be computed by means of the Riemann-Roch theorem. Therefore a straightforward approach to showing that there are nontrivial cusp forms consists in finding lower bounds for the dimension of $S_{k,A}$. Unfortunately, the dimension formula (3.3) involves rather complicated invariants of ρ_A at elliptic and parabolic elements, and it is a non-trivial task to obtain sufficiently strong bounds. In the present paper we resolve this problem. For instance, we obtain the following result (see Theorem 4.5 and Corollary 4.6).

Theorem. *If $\varepsilon > 0$, then*

$$\dim(S_{k,A}) - \dim(M_{2-k,A(-1)}) = |A/\{\pm 1\}| \cdot \left(\frac{k-1}{12} + O_\varepsilon(N_A^{\varepsilon - 1/2}) \right)$$

for every finite quadratic module A and every weight $k \geq 3/2$ with $2k \equiv -\mathrm{sig}(A)$ (mod 4). Here N_A is the level of A, and $A(-1)$ denotes the abelian group A equipped with the quadratic form $-Q$. The constant implied in the Landau symbol is independent of A and k.

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[1]Tables with global realizations can be obtained from [Ehl].

In Corollary 4.7 we conclude that there exist only finitely many isomorphism classes of finite quadratic modules A with bounded number of generators such that $S_{k,A} = \{0\}$ for some weight satisfying the condition of the theorem. In particular, there are only finitely many isomorphism classes of simple lattices. Note that there do exist infinitely many isomorphism classes of 1/2-simple finite quadratic modules, see Remark 4.8.

Since the constant implied in the Landau symbol in the above theorem is large, it is a difficult task to compute the list of all k-simple finite quadratic modules for a bounded number of generators. We develop an efficient algorithm to address this problem. The idea is to first compute all *anisotropic* finite quadratic modules that are k-simple for some k. To this end we derive an explicit formula for $\dim(S_{k,A})$ in terms of class numbers of imaginary quadratic fields and dimension bounds that are strong enough to obtain a classification (Theorem 4.10 and Table 5).

Next we employ the fact that an arbitrary finite quadratic module A has a unique anisotropic quotient A_0, and that there are intertwining operators for the corresponding Weil representations. For the difference $\dim S_{k,A} - \dim S_{k,A_0}$ very efficient bounds can be obtained. This can be used to classify all k-simple finite quadratic modules with a bounded number of generators, see Algorithm 5.5 and the tables in Section 5.

To resolve the problem of finding all simple *lattices* of signature $(2, n)$, it remains to test which of these simple discriminant forms arise as discriminant groups L'/L of even lattices L of signature $(2, n)$. This is done in Section 6 by applying a criterion of [CS]. Finally, in Section 6.1 we explain some applications of our results in the context of Borcherds products.

2. Finite quadratic modules

Let (A, Q) be a finite quadratic module (also called a finite quadratic form or discriminant form in the literature), that is, a pair consisting of a finite abelian group A together with a $\mathbb{Q}/\mathbb{Z}$-valued non-degenerate quadratic form Q on A. We denote the bilinear form corresponding to Q by $(x, y) = Q(x + y) - Q(x) - Q(y)$. Recall that Q is called degenerate if there exists an $x \in A \setminus \{0\}$, such that $(x, y) = 0$ for all $y \in A$. Otherwise, Q is called non-degenerate.

The morphisms in the category of finite quadratic modules are group homomorphisms that preserve the quadratic forms. In particular, two finite quadratic modules (A, Q_A) and (B, Q_B) are isomorphic if and only if there is an isomorphism of groups $\varphi : A \to B$, such that $Q_B \circ \varphi = Q_A$.

In this section we collect some important facts about finite quadratic modules, which are well known among experts but not easily found in the literature. We mainly follow Skoruppa [Sk3]. Other good references include [Ni], [No], [Sk2], [Str].

If L is an even lattice, the quadratic form Q on L induces a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form on the discriminant group L'/L of L. The pair $(L'/L, Q)$ defines a finite quadratic module, which we call the *discriminant module* of L. According to [Ni], any finite quadratic module

can be obtained as the discriminant module of an even lattice L. If (b^+, b^-) then denotes the real signature of L, the difference $b^+ - b^-$ is determined by its discriminant module A modulo 8 by Milgram's formula

$$\frac{1}{\sqrt{|A|}} \sum_{a \in A} e(Q(a)) = e((b^+ - b^-)/8).$$

Here and throughout we abbreviate $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$. We call $\operatorname{sig}(A) := b^+ - b^- \in \mathbb{Z}/8\mathbb{Z}$ the *signature* of A.

We let N be the level of A defined by

$$N = \min\{n \in \mathbb{Z}_{>0} \mid nQ(x) \in \mathbb{Z} \text{ for all } x \in A\}.$$

It is a divisor of $2|A|$.

If (A, Q_A) and (B, Q_B) are two finite quadratic modules then the *orthogonal direct sum* $(A \oplus B, Q_A + Q_B)$ also defines a finite quadratic module. Here $(Q_A + Q_B)(a+b) = Q_A(a) + Q_A(b)$ for $a \in A$ and $b \in B$. We call a finite quadratic module *indecomposable* if it is not isomorphic to such a direct sum with non-zero A and B.

The finite quadratic module A is isomorphic to the orthogonal sum of its p-components $A_p = A \otimes_{\mathbb{Z}} \mathbb{Z}_p$ with p running through the primes.

Next we describe a list of indecomposable finite quadratic modules.

Definition 2.1. Let p be a prime and t be an integer not divisible by p. We define the following elementary finite quadratic modules.

$$\mathcal{A}_{p^k}^t = \left(\mathbb{Z}/p^k\mathbb{Z}, \frac{tx^2}{p^k} \right) \text{ for } p > 2,$$

$$\mathcal{A}_{2^k}^t = \left(\mathbb{Z}/2^k\mathbb{Z}, \frac{tx^2}{2^{k+1}} \right),$$

$$\mathcal{B}_{2^k} = \left(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \frac{x^2 + 2xy + y^2}{2^k} \right),$$

$$\mathcal{C}_{2^k} = \left(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \frac{xy}{2^k} \right),$$

Theorem 2.2.

(1) *The finite quadratic modules listed in Definition 2.1 are indecomposable.*
(2) *Every indecomposable finite quadratic module is isomorphic to a finite quadratic module as in Definition 2.1.*
(3) *Moreover, every finite quadratic module is isomorphic to a direct sum of indecomposable finite quadratic modules.*

Proof. Statement (1) is clear. The other two statements follow from the classification of p-adic lattices. See [Ni] for details, in particular Proposition 1.8.1. $\square$

Consider a decomposition of the p-components of a finite quadratic module A as a direct sum

$$A_p = A_{p,1} \oplus \ldots \oplus A_{p,l_p},$$

where each $A_{p,i}$ is a direct sum of elementary finite quadratic modules $\mathcal{A}_{q_i}^{t_i}, \mathcal{B}_{q_i}, \mathcal{C}_{q_i}$, with $q_i = p^{r_i}$, $r_i \geq 1$ and $q_1 < \ldots < q_r$.

Such a decomposition is called a *Jordan decomposition* of A and the direct summands $A_{p,i}$ are called *Jordan components* of A. We will also call any finite quadratic module that is isomorphic to a direct sum of elementary finite quadratic modules $\mathcal{A}_q^t, \mathcal{B}_q, \mathcal{C}_q$ for a fixed q a Jordan component.

We will now describe a handy notation for such a Jordan decomposition. The symbols we use are essentially those introduced by Conway and Sloane [CS] for the genus of an integral lattice, that is, its class under rational equivalence. The following statement (see [Ni], Corollary 1.16.2) motivates the use of their symbols for us.

Proposition 2.3. *Two even lattices L and M that have the same real signatures have isomorphic finite quadratic modules if and only if L and M are in the same genus.*

The following two lemmas are straightforward to prove (see also [Ni], Proposition 1.8.2).

Lemma 2.4. *Let $p > 2$ be a prime and let $q = p^r$ for a positive integer r.*

(1) *We have $\mathcal{A}_q^t \cong \mathcal{A}_q^s$ if and only if $\left(\frac{s}{p}\right) = \left(\frac{t}{p}\right)$.*

(2) *Suppose that $\left(\frac{2t}{p}\right) = -1$. Then $\mathcal{A}_q^t \oplus \mathcal{A}_q^t \cong \mathcal{A}_q^1 \oplus \mathcal{A}_q^1$.*

(3) *In particular, if A is a Jordan component of the form*

$$A = \bigoplus_{i=1}^{n} \mathcal{A}_q^{t_i},$$

then A is isomorphic to

$$\mathcal{A}_q^t \oplus \bigoplus_{i=1}^{n-1} \mathcal{A}_q^1, \quad \text{with} \quad \prod_{i=1}^{n} \left(\frac{2t_i}{p}\right) = \left(\frac{2t}{p}\right).$$

Lemma 2.5. *Let $q = 2^r$ for a positive integer r. Moreover, let s, t be odd integers.*

(1) *We have $\mathcal{A}_2^s \equiv \mathcal{A}_2^t$ if and only if $s \equiv t \pmod 4$.*

(2) *If $r > 1$, then $\mathcal{A}_q^s \equiv \mathcal{A}_q^t$ if and only if $s \equiv t \pmod 8$.*

(3) *Let $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathbb{Z}$ such that*

$$\sum_{i=1}^{n} s_i \equiv \sum_{i=1}^{n} t_i \pmod 8 \quad \text{and} \quad \prod_{i=1}^{n} \left(\frac{s_i}{2}\right) = \prod_{i=1}^{n} t_i.$$

Then

$$\bigoplus_{i=1}^{n} \mathcal{A}_q^{s_i} \cong \bigoplus_{i=1}^{n} \mathcal{A}_q^{t_i}.$$

(4) *We have $\mathcal{B}_q \oplus \mathcal{B}_q \cong \mathcal{C}_q \oplus \mathcal{C}_q$.*

(5) *Moreover, we have $\mathcal{A}_q^t \oplus \mathcal{B}_q \cong \mathcal{A}_q^{t_1} \oplus \mathcal{A}_q^{t_2} \oplus \mathcal{A}_q^{t_3}$ with $t_1 + t_2 + t_3 \equiv t \pmod 8$ and*

$$\prod_{i=1}^{3} \left(\frac{t_i}{2}\right) = -\left(\frac{t}{2}\right).$$

(6) Finally, $\mathcal{A}_q^t \oplus \mathcal{C}_q \cong \mathcal{A}_q^{t_1} \oplus \mathcal{A}_q^{t_2} \oplus \mathcal{A}_q^{t_3}$ *with* $t_1 + t_2 + t_3 \equiv t \pmod 8$ *and*

$$\prod_{i=1}^3 \left(\frac{t_i}{2}\right) = \left(\frac{t}{2}\right).$$

Definition 2.6. Using the preceding lemmas, we define a symbol for a Jordan decomposition of a finite quadratic module as follows. First of all, by convention, we write 1^{+1} or 1^{-1} for the trivial module $A = \{0\}$ with the 0-map as quadratic form.

Now let A be a Jordan component, p be a prime and $q = p^r$.

(1) If p is odd, the two isomorphism classes of Jordan components in Lemma 2.4, (3) are denoted $q^{\pm n}$, where $\left(\frac{2t}{p}\right) = \pm 1$.

(2) Let $p = 2$.
  (a) We write $q_t^{\pm n}$ if A is isomorphic to $\mathcal{A}_q^{t_1} \oplus \ldots \oplus \mathcal{A}_q^{t_n}$ with $t_1 + \ldots + t_n \equiv t \pmod 8$ and $\left(\frac{t_1}{2}\right) \cdots \left(\frac{t_n}{2}\right) = \pm 1$. We normalize t to be contained in the set $\{1, 3, 5, 7\}$ and if $q = 2$, we take $t \in \{1, 7\}$.
  (b) We write q^{+2n} if A is isomorphic to n copies of $\mathcal{C}_q$.
  (c) And we write q^{-2n} if A is isomorphic to $n-1$ copies of $\mathcal{C}_q$ and one copy of $\mathcal{B}_q$.

For a general finite quadratic module, we concatenate the symbols of the Jordan components as defined above.

Example 2.7. The Jordan decomposition $\mathcal{A}_2^1 \oplus (\mathcal{A}_3^1 \oplus \mathcal{A}_3^1)$ has the symbol $2_1^{+1}3^{+2}$ and $2^{-2}4_3^{+3}3^{-1}$ is the symbol for the Jordan decomposition $\mathcal{B}_2 \oplus (\mathcal{A}_4^1 \oplus \mathcal{A}_4^1 \oplus \mathcal{A}_4^1) \oplus \mathcal{A}_3^1$.

Proposition 2.8. *Let A and B be finite quadratic modules and let $p > 2$ be a prime. If $A_p \cong B_p$, then the corresponding p-components of the genus symbols of A and B coincide.*

Proof. This follows from the uniqueness of the Jordan decomposition in the case of an odd prime p (see, for instance, Theorem 5.3.2 in [Ki]). $\qquad\square$

Remark 2.9. In contrast to Proposition 2.8, note that the symbol (and the Jordan decomposition) for $p = 2$ is not uniquely determined by the isomorphism class. For instance, the Jordan decompositions $2_1^{+1}4_1^{+1}$ and $2_7^{+1}4_3^{-1}$ correspond to isomorphic finite quadratic modules. See also Theorem 2.14 in the next section.

For an integer n we define the n-torsion subgroup of a finite quadratic module A by

$$A[n] = \{\gamma \in A \mid n\gamma = 0\}.$$

Moreover, we let A^n be the image of the multiplication by n map. Then we have the exact sequence

$$0 \longrightarrow A[n] \longrightarrow A \longrightarrow A^n \longrightarrow 0,$$

and A is the orthogonal sum of $A[n]$ and A^n. It follows from the theorem of elementary divisors that

(2.1) $$|A[n]| \leq 2n\frac{|A|}{N}.$$

The quantity $d = d_A := |A/\{\pm 1\}|$ can be expressed in term of the 2-torsion as

$$(2.2) \qquad d = \frac{|A|}{2} + \frac{|A[2]|}{2}.$$

2.1. **Gauss sums and divisor sums.** We now collect some facts about Gauss sums and divisor sums associated to finite quadratic modules which we will need later. For an integer $n \in \mathbb{Z}$ the Gauss sum $G(n, A)$ is defined by

$$(2.3) \qquad G(n, A) = \sum_{\gamma \in A} e(nQ(\gamma)).$$

We have the elementary properties

$$(2.4) \qquad\qquad G(-n, A) = \overline{G(n, A)},$$
$$(2.5) \qquad\qquad G(n + N, A) = G(n, A),$$
$$(2.6) \qquad\qquad G(n, A \oplus B) = G(n, A)G(n, B).$$

The following lemma is a consequence of [Bo3, Lemma 3.1].

Lemma 2.10. *If* $(n, N) = 1$ *then* $|G(n, A)| = \sqrt{|A|}$. *For general* n *we have the estimate*

$$|G(n, A)| \leq \sqrt{|A|}\sqrt{|A[n]|}.$$

We will also need explicit formulas for the Gauss sums in some cases in Section 4.1. These are easily proven by relating $G(n, A)$ to the standard Gauss sums (see, for instance [Str]).

Proposition 2.11. *Let* $p > 2$ *be a prime. We have* $G(n, p^{\pm 1}) = p$ *if* $p \mid n$ *and*

$$G(n, p^{\pm 1}) = \pm\sqrt{p}\left(\frac{p}{2}\right)\left(\frac{n}{p}\right)e\left(\frac{1-p}{8}\right)$$

if $(n, p) = 1$.

Proposition 2.12. *Let* $q = 2^r$. *For* $n \in \mathbb{Z}$ *we put* $n' = n/(n, q)$ *and* $q' = q/(n, q)$. *We have*

$$G(n, q_t^{\pm 1}) = \sqrt{q}\sqrt{|(n, q)|}\left(\frac{tn'}{q'}\right) \cdot \begin{cases} e\left(\frac{tn'}{8}\right), & \text{if } q \nmid n, \\ 0, & \text{if } q \mid\mid n, \\ 1, & \text{if } 2q \mid n. \end{cases}$$

Proposition 2.13. *Let* $q = 2^r$ *with* $r \geq 1$. *We have*

$$G(n, q^{-2}) = q(q, n)\left(\frac{3}{q'}\right)$$

and

$$G(n, q^{+2}) = q(q, n).$$

The following theorem is used later on to decide when two given genus symbols correspond to isomorphic finite quadratic modules.

Theorem 2.14 ([Sk3]). *Let A and B be finite quadratic modules. Then A and B are isomorphic if and only if their underlying abelian groups are isomorphic and*

$$G(n, A) = G(n, B)$$

for all divisors n of the level of A and B.

Proof. It is clear that the condition $G(n, A) = G(n, B)$ is necessary. Using this, it is easy to prove the theorem for p-components with $p > 2$. If A_p and B_p are finite quadratic modules of prime power level p^r with $G(p^i, A_p) = G(p^i, B_p)$ for all $i \in 0, \ldots, r$, then $A_p \cong B_p$ follows from Lemma 2.4 and the explicit formula for the Gauss sum in Proposition 2.11. The general case is treated in [Sk3] in detail. □

For $s \in \mathbb{C}$ we define a divisor sum $\sigma(s, A)$ associated to A by

$$(2.7) \qquad \sigma(s, A) = \sum_{a|N} a^s \sqrt{|A[a]|}.$$

Here the sum runs over all positive divisors a of the level N of A. If B is another finite quadratic module of level N' coprime to N, then

$$\sigma(s, A \oplus B) = \sigma(s, A)\sigma(s, B).$$

Consequently, $\sigma(s, A)$ is the product of the $\sigma(s, A_p)$ for p running through the primes dividing N.

Lemma 2.15. *For $s \in \mathbb{R}$ we have the estimate*

$$\sigma(s, A) \leq \sqrt{\frac{2|A|}{N}} \cdot \sigma_{s+1/2}(N),$$

where $\sigma_s(N) = \sum_{a|N} a^s$ denotes the usual divisor sum.

Proof. This is a direct consequence of the estimate (2.1). □

3. Vector valued modular forms

We write $\mathrm{Mp}_2(\mathbb{Z})$ for the metaplectic extension of $\mathrm{SL}_2(\mathbb{Z})$, realized as the group of pairs $(M, \phi(\tau))$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and ϕ is a holomorphic function on the upper complex half plane $\mathbb{H}$ with $\phi(\tau)^2 = c\tau + d$ (see e. g. [Bo1], [Br1]). It is well known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) \qquad \text{and} \qquad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right).$$

One has the relations $S^2 = (ST)^3 = Z$, where $Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right)$ is the standard generator of the center of $\mathrm{Mp}_2(\mathbb{Z})$.

The Weil representation associated with A is a unitary representation ρ_A of $\mathrm{Mp}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[A]$. If we denote the standard basis of $\mathbb{C}[A]$ by $(\mathfrak{e}_\gamma)_{\gamma \in A}$ then ρ_A can

be defined by the action of the generators $S, T \in \mathrm{Mp}_2(\mathbb{Z})$ as follows (see also [Sk2], [Bo1], [Br1], where the dual of ρ_A is used):

$$(3.1) \qquad \rho_A(T)\mathfrak{e}_\gamma = e(-Q(\gamma))\mathfrak{e}_\gamma,$$

$$(3.2) \qquad \rho_A(S)\mathfrak{e}_\gamma = \frac{e(\mathrm{sig}(A)/8)}{\sqrt{|A|}} \sum_{\delta \in A} e((\gamma,\delta))\mathfrak{e}_\delta.$$

Let $k \in \frac{1}{2}\mathbb{Z}$. We denote by $M_{k,A}$ the vector space of $\mathbb{C}[A]$-valued modular forms of weight k with representation ρ_A for the group $\mathrm{Mp}_2(\mathbb{Z})$. The subspace of cusp forms is denoted by $S_{k,A}$. It is easily seen that $M_{k,A} = 0$, if $2k \not\equiv \mathrm{sig}(A) \pmod 2$.

The dimension of the vector space $M_{k,A}$ can be computed using the Riemann-Roch theorem or the Selberg trace formula. This is carried out in [Fr] and [Fi] in a more general situation. In our special case the following formula holds (see [Bo2] p. 228 and [Fr] Chapter 8.5, Theorem 5.1). For simplicity we assume that $2k \equiv -\mathrm{sig}(A) \pmod 4$, since our application to simple lattices will only concern this case. Then the d-dimensional subspace $W = \mathrm{span}\{\mathfrak{e}_\gamma + \mathfrak{e}_{-\gamma};\ \gamma \in A\}$ of $\mathbb{C}[A]$ is preserved by ρ_A, and $\rho_A(Z)$ acts by multiplication with $e(-k/2)$ on W. We denote by ρ the restriction of ρ_A to W. If M is a unitary matrix of size d with eigenvalues $e(\nu_j)$ and $0 \le \nu_j < 1$ (for $j = 1, \ldots, d$), we define

$$\alpha(M) = \sum_{j=1}^d \nu_j.$$

If $k \ge 3/2$, the dimension of $M_{k,A}$ is given by

$$(3.3) \qquad \dim(M_{k,A}) = d + dk/12 - \alpha\left(e^{\pi i k/2}\rho(S)\right) - \alpha\left(\left(e^{\pi i k/3}\rho(ST)\right)^{-1}\right) - \alpha(\rho(T))$$
$$+ \dim(S_{2-k,A(-1)}).$$

Furthermore, the dimension of $S_{k,A}$ is given by

$$(3.4) \qquad \dim(S_{k,A}) = \text{first line of } (3.3) - |\{\gamma \in A/\{\pm 1\};\ Q(\gamma) \in \mathbb{Z}\}| + \dim(M_{2-k,A(-1)}).$$

Here $A(-1)$ denotes the finite quadratic module given by the abelian group A equipped with the quadratic form $-Q$. If $k > 2$, then $M_{2-k,A(-1)}$ vanishes. If $k = 2$, then $M_{0,A(-1)}$ is equal to the space of $\mathrm{Mp}_2(\mathbb{Z})$-invariants in $\mathbb{C}[A]$ for the dual representation of ρ_A. Finally, when $k = 3/2$, according to the Serre-Stark theorem, the space $M_{1/2,A(-1)}$ is generated by unary theta series. It was explicitly computed by Skoruppa in [Sk1] and [Sk2] as follows. For every non-zero $l \in \mathbb{Z}$ we write $V(l)$ for the finite quadratic module of level $4|l|$ given by $\mathbb{Z}/2l\mathbb{Z}$ equipped with the quadratic form $Q(x) = \frac{1}{4l}x^2$. Let ϵ be the automorphism of $V(l)$ given by multiplication by -1, and write $\mathbb{C}[V(l)]^\epsilon$ for the corresponding space of invariants. According to [Sk2, Theorem 8] we have

$$(3.5) \qquad M_{1/2,A(-1)} \cong \bigoplus_{\substack{l>0,\ 4l|N \\ N/4l \text{ squarefree}}} \left(\mathbb{C}[V(-l)]^\epsilon \otimes \mathbb{C}[A(-1)]\right)^{\mathrm{Mp}_2(\mathbb{Z})}.$$

Here the action of $\mathrm{Mp}_2(\mathbb{Z})$ on the tensor products on the right hand side is given by the Weil representation.

4. Dimension estimates

In this section we derive lower bounds for the dimension of $S_{k,A}$. In view of (3.3) and (3.4) we have to estimate the quantities

$$\alpha_1 := \alpha\left(e^{\pi i k/2}\rho(S)\right),$$
$$\alpha_2 := \alpha\left(\left(e^{\pi i k/3}\rho(ST)\right)^{-1}\right),$$
$$\alpha_3 := \alpha(\rho(T)),$$
$$\alpha_4 := \left|\{\gamma \in A/\{\pm 1\};\ Q(\gamma) \in \mathbb{Z}\}\right|.$$

We begin by recalling some trivial bounds from [Bu, Bemerkungen 2.2.1 and 2.2.5]. We have

$$\alpha_1 \leq \frac{1}{2}d, \qquad \alpha_2 \leq \frac{2}{3}d, \qquad \alpha_3 + \alpha_4 \leq d.$$

If we insert these bounds into (3.4) we obtain the following corollary.

Corollary 4.1. *If $k > 14$ and $2k \equiv -\operatorname{sig}(A) \pmod 4$, then $S_{k,A} \neq \{0\}$.*

Note that this bound on k is sharp, since there are no nontrivial scalar valued cusp forms of weight 14 for $\mathrm{SL}_2(\mathbb{Z})$.

To prove the existence of non-trivial cusp forms for smaller values of k by means of the dimension formula, we need much better estimates for the α_i. The quantities α_1 and α_2 can be expressed in terms of Gauss sums associated with A. By means of the estimate in Lemma 2.10, we obtain the following result (see Lemma 2 and Corollary 3 in [Br2]).

Lemma 4.2. *The quantities α_1 and α_2 satisfy the estimates*

$$(4.1) \qquad |\alpha_1 - d/4| \leq \frac{1}{4}\sqrt{|A[2]|},$$

$$(4.2) \qquad |\alpha_2 - d/3| \leq \frac{1}{3\sqrt{3}}\left(1 + \sqrt{|A[3]|}\right).$$

Lemma 4.3. *We have*

$$|\alpha_4| \leq \frac{|A[2]|}{2} + \frac{\sqrt{|A|}}{2}\sigma(-1, A),$$

where $\sigma(-1, A)$ is the divisor sum defined in (2.7).

Proof. We write α_4 as

$$\alpha_4 = \frac{1}{2}\sum_{\substack{\gamma \in A[2] \\ Q(\gamma) \in \mathbb{Z}}} 1 + \frac{1}{2}\sum_{\substack{\gamma \in A \\ Q(\gamma) \in \mathbb{Z}}} 1.$$

The second term on the right hand side is equal to

$$\frac{1}{2N}\sum_{\gamma \in A}\sum_{\nu\,(N)} e(Q(\gamma)\nu) = \frac{1}{2N}\sum_{\nu\,(N)} G(\nu, A).$$

Using Lemma 2.10, we obtain

$$|\alpha_4| \leq \frac{|A[2]|}{2} + \frac{1}{2N} \sum_{\nu \,(N)} \sqrt{|A|}\sqrt{|A[\nu]|}$$

$$\leq \frac{|A[2]|}{2} + \frac{\sqrt{|A|}}{2N} \sum_{a|N} \sum_{\substack{\mu \,(N/a) \\ (\mu,N/a)=1}} \sqrt{|A[a\mu]|}$$

$$\leq \frac{|A[2]|}{2} + \frac{\sqrt{|A|}}{2N} \sum_{a|N} \frac{N}{a} \sqrt{|A[a]|}$$

$$\leq \frac{|A[2]|}{2} + \frac{\sqrt{|A|}}{2} \sigma(-1, A).$$

This concludes the proof of the lemma. □

Before we consider α_3, we introduce some additional notation. If $x \in \mathbb{R}$, we write $[x] = \max\{n \in \mathbb{Z}; \ n \leq x\}$ for the greatest-integer function. Moreover, we let

(4.3) $$\mathbb{B}(x) = x - \tfrac{1}{2}([x] - [-x]).$$

be the 1-periodic function on $\mathbb{R}$ with $\mathbb{B}(x) = 0$ for $x = 0, 1$ and $\mathbb{B}(x) = x - 1/2$ for $0 < x < 1$. By definition

$$\alpha_3 = \sum_{\gamma \in A/\{\pm 1\}} \left(-Q(\gamma) - [-Q(\gamma)]\right).$$

Using $\mathbb{B}(x)$ and α_4 we may rewrite this in the form

$$\alpha_3 = \frac{d}{2} - \frac{\alpha_4}{2} - \sum_{\gamma \in A/\{\pm 1\}} \mathbb{B}(Q(\gamma))$$

$$= \frac{d}{2} - \frac{\alpha_4}{2} - \frac{1}{2} \sum_{\gamma \in A[2]} \mathbb{B}(Q(\gamma)) - \frac{\beta}{2},$$

where

(4.4) $$\beta = \sum_{\gamma \in A} \mathbb{B}(Q(\gamma)).$$

For $\gamma \in A[2]$ we have $Q(\gamma) \in \frac{1}{4}\mathbb{Z}$, and therefore $|\mathbb{B}(Q(\gamma))| \leq 1/4$. Hence

(4.5) $$|\alpha_3 - d/2 + \alpha_4/2| \leq |A[2]|/8 + |\beta|/2.$$

Lemma 4.4. *The quantity β satisfies*

$$|\beta| \leq \frac{\sqrt{|A|}}{\pi} (3/2 + \ln(N)) \left(\sigma(-1, A) - \frac{\sqrt{|A|}}{N}\right).$$

Proof. Exactly as in the proof of [Br2, Lemma 5], we derive

$$|\beta| \le \frac{\sqrt{|A|}}{\pi} \sum_{\nu=1}^{N-1} \frac{1}{\nu} \sqrt{|A[\nu]|} + \frac{\sqrt{|A|}}{2\pi N} \sum_{\nu=1}^{N-1} \sqrt{|A[\nu]|}.$$

Rewriting the sum over ν, we obtain

$$|\beta| \le \frac{\sqrt{|A|}}{\pi} \sum_{\substack{a|N \\ a\ne N}} \sum_{\substack{\mu=1 \\ (\mu,N/a)=1}}^{N/a} \frac{1}{a\mu} \sqrt{|A[a]|} + \frac{\sqrt{|A|}}{2\pi N} \sum_{\substack{a|N \\ a\ne N}} \sum_{\substack{\mu=1 \\ (\mu,N/a)=1}}^{N/a} \sqrt{|A[a]|}$$

$$\le \frac{\sqrt{|A|}}{\pi} \sum_{\substack{a|N \\ a\ne N}} (1+\ln(N/a))\frac{1}{a} \sqrt{|A[a]|} + \frac{\sqrt{|A|}}{2\pi N} \sum_{\substack{a|N \\ a\ne N}} \frac{N}{a} \sqrt{|A[a]|}$$

$$\le \frac{\sqrt{|A|}}{\pi} (3/2 + \ln(N)) \left(\sigma(-1, A) - \frac{\sqrt{|A|}}{N} \right).$$

Here we have also used the estimate $\sum_{\nu=1}^{n} \frac{1}{\nu} \le 1 + \ln(n)$. □

Putting the above lemmas together, we obtain the following estimate for the dimension of the space $S_{k,A}$.

Theorem 4.5. *If $k \ge 3/2$ and $2k \equiv -\operatorname{sig}(A) \pmod 4$, then*

$$\left| \dim(S_{k,A}) - \dim(M_{2-k,A(-1)}) - \frac{d(k-1)}{12} \right| \le R(A),$$

where

$$R(A) = \frac{\sqrt{|A[2]|}}{4} + \frac{1+\sqrt{|A[3]|}}{3\sqrt{3}} + \frac{3}{8}|A[2]|$$

$$+ \frac{\sqrt{|A|}}{4}\sigma(-1, A) + \frac{\sqrt{|A|}}{2\pi} (3/2 + \ln(N)) \left(\sigma(-1, A) - \frac{\sqrt{|A|}}{N} \right)$$

is independent of k.

Proof. The dimension formula (3.4) states that

$$\dim(S_{k,A}) - \dim(M_{2-k,A(-1)}) = \frac{d(k+12)}{12} - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$$

$$= \frac{d(k-1)}{12} - (\alpha_1 - \frac{d}{4}) - (\alpha_2 - \frac{d}{3}) - (\alpha_3 - \frac{d}{2} + \frac{\alpha_4}{2}) - \frac{\alpha_4}{2}.$$

Employing (4.5), Lemma 4.2, Lemma 4.3, and Lemma 4.4, we obtain the assertion. □

Corollary 4.6. *For every $\varepsilon > 0$ there exists a constant C (independent of k and A) such that*

$$\left| \dim(S_{k,A}) - \dim(M_{2-k,A(-1)}) - \frac{d(k-1)}{12} \right| \le CdN^{\varepsilon-1/2}.$$

for every finite quadratic module A and every weight $k \ge 3/2$ with $2k \equiv -\operatorname{sig}(A) \pmod 4$.

Proof. Using Theorem 4.5, the bound (2.1) for $|A[a]|$, and Lemma 2.15, we find that there are constants $C_1, C_2 > 0$ (independent of k and A) such that

$$R(A) \leq C_1 \frac{|A|}{N} + C_2 \frac{|A|}{\sqrt{N}} \sigma_{-1/2}(N)(1 + \ln(N)).$$

By means of the estimate $\sigma_{-1/2}(N) \ll_\varepsilon N^\varepsilon$ we see that there exists a $C > 0$ (depending on ε) such that

$$R(A) \leq C \cdot dN^{\varepsilon - 1/2}.$$

This proves the assertion. $\square$

Corollary 4.7. *Let $r_0 \in \mathbb{Z}_{\geq 0}$. There exist only finitely many isomorphism classes of finite quadratic modules A with minimal number of generators $\leq r_0$ such that $S_{k,A} = \{0\}$ for some weight $k \geq 3/2$ with $2k \equiv -\operatorname{sig}(A) \pmod 4$.*

Proof. Since for any $N_0 \in \mathbb{Z}_{>0}$ there are only finitely many isomorphism classes of finite quadratic modules A with bounded minimal number of generators and level $N \leq N_0$, we obtain the assertion from Corollary 4.6. $\square$

Remark 4.8. i) In Corollary 4.7, the bound r_0 on the minimal number of generators is essential. For instance, if $A = 3^{\varepsilon n}$ with $n \in \mathbb{Z}_{>0}$ odd and $\varepsilon = (-1)^{\frac{n-1}{2}}$, then $\operatorname{sig}(A) \equiv 2 \pmod 4$ and $S_{3,A} = \{0\}$. This follows for instance from the dimension formula in [Ha], Chapter 5.2.1, p. 93.

ii) Note that if $k = 1/2$, it follows from [Sk2, Theorem 7] that there exist infinitely many isomorphism classes of finite quadratic modules A such that $S_{1/2,A} = \{0\}$. It would be interesting to understand what happens in weight 1.

Under the assumptions of Corollary 4.7 it is possible to make the constants appearing in the proof explicit and to derive an explicit lower bound N_0 such that $S_{k,A}$ is nontrivial for all finite quadratic modules A with level larger than N_0. However, it turns out that such a bound is very large, and therefore not useful for a computer computation of the finite list of simple finite quadratic modules A. For this reason, in the next section, we use a more systematic approach to this computational task. We first compute all *anisotropic* simple finite quadratic modules, and then construct all remaining ones by means of isotropic quotients.

4.1. **Anisotropic finite quadratic modules.** A finite quadratic module (A, Q) is called *isotropic*, if there exists an $x \in A \setminus \{0\}$ such that $Q(x) = 0 \in \mathbb{Q}/\mathbb{Z}$. Otherwise it is called *anisotropic*. In this subsection we now consider anisotropic finite quadratic modules. We show that there are only finitely many isomorphism classes of anisotropic finite quadratic modules A for which $S_{k,A}$ is trivial.

Lemma 4.9. *Let (A, Q) be an anisotropic finite quadratic module of level N. Then $N = 2^t N'$, where N' is an odd square-free number and $t \in \{0, 1, 2, 3\}$. If p is a prime dividing N, then the p-component A_p of A belongs to the finite quadratic modules given in Table 2.*

In the case of an anisotropic finite quadratic module it is also possible to obtain an explicit formula for the quantity β defined in (4.4) in terms of class numbers. Before we can state the precise result we need to introduce some more notation.

TABLE 2. The non-trivial isomorphism classes of anisotropic finite quadratic modules of prime-power order. The isomorphism classes of the finite quadratic modules in the last line depend only on the sum $s + t$. Here, $d(A)$ is the discriminant of A, equal to $|A| \in \mathbb{Q}^\times / (\mathbb{Q}^\times)^2$.

p	genus symbol of A	$\mathrm{sig}(A)$	$d(A)$
$p \equiv 1 \bmod 4$	$p^{\pm 1}$	$4 + 2(1 \pm \left(\frac{p}{2}\right))$	p
	p^{-2}	4	0
$p \equiv 3 \bmod 4$	$p^{\pm 1}$	$\pm 2 \left(\frac{p}{2}\right)$	p
	p^{+2}	4	0
$p = 2$	2^{-2}	4	0
	$2_{nt}^{\pm n}, t = 1, 7, n = 1, 2, 3$	nt	2^n
	$4_t^{\pm 1}, t = 1, 3, 5, 7$	t	0
	$2_s^{\pm 1} 4_t^{\pm 1}, s = 1, 7, t = 1, 3, 5, 7$	$s + t$	2

Let A be an anisotropic finite quadratic module of level N and write $A = \bigoplus_{p \mid N} A_p$ for its decomposition into p-components. For each prime divisor p of N, we denote the minimal number of generators of A_p by r_p. If $A_p = q^{\varepsilon \cdot n}$ with $q = p^r$ we write $\varepsilon_A(p) = \varepsilon$. We define a divisor M of N by

$$M = \prod_{\substack{p \mid N \text{ odd} \\ r_p = 2}} \cdot \begin{cases} 2, & \text{if } A_2 = 2^{-2}, \\ 1, & \text{otherwise.} \end{cases}$$

For $d \mid N$ we define the following quantities.

$$S_1(d) = \{p \text{ prime}; \ p \mid (d, M) \text{ and } p \equiv 1 \bmod 4\},$$
$$\varepsilon_1(d) = (-1)^{|S_1(d)|},$$
$$a_2(d) = \begin{cases} 1, & \text{if } r_2 = 0, \\ 2^{r_2 - 2}, & \text{if } r_2 > 0 \text{ and } d \text{ is odd}, \\ 2^{\left[\frac{r_2}{2}\right] - 1}, & \text{if } r_2 > 0 \text{ and } d \text{ is even}, \end{cases}$$
$$S_3(d) = \left\{p \text{ prime} \ ; \ p \mid \frac{d}{(d, M)}, p \equiv 3 \bmod 4\right\},$$
$$\varepsilon_3(d) = \begin{cases} (-1)^{\frac{|S_3(d)| - 1}{2}}, & \text{if } |S_3(d)| \text{ is odd}, \\ (-1)^{\frac{|S_3(d)|}{2}}, & \text{if } |S_3(d)| \text{ is even}. \end{cases}$$

For the p-components corresponding to odd primes, we define a sign

$$\varepsilon_{\mathrm{odd}}(d) = \prod_{\substack{p | \frac{d}{(d,M)} \text{ odd}}} \varepsilon_A(p) \left(\frac{p}{2}\right) \left(\frac{N/d}{p}\right).$$

We let N_2 be the even part of N and put $N_{2,d} = N_2/(N_2, N/d)$, $N_d = N/(d \cdot (N/d, N_2))$ and $d' = d/(d, M \cdot N_2)$. Note that d' is odd. If d is odd, we let $\varepsilon_2(d) = 1$. For even d, we define $\varepsilon_2(d)$ in Table 3. For simplicity, we also define $\varepsilon(d) = \varepsilon_{\mathrm{odd}}(d)\varepsilon_1(d)\varepsilon_2(d)\varepsilon_3(d)$.

TABLE 3. $\varepsilon_2(d)$ for even d.

A_2	$d' \equiv 1 \pmod 4$	$d' \equiv 3 \pmod 4$
2^{-2}	0	1
$2^{+r_2}_{r_2 t}, 4^{\pm 1}_t$	$\left(\frac{-N_{2,d}}{tN_d}\right)$	$\delta(r_2)\left(\frac{N_{2,d}}{tN_d}\right)$
$2^{+1}_1 4^{+1}_1, 2^{+1}_1 4^{-1}_3$	$\left(\frac{-8}{N/d}\right)$	0
$2^{+1}_1 4^{-1}_5, 2^{+1}_1 4^{+1}_7$	0	$\left(\frac{8}{N/d}\right)$

Theorem 4.10. *Let (A, Q) be an anisotropic finite quadratic module of level N. We have*

$$\beta = - \sum_{\substack{d | N \\ d(d,M) \equiv 0,3 \bmod 4}} \varepsilon(d) a_2(d) \cdot (N/d, M) H(-d(d,M)).$$

Here, $H(-n)$ is equal to the number of classes of positive definite integral binary quadratic forms of discriminant $-n$ for $n > 4$ and $H(-3) = 1/3$ and $H(-4) = 1/2$.

Proof of Theorem 4.10. Using the pointwise convergent Fourier expansion

$$\mathbb{B}(x) = -\frac{1}{2\pi i} \sum_{n \in \mathbb{Z} - \{0\}} \frac{e(nx)}{n}$$

we find

$$\beta = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \Im(G(n, A)) = -\frac{1}{\pi} \sum_{d | N} \sum_{\substack{n \geq 1 \\ (n, N/d) = 1}} \frac{1}{dn} \Im(G(n, A)).$$

First, we assume that N is odd. For a discriminant D, we write χ_D for the quadratic character given by $n \mapsto \left(\frac{D}{n}\right)$. Inserting the explicit formula for $G(n, A)$ from Proposition

2.11, we obtain

$$
\beta = -\frac{\sqrt{N}}{\pi} \sum_{\substack{d|N}} \sum_{\substack{n \geq 1 \\ (n,d)=1}} \frac{(M, N/d)\sqrt{(M,d)}}{n\sqrt{N/d}} \Im\left(\prod_{\substack{p|d \\ r_p=1}} \varepsilon_A(p)\left(\frac{p}{2}\right)\left(\frac{nN/d}{p}\right) e\left(\frac{1-p}{8}\right)\prod_{\substack{p|N/d \\ r_p=2}}(-1)\right)
$$

$$
= -\frac{\sqrt{N}}{\pi} \sum_{\substack{d|N \\ d(d,M)\equiv 3 \bmod 4}} \varepsilon(d)\cdot(M,N/d)\frac{\sqrt{(M,d)}}{\sqrt{N/d}} \sum_{\substack{n\geq 1 \\ (n,d)=1}} \frac{\chi_{-d\cdot(M,d)}(n)}{n}
$$

$$
= -\sum_{\substack{d|N \\ d\cdot(d,M)\equiv 3 \bmod 4}} \varepsilon(d)\cdot(M,N/d)\frac{\sqrt{d\cdot(M,d)}}{\pi} L(\chi_{-d\cdot(M,d)},1).
$$

Here, we used that $e\left(\frac{1-p}{8}\right) = \left(\frac{p}{2}\right)$ for $p \equiv 1 \pmod 4$ and $e\left(\frac{1-p}{8}\right) = \left(\frac{p}{2}\right) i$ for $p \equiv 3 \pmod 4$. Therefore, only divisors congruent to 3 modulo 4 contribute to the sum. Using that $L(\chi_{-D},1)/\pi = H(-D)/\sqrt{D}$ (cf. [Za]) for a negative discriminant $-D$, we obtain the statement of the theorem in this case.

If N is even, we have to consider the different 2-adic components separately. The case $A_2 = 2^{-2}$ is easy to obtain. We give a proof for $A_2 = 2_{rt}^{+r_2}$. The remaining cases are done analogously. Using the same argument as before together with the results in Proposition 2.12, we obtain

$$
\beta = -\frac{\sqrt{N}}{\pi} \sum_{\substack{d|N \\ N/d \text{ odd}}} \varepsilon_{\text{odd}}(d)\varepsilon_1(d)\cdot(M,N/d)\frac{\sqrt{(M,d)}}{\sqrt{N/d}}
$$

$$
\times \sum_{\substack{n\geq 1 \\ (n,d)=1}} \frac{1}{n}\sqrt{2}^{r_2-2}\Im\left(\left(\frac{tnN/d}{2}\right)^{r_2} e\left(\frac{r_2 tnN/d}{8}\right)\prod_{\substack{p|d \text{ odd} \\ r_p=1}}\left(\frac{n}{p}\right)\gamma_p\right)
$$

$$
-\frac{1}{\pi} \sum_{\substack{d|N \\ N/d\equiv 0 \bmod 4}} \varepsilon(d)\cdot(M,N/d)\frac{\sqrt{N\cdot(M,d)}}{\sqrt{N/d}} \sum_{\substack{n\geq 1 \\ (n,d)=1}}\frac{1}{n}\left(\frac{-d\cdot(M,d)}{n}\right).
$$

Here, $\gamma_p = 1$ for $p \equiv 1 \bmod 4$ and $\gamma_p = i$ for $p \equiv 3 \bmod 4$. Using that $\sqrt{2}\left(\frac{m}{2}\right)e\left(\frac{m}{8}\right) = 1 + \left(\frac{-4}{m}\right) i$, we obtain

$$
\sqrt{2}^{r_2-2}\Im\left(\left(\frac{tnN/d}{2}\right)^{r_2} e\left(\frac{r_2 tnN/d}{8}\right)\prod_{\substack{p|d \text{ odd} \\ r_p=1}}\left(\frac{n}{p}\right)\gamma_p\right)
$$

$$
= a_2(N/d)\prod_{\substack{p|d \text{ odd} \\ r_p=1}}\left(\frac{n}{p}\right)\cdot\begin{cases}\left(\frac{-4}{tnN/d}\right), & \text{if } d/(4\cdot(M,d)) \equiv 1 \bmod 4 \\ \delta(r_2)\left(\frac{4}{tnN/d}\right), & \text{if } d/(4\cdot(M,d)) \equiv 3 \bmod 4,\end{cases}
$$

which yields the statement of the theorem for $A_2 = 2_{nt}^{+r_2}$. $\square$

The following upper bound for the class number is well known.

Lemma 4.11. *Let* $-D$ *be a negative discriminant. We have*

$$H(-D) \leq \frac{\sqrt{D} \ln D}{\pi}.$$

Proof. For $D > 4$ this is Lemma 5.6 on page 172 in [Ge]. Note that the bound is also valid for $D = -3$ and $D = -4$ with our normalization that $H(-3) = 1/3$ and $H(-4) = 1/2$. □

Lemma 4.12. *Let* A *be an anisotropic finite quadratic module. We have*

$$|\beta| \leq 1.71 \cdot |A|^{\frac{5}{8}} \ln(2|A|).$$

Proof. We have by Theorem 4.10 and Lemma 4.11 that

$$|\beta| \leq \frac{1}{\pi} \sum_{\substack{d|N \\ d(d,M)\equiv 0,3 \pmod 4}} a_2(d)(N/d, M) \sqrt{d(d, M)} \ln(d(d, M))$$

$$\leq \frac{1}{\pi} M \ln(NM) c_2(A) \sum_{d|N} \sqrt{\frac{d}{(d, M)}},$$

where $c_2(A) = 2$ if $r_2 = 3$ and $c_2(A) = 1$, otherwise. We obtain

$$|\beta| \leq \frac{1}{\pi} M \ln(NM) c_2(A) \sigma_0(M) \sigma_{1/2}(N/M).$$

If the order of A is odd, we have $c_2(A) = 1$ and

$$M \sigma_0(M) \sigma_{1/2}(N/M) = 1.76 \cdot |A|^{5/8}.$$

Moreover, if the order of A is a power of 2, then

$$c_2(A) M \sigma_0(M) \sigma_{1/2}(N/M) \leq 3.05 |A|^{5/8}.$$

Using the multiplicativity of the divisor sum function, we see that

$$c_2(A) M \sigma_0(M) \sigma_{1/2}(N/M) \leq 5.37 \cdot |A|^{5/8}.$$

Finally, using that $NM \leq 2|A|$ implies the statement of the lemma. □

Corollary 4.13. *Let* (A, Q) *be an anisotropic finite quadratic module. If* $k \geq 2$, *then*

$$\dim(S_{k,A}) \geq \frac{(|A| + 1)(k - 1)}{24} - 3.0 - 0.86 \cdot |A|^{5/8} \ln(2|A|).$$

Proof. Since A is anisotropic, we have $\alpha_4 = 1$. Hence the estimate of Theorem 4.5 can be refined to give

$$\left| \dim(S_{k,A}) - \frac{d(k - 1)}{12} \right| \leq R'(A),$$

where

$$R'(A) = \frac{\sqrt{|A[2]|}}{4} + \frac{1 + \sqrt{|A[3]|}}{3\sqrt{3}} + \frac{1}{8}|A[2]| + \frac{1}{2} + \frac{\beta}{2}.$$

Since A is anisotropic, Lemma 4.9 implies that

$$|A[2]| \leq 8,$$
$$|A[3]| \leq 9.$$

Using in addition (2.2) and the estimates $N' \leq |A|$ and $N \leq 2|A|$, we obtain

$$\dim(S_{k,A}) \geq \frac{(|A|+1)(k-1)}{24} - \frac{\sqrt{2}}{2} - \frac{4}{3\sqrt{3}} - \frac{3}{2} - \frac{\beta}{2}.$$

Together with Lemma 4.12 this proves the corollary. □

Corollary 4.14. *Let (A, Q) be an anisotropic finite quadratic module such that $\mathrm{sig}(A) \equiv -2k \pmod 4$. If $|A| \geq 4.71 \cdot 10^7$, then $S_{\frac{3}{2},A} \neq \{0\}$ and if $k \geq 2$, then $S_{k,A} \neq \{0\}$ for $|A| \geq 5.3 \cdot 10^6$.*

We implemented the dimension formula and some of the estimates used here in python using `sage`[S+14]. Note that for the low weights $k = 2$ and $k = 3/2$, we need to calculate the dimension of the invariants of the Weil representation. N. Skoruppa and S. Ehlen wrote a program that determines the invariants explicitly and we included this implementation in our repository [Ehl]. Our complete software package, together with all required libraries, examples and documentation is available online [Ehl].

We used our program to obtain a list of all anisotropic finite quadratic modules such that $S_{k,A} = \{0\}$ for $k \geq \frac{3}{2}$.

Corollary 4.15. *Let (A, Q) be an anisotropic finite quadratic module such that $\mathrm{sig}(A) \equiv -2k \pmod 4$. Then $S_{k,A} = \{0\}$ exactly if A belongs to the lists given in Tables 4 and 5.*

Remark 4.16. The bound in Corollary 4.14 improves a lot for higher weights. However, all of the bounds obtained this way are far away from the correct bounds (the maximal order is 238 for $k = 3/2$ and 60 for $k = 2$) found in Tables 4 and 5.

TABLE 4. The 75 anisotropic finite quadratic modules A with $S_{3/2,A} = \{0\}$. Out of these, 59 have signature 1.

$\mathrm{sig}(A)$	3/2-simple finite quadratic modules
1	$\left(\mathbb{Z}/2N\mathbb{Z}, \frac{x^2}{4N}\right)$ for $1 \leq N < 37$ square-free and
	$N \in \{38, 39, 41, 42, 46, 47, 51, 55, 59, 62, 66, 69, 70, 71, 78, 87, 94, 95, 105, 110, 119\}$,
	$2_3^{+3}3^{+1}, 2_5^{+3}3^{+2}, 2_5^{+3}5^{+1}, 4_7^{+1}3^{-1}, 4_1^{+1}5^{-1}, 4_5^{-1}5^{-2}, 4_3^{-1}7^{-1}, 2_7^{+1}3^{+1}5^{-2}, 2_7^{+1}3^{-1}5^{-1},$
	$2_1^{+1}3^{+1}7^{+1}, 2_7^{+1}3^{+2}7^{-1}, 4_1^{+1}13^{-1}, 4_7^{+1}3^{+1}5^{+1}, 2_7^{+1}5^{-1}7^{+1}, 2_7^{+1}3^{+1}13^{+1}$
5	$4_5^{-1}, 2_7^{+1}3^{+1}, 2_5^{+3}, 2_1^{+1}5^{+1}, 4_7^{+1}3^{+1}, 4_3^{-1}3^{-1}, 2_7^{+1}7^{-1}, 4_5^{-1}5^{-1}, 4_1^{+1}5^{+1}, 2_3^{+3}3^{-1},$
	$2_1^{+1}13^{+1}, 4_1^{+1}3^{+2}, 4_7^{+1}11^{+1}, 2_1^{+1}5^{-2}, 2_3^{+3}7^{+1}, 2_3^{+3}3^{-1}5^{-1}$

TABLE 5. Anisotropic finite quadratic modules A with $S_{k,A} = \{0\}$ for $k \geq 2$.

k	$\mathrm{sig}(A)$	genus symbols
2	0	1^{+1}, 5^{-1}, $2_1^{+1}4_7^{+1}$, $3^{+1}11^{-1}$, $2^{-2}5^{+1}$, $2_2^{+2}3^{+1}$, $2_6^{+2}3^{-1}$, 13^{-1}, $2_6^{+2}7^{+1}$, 17^{+1}, $3^{-1}7^{-1}$, $2_1^{+1}4_1^{+1}3^{+1}$, $2_6^{+2}3^{+1}5^{+1}$
2	4	2^{-2}, 3^{+2}, 5^{+1}, 5^{-2}, $2_1^{+1}4_3^{-1}$, $3^{-1}11^{-1}$, $2^{-2}5^{-1}$, $2_2^{+2}3^{-1}$, $2_6^{+2}3^{+1}$, 13^{+1}, $2_2^{+2}7^{+1}$, 17^{-1}, $3^{+1}7^{-1}$, $2_1^{+1}4_1^{+1}3^{-1}$, $2_2^{+2}3^{-1}5^{-1}$
$\frac{5}{2}$	3	2_3^{+3}, 4_3^{-1}, $4_3^{-1}5^{-1}$, $2_1^{+1}3^{-1}$, $2_7^{+1}3^{+2}$, $2_1^{+1}7^{+1}$, $2_1^{+1}11^{-1}$, $4_1^{+1}7^{+1}$, $2_7^{+1}5^{+1}$, $4_1^{+1}3^{-1}$, $4_5^{-1}3^{+1}$, $2_1^{+1}3^{-1}5^{-1}$
$\frac{5}{2}$	7	2_7^{+1}, 4_7^{+1}, $2_1^{+1}3^{+1}$

k	$\mathrm{sig}(A)$	genus symbols	$\mathrm{sig}(A)$	genus symbols
3	2	3^{-1}, 2_2^{+2}, $2^{-2}3^{+1}$, 7^{+1}, $2_1^{+1}4_1^{+1}$, 11^{-1}, $3^{+1}5^{+1}$, $3^{-1}5^{-1}$, $2_2^{+2}5^{-1}$, 23^{+1}	6	3^{+1}
$\frac{7}{2}$	1	2_1^{+1}, 4_1^{+1}, $2_7^{+1}3^{-1}$, $2_1^{+1}5^{-1}$, $4_3^{-1}3^{+1}$	5	4_5^{-1}
4	0	1^{+1}, 5^{-1}	4	5^{+1}, 2^{-2}
$\frac{9}{2}$	3	4_3^{-1}, $2_1^{+1}3^{-1}$	7	2_7^{+1}
5	2	3^{-1}, 2_2^{+2}, 7^{+1}	6	3^{+1}
$\frac{11}{2}$	1	2_1^{+1}, 4_1^{+1}		
6	0	1^{+1}		
7	2	3^{-1}		
$\frac{15}{2}$	1	2_1^{+1}		
8, 10, 14	0	1^{+1}		

4.2. **Differences of dimensions.** Here we give lower bounds for the difference of the dimensions of $S_{k,A\oplus B}$ and $S_{k,A}$, where A is an arbitrary finite quadratic module and B is an isotropic finite quadratic module of order p^2 for large primes p. We have to estimate the differences of the quantities occurring in the dimension formula (3.3). To indicate the dependency of the finite quadratic module, we write $\alpha_i(A)$ for the quantities α_i associated to A defined at the beginning of Section 4. We will make use of the following principle.

Definition 4.17. If A is a finite quadratic module and $U \subset A$ is a subgroup, we let

$$U^{\perp} = \{a \in A \mid (a, u) = 0 \text{ for all } u \in U\}$$

be the *orthogonal complement* of U.

If U is an isotropic subgroup, that is, we have $Q(u) = 0$ for all $u \in U$, then the pair $(U^\perp/U, Q)$ also defines a finite quadratic module. We have $|A| = |U^\perp/U||U|^2$ and $\mathrm{sig}(A) = \mathrm{sig}(U^\perp/U)$.

Proposition 4.18. *Let A be a finite quadratic module and let $B = U^\perp/U$ for some isotropic subgroup $U \subset A$. We have an injection $S_{k,B} \hookrightarrow S_{k,A}$ given by $f \mapsto F$ with $F_\alpha = 0$ for $\alpha \notin U^\perp$ and $F_\alpha = f_{\alpha+U}$ for $\alpha \in U^\perp$.*

Proof. See Theorem 4.1 in [Sch2]. $\qquad\square$

Proposition 4.19. *If $U \subset A$ is a maximal isotropic subgroup, then $A_0 = U^\perp/U$ is anisotropic. The isomorphism class of A_0 is independent of the choice of U and we call A_0 the* anisotropic reduction *of A.*

Proof. It is easy to see that $U^\perp/U$ is anisotropic for a maximal isotropic subgroup: Suppose that $x \in U^\perp/U$ is isotropic, $x \neq 0$. Then $x = a + U$ for some $a \in U^\perp$ with $Q(a) = 0$. However, since $a \in U^\perp$ and $a \notin U$, this implies that the subgroup U' of A generated by U and a is isotropic and strictly larger than U.

The uniqueness follows from the classification of the anisotropic finite quadratic modules (see Table 2) and the fact that $d(U^\perp/U) = d(A)$ and $\mathrm{sig}(U^\perp/U) = \mathrm{sig}(A)$. $\qquad\square$

Lemma 4.20. *Let A be an arbitrary finite quadratic module, and let B be an isotropic finite quadratic module of order p^2, where p is a prime not dividing $6|A|$. Then*

$$(4.6) \qquad d_{A \oplus B} - d_A = \frac{|A|}{2}(p^2 - 1),$$

and

$$|\alpha_1(A \oplus B) - \alpha_1(A) - \frac{|A|}{8}(p^2 - 1)| \leq \frac{1}{2}\sqrt{|A[2]|},$$

$$|\alpha_2(A \oplus B) - \alpha_2(A) - \frac{|A|}{6}(p^2 - 1)| \leq \frac{2}{3\sqrt{3}}\left(1 + \sqrt{|A[3]|}\right),$$

$$|\alpha_3(A \oplus B) - \alpha_3(A) - \frac{|A|}{4}(p^2 - 1)| \leq \frac{p-1}{2}|A|,$$

$$\alpha_4(A \oplus B) - \alpha_4(A) \leq (p-1)|A|.$$

Proof. To prove (4.6), we use (2.2). Since $p \neq 2$, we have $B[2] = \{0\}$, and therefore

$$d_{A \oplus B} = \frac{|A \oplus B|}{2} + \frac{|A[2] \oplus B[2]|}{2} = \frac{p^2|A|}{2} + \frac{|A[2]|}{2}.$$

This implies the stated formula.

The bounds for the differences of α_1 and α_2 directly follow from (4.6) and Lemma 4.2 combined with the fact that $B[3] = \{0\}$.

Let $r_0(A)$ denote the number of isotropic vectors in A. For α_4 we use that

$$\alpha_4(A) = \frac{1}{2}\left(r_0(A) + r_0(A[2])\right).$$

Since the 2-torsion of B is trivial and since p does not divide $|A|$, we find

$$\alpha_4(A \oplus B) - \alpha_4(A) = \frac{1}{2} \left(r_0(A \oplus B) - r_0(A) \right)$$
$$= \frac{1}{2} r_0(A) \left(r_0(B) - 1 \right).$$

The quantity $r_0(B)$ is bounded by $2p - 1$, and $r_0(A)$ is trivially bounded by $|A|$. This gives the claimed bound.

We now turn to the estimate for α_3. For $x \in \mathbb{R}$ we write $\{x\} = x - [x] \in [0, 1)$ for the fractional part of of x. By definition we have

$$\alpha_3(A) = \sum_{\gamma \in A/\{\pm 1\}} \{-Q(\gamma)\}$$
$$= \frac{1}{2} \sum_{\gamma \in A} \{-Q(\gamma)\} + \frac{1}{2} \sum_{\gamma \in A[2]} \{-Q(\gamma)\}.$$

Consequently,

(4.7) $$\alpha_3(A \oplus B) - \alpha_3(A) = \frac{1}{2} \sum_{\gamma \in A \oplus B} \{-Q(\gamma)\} - \frac{1}{2} \sum_{\gamma \in A} \{-Q(\gamma)\}.$$

For an arbitrary $x \in \mathbb{R}$, we now estimate the sum

(4.8) $$S(x, B) = \sum_{\gamma \in B} \{x - Q(\gamma)\}.$$

If B is the level p isotropic finite quadratic module (which has genus symbol $p^{\varepsilon \cdot 2}$ with $\varepsilon = (-1)^{\frac{p+1}{2}}$), we have

$$S(x, B) = \sum_{a,b \in \mathbb{Z}/p\mathbb{Z}} \{x - \frac{ab}{p}\}$$
$$= p\{x\} + (p - 1) \sum_{b \in \mathbb{Z}/p\mathbb{Z}} \{x + \frac{b}{p}\}$$
$$= p\{x\} + (p - 1) \sum_{b=0}^{p-1} \left(\frac{1}{p}\{px\} + \frac{b}{p} \right)$$
$$= p\{x\} + (p - 1)\{px\} + \frac{(p - 1)^2}{2}$$

Inserting this into (4.7), we get

$$\alpha_3(A \oplus B) - \alpha_3(A) = \frac{1}{2}(p - 1) \sum_{\gamma \in A} \left(\{-Q(\gamma)\} + \{-pQ(\gamma)\} + \frac{(p - 1)}{2} \right).$$

and therefore

(4.9) $$|\alpha_3(A \oplus B) - \alpha_3(A) - \frac{|A|}{4}(p^2 - 1)| \leq \frac{p - 1}{2}|A|.$$

If B is a finite quadratic module of level $q = p^2$, we slightly modify the above argument as follows. Let $\varepsilon \in \mathbb{Z}$ with $\left(\frac{2\varepsilon}{p} \right) = \pm 1$, such that B has the genus symbol $q^{\pm 1}$. In this case we have

$$S(x, B) = \sum_{a \in \mathbb{Z}/p^2\mathbb{Z}} \{ x - \varepsilon \frac{a^2}{p^2} \}$$

$$= \sum_{a \in \mathbb{Z}/p\mathbb{Z}} \sum_{b \in \mathbb{Z}/p\mathbb{Z}} \{ x - \varepsilon \frac{(a + pb)^2}{p^2} \}$$

$$= p\{x\} + \frac{(p-1)^2}{2} + \sum_{a \in (\mathbb{Z}/p\mathbb{Z})^\times} \{ p(x - \varepsilon \frac{a^2}{p^2}) \}.$$

By means of this identity, we obtain the same bound (4.9) as in the earlier case. □

Theorem 4.21. *Let A be an arbitrary finite quadratic module, and let B be an isotropic finite quadratic module of order p^2, where p is a prime not dividing $6|A|$. Then*

$$\dim(S_{k,A \oplus B}) - \dim(S_{k,A}) \geq \frac{|A|(p^2 - 1)}{24} \left(k - 1 - \frac{36p}{p^2 - 1} \right).$$

Proof. We begin by noticing that the difference of dimensions on the left hand side is non-negative because of Proposition 4.18. We use the dimension formula

$$\dim(S_{k,A}) = \frac{d_A(k + 12)}{12} - \alpha_1(A) - \alpha_2(A) - \alpha_3(A) - \alpha_4(A) + \dim(M_{2-k,A(-1)}).$$

Because of Proposition 4.18, we have $\dim(M_{2-k,(A \oplus B)(-1)}) \geq \dim(M_{2-k,A(-1)})$. Employing Lemma 4.20, we obtain

$$\dim(S_{k,A \oplus B}) - \dim(S_{k,A}) \geq \frac{|A|(p^2 - 1)}{24}(k - 1) - \frac{\sqrt{|A[2]|}}{2} - \frac{2 + 2\sqrt{|A[3]|}}{3\sqrt{3}} - \frac{3}{2}(p - 1)|A|.$$

The claim now follows by the trivial estimates $|A[2]| \leq |A|$ and $\frac{2 + 2\sqrt{|A[3]|}}{3\sqrt{3}} \leq |A|$. □

Corollary 4.22. *With the same assumptions as in Theorem 4.21, we have $S_{k,A \oplus B} \neq \{0\}$ for $p \geq p_k$ given in Table 6.*

TABLE 6. Bounds on p in Corollary 4.22.

k	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2} \leq k \leq \frac{13}{2}$	$7 \leq k \leq 9$	$k \geq \frac{19}{2}$
p_k	73	37	29	19	17	13	11	7	5

5. Simple finite quadratic modules

If A is a finite quadratic module and k is an integer, we say that A is k-simple if $S_{k,A} = \{0\}$. We will now develop an algorithm that allows us to easily iterate over all finite quadratic modules starting from anisotropic ones.

For a finite quadratic module A and an integer n, consider the finite set of finite quadratic modules

$$B(A,n) = \{A' \mid A = U^{\perp}/U \text{ for an isotropic subgroup } U \subset A' \text{ with } |U| = n\}.$$

For simplicity, we define a subset $C(A,n) \subset B(A,n)$ using the following formal rules. Let p be an odd prime and $r \geq 0$ be an integer.

(To) $1^{+1} \mapsto p^{\epsilon_p \cdot 2}$, where $\epsilon_p = 1$ if $p \equiv 1 \pmod 4$ and $\epsilon_p = -1$, otherwise,

(O) $(p^r)^{\pm 1} \mapsto (p^{r+2})^{\pm 1}$.

A rule can be applied to A if the module on the left hand side of the rule is a direct summand of A. It is important to note that we can always apply the rule starting with the trivial module 1^{+1}. We should also remark that if $r = 0$ above, the left hand side of the rule (O) is the trivial module. Therefore, for $r = 0$, we have the rules $1^{+1} \mapsto (p^2)^{+1}$ and $1^{+1} \mapsto (p^2)^{-1}$. The application of any of these rules to the genus symbol of A yields the genus symbol of a finite quadratic module in $B(A,p)$.

Example 5.1. Consider the finite quadratic module A given by the genus symbol $3^{+1}9^{-1}7^{-3}$. Applying rule (To) to A for $p = 3$, we obtain $3^{-3}9^{-1}7^{-3}$. Note that we can also apply rule (O) for both signs for $p = 7$ by writing $3^{+1}9^{-1}7^{-3} = 3^{+1}9^{-1}7^{+2}7^{-1} \mapsto 3^{+1}9^{-1}7^{+2}343^{-1}$ and similarly $3^{+1}9^{-1}7^{-3} = 3^{+1}9^{-1}7^{-2}7^{+1} \mapsto 3^{+1}9^{-1}7^{-2}343^{+1}$. By applying both rules once in all possible cases, we obtain

$$C(A,3) = \{3^{-3}9^{-1}7^{-3}, 3^{+1}9^{-2}7^{-3}, 3^{+1}9^{+2}7^{-3}, 9^{-1}27^{+1}7^{-3}, 3^{+1}81^{-1}7^{-3}\}$$

$$C(A,7) = \{3^{+1}9^{-1}7^{+5}, 3^{+1}9^{-1}7^{-3}49^{+1}, 3^{+1}9^{-1}7^{-3}49^{-1}, 3^{+1}9^{-1}7^{-2}343^{+1}, 3^{+1}9^{-1}7^{+2}343^{-1}\},$$

$$C(A,p) = \{3^{+1}9^{-1}7^{-3}p^{\epsilon_p \cdot 2}, 3^{+1}9^{-1}7^{-3}(p^2)^{+1}, 3^{+1}9^{-1}7^{-3}(p^2)^{-1}\} \text{ for } p \notin \{3,7\}.$$

For $p = 2$, the rules we require are more complicated. Let $q = 2^r$ with $r \geq 1$.

(Te1) $1^{+1} \mapsto 2^{+2}$,

(Te2) $1^{+1} \mapsto 2_0^{+2}$,

(E1) $q^{+2} \mapsto (2q)^{+2}$,

(E2) $q_4^{-2} \mapsto (2q)^{-2}$,

(E3) $q_t^{\pm 1} \mapsto (4q)_t^{\pm 1}$,

(E4) $2^{+2} \mapsto 4_0^{+2}$,

(E5) $2^{-2} \mapsto 4_4^{-2}$,

(E6) $2_{2t}^{+2} \mapsto 4_{2t}^{+2}$ for $t \in \{1,7\}$,

(E7) $2_{2t}^{+2} \mapsto 4_{2t}^{-2}$ for $t \in \{1,7\}$.

Remark 5.2. Note that $2_0^{+2} \cong 2_4^{-2}$ and therefore rule (E2) applies to 2_0^{+2}, as well.

Definition 5.3. We define $C(A,p)$ to be the set of finite quadratic modules obtained from A after application of a single rule as listed above, only involving operations for p. For a prime power $n = p^r$, we define $C(A,p^r)$ to be the set that is obtained from r consecutive

applications of rules only involving p. Finally, we define $C(A, n)$ for any positive integer n by induction on the number of different primes dividing n by putting

$$C(A, p^r m) = \bigcup_{B \in C(A, m)} C(B, p^r)$$

for $(m, p) = 1$.

We use these formal rules because it is very easy to implement them on a computer.

Theorem 5.4. *Let A be a finite quadratic module and let A_0 be its anisotropic reduction. Then A can be obtained from A_0 in finitely many steps using the rules given above. More precisely, we have $A \in C(A_0, n)$ for $n^2 = |A|/|A_0|$.*

Proof. It is enough to prove the claim for a p-module, that is a finite quadratic module of prime-power order p^n. Let us first assume that p is odd and that A has a genus symbol of the form $q^{\pm n}$ with $q = p^r$. Then it is easy to see that if r is even, we can obtain the symbol $q^{\pm 1}$ starting from the trivial finite quadratic module

$$1^{+1} \mapsto (p^2)^{\pm 1} \mapsto \ldots \mapsto (p^r)^{\pm 1}.$$

Applying the same rule n times, we obtain the symbol $q^{\pm n}$. If r is odd instead, we start with the anisotropic symbol $p^{\pm 1}$. We obtain

$$p^{\pm 1} \mapsto (p^3)^{\pm 1} \mapsto \ldots \mapsto (p^r)^{\pm 1}.$$

We have now seen that we can obtain any finite quadratic p-module from a symbol of the form $p^{\pm n}$. Applying rule (To) several times reduces this symbol either to the trivial module or to the anisotropic finite quadratic module $p^{-\varepsilon_p \cdot 2}$.

For $p = 2$ we have to distinguish a few more cases. Suppose we are given a symbol of the form $q_t^{\pm r}$. We can obtain $q_t^{\pm r}$ from a symbol that is a direct sum of symbols of the form $2_s^{\pm r'}$ or $4_s^{\pm r'}$ by applying rule (E3). Using rules (E4-E7), any even number of odd summands of level 8 can be reduced to level 4, leaving only a rank one odd component of level 8.

Now let $2_{t_1}^{+1} \ldots 2_{t_r}^{+1}$ be any odd discriminat form of level 4 with $t_1, \ldots, t_r \in \{1, 7\}$. If $\{1, 7\} \subset \{t_1, \ldots, t_r\}$, then $2_0^{+2} = 2_1^{+1} 2_7^{+1}$ is a summand. Now suppose that the rank is at least equal to four and the symbol does not contain both, 2_1^{+1} *and* 2_3^{-1}. Then 2_4^{+4} is contained. However, $2_4^{+4} \cong 2^{-2} 2_4^{-2} \cong 2^{-2} 2_0^{+2}$. Then, we can apply (Te2) to reduce the rank of the level 4 part to at most 3.

Finally, if we are given an even 2-adic symbol $(2^n)^{\pm r}$ which is not anisotropic (i.e. is not 2^{-2}), it always contains $(2^n)^{+2}$ or $(2^n)^{-2}$ as a direct summand. Therefore, using rules (E1), (E2) and (E5), we can reduce to the case of level 2 or 1. Combining this with the strategy for the odd symbols gives the result. $\square$

We now describe the algorithm used to compute all simple lattices.

Algorithm 5.5. Given integers r, s and a half-integer k, the following algorithm determines the isomorphism classes of all k-simple finite quadratic modules of signature s with a minimal number of generators less than or equal to r.

   (A) Compute all anisotropic k-simple finite quadratic modules satisfying the conditions (see Table 5).

(B) For each previously computed k-simple finite quadratic module A compute the set $C(A, p)$ for all primes $p \leq p_k$ with p_k given in Table 6.

(C) Repeat step (B) until no further k-simple finite quadratic modules have been found.

The correctness of the algorithm follows from Theorem 5.4 and Proposition 4.18 together with the results from the last sections. Moreover, that the algorithm terminates follows from Corollary 4.7.

Remark 5.6. In each iteration, the bound on the primes can be reduced to the maximal prime such that there is a newly discovered finite quadratic module A' in $C(A, p)$ for some k-simple finite quadratic module obtained one iteration earlier.

The algorithm can be nicely illustrated in a graph. Figure 1 shows the output for $k = 3, r = 6$ and $s = 6$. This input corresponds to the parameters for lattices of signature $(2, 4)$.

In Tables 7-8, we list all $\frac{2-n}{2}$-simple finite quadratic modules of signature $2 - n$ with minimal number of generators $r \leq 2 + n$ for $n \geq 2$.

For $n = 1$, a large family of $3/2$-simple finite quadratic modules is given by the cyclic finite quadratic modules $A_0(N) = (\mathbb{Z}/2N\mathbb{Z}, x^2/4N)$ for $N \in \mathbb{Z}_{>0}$. The corresponding orthogonal modular varieties are the modular curves $\Gamma_0(N) \backslash \mathbb{H}^*$, where $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$. Moreover, each of these finite quadratic modules has a global realization given by $L_N = \mathbb{Z}^3$ with quadratic form $N x_1^2 + x_2 x_3$. We have that L_N is simple if and only if $1 \leq N \leq 36$ or if N is in the following list:

$$38, 39, 40, 41, 42, 44, 45, 46, 47, 48, 49, 50, 51, 52, 54, 55, 56, 59, 60, 62, 63,$$
$$64, 66, 68, 69, 70, 71, 72, 75, 76, 78, 80, 81, 84, 87, 90, 94, 95, 96, 98,$$
$$100, 104, 105, 108, 110, 119, 120, 126, 132, 140, 144, 150, 168, 180.$$

It is interesting to observe that for N squarefree, L_N is simple if and only if $X_0^*(N) = \Gamma_0^*(N) \backslash \mathbb{H}^*$ has genus zero. Here, $\Gamma_0^*(N)$ is the extension of $\Gamma_0(N)$ by all Atkin-Lehner involutions. The situation for non-squarefree N is more complicated. It would be interesting to find a similar geometric interpretation in the general case.

In Table 9, we list the remaining $3/2$-simple finite quadratic modules of signature 1 and minimal number of generators at most 3.

6. SIMPLE LATTICES

Let L be an even lattice of signature $(2, n)$. We will say that L is *simple* if $S_{k, L'/L} = \{0\}$ for $k = \frac{2+n}{2}$. In this section we determine all isomorphism classes of simple lattices of signature $(2, n)$. If L is simple, then the finite quadratic module L'/L is k-simple for $k = \frac{2+n}{2}$. Thus, we are interested in all k-simple finite quadratic modules with minimal number of generators $r \leq 2 + n$ that actually correspond to a lattice of signature $(2, n)$.

Proposition 6.1. *Let A be a finite quadratic module and write ε_q for the sign of the Jordan component of A of order q. Let r_p be the minimal number of generators of A_p. There is an even lattice L of signature (r, s) with $L'/L = A$ if and only if the following conditions hold.*
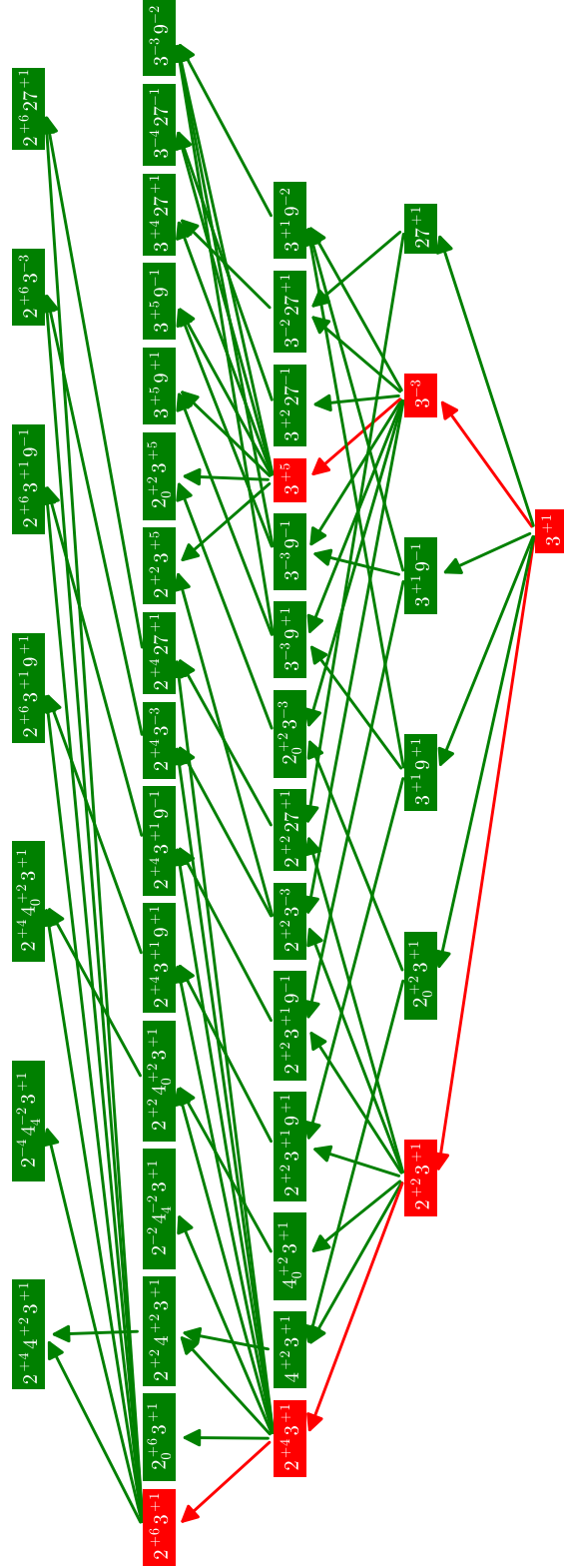
FIGURE 1. Illustration of Algorithm 5.5 for $k = 3, r = 6$ and $s = 6$. The 3-simple finite quadratic modules are colored in red.

TABLE 7. The table shows the 70 finite quadratic modules A of signature 0 with minimal number of generators $r \leq 4$ and $S_{2,A} = \{0\}$.

level	genus symbols	level	genus symbols
1	1^{+1}	18	$2^{+2}9^{+1}$, $2^{+2}9^{-1}$, $2^{+4}9^{+1}$
2	2^{+2}, 2^{+4}	20	$2_0^{+2}5^{-1}$
3	3^{-2}, 3^{+4}	21	$3^{-1}7^{-1}$
4	2_0^{+2}, 4^{-2}, 4^{+2}, 2_0^{+4}, $2^{+2}4^{+2}$, $[2^{+2}4^{-2}]$, $2_0^{+2}4^{+2}$, $[4^{-4}]$, 4^{+4}	24	$2_1^{+1}4_1^{+1}3^{+1}$, $4_2^{-2}3^{+1}$
5	5^{-1}, 5^{+2}, 5^{-3}, 5^{+4}	25	25^{+1}, 25^{-1}
6	$2^{+2}3^{-2}$, $2^{+4}3^{-2}$, $2^{+2}3^{+4}$	27	$3^{+1}27^{-1}$
7	7^{-2}	28	$2_6^{+2}7^{+1}$
8	$2_1^{+1}4_7^{+1}$, 4_0^{+2}, $2_1^{+3}4_7^{+1}$, $2^{+2}4_0^{+2}$, 8^{+2}, $2^{+2}8^{+2}$	32	$4_7^{+1}16_1^{+1}$
9	9^{+1}, 9^{-1}, $3^{-2}9^{-1}$, 9^{-2}	33	$3^{+1}11^{-1}$
10	$2^{-2}5^{+1}$, $2^{+2}5^{-1}$, $[2^{+4}5^{-1}]$, $2^{-4}5^{+1}$, $2^{+2}5^{+2}$	36	$2_0^{+2}9^{-1}$
12	$2_2^{+2}3^{+1}$, $2_6^{+2}3^{-1}$, $2_0^{+2}3^{-2}$, $2_6^{+4}3^{-1}$, $2_6^{+2}3^{+3}$, $4^{+2}3^{-2}$	45	$9^{+1}5^{-1}$
13	13^{-1}	48	$2_3^{-1}8_3^{-1}3^{-1}$
16	$2_7^{+1}8_1^{+1}$, $2_5^{-1}8_3^{-1}$, $4_7^{+1}8_1^{+1}$, $2_7^{+3}8_1^{+1}$, 8_0^{+2}, $2_7^{+1}4^{+2}8_1^{+1}$	49	49^{+1}
17	17^{+1}	60	$2_6^{+2}3^{+1}5^{+1}$
		64	$2_7^{+1}32_1^{+1}$

(1) We have $\operatorname{sig}(A) \equiv r - s \pmod 8$.

(2) For all primes p, we have $r + s \geq r_p$.

(3) Let p be an odd prime and write $(-1)^s |A| = p^\alpha a$ with $(a, p) = 1$. If $r + s = r_p$, we have

(6.1)
$$\prod_q \varepsilon_q = \left(\frac{a}{p}\right),$$

where the product runs over all powers q of p.

(4) If $r+s = r_2$ and A_2 does not contain a direct summand of the form $2_t^{\pm m}$ with $m \geq 1$, then (6.1) holds for $p = 2$ and $(-1)^s |A| = 2^\alpha a$ with $(a, 2) = 1$, as well.

Proof. See [Ni], Theorem 1.10.1. □

Using Proposition 6.1, we determined all genus symbols that do not correspond to lattices of signature $(2, n)$. We enclosed them in parentheses $[\cdot]$ in Tables 7-9.

Recall that the genus symbol does only determine a lattice up to rational equivalence. A genus consists of finitely many integral isometry classes of lattices. However, the following

TABLE 8. The table shows all finite quadratic modules of signature $2 - n$ with minimal number of generators $r \leq 2 + n$ such that $S_{\frac{2+n}{2}} = \{0\}$

n	**level**: genus symbols
3	$\mathbf{4} : 2_7^{+1}, 2_7^{+3}, 2_7^{+1}4^{+2}, 2_7^{+5}, 2_7^{+3}4^{+2}, 2_7^{+1}4^{+4}$
	$\mathbf{8} : 4_7^{+1}, 2^{+2}4_7^{+1}, 2^{+4}4_7^{+1},\quad \mathbf{12} : 2_1^{+1}3^{+1}, 2_7^{+1}3^{-2}, 2_7^{+1}3^{+4},$
	$\mathbf{16} : 8_3^{-1}, 8_7^{+1}, 2^{+2}8_3^{-1}, [2^{+4}8_3^{-1}]$
4	$\mathbf{3} : 3^{+1}, 3^{-3}, 3^{+5},\quad \mathbf{6} : 2^{+4}3^{+1}, [2^{+6}3^{+1}], 2^{+2}3^{+1}$
5	$\mathbf{8} : 4_5^{-1}, 2^{+2}4_5^{-1}, 2^{+4}4_5^{-1}, [2^{+6}4_5^{-1}]$
6	$\mathbf{2} : 2^{-2}, 2^{-4}, 2^{-6}, [2^{-8}],\quad \mathbf{5} : 5^{+1}$
7	$\mathbf{8} : 4_3^{-1},\quad \mathbf{12} : 2_1^{+1}3^{-1}$
8	$\mathbf{3} : 3^{-1},\quad \mathbf{4} : 2_2^{+2},\quad \mathbf{7} : 7^{+1}$
9	$\mathbf{4} : 2_1^{+1},\quad \mathbf{8} : 4_1^{+1},\quad \mathbf{16} : 8_1^{+1}$
10	$\mathbf{1} : 1^{+1},\quad \mathbf{2} : 2^{+2}$
18, 26	$\mathbf{1} : 1^{+1}$

proposition gives a full classification of all isomorphism classes of lattices of signature $(2, n)$ for $n \geq 1$.

Proposition 6.2. *If L is a simple lattice of signature $(2, n)$ and L is not contained in the genus $2_1^{+1}5^{-1}25^{-1}$, then its genus contains a unique isomorphism class. The genus $2_1^{+1}5^{-1}25^{-1}$ contains two isomorphism classes.*

Proof. For $n \geq 2$, Corollary 22 in Chapter 15 of [CS] states that if there is more than one class in the genus of an indefinite lattice L, then $|\det(L)| \geq 5^6$. There is no finite quadratic module of this size in our list in Tables 7-8.

The lattices in signature $(2, 1)$ are slightly more complicated to treat. Using Theorem 21 in Chapter 15 of [CS], we find that only the lattices of discriminant d with $4d$ divisible by 5^3 or 8^3 might contain more than one class in their genus. Theorem 19 ibid. finally leaves us with the following list of genera that might contain more than one class:

$$2_7^{+1}4_1^{+1}32_1^{+1}, \quad 2_1^{+1}4_7^{+1}16_1^{+1}, \quad 2_7^{+1}8_1^{+1}16_1^{+1}, \quad 2_1^{+1}5^{-1}25^{-1}.$$

For the first 3 genera, we can use [EH], Theorem 2.2, to see that these also only contain one class. The last one is treated in [Wa] in Chapter 7, Section 5. This genus contains two isomorphism classes represented by the integral ternary quadratic forms

$$\phi_1(x_1, x_2, x_3) = x_1^2 + x_1 x_2 - x_2^2 + 25x_3^2,$$
$$\phi_2(x_1, x_2, x_3) = 5(x_1^2 + x_1 x_2 - x_2^2) + x_3^2.$$

$\square$

6.1. **Applications to Borcherds products.** Let (L, Q) be an even lattice of signature $(2, n)$ and let $A = L'/L$ be its the discriminant module. We write $O(L)$ for the orthogonal group of L and $O(L)^+$ for the subgroup of index 2 consisting of those elements whose determinant has the same sign as the spinor norm. We consider the kernel Γ_L of the natural homomorphism $O(L)^+ \to \mathrm{Aut}(A)$, sometimes referred to as the stable orthogonal group of L.

Let D be the hermitian symmetric space associated to the group $O(L \otimes_{\mathbb{Z}} \mathbb{R})$. It can be realized as a tube domain in $\mathbb{C}^n$. The group Γ_L acts on D and the quotient

$$X_L = \Gamma_L \backslash D$$

has the structure of a quasi-projective algebraic variety. For suitable choices of L, important families of classical modular varieties can be obtained in this way, including Shimura curves, Hilbert modular surfaces and Siegel modular threefolds.

For every $\mu \in A$ and every negative $m \in \frac{1}{N}\mathbb{Z}$, there is a Heegner divisor $Z(m, \mu)$ on X_L (sometimes also referred to as special divisor or rational quadratic divisor), see e.g. [Bo2], [Br1]. We denote by $\mathrm{Pic}_{\mathrm{Heeg}}(X_L)$ the subgroup of the Picard group $\mathrm{Pic}(X_L)$ of X_L generated by all such Heegner divisors.

If L is simple, then for *every* pair (m, μ) as above there is a weakly holomorphic modular form $f_{m,\mu} \in M^!_{1-n/2, A(-1)}$ of weight $1 - n/2$ whose Borcherds lift $\Psi(f_{m,\mu})$ (in the sense of Theorem 13.3 in [Bo1]) is a meromorphic modular form for Γ_L whose divisor is $Z(m, \mu)$. In particular, the vector space $\mathrm{Pic}_{\mathrm{Heeg}}(X_L) \otimes_{\mathbb{Z}} \mathbb{Q}$ is one-dimensional and generated by the Hodge bundle, hence it is as small as it can be.

The weight of the Borcherds product $\Psi(f_{m,\mu})$ is given by half of the constant term of the component of $f_{m,\mu}$ corresponding to the characteristic function ϕ_0 of the zero element of A. Equivalently, it can be expressed in terms of the coefficient of index $(-m, \mu)$ of the unique normalized Eisenstein series $E_{1+n/2, A} \in M_{1+n/2, A}$ whose constant term is ϕ_0, see e.g. Theorem 12 of [BK]. The coefficients of such Eisenstein series can be explicitly computed, see Theorem 7 of [BK] or [KY]. It would be interesting to use the list of simple lattices to search systematically for holomorphic Borcherds products of singular weight $n/2 - 1$ for Γ_L. Such Borcherds products are often denominator identities of generalized Kac-Moody algebras, see [Sch1].

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Appendix

Global realizations. In this section we provide a list of global realizations of all simple lattices of signature $(2,n)$.

We write $\mathcal{O}_D$ for the 2-dimensional lattice given by the order of discriminant D in $\mathbb{Q}(\sqrt{D})$ with the norm as quadratic form. We write $\mathbb{Z}$ for the integers with quadratic form x^2.

n = 1:

We start with the anisotropic lattices (anisotropic over $\mathbb{Q}$) of signature $(2,1)$ These correspond to *Shimura curves*. Those which have an anisotropic discriminant form can be realized as follows.

(1) $2_3^{+3}3^{+1}$ $\qquad\qquad\qquad$ $\mathbb{Z} \oplus \mathcal{O}_{12}$
(2) $2_5^{+3}3^{+2}$ $\qquad\qquad\qquad$ $\mathbb{Z}(3) \oplus \mathcal{O}_{12}(-1)$
(3) $2_5^{+3}5^{+1}$ $\qquad\qquad\qquad$ $\mathbb{Z} \oplus \mathcal{O}_5(2)$
(4) $4_7^{+1}3^{-1}$ $\qquad\qquad\qquad$ $\mathbb{Z}(-2) \oplus \mathcal{O}_{-3}$
(5) $4_1^{+1}5^{-1}$ $\qquad\qquad\qquad$ $\mathbb{Z}(2) \oplus \mathcal{O}_5$
(6) $4_5^{-1}5^{-2}$ $\qquad\qquad\qquad$ $\mathbb{Z}(10) \oplus \mathcal{O}_5$
(7) $4_3^{-1}7^{-1}$

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & -6 \end{pmatrix}$$

(8) $2_7^{+1}3^{+1}5^{-2}$ $\qquad\qquad$ $\mathbb{Z}(-5) \oplus \mathcal{O}_{-15}$
(9) $2_7^{+1}3^{-1}5^{-1}$ $\qquad\qquad$ $\mathbb{Z}(-5) \oplus \mathcal{O}_{-3}$
(10) $2_1^{+1}3^{+1}7^{+1}$ $\qquad\qquad$ $\mathbb{Z} \oplus \mathcal{O}_{21}$
(11) $2_7^{+1}3^{+2}7^{-1}$ $\qquad\qquad$ $\mathbb{Z}(3) \oplus \mathcal{O}_{-3}(-1)$
(12) $4_1^{+1}13^{-1}$ $\qquad\qquad\quad$ $\mathbb{Z}(2) \oplus \mathcal{O}_{13}$
(13) $4_7^{+1}3^{+1}5^{+1}$ $\qquad\qquad$ $\mathbb{Z}(-2) \oplus \mathcal{O}_{15}$
(14) $2_7^{+1}5^{-1}7^{+1}$ $\qquad\qquad$ $\mathbb{Z}(-5) \oplus \mathcal{O}_{-7}$
(15) $2_7^{+1}3^{+1}13^{+1}$ $\qquad\qquad$ $\mathbb{Z}(-1) \oplus \mathfrak{a}$, where $\mathfrak{a} \subset \mathcal{O}_{-39}$ is an ideal representing the non-trivial class

There are in total 26 lattices (i.e. these 15 and 11 more corresponding to isotropic modules) that are anisotropic over $\mathbb{Q}$. These 11 more lattices correspond to certain sublattices of the ones given above.

We now turn to those lattices of signature $(2,1)$ which are isotropic over $\mathbb{Q}$, that is, that correspond to *modular curves*. We have seen that $L_N = \mathbb{Z}^3$ with quadratic form $Nx^2 + yz$ is simple for exactly 90 values of N and we listed these values on page 25. The following sublattices yield the remaining simple lattices of signature $(2,1)$. We write $L_{a,b}$ for the lattice $\mathbb{Z}^3$ with quadratic form $ax^2 + byz$ with $a, b \in \mathbb{Z}$. Then $L_{N,1} = L_N$. The following list of 69 pairs (a,b) corresponds to a system of representatives for the isomorphism classes of lattices $L_{a,b}$ with $b > 1$ that are simple.

$$(2,2), (2,3), (2,4), (2,5), (2,6), (2,8), (2,9), (2,10), (2,16), (3,3),$$
$$(3,4), (3,7), (3,12), (4,2), (4,3), (4,4), (4,6), (4,8),$$
$$(4,5), (5,2), (5,3), (5,5), (6,2), (6,3), (6,4), (6,6), (6,9), (7,4), (7,7),$$
$$(8,2), (8,3), (8,4), (8,8), (9,2), (9,3), (9,4), (9,9), (10,2), (10,5),$$
$$(11,3), (12,2), (12,3), (12,4), (12,6), (14,2), (15,3), (15,4), (16,2), (16,4), (16,4),$$
$$(18,2), (18,3), (20,2), (20,5), (22,2), (24,2), (24,3), (24,4), (27,3), (28,2),$$
$$(30,2), (32,2), (36,2), (36,3), (36,4), (40,2), (48,2), (60,2), (72,2)$$

The remaining ones are also all sublattices of L_N for some N.

n = 2:

We write H for the hyperbolic plane $\mathbb{Z}^2$ with quadratic form xy.

Witt rank 1

(1) $2_0^{+2}5^{-1}$ $\qquad\qquad$ $\mathbb{Z}(-1) \oplus \mathbb{Z}(5) \oplus H$

(2) 5^{-1} $\qquad\qquad$ $\mathcal{O}_5 \oplus H$

(3) 5^{-3} $\qquad\qquad$ $\mathcal{O}_5 \oplus H(5)$

(4) $2_1^{+1}4_7^{+1}$ $\qquad\qquad$ $\mathbb{Z} \oplus \mathbb{Z}(-2) \oplus H$

(5) $2_1^{+3}4_7^{+1}$ $\qquad\qquad$ $\mathbb{Z} \oplus \mathbb{Z}(-2) \oplus H(2)$

(6) $2^{-2}5^{+1}$ $\qquad\qquad$ $\mathcal{O}_5(2) \oplus H$

(7) $2^{+2}5^{-1}$ $\qquad\qquad$ $\mathcal{O}_5 \oplus H(2)$

(8) $2^{-4}5^{+1}$ $\qquad\qquad$ $\mathcal{O}_5(2) \oplus H(2)$

(9) $2_2^{+2}3^{+1}$ $\qquad\qquad$ $\mathbb{Z} \oplus Z(-3) \oplus H$

(10) $2_6^{+2}3^{-1}$ $\qquad\qquad$ $\mathbb{Z}(-1) \oplus Z(3) \oplus H$

(11) $2_6^{+4}3^{-1}$ $\qquad\qquad$ $\mathbb{Z} \oplus Z(-3) \oplus H(2)$

(12) $2_6^{+2}3^{+3}$ $\qquad\qquad$ $\mathbb{Z} \oplus Z(-3) \oplus H(6)$

(13) 13^{-1} $\qquad\qquad$ $\mathcal{O}_{13} \oplus H$

(14) $2_1^{+1}8_3^{-1}$ $\qquad\qquad$ $\mathbb{Z} \oplus S_8$, where

$$S_8 = \begin{pmatrix} -8 & -4 & 0 \\ -4 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

(15) $4_7^{+1}8_1^{+1}$ $\qquad\qquad$ $\mathbb{Z}(-2) \oplus \mathbb{Z}(4) \oplus H$

(16) 17^{+1} $\qquad\qquad$ $\mathcal{O}_{17} \oplus H$

(17) $3^{-1}7^{-1}$ $\qquad\qquad$ $\mathcal{O}_{-3} \oplus \mathcal{O}_{-7}(-1)$

(18) $2_1^{+1}4_1^{+1}3^{+1}$

$$\mathbb{Z} \oplus \begin{pmatrix} -4 & -2 & 0 \\ -2 & -2 & -3 \\ 0 & -3 & -6 \end{pmatrix}$$

(19) $2_6^{+2}7^{+1}$ $\qquad\qquad$ $\mathbb{Z}(-1) \oplus \mathbb{Z}(-1) \oplus \mathcal{O}_{-7}$

(20) $3^{+1}11^{-1}$ $\qquad\qquad$ $\mathcal{O}_{-3}(-1) \oplus \mathcal{O}_{-11}$

(21) $9^{+1}5^{-1}$ $\qquad\qquad$ $S_9 \oplus \mathcal{O}_5$

(22) $2_7^{+1}8_3^{-1}3^{-1}$ $\qquad\qquad$ $\mathbb{Z}(3) \oplus S_8$

(23) $4_2^{-2}3^{+1}$ $\qquad\qquad$ $\mathbb{Z}(-2) \oplus \mathbb{Z}(6) \oplus H$

(24) $2_6^{+2}3^{+1}5^{+1}$ $\qquad\qquad$ $\mathcal{O}_{15} \oplus H$

Witt rank 2

(1) 1^{+1} $\qquad\qquad$ $H \oplus H$

(2) 2^{+2} $\qquad\qquad$ $H \oplus H(2)$

(3) 2^{+4} $\qquad\qquad$ $H(2) \oplus H(2)$

(4) 3^{-2} $\qquad\qquad$ $\mathcal{O}_{-3} \oplus \mathcal{O}_{-3}(-1)$

(5) 3^{+4} $\qquad\qquad$ $H(3) \oplus H(3)$

(6) 2_0^{+2} $\qquad\qquad$ $\mathbb{Z} \oplus \mathbb{Z}(-1) \oplus H$

(7) 4^{-2} V_4, where (Witt rank 2)

$$V_4 = \begin{pmatrix} 8 & 4 & -8 & 0 \\ 4 & 8 & 0 & 0 \\ -8 & 0 & -2 & -5 \\ 0 & 0 & -5 & -2 \end{pmatrix}$$

(8) 4^{+2} $H \oplus H(4)$
(9) 2_0^{+4} $\mathbb{Z} \oplus \mathbb{Z}(-1) \oplus H(2)$
(10) $2^{+2}4^{+2}$ $H(2) \oplus H(4)$
(11) $2_0^{+2}4^{+2}$ $\mathbb{Z} \oplus \mathbb{Z}(-1) \oplus H(4)$
(12) 4^{+4} $H(4) \oplus H(4)$
(13) 5^{+2} $H \oplus H(5)$
(14) 5^{+4} $H(5) \oplus H(5)$
(15) $2^{+2}3^{-2}$ $H(2) \oplus H(3)$
(16) $2^{+4}3^{-2}$ $H(2) \oplus H(6)$
(17) $2^{+2}3^{+4}$ $H(6) \oplus H(6)$
(18) 7^{-2} $H \oplus H(7)$
(19) 4_0^{+2} $\mathbb{Z}(2) \oplus \mathbb{Z}(-2) \oplus H$
(20) $2^{+2}4_0^{+2}$ $\mathbb{Z}(2) \oplus \mathbb{Z}(-2) \oplus H(2)$
(21) 8^{+2} $H \oplus H(8)$
(22) $2^{+2}8^{+2}$ $H(2) \oplus H(8)$
(23) 9^{+1} $S_9 \oplus H$, where

$$S_9 = \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}$$

(24) 9^{-1} $S_9(-1) \oplus H$
(25) $3^{-2}9^{-1}$ $S_9(-1) \oplus H(3)$
(26) 9^{-2} $H \oplus H(9)$
(27) $2^{+2}5^{+2}$ $H(2) \oplus H(5)$
(28) $2_0^{+2}3^{-2}$ $\mathbb{Z} \oplus Z(-1) \oplus H(3)$
(29) $4^{+2}3^{-2}$ $H(3) \oplus H(4)$
(30) $2_7^{+1}8_1^{+1}$ $\mathbb{Z}(-1) \oplus Z(4) \oplus H$
(31) $2_7^{+3}8_1^{+1}$ $\mathbb{Z}(-1) \oplus \mathbb{Z}(4) \oplus H(2)$
(32) 8_0^{+2} $\mathbb{Z}(4) \oplus \mathbb{Z}(-4) \oplus H$
(33) $2_7^{+1}4^{+2}8_1^{+1}$ $\mathbb{Z}(-1) \oplus \mathbb{Z}(4) \oplus H(4)$
(34) $2^{+2}9^{+1}$ $S_9 \oplus H(2)$
(35) $2^{+2}9^{-1}$ $S_9(-1) \oplus H(2)$
(36) $2^{+4}9^{+1}$ $S_9(-2) \oplus H(2)$
(37) 25^{+1} $S_{25} \oplus H$, where

$$S_{25} = \begin{pmatrix} 2 & 1 \\ 1 & -12 \end{pmatrix}$$

(38) 25^{-1} $\qquad$ $S_{25}(-1) \oplus H$

(39) $3^{+1}27^{-1}$ $\qquad$ $\mathcal{O}_{-3}(-1) \oplus S_{27}$, where

$$S_{27} = \begin{pmatrix} 18 & -3 \\ -3 & 2 \end{pmatrix}$$

(40) $4_7^{+1}16_1^{+1}$ $\qquad$ $\mathbb{Z}(-2) \oplus \mathbb{Z}(8) \oplus H$

(41) $2_0^{+2}9^{-1}$ $\qquad$ $\mathbb{Z} \oplus \mathbb{Z}(-1) \oplus S_9(-1)$

(42) 49^{+1} $\qquad$ $S_{49} \oplus H$, where

$$S_{49} = \begin{pmatrix} -2 & -1 \\ -1 & 24 \end{pmatrix}$$

(43) $2_7^{+1}32_1^{+1}$ $\qquad$ $\mathbb{Z}(-1) \oplus \mathbb{Z}(16) \oplus H$

$n \geq 3$. **n = 4:** All lattices from here on have Witt rank 2.

(1) 2_7^{+1} $\qquad$ $\mathbb{Z}(-1) \oplus H \oplus H$

(2) 2_7^{+3} $\qquad$ $\mathbb{Z}(-1) \oplus H(2) \oplus H$

(3) $2_7^{+1}4^{+2}$ $\qquad$ $\mathbb{Z}(-1) \oplus H(4) \oplus H$

(4) 2_7^{+5} $\qquad$ $\mathbb{Z}(-1) \oplus H(2) \oplus H(2)$

(5) $2_7^{+3}4^{+2}$ $\qquad$ $\mathbb{Z}(-1) \oplus H(2) \oplus H(4)$

(6) $2_7^{+1}4^{+4}$ $\qquad$ $\mathbb{Z}(-1) \oplus H(4) \oplus H(4)$

(7) 4_7^{+1} $\qquad$ $\mathbb{Z}(-2) \oplus H \oplus H$

(8) $2^{+2}4_7^{+1}$ $\qquad$ $\mathbb{Z}(-2) \oplus H(2) \oplus H$

(9) $2^{+4}4_7^{+1}$ $\qquad$ $\mathbb{Z}(-1) \oplus H(2) \oplus H(2)$

(10) $2_1^{+1}3^{+1}$ $\qquad$ $\mathbb{Z}(-3) \oplus H \oplus H$

(11) $2_7^{+1}3^{-2}$ $\qquad$ $\mathbb{Z}(-3) \oplus \mathcal{O}_{-3} \oplus H$

(12) $2_7^{+1}3^{+4}$ $\qquad$ $\mathbb{Z}(-3) \oplus \mathcal{O}_{-3} \oplus H(3)$

(13) 8_3^{-1} $\qquad$ $S_8 \oplus H$

(14) 8_7^{+1} $\qquad$ $Z(-4) \oplus H \oplus H$

(15) $2^{+2}8_3^{-1}$ $\qquad$ $S_8 \oplus H(2)$

Here,

$$S_8 = \begin{pmatrix} -8 & -4 & 0 \\ -4 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

n = 4:

(1) 3^{+1} $\qquad$ $\mathcal{O}_{-3}(-1) \oplus H \oplus H$

(2) 3^{-3} $\qquad$ $\mathcal{O}_{-3}(-1) \oplus H \oplus H(3)$

(3) 3^{+5} $\qquad$ $\mathcal{O}_{-3}(-1) \oplus H(3) \oplus H(3)$

(4) $2^{+2}3^{+1}$ $\qquad$ $\mathcal{O}_{-3}(-1) \oplus H(2) \oplus H(2)$

(5) $2^{+4}3^{+1}$ $\qquad$ $\mathcal{O}_{-3}(-1) \oplus H \oplus H(2)$

n = 5: Let

$$S_4 = \begin{pmatrix} -4 & -2 & 0 \\ -2 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

(1) 4_5^{-1} $S_4 \oplus H \oplus H$
(2) $2^{+2}4_5^{-1}$ $S_4 \oplus H \oplus H(2)$
(3) $2^{+4}4_5^{-1}$ $S_4 \oplus H(2) \oplus H(2)$

n = 6:

(1) 2^{-2} $T_4 \oplus H \oplus H$
(2) 2^{-4} $T_4 \oplus H(2) \oplus H$
(3) 2^{-6} $T_4 \oplus H(2) \oplus H(2)$
(4) 5^{+1} $S_5 \oplus H \oplus H$

Here,

$$S_5 = \begin{pmatrix} -2 & -1 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & -2 & -1 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

and

$$T_4 = \begin{pmatrix} -4 & -2 & -2 & 0 \\ -2 & -4 & 0 & 0 \\ -2 & 0 & -2 & -1 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

n = 7:

(1) 4_3^{-1}

$$\begin{pmatrix} -4 & -2 & 0 & 0 & 0 \\ -2 & -2 & -1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & -1 & -2 \end{pmatrix} \oplus H \oplus H$$

(2) $2_1^{+1}3^{-1}$

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \oplus H \oplus H$$

n = 8:

(1) 3^{-1} $\mathcal{O}_{-3} \oplus E_8(-1)$
(2) 2_2^{+2} $\mathbb{Z} \oplus \mathbb{Z} \oplus E_8(-1)$
(3) 7^{+1} $\mathcal{O}_{-7} \oplus E_8(-1)$

n = 9:

  (1) 2_1^{+1} $\mathbb{Z} \oplus E_8(-1) \oplus H$
  (2) 4_1^{+1} $\mathbb{Z}(2) \oplus E_8(-1) \oplus H$
  (3) 8_1^{+1} $\mathbb{Z}(4) \oplus E_8(-1) \oplus H$

n = 10:

  (1) 1^{+1} $E_8(-1) \oplus H \oplus H$
  (2) 2^{+2} $E_8(-1) \oplus H \oplus H(2)$

n = 18: $E_8(-1) \oplus E_8(-1) \oplus H \oplus H$

n = 26: $E_8(-1) \oplus E_8(-1) \oplus E_8(-1) \oplus H \oplus H$

Table 9: The table lists the 169 finite quadratic modules of signature 1 which $\frac{3}{2}$-simple and not of type $A_0(N)$ (see page 25).

level	$\frac{3}{2}$-simple modules
4	2_1^{+3}, $2_1^{+1}4^{+2}$
8	$2^{+2}4_1^{+1}$, $2_0^{+2}4_1^{+1}$, $2_1^{+1}4_0^{+2}$, 4_1^{+3}, $2_1^{+1}8^{+2}$, $4_1^{+1}8^{+2}$
12	$2_1^{+1}3^{-2}$, $2_3^{+3}3^{+1}$, $2_7^{+3}3^{-1}$, $2_7^{+1}3^{+3}$, $2_5^{+3}3^{+2}$, $2_1^{+3}3^{-2}$, $2_7^{+1}4^{+2}3^{-1}$, $2_7^{+3}3^{+3}$, $2_1^{+1}4^{+2}3^{-2}$, $2_7^{+1}4^{+2}3^{+3}$
16	8_5^{-1}, $[2^{+2}8_5^{-1}]$, $2^{+2}8_1^{+1}$, $2_0^{+2}8_1^{+1}$, $2_6^{-2}8_3^{-1}$, $2_7^{+1}4_1^{+1}8_1^{+1}$, $4^{+2}8_1^{+1}$, $[4^{+2}8_5^{-1}]$, $2_5^{-1}8_4^{-2}$, $4_2^{-2}8_3^{-1}$, $2_7^{+1}8_2^{+2}$, $4_1^{+1}8_0^{+2}$, $2_1^{+1}16^{+2}$, 8_1^{+3}, $4_1^{+1}16^{+2}$
20	$2_5^{+3}5^{+1}$, $2_1^{+3}5^{-1}$, $2_1^{+1}5^{+2}$, $2_1^{+3}5^{+2}$, $2_1^{+1}5^{-3}$, $2_1^{+1}4^{+2}5^{+2}$
24	$4_7^{+1}3^{-1}$, $4_1^{+1}3^{-2}$, $2_0^{+2}4_3^{-1}3^{+1}$, $[2^{+2}4_7^{+1}3^{-1}]$, $2^{+2}4_3^{-1}3^{+1}$, $2_0^{+2}4_7^{-1}3^{-1}$, $2_3^{-1}4_4^{-2}3^{+1}$, $4_3^{-1}3^{-3}$, $2_0^{+2}4_1^{+1}3^{-2}$, $2^{+2}4_1^{+1}3^{-2}$, $4_3^{-3}3^{+1}$, $2_3^{-1}8^{-2}3^{+1}$, $2^{+2}4_3^{-1}3^{-3}$
28	$2_7^{+3}7^{+1}$, $2_1^{+1}7^{-2}$, $2_7^{+1}4^{+2}7^{+1}$, $2_7^{+1}7^{-3}$
32	$2_0^{+2}16_1^{+1}$, $2^{+2}16_1^{+1}$, $2_1^{+1}4_7^{+1}16_1^{+1}$, $4_0^{+2}16_1^{+1}$, $4^{+2}16_1^{+1}$, $2_7^{+1}8_1^{+1}16_1^{+1}$, $8^{+2}16_1^{+1}$
36	$2_1^{+1}9^{+1}$, $2_7^{+1}3^{-1}9^{-1}$, $2_1^{+3}9^{-1}$, $2_1^{+3}9^{+1}$, $2_1^{+1}9^{-2}$, $2_1^{+1}9^{+2}$, $2_1^{+1}3^{-2}9^{-1}$, $2_7^{+3}3^{-1}9^{-1}$, $2_1^{+1}4^{+2}9^{-1}$, $2_1^{+1}9^{+3}$
40	$4_1^{+1}5^{-1}$, $2^{+2}4_5^{-1}5^{+1}$, $2_0^{+2}4_1^{+1}5^{-1}$, $4_5^{-1}5^{-2}$, $4_1^{+1}5^{+2}$, $4_1^{-3}5^{-1}$, $2^{+2}4_1^{+1}5^{+2}$, $4_5^{-1}5^{+3}$
44	$2_7^{+3}11^{-1}$
48	$8_7^{+1}3^{-1}$, $8_1^{+1}3^{-2}$, $2^{+2}8_3^{-1}3^{-1}$, $2_4^{-2}8_3^{-1}3^{-1}$, $2_6^{+2}8_1^{+1}3^{-1}$, $8_3^{-1}3^{+3}$, $2^{+2}8_1^{+1}3^{-2}$, $2_2^{-2}8_3^{-1}3^{+2}$, $4^{+2}8_3^{-1}3^{-1}$, $2_3^{-1}8_4^{-2}3^{-1}$, $2^{+2}8_3^{-1}3^{+3}$
52	$2_1^{+1}13^{+2}$

Continued on next page

Table 9 – continued from previous page

level	$\frac{3}{2}$-simple modules
56	$4_3^{-1}7^{-1}$, $2_0^{+2}4_7^{+1}7^{+1}$, $2^{+2}4_7^{+1}7^{+1}$
60	$2_7^{+1}3^{-1}5^{-1}$, $2_1^{+1}3^{-2}5^{-1}$, $2_7^{+3}3^{+1}5^{+1}$, $2_7^{+3}3^{-1}5^{-1}$, $2_7^{+1}3^{+1}5^{-2}$, $2_7^{+1}3^{-3}5^{+1}$, $2_7^{+1}4^{+2}3^{+1}5^{+1}$
64	32_5^{-1}, $2_0^{+2}32_1^{+1}$, $2_6^{-2}32_3^{-1}$, $2^{+2}32_1^{+1}$, $2_0^{+2}32_5^{-1}$, $2_7^{+1}4_1^{+1}32_1^{+1}$, $4_2^{-2}32_3^{-1}$, $4^{+2}32_1^{+1}$, $4^{-2}32_5^{-1}$, $[4^{-2}32_1^{+1}]$
72	$4_1^{+1}9^{-1}$, $4_7^{+1}3^{-1}9^{+1}$, $4_3^{-1}3^{+1}9^{+1}$, $4_3^{-1}3^{+1}9^{-1}$, $2^{+2}4_1^{+1}9^{-1}$, $2^{+2}4_1^{+1}9^{+1}$, $4_1^{+1}3^{-2}9^{+1}$, $4_1^{+1}9^{-2}$, $4_3^{-1}3^{+1}9^{-2}$
80	$8_1^{+1}5^{-1}$, $2_2^{-2}8_3^{-1}5^{+1}$, $2^{+2}8_5^{-1}5^{-1}$, $2_0^{+2}8_1^{+1}5^{-1}$, $8_5^{-1}5^{+2}$, $8_1^{+1}5^{+2}$, $8_5^{-1}5^{-3}$
84	$2_1^{+1}3^{+1}7^{+1}$, $2_7^{+1}3^{+2}7^{-1}$, $2_7^{+1}3^{-1}7^{-2}$
88	$2^{+2}4_3^{-1}11^{+1}$
92	$2_7^{+3}23^{+1}$
96	$16_7^{+1}3^{-1}$, $16_1^{+1}3^{-2}$, $2^{+2}16_3^{-1}3^{+1}$, $2_0^{+2}16_3^{-1}3^{+1}$, $16_3^{-1}3^{-3}$, $4^{+2}16_3^{-1}3^{+1}$
100	$2_1^{+1}25^{+1}$, $2_1^{+1}5^{-1}25^{-1}$
104	$4_1^{+1}13^{-1}$
108	$2_1^{+1}3^{+1}27^{-1}$, $2_7^{+1}3^{-2}27^{-1}$
112	$2^{+2}8_7^{+1}7^{+1}$
120	$4_7^{+1}3^{+1}5^{+1}$, $2^{+2}4_7^{+1}3^{-1}5^{-1}$
128	$2^{+2}64_1^{+1}$
132	$2_7^{+1}3^{-2}11^{-1}$
140	$2_7^{+1}5^{-1}7^{+1}$
144	$8_5^{-1}9^{+1}$, $8_5^{-1}9^{-1}$, $8_1^{+1}9^{+1}$, $2^{+2}8_1^{+1}9^{-1}$, $2_0^{+2}8_1^{+1}9^{-1}$, $8_1^{+1}3^{-2}9^{-1}$, $4^{+2}8_1^{+1}9^{-1}$
156	$2_7^{+1}3^{+1}13^{+1}$
160	$2^{+2}16_5^{-1}5^{+1}$
176	$8_7^{+1}11^{-1}$
180	$2_7^{+1}3^{+1}9^{-1}5^{+1}$
192	$2^{+2}32_3^{-1}3^{-1}$
196	$2_1^{+1}49^{-1}$
200	$4_1^{+1}25^{-1}$
224	$16_3^{-1}7^{-1}$
240	$2^{+2}8_7^{+1}3^{+1}5^{+1}$
288	$2^{+2}16_1^{+1}9^{+1}$

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