Singularity Functions

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1 Description of Singularity Functions

To handle the discontinuities in $V(x)$ and $M(x)$ curves we introduce a family of functions called *singularity functions*. We define the function as:

$$f(x) \equiv < x - a >^n = \begin{cases} (x-a)^n & x \geq a, \\ 0 & x < a. \end{cases} \tag{1}$$

The basic singularity functions are shown in Figure 1. They are (from top to bottom): *unit doublet, unit impulse, unit step, unit ramp, and unit acceleration*. The unit impulse is sometimes referred to as the *Dirac delta* function. The integration rule for singularity functions is:

$$\int_{-\infty}^{x} < x - a >^n dx = \begin{cases} < x-a >^{-1} & n = -2, \\ < x-a >^{0} & n = -1, \\ \frac{< x-a >^{n+1}}{n+1} & n \geq 0. \end{cases} \tag{2}$$

The loading of beams can be determined from a superposition of singularity functions for the load distribution function $q(x)$. The unit doublet is the distribution function representation for the applied moment and the unit impulse is the representation for an applied load. For example, an applied torque of M_0 at point $x = a$ is $q(x) = M_0 < x - a >^{-2}$. Once we have the

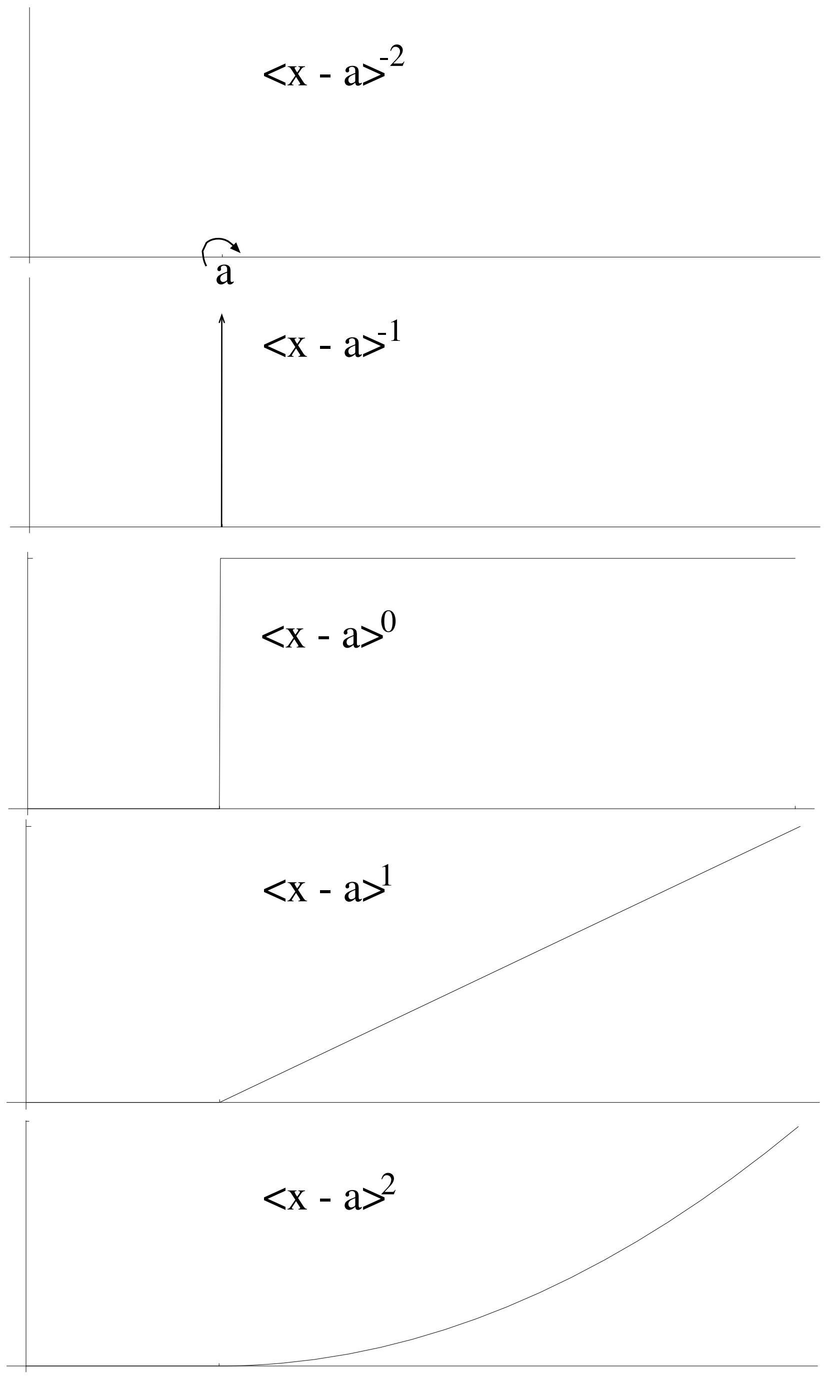


Figure 1: Basic Singularity Functions

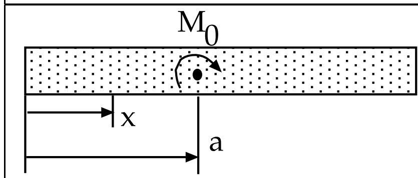
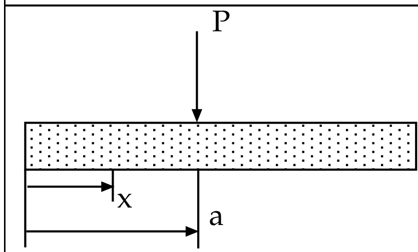
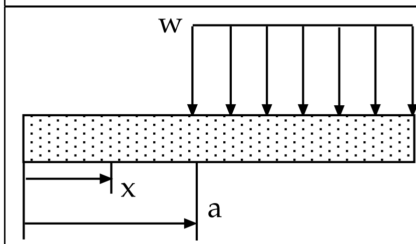
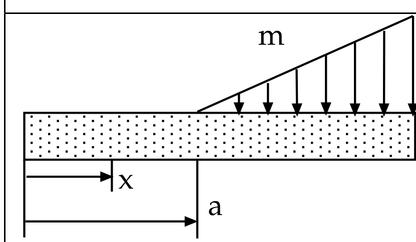
distribution function $q(x)$, we can integrate to get the shear $V(x)$ and the moment $M(x)$ functions.

$$V(x) = -\int q(x) + C_1 \tag{3}$$

$$M(x) = -\int V(x) + C_1 < x >^0 + C_2 \tag{4}$$

Integration of the singularity functions in the load distribution and the resulting internal shear force distribution and internal bending moment distribution is summarised in Table 1.

Table 1: Summary of Singularity Functions

Loading	Distribution q(x)	Shear $= -\int q(x)dx$	Moment $= -\int q(x)dx$
	$q = M_0 < x-a >^{-2}$	$V = -M_0 < x-a >^{-1}$	$M = M_0 < x-a >^{0}$
	$q = -P < x-a >^{-1}$	$V = P < x-a >^{0}$	$M = -P < x-a >^{1}$
	$q = -w < x-a >^{0}$	$V = w < x-a >^{1}$	$M = -\frac{w}{2} < x-a >^{2}$
	$q = -m < x-a >^{1}$	$V = \frac{m}{2} < x-a >^{2}$	$M = -\frac{m}{6} < x-a >^{3}$

2 Example 1

To illustrate the use of singularity functions in getting the shear and moment equations, consider this simple example. Solving for the reactions:

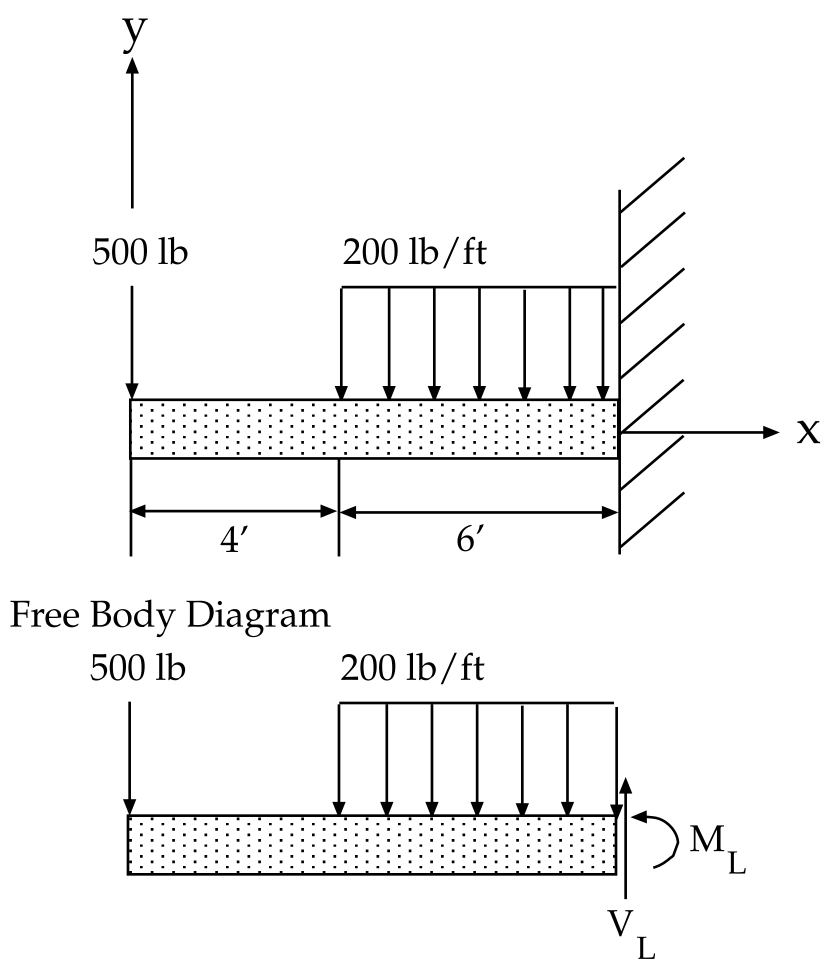


Figure 2: Simple Loading Case

$$V_L = 500 + 200(6) = 1700lb. \tag{5}$$

$$M_L = -500(10) - 1200(3) = -8600lb. \cdot ft. \tag{6}$$

First, find q(x):

$$q(x) = -500 < x >^{-1} -200 < x-4 >^{0} +1700 < x-10 >^{-1} +8600 < x-10 >^{-2} \tag{7}$$

To get the shear, we integrate the loading according as shown below:

$$V(x) = -\int_0^x q(x)dx \tag{8}$$

substituting the loading and integrating:

$$V(x) = 500 < x >^{0} +200 < x-4 >^{1} -1700 < x-10 >^{0} -8600 < x-10 >^{-1} \tag{9}$$

We can now integrate the shear to get the bending moment equation as shown below:

$$M(x) = -\int_0^x V(x)dx \tag{10}$$

$$M(x) = -500 < x >^{1} - \frac{200}{2} < x-4 >^{2} +1700 < x-10 >^{1} +8600 < x-10 >^{0} \tag{11}$$

$$M(x) = -500x - 100 < x-4 >^{2} +8600 < x-10 >^{0} \tag{12}$$

The shear and bending moment equations can be represented in graphical form:

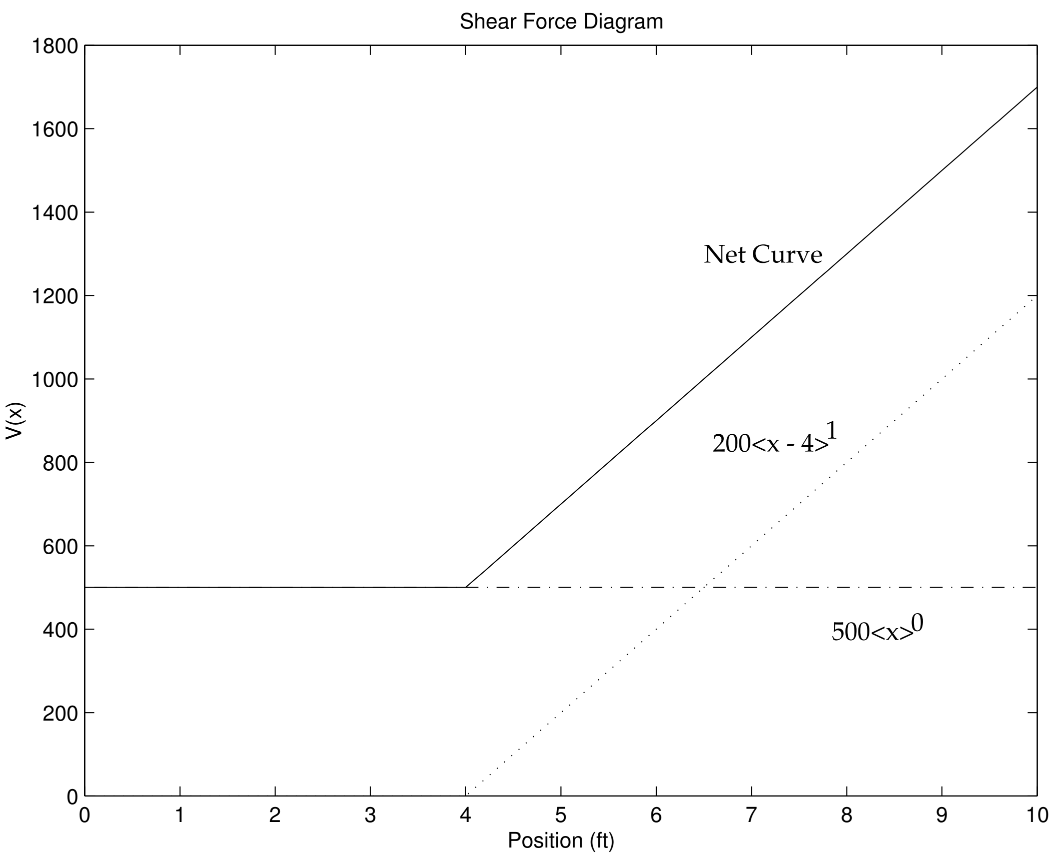


Figure 3: Shear Force Diagram for Example 1

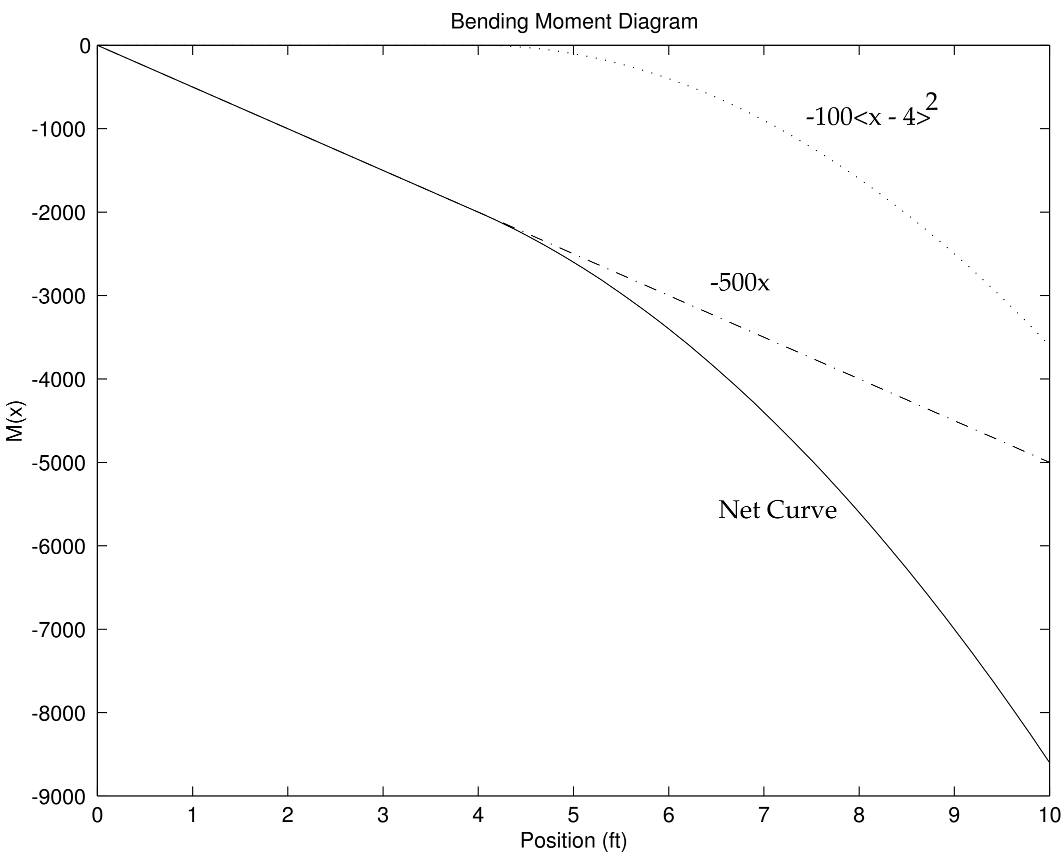


Figure 4: Bending Moment Diagram for Example 1

3 Example 2

To compare and contrast the two techniques for getting the shear and moment equations we will look at the following example.

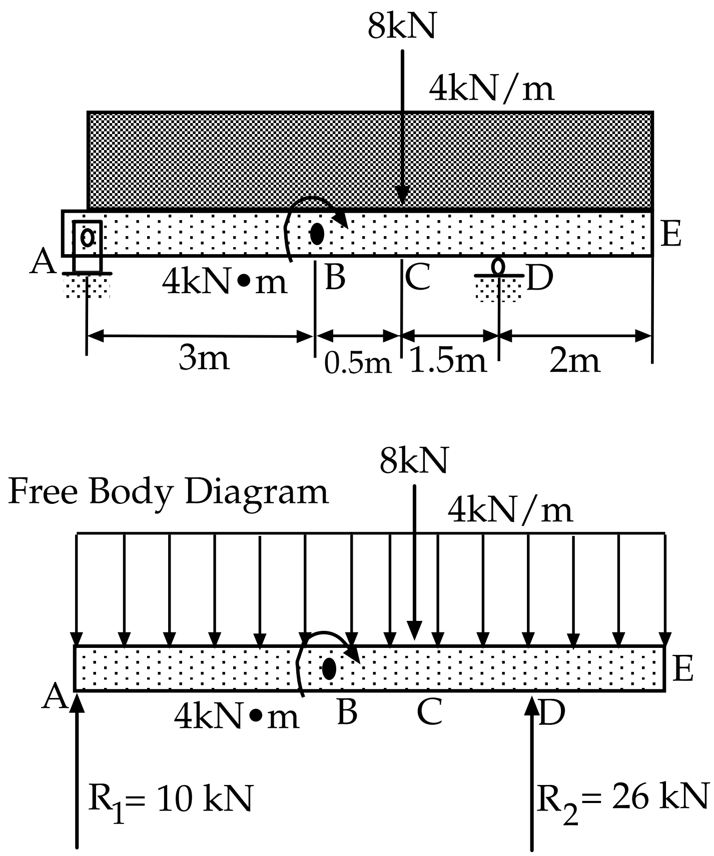


Figure 5: Multiple Loadings

Total Length:

$$L = 3 + \frac{1}{2} + 1\frac{1}{2} + 2 = 7m \tag{13}$$

3.1 Method of Sections

Sections between A and B: $0 \leq x < 3$

$$V_{AB} + R_1 - 4x = 0 \tag{14}$$

$$V_{AB} = 4x - 10 \tag{15}$$

$$M_{AB} - 10x + \int_0^x 4x dx = 0 \tag{16}$$

$$M_{AB} = -\frac{1}{2}4x^2 + 10x \tag{17}$$

$$M_{AB} = -2x^2 + 10x \tag{18}$$

Sections from B to C: $3 \leq x < 3.5$

$$V_{BC} = V_{AB} = 4x - 10 \tag{19}$$

$$M_{BC} - 4 - 10x + \int_0^x 4x dx = 0 \tag{20}$$

$$M_{BC} = -2x^2 + 10x + 4 \tag{21}$$

Sections from C to D: $3.5 \leq x < 5$

$$V_{CD} + 10 - 4x - 8 = 0 \tag{22}$$

$$V_{CD} = 4x - 2 \tag{23}$$

$$M_{CD} - 4 - 10x + 8(x - 3.5) + \int_0^x 4x dx = 0 \tag{24}$$

$$M_{CD} = 4 + 10x - 8x + 28 - 2x^2 \tag{25}$$

$$M_{CD} = -2x^2 + 2x + 32 \tag{26}$$

Sections from D to E: $5 \leq x \leq 7$

$$V_{DE} + 10 + 26 - 8 - 4x = 0 \tag{27}$$

$$V_{DE} = 4x - 28 \tag{28}$$

$$M_{DE} - 4 - 10x + 8(x - 3.5) - 26(x - 5) + \int_0^x 4x dx = 0 \tag{29}$$

$$M_{DE} = -2x^2 + 28x - 98 \tag{30}$$

3.2 Singularity Functions

$$q(x) = 10 < x >^{-1} + 4 < x - 3 >^{-2} - 8 < x - 3.5 >^{-1} + 26 < x - 5 >^{-1} - 4 < x >^0 \tag{31}$$

Integrating to get the shear:

$$\begin{aligned} V(x) &= -\int_{-\infty}^{x} q(x) dx & (32) \\ &= -10 < x >^0 - 4 < x - 3 >^{-1} + 8 < x - 3.5 >^0 - 26 < x - 5 >^0 + 4 < x >^1 + V_0 & (33) \end{aligned}$$

Since we have no unknown shear forces, V_0 is zero, so our shear force equation is:

$$V(x) = -10 < x >^0 - 4 < x - 3 >^{-1} + 8 < x - 3.5 >^0 - 26 < x - 5 >^0 + 4 < x >^1 \tag{34}$$

Plotting the shear force diagram:

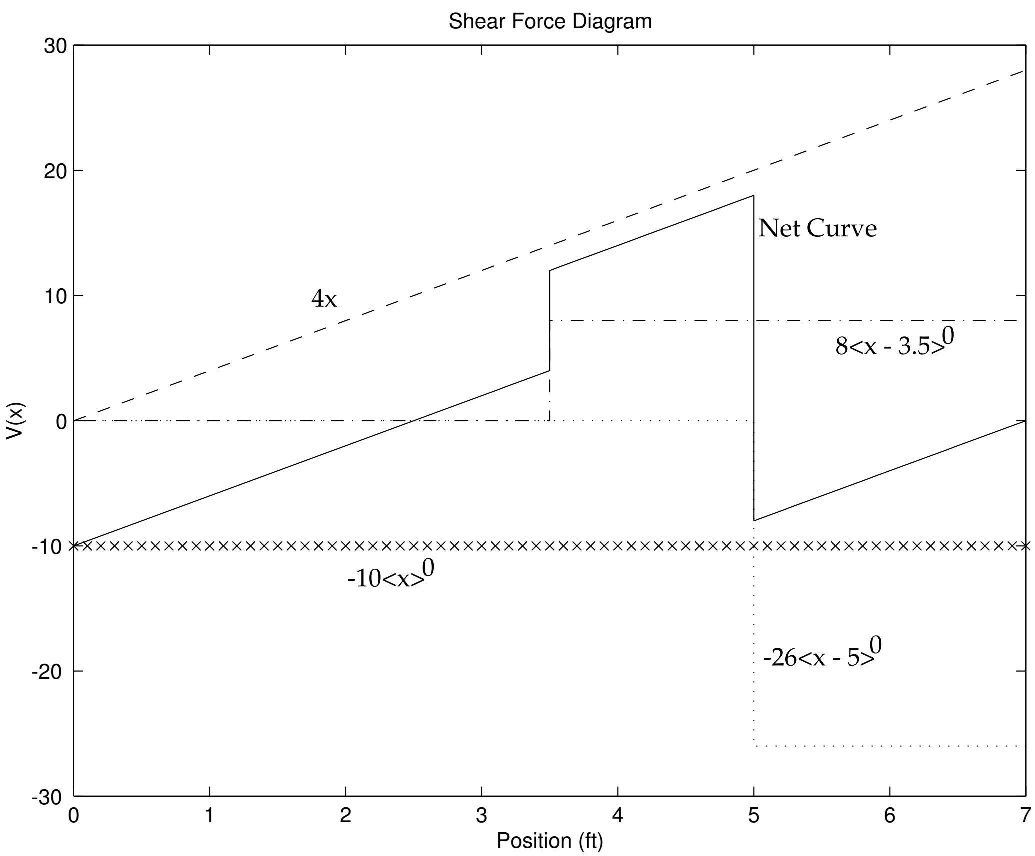


Figure 6: Shear Force Diagram for Example 2

Integrating the shear force equation:

$$M(x) = -\int_{-\infty}^{x} V(x)dx \tag{35}$$

$$= 10 < x >^1 +4 < x-3 >^0 -8 < x-3.5 >^1 +26 < x-5 >^1 -\frac{4}{2} < x >^2 +M_0 \tag{36}$$

We have no unknown moments so, M_0 is zero. Our bending moment equation becomes:

$$M(x) = 10 < x >^1 +4 < x-3 >^0 -8 < x-3.5 >^1 +26 < x-5 >^1 -2 < x >^2 \tag{37}$$

Finally, plotting the bending moment diagram:

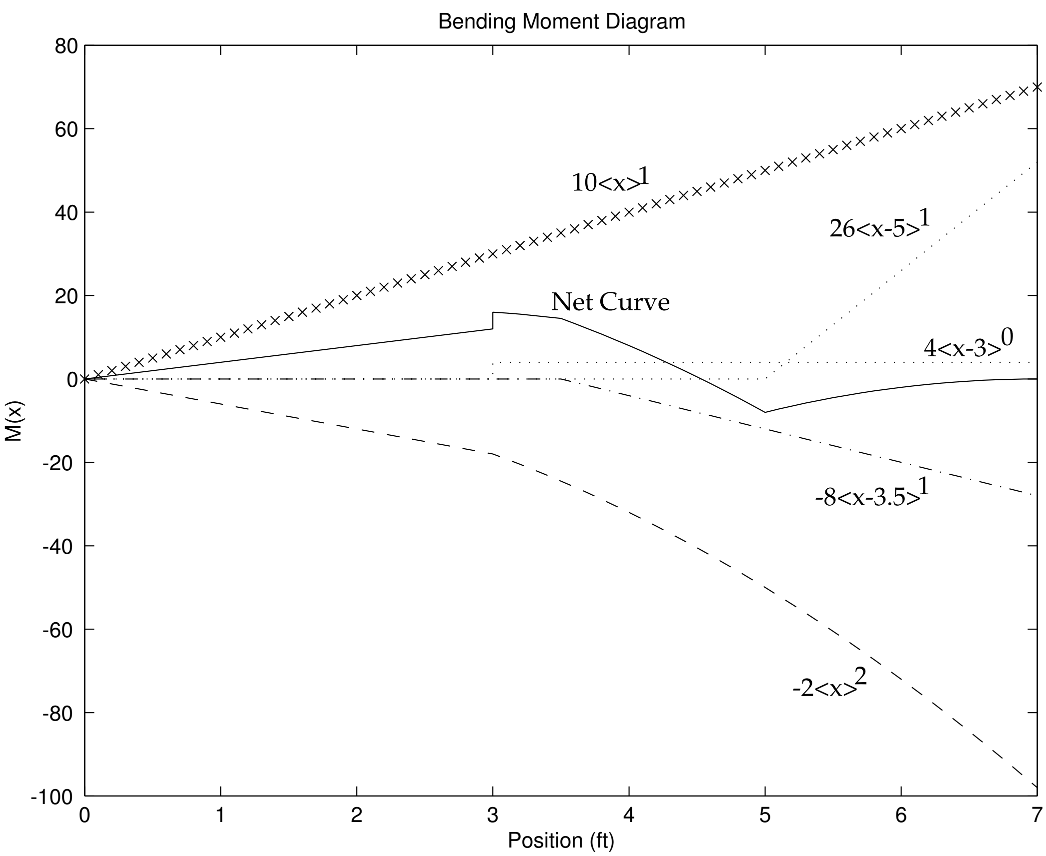


Figure 7: Bending Moment Diagram for Example 2