RESEARCH STATEMENT

SCOTT SCHMIEDING

My research lies in the realm of dynamical systems, specifically in topological dynamics, symbolic dynamics, and ergodic theory, with connections to aperiodic tilings, combinatorics, and K-theory.

Broadly speaking, dynamical systems concerns the behavior of systems evolving in time. For topological dynamics, this often takes the following form: given a continuous map $f\colon X \to X$ of some topological space X, what is the structure of the orbits of points under iteration of f?

Much of my work concerns the class of dynamical systems called symbolic systems. Historically symbolic systems grew out of the notion of *coding*: given $f\colon X \to X$ and a partition of X into finitely many labeled pieces, the f-orbit $\{f^i(x)\}$ of a point x generates a sequence of symbols corresponding to which piece of the partition $f^i(x)$ lies in. Coding allows one to study general dynamical systems via symbolic ones, deducing dynamical properties from an associated symbolic system. The study of symbolic systems has grown considerably, having strong connections to both topological dynamics and ergodic theory, as well as C^*-algebras and automata theory.

A broad goal in my research is studying the rich interplay between algebraic structures and dynamical phenomena arising from topological dynamical systems. Two areas of my work exemplifying this are automorphism groups arising from symbolic systems, and problems concerning shifts of finite type and their connections to algebraic K-theory. I'll focus here on three areas of my research: **Symmetry groups of dynamical systems, Shifts of finite type and algebraic K-theory**, and **Aperiodic tilings**, very briefly described below. More detailed descriptions follow after.

The first area (**Symmetry groups of dynamical systems** below) concerns various groups of self-symmetries associated to dynamical systems, which have been of interest from both a dynamical perspective and a group theoretic perspective. A long standing problem in the area has been to determine when the automorphism groups of full shifts on different alphabet sizes are isomorphic; in particular, can the automorphism groups of full shifts on two symbols and three symbols be isomorphic? Recently I introduced a stabilized setting for studying these groups, first appearing in [21]. I then completely resolved this problem in the stabilized setting, introducing a new entropy-like quantity for groups, called local $\mathcal{P}$ entropy, which can be used as an invariant to distinguish groups up to isomorphism [42]. One of my main results from [42] is the following.

Theorem 1 ([42]). *Suppose (X,σ_X) is a mixing shift of finite type. There is a class of finite groups $\mathcal{P}_X$ such that the local $\mathcal{P}_X$ entropy of the stabilized automorphism group $\left(\mathrm{Aut}^{(\infty)}(\sigma_X),\sigma_X\right)$ is equal to the topological entropy of the system (X,σ_X). If (Y,σ_Y) is a shift of finite type such that the groups $\mathrm{Aut}^{(\infty)}(\sigma_X)$ and $\mathrm{Aut}^{(\infty)}(\sigma_Y)$ are isomorphic, then the ratio of the topological entropies $\frac{h_{top}(\sigma_X)}{h_{top}(\sigma_Y)}$ is rational. In particular, the stabilized automorphism groups of full shifts on m and n symbols are isomorphic if and only if $m^k = n^j$ for some k,j.*

The second area (**Shifts of finite type and K-theory** below) concerns my work on K-theoretic connections with shifts of finite type, which stems from the (still very open) classification problem: to determine when two shifts of finite type are conjugate. Shifts of finite type have classically had a strong connection to linear algebra, with dynamical invariants being computable in terms of algebraic data coming from associated matrices. A matrix framework for the classification problem was given by Williams in [50], introducing a pair of equivalence relations on matrices called shift equivalence and strong shift equivalence (definitions given in the relevant section below). These relations are also of interest algebraically. Together with Mike Boyle, in [6] we connected the algebraic component of these relations to algebraic K-theory, and used our results to resolve various other open problems [1, 5]. One of our main results was the classification, in terms of certain K-theoretic groups, of the refinement of shift equivalence by strong shift equivalence over an arbitrary ring $\mathcal{R}$.

Theorem 2 (Boyle-Schmieding [6])**.** *For any ring $\mathcal{R}$ and square matrix A over $\mathcal{R}$, the set of strong shift equivalence classes of matrices shift equivalent to A is in bijective correspondence with a quotient of the algebraic K-group $NK_1(\mathcal{R})$.*

The third area (**Aperiodic tilings** below) is my recent work in aperiodic tilings, including studying the deviations of ergodic averages for various classes of aperiodic tilings in $\mathbb{R}^d$ and mapping class groups of tiling spaces. Together with Trevino we proved deviation results for a class of random tilings, obtained via random iterations of substitution procedures. The work on mapping class groups overlaps with the previously mentioned symmetry groups, and includes, in recent work with Yang [46], a characterization of the mapping class groups of one-dimensional substitutions tiling spaces.

Symbolic dynamics. The central object of study in symbolic dynamics is that of a subshift[1]: a dynamical system comprised of the set of bi-infinite strings over a finite alphabet, together with the map which shifts the symbols in a string one place to the left.

More formally, for a finite set $\mathcal{A}$ (an *alphabet*) we may equip $\mathcal{A}^{\mathbb{Z}}$ with a metric making it into a compact metric space; we can identify points $x \in \mathcal{A}^{\mathbb{Z}}$ with bi-infinite sequences $x = (x_i)_{i \in \mathbb{Z}}$ of symbols from $\mathcal{A}$. The space $\mathcal{A}^{\mathbb{Z}}$ carries a shift map $\sigma \colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$ which is a homeomorphism of $\mathcal{A}^{\mathbb{Z}}$ to itself. When $\mathcal{A}$ has n elements we call this system a *full shift on n symbols* and denote it by (X_n, σ_n). A *subshift* (X, σ) is a system obtained as the restriction of the shift map σ to some closed σ-invariant subspace X of a full shift. Subshifts are precisely the class of expansive zero-dimensional dynamical systems, and exhibit an enormously wide range of dynamical behavior.

Of particular importance among subshifts are the *shifts of finite type*, which are subshifts obtained by forbidding a finite set of finite words from appearing. Along with other areas, shifts of finite type arise from codings associated to Markov partitions, and may be characterized as precisely the class of zero-dimensional hyperbolic systems.

A theme throughout some of my work on symbolic systems is using various notions of stabilization, which can be seen in the following parts **1.** and **2.**. Very informally speaking, here by stabilization we mean enlarging a considered class of objects to some stable version, often yielding more desirable structural properties. (For a general exposition on some uses of stabilization in

[1]We will only discuss here symbolic systems over $\mathbb{Z}$.

symbolic dynamics, see [7]). Stabilization techniques have proven to be a powerful way to approach what are otherwise intractable problems. Some of our forms of stabilization are closely (and sometimes directly) related to, and influenced by, the process of stabilizing inherent in algebraic K-theory.

1. Symmetry groups of dynamical systems The study of various symmetry groups associated to dynamical systems has a rich history. Among such groups in topological dynamics, one of the most fundamental is the automorphism group of a system. For a homeomorphism $f\colon X \to X$, an automorphism of the system (X, f) is a homeomorphism $\phi\colon X \to X$ such that $\phi f = f\phi$. The set of automorphisms of (X, f) forms a group we denote $\mathrm{Aut}(X, f)$.

A broad question guiding some of my work in this area is the following: how does the algebraic structure of the symmetry group (e.g. the automorphism group) depend on the dynamics of the system in question?

Automorphism groups have been of considerable interest especially in the symbolic setting (e.g. [4, 41, 36, 20, 25, 24, 23, 37, 38, 12, 13, 14, 15, 9, 11, 10] to list just some), with pioneering work going back to Hedlund [22] and others who initiated the study of the automorphism groups of full shifts. The Curtis-Hedlund-Lyndom Theorem [22] implies the automorphism group of a subshift is always countable, and the enormous range of possible dynamics achievable by subshifts is complemented by a wide range of group structures that can appear as automorphism groups of symbolic systems.

Arguably the most heavily studied are the automorphism groups of shifts of finite type, especially the automorphism groups of full shifts(e.g. [24, 23, 25, 20, 36, 4, 41, 37, 38] to name a few). It is known that the automorphism group of a (non-trivial) shift of finite type is always 'large': it contains all finite groups, and free groups [4]. But many properties remain unknown, and in general the automorphism groups of shifts of finite type, especially full shifts, remain a fundamental and mysterious class of groups that have been the object of intense study (a survey of one aspect of this can be found in Sec. 7 of [7]).

A fundamental open question in this area is the following.

Question 1: If $m, n \geq 2$ are natural numbers and the automorphism groups $\mathrm{Aut}(X_m, \sigma_m)$ and $\mathrm{Aut}(X_n, \sigma_n)$ are isomorphic, must $m = n$?

Question 1 is the most important and fundamental case of the following more general problem.

Question 2: If (X, σ_X) and (Y, σ_Y) are shifts of finite type for which the groups $\mathrm{Aut}(X, \sigma_X)$ and $\mathrm{Aut}(Y, \sigma_Y)$ are isomorphic, must (X, σ_X) be topologically conjugate[2] to either (Y, σ_Y) or (Y, σ_Y^{-1})?

The only cases for which Question 2 has been addressed use early techniques, first pointed out in [4]: for example, it was known since the '80s that a theorem of Ryan [32, 33] implies for any prime p that $\mathrm{Aut}(\sigma_p)$ and $\mathrm{Aut}(\sigma_{p^2})$ are not isomorphic groups.

[2]Two systems $(X, f), (Y, g)$ are topologically conjugate if there is a homeomorphism $h\colon X \to Y$ such that $hf = gh$; one should think of topological conjugacy as isomorphism in the setting of topological dynamics.

We note a problem analogous to Question 2 has been studied in the context of minimal Cantor systems: in [18], Giordano, Putnam, and Skau showed the topological full group is a complete invariant of a Cantor minimal system up to flip conjugacy [18].

Recently in [21] we introduced a stabilized approach to the study of automorphism groups. For a system (X,T) we defined the *stabilized automorphism group* of (X,T) to be the group

$$\mathrm{Aut}^{(\infty)}(X,T) = \bigcup_{k=1}^{\infty} \mathrm{Aut}(X,T^k).$$

In other words, the stabilized automorphism group of (X,T) consists of all homeomorphisms that commute with *some* power of T. The stabilized automorphism group always contains the classical automorphism group, but in general can be much larger.

In [21] together with Hartman and Kra we studied the stabilized automorphism group for shifts of finite type, showing they are more accessible than the classical ones. We showed they are countable, centerless, non-residually finite groups which acts minimally and uniquely ergodic on their shift space. Furthermore, we showed that a certain linear representation of the automorphism group (known as the dimension representation) extends to a stabilized linear representation, and this stabilized linear representation in fact coincides with the abelianization of the stabilized group. A condensed form of our results proved in [21] is the following.

Theorem 3 (Hartman-Kra-Schmieding, [21])**.** *For a full shift (X_n,σ_n) on n symbols, the abelianization of $\mathrm{Aut}^{(\infty)}(\sigma_n)$ is isomorphic to $\mathbb{Z}^{\omega(n)}$ (where $\omega(n)$ denotes the number of distinct prime factors of n). In particular, if $\omega(m) \neq \omega(n)$ then $\mathrm{Aut}^{(\infty)}(\sigma_m)$ and $\mathrm{Aut}^{(\infty)}(\sigma_n)$ are not isomorphic. Moreover, the commutator subgroup of $\mathrm{Aut}^{(\infty)}(\sigma_n)$ is a simple group.*

We also proved results concerning the abelianization of the stabilized automorphism group for the general class of mixing shifts of finite type.

The proof of simplicity of the commutator subgroup of $\mathrm{Aut}^{(\infty)}(\sigma_n)$ required very different techniques than what have normally been used to prove simplicity of certain dynamically defined groups in the Cantor setting.[3] The techniques should have more to offer in the realm of studying groups of dynamical origin. Also, it would be very interesting to know whether an analogous simplicity result holds for every mixing shift of finite type[4].

The results in Theorem 3 left open the most important cases of distinguishing the stabilized automorphism groups of full shifts: when $\omega(m) = \omega(n)$. To try to illustrate why these cases are harder, we note that $\mathrm{Aut}^{(\infty)}(\sigma_2)$ and $\mathrm{Aut}^{(\infty)}(\sigma_3)$ are each respectively an extension of $\mathbb{Z}$ by some infinite simple groups. Subsequent to [21] I resolved this case, and gave in [42] a complete classification, up to isomorphism, of the stabilized automorphism groups of full shifts.

The key idea in [42] was the introduction of new entropy-like quantity for groups[5] called local $\mathcal{P}$ entropy. Local $\mathcal{P}$ entropy is defined with respect to a chosen class $\mathcal{P}$ of finite groups which is closed under isomorphism. As a rough idea of what local $\mathcal{P}$ entropy measures, fix such a class $\mathcal{P}$,

[3] A key reason highlighting the difference is the following: if $\phi \neq \mathrm{id}$ is an automorphism of an irreducible shift of finite type, then ϕ can not be the identity on any nontrivial open set.

[4] For general mixing shifts of finite type, the commutator subgroup of the stabilized automorphism group may not be simple; but there is a natural candidate for simplicity called the subgroup of stabilized inert automorphisms.

[5] More precisely, it is defined for *pointed groups*, i.e. pairs (G,g) where g is a distinguished element in G.

consider some group G with some distinguished element $g \in G$, and consider the conjugation map $C_g \colon G \to G$ given by $C_g(h) = g^{-1}hg$. One can try to measure the growth rate of the C_g-periodic point sets $\mathrm{Fix}(C_{g^n})$, which are precisely the centralizers of g^n in G; but these sets may be infinite. To proceed, we instead approximate the centralizers using groups belonging to the chosen class $\mathcal{P}$ (which are by definition finite), and then consider the doubly exponential[6] growth rate (in n) of such $\mathcal{P}$-approximations. This leads to a nonnegative quantity $h_{\mathcal{P}}(G,g)$ called the local $\mathcal{P}$ entropy of the pair (G,g). A key thing I proved in [42] is that the local $\mathcal{P}$ entropy of a pair (G,g) is an invariant of isomorphism of the pair: if there is an isomorphism of groups $G \xrightarrow{\cong} H$ taking $g \in G$ to $h \in H$, then assuming the local $\mathcal{P}$ entropies exist, we have $h_{\mathcal{P}}(G,g) = h_{\mathcal{P}}(H,h)$[7].

In [42] I showed that for any mixing shift of finite type, the local $\mathcal{P}$ entropy of the stabilized automorphism group recovers the topological entropy of the respective shift system. In particular, I proved the following.

Theorem 4 (Schmieding, [42])**.** *If (X_A, σ_A) is any non-trivial mixing shift of finite type then each of the following hold:*

(1) There is a class $\mathcal{P}_A$ of finite groups such that the local $\mathcal{P}_A$ entropy of the pair $(\mathrm{Aut}^{(\infty)}(\sigma_A), \sigma_A)$ is equal to the topological entropy of (X_A, σ_A).

(2) If (X_B, σ_B) is any other shift of finite type such that the stabilized automorphism groups $\mathrm{Aut}^{(\infty)}(\sigma_A)$ and $\mathrm{Aut}^{(\infty)}(\sigma_B)$ are isomorphic, then the ratio of the topological entropies $\frac{h_{top}(\sigma_A)}{h_{top}(\sigma_B)}$ is rational.

One consequence of the above is a complete classification of the stabilized automorphism groups of full shifts, answering Question 1 in the stabilized setting.

Corollary 5 (Schmieding, [42])**.** *For natural numbers $m, n \geq 2$, the stabilized automorphism groups $\mathrm{Aut}^{(\infty)}(\sigma_m)$ and $\mathrm{Aut}^{(\infty)}(\sigma_n)$ are isomorphic if and only if there exists natural numbers k, j such that $m^k = n^j$.*

Some of our motivation for introducing the stabilized automorphism groups comes from algebraic K-theory, in which the notion of stabilization plays a fundamental role. There are intriguing analogies between the stabilized linear group of a ring and the stabilized automorphism group $\mathrm{Aut}^{(\infty)}(\sigma_X)$ of a shift of finite type: for example, our results on the structure of the commutator subgroup of $\mathrm{Aut}^{(\infty)}(\sigma_n)$ parallel Whitehead's Theorem regarding the commutator of the general linear group of a ring.

Future work in this area involves developing and expanding on the theory of local $\mathcal{P}$ entropy, and applying it to other groups of dynamical origin. In addition, I propose studying local $\mathcal{P}$ entropy in the context of geometric group theory. Finally, a next step is seeing to what extent results and techniques from the stabilized setting can be applied to the classical non-stabilized problems.

There has also been recent interest in automorphism groups of other systems [16, 35, 9], especially low complexity subshifts (here low complexity means the growth of admissible words in the language of the subshift is 'slow'). Broadly speaking, a trend has started to emerge in structural

[6] A related quantity is defined by considering just exponential growth; here we'll consider only the one using doubly exponential.

[7] It is also proved in the same paper that if there is an injective homomorphism $G \to H$ taking g to h, then $h_{\mathcal{P}}(G,g) \leq h_{\mathcal{P}}(H,h)$.

results for these groups: when (X,σ) is a shift of finite type, $\mathrm{Aut}(X,\sigma)$ is quite complicated (not amenable, contains every finite group; see [4]), while recent results have indicated that when (X,σ) is a subshift of lower complexity, the structure of $\mathrm{Aut}(\sigma)$ is more constrained (e.g. [12, 13, 14, 15]). In recent work with Pavlov [27] we showed that in the very low complexity regime, the group $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ must be locally finite; in fact we showed that the class of locally finite groups is precisely the class of groups that can appear as $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ for some subshift (X,σ) of complexity $n(\log\log n)^{o(1)}$.

Finally, other symmetry groups are also of interest, including mapping class groups for symbolic systems. The mapping class group of a symbolic system is the group of isotopy classes of (orientation preserving) self-homeomorphisms of the mapping torus of the system, and has been studied in [3, 39, 40]. In [46], Yang and I studied the mapping class group for minimal subshifts, giving structural results in the low complexity setting. Our work there also connects with the study of aperiodic tilings mentioned in **3.** later.

2. Shifts of finite type and algebraic K-theory In [6], Boyle and I gave a more unified framework connecting shifts of finite type[8] to algebraic K-theory. To briefly describe how this connection works, to any shift of finite type one can associated a square matrix over the nonnegative integers $\mathbb{Z}_+$. In the foundational paper [50] Williams proved that two shifts of finite type are topologically conjugate if and only if their respective matrices belong to the same class of an equivalence relation on square matrices called strong shift equivalence. Two square matrices A,B over $\mathbb{Z}_+$ are elementary strong shift equivalent (over $\mathbb{Z}_+$) if there are matrices R,S over $\mathbb{Z}_+$ such that $A=RS, B=SR$, and strong shift equivalence is the transitive closure of elementary strong shift equivalence. Williams at the same time introduced a coarser equivalence relation called shift equivalence: two square matrices A,B are shift equivalent if there exists $l\geq 1$ and matrices R,S such that $A^l=RS, B^l=SR$, $AR=RB$, $BS=SA$. The strong shift and shift equivalence relations can be defined over any semiring, and are of interest from the point of view of both dynamical systems and algebra. To what degree shift equivalence implies strong shift equivalence over $\mathbb{Z}_+$ is a fundamental problem in symbolic dynamics rooted in the goal of classifying shifts of finite type up to topological conjugacy.

In [6], we developed a strong connection between the shift and strong shift equivalence relations and algebraic K-theory; in particular, we showed the difference between shift equivalence and strong shift equivalence over a ring is determined by certain algebraic K-groups associated to the ring. A version of one our main results is the following.

Theorem 6 (Boyle-Schmieding, [6]). *For any ring $\mathcal{R}$ and square matrix A over $\mathcal{R}$, the set of strong shift equivalence classes of matrices shift equivalent to A is in bijective correspondence with a quotient of the algebraic K-group $NK_1(\mathcal{R})$.*

It had been an open question of whether these relations were always the same, and our machinery not only showed they are not, but parametrizes the difference in terms of K-theoretic data. We subsequently used our results from [6] in other works. In [1], working in the setting where $\mathcal{R}=\mathbb{Z}G$ is the integral group ring of a finite group G, we resolved a conjecture of Bill Parry concerning group extensions of shifts of finite type. In [5], we used our results to address conjectures related to the spectrum of positive matrices over rings.

[8] Along with other symbolic systems; see [6].

In many ways, this K-theory framework associated to shifts of finite type is still in its infancy. Particular future work includes understanding the relationship with the positive K-theory outlined in [2], and Wagoner's work regarding the group K_2 in connection with shifts of finite type [48, 47, 49]. There are also several intriguing avenues considering the stabilized automorphism groups from a K-theoretic point of view. For example, there are strong analogies suggesting the commutator $[\mathrm{Aut}^{(\infty)}(\sigma_n), \mathrm{Aut}^{(\infty)}(\sigma_n)]$ of the stabilized automorphism group $\mathrm{Aut}^{(\infty)}(\sigma_n)$ might play the role of a stabilized elementary group of some unknown ring. One way to define K_2 of a ring $\mathcal{R}$ is $H_2(E(\mathcal{R}), \mathbb{Z})$, where $E(\mathcal{R})$ is the group of stabilized elementary matrices over $\mathcal{R}$. One might define an analogue of the algebraic K-group K_2 in the symbolic setting by considering $H_2([\mathrm{Aut}^{(\infty)}(\sigma_n), \mathrm{Aut}^{(\infty)}(\sigma_n)], \mathbb{Z})$.

3. Aperiodic tilings Central to the study of aperiodic order is the concept of an aperiodic tiling. Aperiodic tilings are intimately related to Delone sets in $\mathbb{R}^d$, which are uniformly discrete and relatively dense point sets in $\mathbb{R}^d$. While historically Delone sets are more closely connected to the theory of diffraction and quasicrystals, results may be transferred between the tiling world and the Delone world, and the choice of language is often a matter of convenience for the topic in question; we will state things here for Delone sets.

There is a metric d on the collection of Delone sets: two Delone sets are close if a small translate of one agrees with the other in a large ball around the origin (see [43] for a precise definition). To study the properties of a point pattern Λ in $\mathbb{R}^d$ it is fruitful to consider the hull associated to Λ

$$\Omega_\Lambda = \{\text{Delone sets } \Lambda' \mid \text{every patch of } \Lambda' \text{ is a translate of a patch of } \Lambda\}.$$

Most often some mild conditions on Λ are assumed under which Ω_Λ, together with the metric d, becomes a compact metric space, carrying an $\mathbb{R}^d$ action given by translation for which the system $(\Omega_\Lambda, \mathbb{R}^d)$ is uniquely ergodic (has only one ergodic probability measure) and minimal (there are no proper closed invariant subsets). This unique ergodic measure μ on Ω_Λ is closely connected to the theory of diffraction for point sets, and more generally, one may determine properties of Λ by studying the system $(\Omega_\Lambda, \mathbb{R}^d)$ using tools from both topology and dynamical systems.

In recent work, Trevino and I have studied the growth of norms of ergodic integrals for tiling systems coming from certain classes of Delone sets. In [43], we considered a class of tilings we call renormalizable of finite type, which includes all those associated to substitution tilings, as well as many more. A special case of our results is the following.

Theorem 7. [43] *There is $C > 0$, complex numbers $\{\nu_1, \ldots, \nu_k\}$ and k $\mathbb{R}^d$-invariant distributions $\mathcal{D}_1, \ldots, \mathcal{D}_k$ such that for any leaf-wise smooth transversally locally constant function f on Ω_Λ, if $\mathcal{D}_i(f) \neq 0$ and $\mathcal{D}_j(f) = 0$ for all $j < i$, then for some $\epsilon_i \in \{0,1\}$, any large enough ball B around the origin and any $\Lambda_0 \in \Omega_\Lambda$,*

$$\left| \int_B f(\Lambda_0 - s)\, ds \right| \leq C (\log T)^{\epsilon_i} T^{d \frac{\log |v_i|}{\log |v_1|}} ||f||_\infty.$$

This result paralleled work of Sadun in [34], and also Bufetov-Solomyak in [8] on limit theorems for substitution tilings; however the functions considered in [8] are different than ours. In [43] we also spelled out the connection between the asymptotic cycle and the *autocorrelation measure* associated to Λ. The autocorrelation measure was used by Dworkin in [17] to give a strong connection between the ergodic data of Ω_Λ and the diffraction spectrum of the Delone set Λ, a key concept in the atomic theory associated to Λ.

In [44], Trevino and I used our Theorem 7 to give a refinement of the convergence of Shubin's formula relating the trace per unit volume and the integrated density of states.

More recently, in [45], Trevino and I obtained an analogous result to Theorem 7 for a more general class of Delone sets built from a *random substitution procedure*. Random substitutions systems have been studied by many authors (for example [26, 19, 31, 30]) and has many parallels to the fusion framework presented in [28, 29]. In this framework, one starts with a finite collection of substitution rules and constructs an aperiodic tiling (and hence, a Delone set) by applying the substitutions according to a random sequence. Such examples are a departure from the classic stationary substitution systems, where the same substitution is used at every stage, and allow us to consider aperiodic sets that could be considered more 'generic'.

References

[1] M. Boyle and Scott Schmieding. Finite group extensions of shifts of finite type: K-theory, Parry and Livšic. *Ergodic Theory Dynam. Systems*, 37(4):1026–1059, 2017.

[2] M. Boyle and J. B. Wagoner. Positive algebraic K-theory and shifts of finite type. In *Modern dynamical systems and applications*, pages 45–66. Cambridge Univ. Press, Cambridge, 2004.

[3] Mike Boyle and Sompong Chuysurichay. The mapping class group of a shift of finite type. *J. Mod. Dyn.*, 13:115–145, 2018.

[4] Mike Boyle, Douglas Lind, and Daniel Rudolph. The automorphism group of a shift of finite type. *Trans. Amer. Math. Soc.*, 306(1):71–114, 1988.

[5] Mike Boyle and Scott Schmieding. Strong shift equivalence and the generalized spectral conjecture for nonnegative matrices. *Linear Algebra Appl.*, 498:231–243, 2016.

[6] Mike Boyle and Scott Schmieding. Strong shift equivalence and algebraic K-theory. *J. Reine Angew. Math.*, 752:63–104, 2019.

[7] Mike Boyle and Scott Schmieding. Symbolic dynamics and the stable algebra of matrices. *arXiv e-prints*, page arXiv:2006.01051, June 2020.

[8] Alexander I. Bufetov and Boris Solomyak. Limit theorems for self-similar tilings. *Comm. Math. Phys.*, 319(3):761–789, 2013.

[9] Ethan M. Coven. Endomorphisms of substitution minimal sets. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 20:129–133, 1971/72.

[10] Van Cyr, John Franks, and Bryna Kra. The spacetime of a shift endomorphism. *Trans. Amer. Math. Soc.*, 371(1):461–488, 2019.

[11] Van Cyr, John Franks, Bryna Kra, and Samuel Petite. Distortion and the automorphism group of a shift. *J. Mod. Dyn.*, 13:147–161, 2018.

[12] Van Cyr and Bryna Kra. The automorphism group of a shift of linear growth: beyond transitivity. *Forum Math. Sigma*, 3:e5, 27, 2015.

[13] Van Cyr and Bryna Kra. The automorphism group of a minimal shift of stretched exponential growth. *J. Mod. Dyn.*, 10:483–495, 2016.

[14] Van Cyr and Bryna Kra. The automorphism group of a shift of subquadratic growth. *Proc. Amer. Math. Soc.*, 144(2):613–621, 2016.

[15] Sebastián Donoso, Fabien Durand, Alejandro Maass, and Samuel Petite. On automorphism groups of low complexity subshifts. *Ergodic Theory Dynam. Systems*, 36(1):64–95, 2016.

[16] Sebastian Donoso, Fabien Durand, Alejandro Maass, and Samuel Petite. On automorphism groups of Toeplitz subshifts. *Discrete Anal.*, pages Paper No. 11, 19, 2017.

[17] Steven Dworkin. Spectral theory and x-ray diffraction. *J. Math. Phys.*, 34(7):2965–2967, 1993.

[18] Thierry Giordano, Ian F. Putnam, and Christian F. Skau. Full groups of Cantor minimal systems. *Israel J. Math.*, 111:285–320, 1999.

[19] Philipp Gohlke, Dan Rust, and Timo Spindeler. Shifts of finite type and random substitutions. *arXiv e-prints*, page arXiv:1712.05340, December 2017.

[20] Pierre Guillon and Ville Salo. Distortion in one-head machines and cellular automata. In *Cellular automata and discrete complex systems*, volume 10248 of *Lecture Notes in Comput. Sci.*, pages 120–138. Springer, Cham, 2017.

[21] Yair Hartman, Bryna Kra, and Scott Schmieding. The Stabilized Automorphism Group of a Subshift. *International Mathematics Research Notices*, to appear, 2021, arXiv:2001.09530.

[22] G. A. Hedlund. Endomorphisms and automorphisms of the shift dynamical system. *Math. Systems Theory*, 3:320–375, 1969.

[23] K. H. Kim and F. W. Roush. On the automorphism groups of subshifts. *Pure Math. Appl. Ser. B*, 1(4):203–230 (1991), 1990.

[24] K. H. Kim and F. W. Roush. On the structure of inert automorphisms of subshifts. *Pure Math. Appl. Ser. B*, 2(1):3–22, 1991.

[25] Johan Kopra. Glider automata on all transitive sofic shifts. *arXiv e-prints*, page arXiv:2002.05964, February 2020.

[26] Eden Miro, Dan Rust, Lorenzo Sadun, and Gwendolyn S. Tadeo. Topological Mixing of Random Substitutions. *arXiv e-prints*, page arXiv:2103.02361, March 2021.

[27] Ronnie Pavlov and Scott Schmieding. Local finiteness and automorphism groups of low complexity subshifts. *arXiv e-prints*, page arXiv:2107.06062, July 2021.

[28] Natalie Priebe Frank and Lorenzo Sadun. Fusion: a general framework for hierarchical tilings of $\mathbb{R}^d$. *Geom. Dedicata*, 171:149–186, 2014.

[29] Natalie Priebe Frank and Lorenzo Sadun. Fusion tilings with infinite local complexity. *Topology Proc.*, 43:235–276, 2014.

[30] Dan Rust. Periodic points in random substitution subshifts. *Monatsh. Math.*, 193(3):683–704, 2020.

[31] Dan Rust and Timo Spindeler. Dynamical systems arising from random substitutions. *Indag. Math. (N.S.)*, 29(4):1131–1155, 2018.

[32] J. Patrick Ryan. The shift and commutativity. *Math. Systems Theory*, 6:82–85, 1972.

[33] J. Patrick Ryan. The shift and commutivity. II. *Math. Systems Theory*, 8(3):249–250, 1974/75.

[34] Lorenzo Sadun. Exact regularity and the cohomology of tiling spaces. *Ergodic Theory Dynam. Systems*, 31(6):1819–1834, 2011.

[35] Ville Salo. Toeplitz subshift whose automorphism group is not finitely generated. *Colloq. Math.*, 146(1):53–76, 2017.

[36] Ville Salo. A note on subgroups of automorphism groups of full shifts. *Ergodic Theory Dynam. Systems*, 38(4):1588–1600, 2018.

[37] Ville Salo. No Tits alternative for cellular automata. *Groups Geom. Dyn.*, 13(4):1437–1455, 2019.

[38] Ville Salo. Conjugacy of reversible cellular automata. *arXiv e-prints*, page arXiv:2011.07827, November 2020.

[39] Ville Salo. Thompson's V in MCG of mixing SFT by PW-linear homeos. *arXiv e-prints*, page arXiv:2103.15539, March 2021.

[40] Ville Salo. Veelike actions and the MCG of a mixing SFT. *arXiv e-prints*, page arXiv:2103.15505, March 2021.

[41] Scott Schmieding. Automorphisms of the shift: Lyapunov exponents, entropy, and the dimension representation. *Ergodic Theory Dynam. Systems*, 40(9):2552–2570, 2020.

[42] Scott Schmieding. Local $\mathcal{P}$ entropy and stabilized automorphism groups of subshifts. *Inventiones Mathematicae*, to appear, 2021, arXiv:2007.02183.

[43] Scott Schmieding and Rodrigo Treviño. Self affine Delone sets and deviation phenomena. *Comm. Math. Phys.*, 357(3):1071–1112, 2018.

[44] Scott Schmieding and Rodrigo Treviño. Traces of random operators associated with self-affine Delone sets and Shubin's formula. *Ann. Henri Poincaré*, 19(9):2575–2597, 2018.

[45] Scott Schmieding and Rodrigo Treviño. Random substitution tilings and deviation phenomena. *Discrete Contin. Dyn. Syst.*, 41(8):3869–3902, 2021.

[46] Scott Schmieding and Kitty Yang. The mapping class group of a minimal subshift. *Colloq. Math.*, 163(2):233–265, 2021.

[47] J. B. Wagoner. Gyration numbers, fixed points and K_2. *Bull. London Math. Soc.*, 19(5):474–480, 1987.

[48] J. B. Wagoner. Markov partitions and K_2. *Inst. Hautes Études Sci. Publ. Math.*, (65):91–129, 1987.

[49] J. B. Wagoner. Strong shift equivalence theory and the shift equivalence problem. *Bull. Amer. Math. Soc. (N.S.)*, 36(3):271–296, 1999.
[50] R. F. Williams. Classification of subshifts of finite type. *Ann. of Math. (2)*, 98:120–153, 1973; erratum, ibid. 99:380–381, 1974.